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Gravitational and cosmological considerations based on the Finsler and Lagrange metric structures

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ABSTRACT

In this work, we study Finslerian and Langrangian structures in Gravitational and Cosmological problems of an anisotropic perfect fluid. The wave vector of light rays is incorporated in the geometrical structure of the tangent bundle of a Finsler-Randers space–time. A Raychaudhuri equation is developed in the framework of a Finslerian tangent bundle in virtue of generalized expansion, shear and vorticity.

Finally, the Einstein field equations and the energy conditions for a perfect fluid are given in our consideration.

1. Introduction

In the field of Applications of Generalized metric structures Finsler, Lagrange, Hamilton (FLH) spaces, significant results from many authors have been given. Among them are Antonelli [1], Asanov [2], Balan [3], Beil [4], Bogoslovsky [5], Brandt [6], Ikeda [7], Miron [8], Stavrinos [9], Vacaru [10] et al. have studied properties F–L spaces in the fields of Ecology, Seismology, Gravitation, Relativity, Gauge theory, String theory and Cosmology. The interest of Applications has been increased during the last years. Applications of F–L spaces have inspired authors from many countries. Quantum Gravity, Violation of Special Relativity, Fermat Principle etc. can be studied in the framework of Finsler Geometry. Finsler, Lagrange and Hamilton geometries constitute a significant branch of metrical differential geometry, extending with its Applications the topics of Theoretical Physics, General Relativity, gravitational waves and Cosmology including in their internal structures anisotropic variables, such as velocity(direction), accelaration, scalar, spinor etc. For a unified geometrical description of the gravitational field the study can be done in the total space of a tangent bundle. Applications of FLH spaces have been developed in the spinor theory by Vacaru and Stavrinos utilizing Clifford Algebras and spinor variables [11]. Further applications to Relativity, Strings Theory and Theoretical Physics can be found in a recent monograph [12]. During the last years, observational investigations of the increased values of anisotropy of microwave cosmic background radiation [13,14] suggest the introduction of an anisotropic metric structure in the underlying geometry of space–time. In some theories of anisotropies the basic philosophy for the study of an anisotropic space–time is supported in perturbations of a spatial homogeneous isotropic universe. These are investigated in terms of small variations of the curvature. As the universe expands the particle energies and momenta will eventually move into a regime where non-trivial interactions take place and the particle distribution anisotropy is then communicated to the space–time geometry. The above mentioned consideration retains the geometrical concepts of the homogeneous isotropic case. A candidate geometry for the study of generalized field equations with respect to the density and pressure of fluids moving in anisotropic gravitational fields is Finsler Geometry. It has a fundamentally different geometrical character for the study of locally anisotropic cosmological phenomena, i.e. it intrinsically incorporates the anisotropy in the geometry of space [9].

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2. Preliminaries

In the following we mention some fundamental geometrical concepts from the theory of Finsler Spaces. We consider a smooth 4-dimensional pseudoriemannian manifold M, (TM, π, M) its tangent bundle and $\tilde{TM} = TM \setminus \{0\}$, where 0 means the image of the null cross-section of the projection $\pi : TM \to M$. We also consider a local system of coordinates (x^i), $i = 0, 1, 2, 3$ and U a chart of M. Then the couple (x^i, y^a) is a local system of coordinates on $\pi^{-1}(U)$ in TM. A coordinate transformation on the total space TM is given by

$$\tilde{x}^i = \tilde{x}^i(x^0, \dots, x^3), \qquad \det\left|\frac{\partial \tilde{x}^i}{\partial x^j}\right| \neq 0,$$
$$\tilde{y}^a = \frac{\partial \tilde{x}^a}{\partial x^b} y^b, \qquad x^a = \delta^a_i x^i. \tag{1}$$

A Finsler metric on M is a function $F : TM \to \mathbb{R}$ having the properties:

1. The restriction of F to $\tilde{TM}$ is of the class C^∞ and F is only continuous on the image of the null cross section in the tangent bundle to M.
2. The restriction of F to $\tilde{TM}$ is positively homogeneous of degree 1 with respect to (y^a).
   $F(x, ky) = kF(x, y), \quad k \in \mathbb{R}^*_+$.
3. The quadratic form on $\mathbb{R}^n$ with the coefficients
   $$f_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} \tag{2}$$
   defined on $\tilde{TM}$ is non degenerate $(\det(f_{ij}) \neq 0)$, with $\mathrm{rank}(f_{ij}) = 4$.

A non linear connection N on TM is a distribution on TM, supplementary to the vertical distribution V on TM:

$$T_{(x,y)}(TM) = N_{(x,y)} \oplus V_{(x,y)}.$$

In our case a non linear connection can be defined by

$$N^a_j = \frac{\partial G^a}{\partial y^j} \tag{3}$$

where G^a are defined from

$$G^a = \frac{1}{4} f^{aj}\left(\frac{\partial^2 F}{\partial y^j \partial x^k} y^k - \partial_j F\right) \tag{4}$$

and the relation

$$\frac{dy^a}{ds} + 2G^a(x, y) = 0 \tag{5}$$

yields from the Euler–Lagrange equations:

$$\frac{d}{ds}\left(\frac{\partial F}{\partial y^a}\right) - \frac{\partial F}{\partial x^a} = 0. \tag{6}$$

The transformation rule of the non-linear connection coefficients is

$$\tilde{N}^a_i = \frac{\partial \tilde{x}^a}{\partial x^b}\frac{\partial x^j}{\partial \tilde{x}^i} N^b_j(x, y) + \frac{\partial \tilde{x}^a}{\partial x^h}\frac{\partial^2 x^h}{\partial \tilde{x}^i \partial \tilde{x}^b} y^b \tag{7}$$

also

$$\frac{\delta}{\delta \tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\delta}{\delta x^j} \qquad \frac{\partial}{\partial \tilde{y}^a} = \frac{\partial x^b}{\partial \tilde{x}^a}\frac{\partial}{\partial y^b}$$
$$d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} dx^j \qquad \delta \tilde{y}^a = \frac{\partial \tilde{x}^a}{\partial x^b}\delta y^b.$$

A local basis of $T_{(x,y)}(TM)$, $(\delta_i, \dot{\partial}_a)$ adapted to the horizontal distribution N is

$$\delta_i = \partial_i - N^a_i(x, y)\dot{\partial}_a, \tag{8}$$

where

$$\partial_i = \frac{\partial}{\partial x^i}, \qquad \dot{\partial}_a = \frac{\partial}{\partial y^a}$$

where $N^a_i(x, y)$ are the coefficients of the non-linear Cartan connection N as we mentioned above.

The concept of non-linear connection is fundamental in the geometry of vector bundles and anisotropic spaces. It is a powerful tool for unification of fields. For example, in the case of gravitational field, the non-linear connection in the framework of tangent bundle unifies the external and internal spaces, i.e. the position space (the base manifold M) with the tangent space T_pM. In other words it is connected with the local anisotropic structure of space–time (depends on the velocities).

The dual local basis is

$$\{d^i = dx^i,\ \delta y^a = dy^a + N_j^a dx^j\}_{i,a=\overline{0,3}}. \tag{9}$$

A d-connection on tangent bundle TM of space–time is a linear connection on TM which preserves by parallelism the horizontal distribution N and the vertical distribution V on TM. A covariant derivation associated to a d-connection becomes d-covariant. In our study, we use the d-connection in order to preserve the horizontal and vertical distribution of the field of anisotropy with respect to the anisotropic axis.

Generally an h-v metric on the tangent bundle (TM, π, M) is given by

$$G = f_{ij}(x, y)dx^i \otimes dx^j + h_{ab}\delta y^a \otimes \delta y^b. \tag{10}$$

We consider a metrical d-connection $C\Gamma = (N_j^a, L_{jk}^i, C_{jk}^i)$ with the property

$$f_{ij|k} = \delta_k f_{ij} - L_{ik}^h f_{hj} - L_{jk}^h f_{ih} = 0 \tag{11}$$

$$f_{ij}|_k = \dot{\partial}_k f_{ij} - C_{ik}^h f_{hj} - C_{jk}^h f_{ih} = 0 \tag{12}$$

where

$$L_{jk}^i = \frac{1}{2} f^{ir}\left(\delta_j f_{rk} + \delta_k f_{jr} - \delta_r f_{jk}\right) \tag{13}$$

$$C_{jk}^i = \frac{1}{2} f^{ir}(\dot{\partial}_j f_{rk} + \dot{\partial}_k f_{jr} - \dot{\partial}_r f_{jk}). \tag{14}$$

The coordinate transformation of the objects L_{jk}^i and C_{jk}^i is:

$$\bar{L}_{jk}^i = \frac{\partial \bar{x}^i}{\partial x^h}\frac{\partial x^l}{\partial \bar{x}^j}\frac{\partial x^r}{\partial \bar{x}^k} L_{lr}^h(x, y) + \frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial^2 x^r}{\partial \bar{x}^j \partial \bar{x}^k} \tag{15}$$

$$\bar{C}_{jk}^i = \frac{\partial \bar{x}^i}{\partial x^h}\frac{\partial x^l}{\partial \bar{x}^j}\frac{\partial x^r}{\partial \bar{x}^k} C_{lr}^h(x, y). \tag{16}$$

The Cartan torsion coefficients C_{ijk} are given by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_k f_{ij} \tag{17}$$

while the Christoffel symbols of the first and second kind for the metric f_{ij} are respectively:

$$\gamma_{ijk} = \frac{1}{2}\left(\frac{\partial f_{kj}}{\partial x^i} + \frac{\partial f_{ik}}{\partial x^j} - \frac{\partial f_{ij}}{\partial x^k}\right) \tag{18}$$

$$\gamma_{ij}^l = \frac{1}{2} f^{lk}\left(\frac{\partial f_{kj}}{\partial x^i} + \frac{\partial f_{ik}}{\partial x^j} - \frac{\partial f_{ij}}{\partial x^k}\right). \tag{19}$$

The torsions and curvatures which we use are given by

$$T_{kj}^i = 0 \qquad S_{kj}^i = 0 \tag{20}$$

$$R_{jk}^i = \delta_k N_j^i - \delta_j N_k^i \qquad P_{jk}^i = \dot{\partial}_k N_j^i - L_{kj}^i \tag{21}$$

$$P_{jk}^i = f^{im} P_{mjk} \qquad P_{ijk} = C_{ijk|l}\, y^l \tag{22}$$

$$R_{jkl}^i = \delta_l L_{jk}^i + \delta_k L_{jl}^i + L_{jk}^h L_{hl}^i - L_{jl}^h L_{hk}^i + C_{jc}^i R_{kl}^c \tag{23}$$

$$S_{jikh} = C_{iks} C_{jh}^s - C_{ihs} C_{jk}^s \tag{24}$$

$$P_{ihkj} = C_{ijk|h} - C_{hjk|i} + C_{hj}^r C_{rik|l}\, y^l - C_{ij}^r C_{rkh|l}\, y^l \tag{25}$$

$$S_{ikh}^l = f^{lj} S_{jikh} \tag{26}$$

$$P_{ikh}^l = f^{lj} P_{jikh}. \tag{27}$$

The Ricci identities are

$$X^{i}{}_{|k|h} - X^{i}{}_{|h|k} = X^{r} R_{r}{}^{i}{}_{kh} - X^{i}{}_{|r} R^{r}{}_{kh} \tag{28}$$

$$X^{i}{}_{|k|h} - X^{i}{}_{|h|k} = X^{r} P_{r}{}^{i}{}_{kh} - X^{i}{}_{|r} C^{r}_{kh} - X^{i}{}_{|r} P^{r}{}_{kh} \tag{29}$$

$$X^{i}|_{k}|_{h} - X^{i}|_{h}|_{k} = X^{r} S_{r}{}^{i}{}_{kh}. \tag{30}$$

3. Finsler geometry associated with the gravity

As is well known in general relativity, Einstein used the principle of equivalence as the basis for a geometrical description of gravity. In the four-dimensional world space–time the trajectory of a particle falling freely in a gravitational field is a certain fixed curve. Its direction at any point depends on the velocity of the particle. The equivalence principle implies that there is a preferred set of curves in space–time: at any point, pick any direction and there is a unique curve in that direction that will be the trajectory of any particle starting with that velocity.

On the other hand, in the Finslerian framework the gravitational field is defined in a space that is spanned by the vectors y, with $y^i = \frac{dx^i}{dt}$ $(y^i/i = 0, 1, 2, 3)$ which is attached to each point x $(x^k/k = 0, 1, 2, 3)$ as an independent internal variable. Namely the independent variables become (x^k, y^i). In the terms of E. Cartan the couple (x^k, v^i) (position-velocity or direction) is called an *element of support*.

The intrinsic behavior of the internal variable y $(y = v = \frac{dx}{dt})$ of the field in a space–time can be considered as a property of the field itself. Consequently it is plausible to consider a Finsler space as the basic structure for the study of the gravitational field. This y-dependence has been combined with the concept of *anisotropy*.

The motion of a particle in a Finslerian space–time F_4 is described by a pair (x, V) where $x \in F_4$ and $V = \frac{dx}{d\tau}$ is the 4-velocity of the particle (τ is proper time) which represents the tangent of its world-line expressing the motion of typical observers in the Finslerian anisotropic universe.

Along this line the Finslerian gravitational field may be regarded as a gravitational field spanned by the line-elements (x, y) over the total space of the tangent bundle. This standpoint constitutes a unified description of field between the external x-field spanned by points x (e.g. Riemannian description of the gravitational field) and the internal y-field spanned by vectors y. Such a model fulfils the assumptions of a geometrical interpretation of anisotropic distribution of matter.

In a graviational theory of a Finslerian tangent bundle curvature effects can be considered as tidal forces. Possible modifications of them are ought to the non-linear character.

Suppose $(F_4, f_{ij}(x, y))$ is a four dimensional differentiable manifold and $f_{ij}(x, y)$ the anisotropic Finslerian metric is assumed to have the signature $(+, -, -, -)$ for any (x, y). The square of the length of an arbitrary contravariant vector X^i is to be defined by the quadratic form

$$|X|^2 = f_{ij}(x, y) X^i X^j.$$

The time-like, null and space-like curves can be defined in the Finslerian framework by the following relations.

time-like $\quad f_{ij}(x, y) V^i V^j > 0$
null-like $\quad f_{ij}(x, y) V^i V^j = 0$
space-like $\quad f_{ij}(x, y) V^i V^j < 0.$

Finslerian geodesics satisfy the Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial F(x, y)}{\partial y^i}\right) - \frac{\partial F(x, y)}{\partial x^i} = 0. \tag{31}$$

This is the equation of Finslerian geodesics associated with the Lagrangian $F(x, y)$. In the case where the vector $y^i = \dot{x}^i = \frac{dx^i}{ds}$ has unit Finslerian length, then the Finslerian geodesics take on the form

$$\frac{d^2 x^i}{ds^2} + \gamma^i_{jk}(x, \dot{x}) \dot{x}^j \dot{x}^k = 0 \tag{32}$$

where $\gamma^i_{jk}(x, \dot{x})$ are the Finslerian Christofell symbols. This equation shows the dependence of geodesic equation on velocity, which is incorporated in the principle of equivalence. This approach extends the Riemannian general relativity without any contradiction with its basic principles.

The above mentioned consideration reveals a profound relation between the principle of equivalence of the Riemannian prototype and the space–time paths (geodesics) in Finslerian space–time.

If we consider a Finsler-Randers (F–R) type of 4-dimensional space, where the electromagnetic field is included, we obtain the generalized form of geodesics equation, where the electromagnetic field is intrinsically incorporated in the geometry. The equation of geodesics satisfies the Lorentz equation and the curvature tensor of space is then written as

$$H^i_{hjk}(x, y) = R^i_{hjk} + E^i_{hjk} \tag{33}$$

where R^i_{hjk} represents the Riemannian curvature and E^i_{hjk} the "electromagnetic curvature".

In many cases we consider a convenient Finsler metric for the research on gravitation theories. A metric which is connected with a Riemannian one, $a_{ij}(x)$, is called osculating Riemannian metric and is defined over a region U of F_n

$$a_{ij}(x) = f_{ij}(x, y(x)). \tag{34}$$

A generalized Finslerian metric tensor can be defined, as well, by the underlying Riemannian metric

$$f_{ij}(x, v) = g_{ij}(x) + \frac{1}{2}\frac{\partial f_{ij}}{\partial v^k}v^k + O(v^2) \tag{35}$$

where $g_{ij}(x)$ is the Riemannian metric and $y^i(x) = v^i(x) = \frac{dx^i}{ds}$. We can take into account the anisotropic *torsion tensor of Cartan*, $C_{ijk}(x) = \frac{1}{2}\frac{\partial f_{ij}}{\partial v_k}$. Then, the equation of geodesics becomes

$$\frac{dv^i}{ds} + a^i_{kj}(x)v^k v^j = 0 \tag{36}$$

where

$$a^i_{kj}(x) = \gamma^i_{jk}(x, v) + C^i_{kl}\frac{\partial \xi^l}{\partial x_k} + C^i_{jl}\frac{\partial \xi^l}{\partial x_j} - f^{ih}(x, v)C_{jkl}(x, \xi)\frac{\partial \xi^l}{\partial x^h}. \tag{37}$$

Osculating Riemannian metrics are useful for studing problems in the framework of Finslerian Relativity [2,9].

A weak metric can be introduced into studying gravitational waves in the Finslerian space–time by assuming a weak gravitational field in the Riemannian framework, where the weak Riemannian metric is analyzed in the form

$$g_{ij}(x) = \eta_{ij} + h_{ij}, \quad |h_{ij}| \ll 1. \tag{38}$$

By inserting (38) in (35) we have a Finslerian weak metric

$$f_{ij}(x, v) = (\eta_{ij} + h_{ij}) + C_{ijk}v^k + O(v^2).$$

Weak gravitational models of this type have been studied [15]. This analysis can be used for the anisotropic description of the polarization of waves. It is closely connected to the equation of deviation of geodesics to detect gravitational waves.

In order to study the weak field of a F–R space we relate it to the deviation of two moving charged particles. It is necessary to take into account the relation

$$\frac{\delta^2 z^i}{\delta u^2} + E^i_{jkl}v^j v^k z^l = 0 \tag{39}$$

where z^i expresses the deviation vector between two charged particles and v^k their 4-velocity. The operator $\frac{\delta}{\delta u}$ represents the covariant derivative in the framework of Finsler geometry. This is reasonable since in order to detect a gravitational wave at least two particles are needed. So the deviation of geodesics of the weak F–R space is written in the form

$$\frac{D^2\xi^i}{d\tau^2} + (\epsilon^i_{ljm} + h^i_{ljm})\xi^j\frac{dx^l}{d\tau}\frac{dx^m}{d\tau} = 0 \tag{40}$$

where ϵ^i_{ljm} expresses the linearized Riemannian curvature tensor and h^i_{ljm} are functions of the electromagnetic field.

The relation (40) in a first approximation of h_{ij} takes the form

$$\frac{\partial^2\xi^i}{\partial\tau^2} = -\left(\frac{1}{2}\frac{\partial^2\epsilon^i_j}{\partial t^2} + 2F^i_0F^0_j + u_j\frac{\partial F^i_0}{\partial\tau}\right)\xi^j. \tag{41}$$

The Eq. (41) coincides with the corresponding equation for a weak field of the Riemannian space–time. The difference between these two is that in the Riemannian case the electromagnetic field has been introduced ad hoc. In Eq. (39) the electromagnetic field is incorporated in the geometry and the two particles are moved in geodesics (potential lines) of the Finsler space. In virtue of Eq. (41) their relative acceleration is governed by the curvature of the electromagnetic field that is produced by one part of the energy–momentum tensor.

4. Finslerian cosmological models of anisotropic geometrical field. Propagation of light rays

Some applications of Finsler spaces to Cosmology started in 2004 [16]. The motivation to this direction arises by some observational astrophysical results concerning the degree of anisotropy of Cosmic Background Radiation (CMB). Anisotropic direction dependent expansion may be present if the underling geometry of the universe is anisotropic. In this case the isotropic Robertson–Walker metric is no longer valid [14], or if anisotropic non gravitational forces are present such as large scale primordial magnetic field. Finslerian or Lagrangian geometrical models of the form

$$L(x, y) = \sqrt{a_{ij}(x)y^i y^j} + \phi(x)\hat{\ell}_\alpha y^\alpha \tag{42}$$

may express a generalized geometric model for the anisotropic structure of Universe. In our situation the replacement of a vector y^a by $-y^a$ is not valid.

However, a Finslerian geometrical structure of models can correspond to anisotropic structures of space–time regions (radius less or equal 10^8 light years).The function $\phi(x) \in \mathbb{R}$ plays the role of "length", $\hat{\ell}_\alpha$ is the unit vector of the axis of anisotropy. In this model the equation of geodesics is given by(*rotational geodesics*)

$$\frac{d^2x^l}{ds^2} + \overset{(a)}{\Gamma^l}_{ij}y^iy^j + \sigma a^{lm}\left((\partial_j\phi)\hat{\ell}_m - (\partial_m\phi)\hat{\ell}_j\right)y^j = 0. \tag{43}$$

A generalized cosmological Friedman – like Robertson–Walker anisotropic model can be derived by (42) if we consider the Riemannian part in the form

$$\sigma(x, y) = \left(\alpha_{\kappa\ell}(x)y^\kappa y^\ell\right)^{1/2} \tag{44}$$

with

$$\alpha_{\kappa\ell}(x) = \operatorname{diag}\left(1, -\frac{\alpha^2}{1-\kappa r^2}, -\alpha^2r^2, -\alpha^2r^2\sin^2\theta\right) \tag{45}$$

represents the Robertson–Walker metric. This form of metric has been studied in relation with the anisotropy of a primeordial magnetic field and the microwave cosmic radiation (CMB) in [9]. In the case of electromagnetic waves the direction of propagation of light rays is determined by the wave vector tangent to the ray. The geodesic equation of propagation has the form

$$\frac{dk^l}{d\lambda} + \overset{(a)}{\Gamma^l}_{ij}k^ik^j + \sigma a^{lm}(\partial_j\phi\hat{k}_m - \partial_m\phi\hat{k}_j)k^j = 0 \tag{46}$$

where λ is an affine parameter, $\overset{(a)}{\Gamma^l}_{ij}$ are the Christofel symbols of the Riemannian space and k^i a wave vector. The frequence ω of the wave in a Finslerian space–time can be defined by $\omega = u^ik_i$, where $u^i = \frac{dx^i}{d\tau}$ is the 4-velocity for an observer with proper time τ moving in a world-line $x^i(\tau)$. The integral curves $x^i(\lambda)$ of the vector k^i defined by $\frac{dx^i}{d\lambda} = k^i$ are called light rays. In the natural lift to TM the form of light rays is $\tilde{C}(\lambda) = \left(x^i(\lambda), k^i\right)$.

The Christoffel symbols correspond to the Riemann metric $a_{ij}(x)$ and the additional term

$$\sigma a^{lm}\left((\partial_j\phi)\hat{\ell}_m - (\partial_m\phi)\hat{\ell}_j\right)y^j \tag{47}$$

expresses a rotation of anisotropy direction(axis). Also a physical interpretation of this type of rotation can be given in relation with spin densities of the angular momenta of galaxies. The non-linear connection in this model has the form

$$N^l_k(x, y) = \overset{(a)}{\Gamma^l}_{jk}(x)y^j + \sigma a^{ml}\mathrm{A}_{km}\left[\partial_k(\phi(x))\hat{\ell}_m\right] + \frac{1}{\sigma}a^{ml}y^j\mathrm{A}_{jm}\left[(\partial_j\phi(x))\hat{\ell}_m\right]y_k. \tag{48}$$

The wave vector k^l of light rays has the form

$$\frac{dk^l}{d\lambda} + N^l_\ell(x, k)k^\ell = 0 \tag{49}$$

where N^l_ℓ is given by (48). The invariant equivalent form of (49) is

$$\nabla k^i = 0. \tag{50}$$

The S-curvature can be computed by the formula

$$S_{ih} = -\frac{3\left(m\sigma^2\phi^2 - \beta^2\phi^2\right)}{4\mathrm{F}^2\sigma^2}a_{ih} - \frac{\phi^2}{4\mathrm{F}^2} + \frac{\beta\phi^2}{2\mathrm{F}^2\sigma^2}\mathrm{S}_{ih}(\hat{k}_iy_h) + \frac{3m\sigma^2\phi^2 - 4\beta^2}{4\mathrm{F}^2\sigma^2} \tag{51}$$

where

$$m = \hat{\ell}_\alpha\hat{\ell}^\alpha = 0, \pm 1 \qquad \sigma^2 = a_{ij}y^iy^j \qquad \beta = \hat{\ell}_\alpha y^\alpha. \tag{52}$$

The S-curvature can be considered as a curvature parameter of anisotropy since $S = 0 \Longleftrightarrow \phi = 0$.

For the case where our model includes an electromagnetic field with electromagnetic potential, the Lagrangian can be modified to the form

$$\mathrm{L}(x, y) = \sqrt{a_{ij}(x)y^iy^j} + \phi(x)\hat{\ell}_\alpha y^\alpha + KA_\alpha y^\alpha \tag{53}$$

which produces the equation of motion

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}y^jy^k + \left[a^{im}\left((\partial_j\phi)\hat{\ell}_m - (\partial_m\phi)\hat{\ell}_j\right) - KF^i_j\right]y^j = 0. \tag{54}$$

In this case $KF^i_jy^j$ governs the equation of motion of rotation geodesics. The constant $K \to \frac{q}{mc^2}$ can be associated with specific charged particles. The geodesic equation of a wave vector in this case is analogous to the (46). In the tangent bundle of a general Finsler space the wave vector k^i for light rays will satisfy the equation

$$\frac{dk^i}{d\lambda} + \Gamma^i_{j\ell}(x,k)k^jk^\ell + C^i_{j\ell}(x,k)k^j\frac{dk^\ell}{d\lambda} = 0 \tag{55}$$

where $\Gamma^i_{j\ell}$, $C^i_{j\ell}$ represent the Cartan connection coefficienes. By the relations (42) and (43) the null-condition of light rays is provided because of the nullity of the second term (46). In the framework of a curved space–time for light rays, locally plane waves are presented by approximate solutions of Maxwel's equations [17]. In the form of the Finslerian ansatz these remain invariant [16] and they have same form as in classical case.

5. Modified connection structures

In the total space, the adapted frame is set as (8).

The intrinsic connection δy represents the intrinsic behavior of the internal variable y: For example, if the intrinsic behavior of y is related to $\bar{y}^\alpha = K^\alpha_j(x)y^j$, $K^\alpha_j(x)$ being the rotation matrix, then δy is given by

$$\begin{aligned} &\delta y^\alpha = dy^\alpha + N^\alpha_\lambda dx^\lambda (= 0) \\ &dy^\alpha = \bar{y}^\alpha - K^\alpha_j(0)y^j \\ &N^\alpha_\lambda = -\frac{\partial K^\alpha_j(x)}{\partial x^\lambda}y^j \end{aligned} \tag{56}$$

where we have put $K^\alpha_j(x) = K^\alpha_j(0) + \frac{\partial K^\alpha_j(x)}{\partial x^\lambda}dx^\lambda$.

The intrinsic parallelism of δy^i produces the non-linear connection [18].

From a physical point of view we may apply the above mentioned form of non-linear connection in the framework of the observed anisotropy of cosmic background radiation (CMB). As it has been studied in a Finslerian ansantz (42), it can be represented by a vector $\hat{\ell}^i(x)$, which is incorporated in a metric model of of Finsler geometry, as a result of our motion with respect to some local frame in the universe. The rotation of a vector ℓ^α from anisotropy axis ℓ^β is given in virtue of the rotation group $\Lambda^\alpha_\beta(x)$. In this case the non-linear connection

$$N^\alpha_\lambda = -\frac{\partial \Lambda^\alpha_\beta(x)}{\partial x^\lambda}\ell^\beta \tag{57}$$

expresses the variation of the rotation group with respect to the anisotropy axis [16]. The rotation group Λ^0_0 can be used instead of the function $\phi(x)$ in the right-hand side of (42), giving a profound geometrical and physical meaning to the concept of non-linear connection. Then, if we choose $N^\alpha_\lambda = \frac{\partial\phi(x)}{\partial x^\lambda}\ell^\alpha$, we can see that the non-linear connection expresses the variation of spin density.

In addition the Berwald type form of the non-linear connection is given by

$$N^\alpha_{\lambda\kappa} = \frac{\partial N^\alpha_\lambda}{\partial \ell^\kappa} = -\frac{\partial \Lambda^\alpha_\kappa(x)}{\partial x^\lambda}. \tag{58}$$

The tensor field

$$\Lambda^\alpha_{\kappa\lambda} = N^\alpha_{\kappa\lambda} - N^\alpha_{\lambda\kappa} = \frac{\partial \Lambda^\alpha_\kappa}{\partial x^\lambda} - \frac{\partial \Lambda^\alpha_\lambda}{\partial x^\kappa} \tag{59}$$

denotes a torsion of variation of the rotation group Λ^α_κ.

The adapted frame (8) can be modified in the form,

$$\begin{cases} dx^A = (\delta x^k = dx^k + \Pi^k_i\delta y^i, \delta y^i = dy^i + N^i_\lambda dx^\lambda) \\ \dfrac{\partial}{\partial x^A} = \left(\dfrac{\delta}{\delta x^\lambda} = \dfrac{\partial}{\partial x^\lambda} - N^i_\lambda\dfrac{\partial}{\partial y^i}, \dfrac{\delta}{\delta y^i} = \dfrac{\partial}{\partial y^i} - \Pi^\lambda_i\dfrac{\delta}{\delta x^\lambda}\right) \end{cases} \tag{60}$$

where we have used $A = (k, i) = (1, 2 \cdots 8)$ for x and y fields and Π^λ_i another non linear connection.

On the basis of the adapted frame (8), the Finslerian connection structure is introduced by

$$\begin{aligned} &\nabla_{\frac{\partial}{\partial x^C}}\frac{\partial}{\partial x^B} = \Gamma^A_{BC}\frac{\partial}{\partial x^A} \\ &\Gamma^A_{BC} = (L^k_{\lambda\mu}, L^i_{j\mu}, C^k_{\lambda l}, C^i_{jl}). \end{aligned} \tag{61}$$

Namely, four connection coefficients appear. Then, the following four kinds of covariant derivatives can be defined, for an arbitrary vector $V^A = (V^k, V^i)$:

$$\begin{cases} V^k{}_{|\mu} = \dfrac{\delta V^k}{\delta x^\mu} + L^k_{\lambda\mu} V^\lambda \\ V^k|_l = \dfrac{\partial V^k}{\partial y^l} + C^k_{\lambda l} V^\lambda \\ V^i{}_{|\mu} = \dfrac{\delta V^k}{\delta x^\mu} + L^i_{j\mu} V^j \\ V^i|_l = \dfrac{\delta V^i}{\delta y^l} + C^i_{jl} V^j. \end{cases} \tag{62}$$

We shall first consider the case where (x^k, y^i) are changed to (x^k, y^0), y^0 being a scalar. Then, the adapted frame is reduced to

$$\begin{cases} dx^A = (dx^k, \delta y^0 = dy^0 + N^0_\lambda dx^\lambda) \\ \dfrac{\partial}{\partial x^A} = \left(\dfrac{\delta}{\delta x^\lambda} = \dfrac{\partial}{\partial x^\lambda} - N^0_\lambda \dfrac{\partial}{\partial y^0}, \dfrac{\partial}{\partial y^0} \right). \end{cases} \tag{63}$$

In this case, the connection structure and the metrical structure are given by

$$\Gamma^A_{BC} = (L^k_{\lambda\mu}, L^0_{0\mu}, C^k_{\lambda 0}, C^0_{00}) \tag{64}$$

and

$$G = g_{\lambda k} dx^k \otimes dx^\lambda + g_{00} \delta y^0 \otimes \delta y^0 \tag{65}$$

respectively.

If we choose the non-linear connection in the form $N^0_\lambda = -\frac{\partial A^0_0}{\partial x^\lambda} y^0$ with $A^0_0(x) = \phi(x)$ and $y^0 = a(t)$, then we get

$$N^0_\lambda = -\frac{\partial \phi(x)}{\partial x^\lambda} y^0 = -\phi_{,\lambda}(x) y^0 = -\phi_{,\lambda}(x) a(t)$$

that means that N^0_λ takes the form of a covector caused by the fluctuations of material fields, where $\alpha(t)$ denotes the scale factor.

If y^0 is constant, then $dy^0 = 0$, $\delta y^0 = N^0_\lambda dx^\lambda$ and $\frac{\delta}{\delta\lambda} = \frac{\partial}{\partial x^\lambda}$, $\frac{\partial}{\partial y^0} = 0$. Therefore $\Gamma^A_{BC} = (L^k_{\lambda\mu}, L^0_{0\mu}, C^k_{\lambda 0} = 0, C^0_{00} = 0)$. ($g_{00}(x^k, y^0)$ is not a constant, in general. If g_{00} is a constant, then $L^0_{0\mu} = 0$).

Next, we shall take up the case where (x^k, y^i) are changed to (x^0, y^i), x^0 being the time axis. This case is dual to the above mentioned case. The adapted frame becomes

$$\begin{cases} dx^A = (dx^0, \delta y^i = dy^i + N^i_0 dx^0) \\ \dfrac{\partial}{\partial x^A} = \left(\dfrac{\delta}{\delta x^0} = \dfrac{\partial}{\partial x^0} - N^i_0 \dfrac{\partial}{\partial y^i}, \dfrac{\partial}{\partial y^i} \right). \end{cases} \tag{66}$$

The connection structure and the metrical structure are given by, respectively,

$$\Gamma^A_{BC} = (L^0_{0\mu}, L^i_{j0}, C^0_{0k}, C^i_{jk}) \tag{67}$$

and

$$G = g_{00} dx^0 \otimes dx^0 + g_{ij} \delta y^i \otimes \delta y^j. \tag{68}$$

6. Finslerian congruences of anisotropic flows and Raychaudhuri equation

In the following we shall consider the concept of expansion, shear and vorticity of time-like flows as these are defined in the Finslerian context and we shall use them to derive the Finslerian Raychaudhuri equation. This equation plays an important role in the Riemannian prototype of general relativity [19]. In virtue of Raychaudhuri equation the concept of deviation of geodesics is extended.

A *smooth congruence* in an open coordinate neighborhood U of F^4 can be represented by a preferred family of world lines (time-like curves) such that through each couple $(x, V) \in U$ there passes precisely one curve in this family in which V is the tangent vector of this curve to that point x. This consideration is analogous to the Riemannian context.

The metric of Finslerian space–time is described by the relation

$$ds^2 = F^2(x, y) = f_{ij} y^i y^j.$$

In the following we assume Finslerian fluid congruences that the matter flow lines of the fluid are time-like geodesics and are parameterized by the proper time τ so that a vector field $V^i(x)$ of tangents is normalized to the unit length $V^i = \frac{y^i}{F}$. We denote by $A_{ik} = V_{i;k}$ the Finslerian δ-covariant derivative with respect to the direction of $V(x)$.

We notice that A_{ik} belongs to the normal subspace of the tangent space

$$A_{ik}V^k = 0, \quad A_{ik}V^i = 0. \tag{69}$$

These are followed because of geodesic condition and the relation of normalization that means A_{ik} is a "spatial" vector.

A physical and geometrical interpretation can be given if we consider a smooth one parameter $C_s(\tau)$ congruence of Finslerian geodesics. Because of the equation of geodesics deviation, the deviation vector z^i provide us the separation from a geodesics C_0 to a nearby one of the family.

From the condition

$$\mathrm{L}_V z^i = 0 \tag{70}$$

we get

$$z^i_{;\ell}V^\ell = V^i_{;\ell}z^\ell - \frac{\partial z^i}{\partial y^n}(V^n_{;\ell}y^\ell) = A^i_\ell z^\ell - \frac{\partial z^i}{\partial y^n}A^n_\ell y^\ell \tag{71}$$

where L represents the Finslerian Lie variation.

The tensor field A^j_i measures the change of z^i to be parallel -transported along- a Finslerian stream line. From a physical point of view an observer moving along the geodesic C_0 would find the adjacent geodesics surrounding him to be stretched and rotated by the field A^j_i. We write down the angular metric h_{ij} in the Finslerian framework

$$h_{ij} = f_{ij} - V_iV_j$$

where V^i is the unit tangent vector. This tensor has the property

$$h_{ij}V^i = 0. \tag{72}$$

Using the δ-differentiation in the direction of $V^i(x)$ for a congruence of fluid lines (not necessarily geodesics) are defined by the expansion, vorticity and the shear [2] by the following forms:

$$\widetilde{\theta} = A_{ij}h^{ij} = V^i_{|i} - C^i_{im}\dot{V}^m \tag{73}$$

$$\tilde{\omega}_{ik} = A_{[ik]} + \dot{V}_iV_k - \dot{V}_kV_i \tag{74}$$

$$\tilde{\sigma}_{ik} = A_{(ik)} - \frac{1}{3}\widetilde{\theta}h_{ik} - 2C_{ikm}V^m - \dot{V}_iV_k - \dot{V}_kV_i \tag{75}$$

where $\dot{V}^i = V^i_{|k}V^k = A^i_kV^k$ and " " denotes the Riemannian covariant derivative associated with the osculating Riemannian metric $a_{ij}(x) = g_{ij}(x, V(x))$. The symbols "[ ]", "( )" denote the antisymmetrization and symmetrization of A_{ik} respectively. The tensor $C_{ijk} = \frac{1}{2}\frac{\partial f_{ij}(x,y)}{\partial y^k}$ is symmetric in all its subscripts. Therefore the first extended Finslerian covariant derivative of V can be expressed by

$$A_{ik} = \frac{1}{3}\widetilde{\theta}h_{ik} + \tilde{\sigma}_{ik} + \tilde{\omega}_{ik} + \dot{V}_iV_k. \tag{76}$$

The proper time derivative of any tensor T^{ij}_{kl} along the fluid flow lines can be given by

$$\dot{T}^{ij}_{kl} = T^{ij}_{kl;m}V^m.$$

Remark. The consideration of a Finslerian incoherent fluid provides that the fluid lines are geodesics and $\dot{V}^i = A^i_kV^k = 0$. In this case the Finslerian geodesics coincide with the Riemannian ones of a V-Riemannian space (osculating Riemannian).

$$\frac{d\widetilde{\theta}}{d\tau} = -\frac{1}{3}\widetilde{\theta}^2 - \tilde{\sigma}_{ik}\tilde{\sigma}^{ik} + \tilde{\omega}_{ik}\tilde{\omega}^{ik} - L_{i\ell}V^iV^\ell + \dot{V}^i_{;i}. \tag{77}$$

This is *Raychaudhuri's* equation of the Finslerian space–time [20]. In the Riemannian approach of general relativity this equation plays a crucial role in the theorems of singularities. The variation of expansion which is expressed by $\frac{d\widetilde{\theta}}{d\tau}$ depends *on the V-anisotropic behavior of Cartan tensor C^i_{jk} along the matter flow lines.*

By a physical viewpoint the anisotropic Cartan tensor is introduced in the geometry of space–time because of a primordial vector field.

In the case that $\dot{\widetilde{\theta}} = 0$, $\tilde{\sigma}_{ij} = 0$, $\tilde{\omega}_{ij} = const$ from the relation (77) the tidal field $L_{i\ell}V^iV^\ell$ is due to the vorticity $\tilde{\omega}$ which play the role of vacuum energy (cosmological constant). It is analogous to a centrifugal field of the Newtonian theory. It counterbalances the tidal field.

When we consider an incoherent fluid, the fluid-lines are geodesics and the last term of right hand side of (77) is $\dot{V}^i = 0$. In this case the Raychaudhuri equation is reduced to the form of a V-Riemannian metric space associated with the congruence of geodesics.

A perfect fluid in the Finslerian space time case has the form

$$T_{ij}(x, V(x)) = (\mu + p)V_i(x)V_j(x) + pa_{ij} \tag{78}$$

where $p = p(x)$, $\mu = \mu(x)$ represent the pressure and the density of the fluid respectively.

The Einstein equations can be written in the form

$$L_{ij}(x, V(x)) = K\left(T_{ij}(x, V(x)) - \frac{1}{2}T^k_k a_{ij}\right), \quad K : \text{constant} \tag{79}$$

where the Ricci tensor L_{ij} is directly determined by the matter energy–momentum tensor T_{ij} at each point, associated with the osculating Riemannian metric tensor $a_{ij}(x) = g_{ij}(x, V(x))$. Substitution of (78) to (79) gives

$$L_{ij}V^iV^j = \frac{1}{2}K(\mu + 3p). \tag{80}$$

The term $L_{i\ell}V^iV^\ell$ corresponds to an anisotropic gravitational influence of the matter along the world lines of the fluid and it expresses the tidal force of the field.

The form of Raychaudhuri equation in the case of perfect fluids is given by

$$\dot{\tilde{\theta}} = \frac{d\tilde{\theta}}{d\tau} = -\frac{1}{3}\tilde{\theta}^2 - \tilde{\sigma}_{ik}\tilde{\sigma}^{ik} + \tilde{\omega}_{ik}\tilde{\omega}^{ik} - \frac{1}{2}K(\mu + 3p) + \dot{V}^i_{;i}. \tag{81}$$

The condition

$$L_{i\ell}V^iV^\ell > 0 \tag{82}$$

provides us with the so called strong energy condition for every time-like vector V^α tangent to time-like geodesics. We notice that the fluid energy μ and pressure p satisfy the energy condition $\mu + p > 0$. This condition uniquely defines the Finslerian world lines (congruences) of the fluid with $V(x)$ tangent vector field analogous to that of Riemannian framework. The term $L_{il}V^iV^l > 0$ can be considered as a key for the existence of conjugate points in the Finslerian space–time structure.

The concepts of expansion, shear and vorticity for fluids were introduced for congruences in a Finslerian anisotropic space–time [2]. The definitions of these cosmological quantities led to the generalized Raychaudhuri equation which is related to the attractive nature of classical gravity [20]. The form of the Finslerian Raychaudhuri equation is given by

$$\frac{d\tilde{\theta}}{d\tau} = -\frac{1}{3}\tilde{\theta}^2 - \tilde{\sigma}_{ik}\tilde{\sigma}^{ik} + \tilde{\omega}_{ik}\tilde{\omega}^{ik} - L_{ir}V^iV^r + \dot{V}^i_{;i} \tag{83}$$

where $\tilde{\theta}, \tilde{\sigma}, \tilde{\omega}$ are the expansion, shear and vorticity, the ";" stands for the δ-covariant derivative, $\tilde{\theta} = V^i_{|i} - C^i_{im}\dot{V}^m$ and $\dot{V}^i = V^i_{|k}V^k$. The operator " " is the Riemann covariant derivative with respect to osculating Riemannian metric $a_{ij}(x) = g_{ij}(x)(x, V(x))$. Despite the x-dependence of the metric the Cartan torsion tensor

$$C_{ijk} = \frac{1}{2}\frac{\partial f_{ij}}{\partial V^k}(x, V(x)) \tag{84}$$

does not generally vanish at the Christoffel symbols' expression and affects both the geometry and dynamics.

The cosmological Hubble parameter in the Finslerian ansatz can be defined by the anisotropic scale factor $\tilde{\alpha}$ related to anisotropic expansion:

$$\dot{\tilde{\alpha}} = \frac{d\tilde{\alpha}(\xi^i(s))}{ds} \tag{85}$$

$$\tilde{H} = \frac{\dot{\tilde{\alpha}}}{\tilde{\alpha}} = \frac{1}{3}\tilde{\theta} \tag{86}$$

where $\tilde{\alpha} = \tilde{\alpha}(\xi^i(s))$, with $\xi^i(s))$ the unit tangent vector along the flow lines.

The anisotropic Hubble parameter has been used for cosmological problems in the Finslerian ansatz [9].

Expansion on the Finslerian tangent bundle

The Finslerian expansion shear and vorticity can be extended on the tangent bundle. We consider $V = (V^i, V^m)$ a unit tangent vector along the Finslerian stream lines of the cosmological fluid. We assume that the tangent vector field V is along a congruence of horizontal and vertical geodesics(tangent to the flow lines of an *incoherent fluid*). The Finslerian space–time has to be affinely connected i.e. R-torsion free, $R^{\alpha}_{jk} = 0$. We assume that a cosmological fluid in both cases is determined by horizontal and vertical paths parameterized by the cosmic proper time τ.[1] The tangent vector field for each case is given by

$$V^i = \frac{dx^i}{d\tau}, \qquad V^{\mu} = \frac{\delta y^{\mu}}{\delta\tau} \tag{87}$$

hence we can impose the following conditions

$$\begin{aligned} &V^iV_i = 1, \qquad V^i_{|j}V^j = 0 \\ &V^{\mu}V_{\mu} = 1, \qquad V^{\mu}|_{\nu}V^{\nu} = 0. \end{aligned} \tag{88}$$

We emphasize that for the vertical case the interaction which is constructed by physical and geometrical quantities on a specific fiber $\pi^{-1}(\{x_0\}) = T_{x_0}M$ attributes the relations

$$\Lambda_{ij} = V_{i|j}, \qquad \Lambda_{\mu\nu} = V_{\mu}|_{\nu} \tag{89}$$

and the conditions (88) can be re-expressed as

$$\begin{aligned} &\Lambda_{ij}V^j = \Lambda_{ij}V^i = 0 \\ &\Lambda_{\mu\nu}V^{\mu} = \Lambda_{\mu\nu}V^{\nu} = 0 \end{aligned} \tag{90}$$

stating that the tensors Λ_{ij}, $\Lambda_{\mu\nu}$ are purely spatial.

The Finslerian expansion on a bundle can be naturally defined as

$$\begin{aligned} &\tilde{\theta}_1 = V^k_{|k} \\ &\tilde{\theta}_2 = V^{\mu}|_{\mu} \end{aligned} \tag{91}$$

together with the vorticity

$$\begin{aligned} &\tilde{\omega}_{il} = \Lambda_{[il]} + \dot{V}_iV_l - \dot{V}_lV_i \\ &\tilde{\omega}_{\kappa\mu} = \Lambda_{[\kappa\mu]} + \dot{V}_{\kappa}V_{\mu} - \dot{V}_{\mu}V_{\kappa} \end{aligned} \tag{92}$$

and the shear

$$\begin{aligned} &\tilde{\sigma}_{ij} = \Lambda_{(ij)} - \frac{1}{3}\tilde{\theta}_1 h_{ij} - 2C_{ijl}V^l - \dot{V}_iV_j - \dot{V}_jV_i \\ &\tilde{\sigma}_{\kappa\mu} = \Lambda_{(\kappa\mu)} - \frac{1}{3}\tilde{\theta}_2 h_{\kappa\mu} - 2C_{\kappa\mu\lambda}V^{\lambda} - \dot{V}_{\kappa}V_{\mu} - \dot{V}_{\mu}V_{\kappa} \end{aligned} \tag{93}$$

where

$$\dot{V}^{\mu} = V^{\mu}|_{\nu}V^{\nu}, \dot{V}^i = V^i_{|k}V^k. \tag{94}$$

Therefore we can express the derivatives $V_{i|j}$, $V_{\kappa}|_{\mu}$ by virtue of θ, σ, ω. Indeed, the combination of (92) and (93) implies

$$\begin{aligned} &\Lambda_{il} = \frac{1}{3}\tilde{\theta}_{(1)}h_{il} + \tilde{\omega}_{il} + \tilde{\sigma}_{il} + 2\dot{V}_iV_l + 2C_{ilm}V^m \\ &\Lambda_{\kappa\mu} = \frac{1}{3}\tilde{\theta}_{(2)}h_{\kappa\mu} + \tilde{\omega}_{\kappa\mu} + \tilde{\sigma}_{\kappa\mu} + 2\dot{V}_{\kappa}V_{\mu} + 2C_{\kappa\mu\lambda}V^{\lambda}. \end{aligned} \tag{95}$$

The rate of expansion for both $\tilde{\theta}_1$, $\tilde{\theta}_2$ can be implied from the commutation relations and the properties $V^k_{|i}V^i = V^{\mu}|_{\nu}V^{\nu} = 0$

$$\begin{aligned} &\frac{d\tilde{\theta}_1}{d\tau} = -V^i_{|k}V^k_{|i} - R_{im}V^iV^m - V^k|_{\mu}R^{\mu}_{ki}V^i \\ &\frac{d\tilde{\theta}_2}{d\tau} = -V^{\mu}|_{\nu}V^{\nu}|_{\mu} - S_{\rho\sigma}V^{\rho}V^{\sigma} \end{aligned} \tag{96}$$

where $\frac{d\tilde{\theta}_1}{d\tau} = \tilde{\theta}_{1,k}V^k = \tilde{\theta}_{1|k}V^k$ and $\frac{d\tilde{\theta}_2}{d\tau} = \frac{\partial\tilde{\theta}_2}{\partial y^{\mu}}V^{\mu} = \tilde{\theta}_2|_{\mu}V^{\mu}$. Since $\dot{V}^i = \dot{V}^{\mu} = 0$. The contractions $\Lambda^{ik}\Lambda_{ki}$, $\Lambda^{\mu\nu}\Lambda_{\mu\nu}$ are simplified to the form

$$\begin{aligned} &\Lambda^{ik}\Lambda_{ki} = \frac{1}{3}\tilde{\theta}^2_{(1)} + \tilde{\sigma}^{ik}\tilde{\sigma}_{ki} + \tilde{\omega}^{ik}\tilde{\omega}_{ki} \\ &\Lambda^{\mu\nu}\Lambda_{\nu\mu} = \frac{1}{3}\tilde{\theta}^2_{(2)} + \tilde{\sigma}^{\mu\nu}\tilde{\sigma}_{\nu\mu} + \tilde{\omega}^{\mu\nu}\tilde{\omega}_{\nu\mu} \end{aligned} \tag{97}$$

thus we can generalize the Raychaudhuri equation in a tangent bundle to the form

[1] We note that the cosmic parameters referring to the horizontal and vertical geodesics are affine parameters of the arc-length.

$$\frac{d\tilde{\theta}_{(1)}}{d\tau} = -\frac{1}{3}\tilde{\theta}^2_{(1)} - \tilde{\sigma}^{ik}\tilde{\sigma}_{ik} + \tilde{\omega}^{ik}\tilde{\omega}_{ik} - R_{im}V^iV^m \tag{98}$$

$$\frac{d\tilde{\theta}_{(2)}}{d\tau} = -\frac{1}{3}\tilde{\theta}^2_{(2)} - \tilde{\sigma}^{\mu\nu}\tilde{\sigma}_{\mu\nu} + \tilde{\omega}^{\mu\nu}\tilde{\omega}_{\mu\nu} - S_{\rho\sigma}V^\rho V^\sigma. \tag{99}$$

For a case that the horizontal and vertical congruences are rotation free, we have

$$\tilde{\omega}_{ij} = 0, \qquad \tilde{\omega}_{\mu\nu} = 0. \tag{100}$$

In case we have to study the evolution of the congruence's expansion we should note the inequalities

$$\tilde{\sigma}_{ij}\tilde{\sigma}^{ij} = (\tilde{\sigma})^2 \geq 0, \qquad \tilde{\sigma}_{\mu\nu}\tilde{\sigma}^{\mu\nu} = (\tilde{\sigma})^2 \geq 0. \tag{101}$$

7. The Einstein field equation and the energy momentum tensor of perfect fluids

The energy–momentum tensor can be decomposed to horizontal and vertical part [12]

$$T(x, V) = T^h(x, V) + T^v(x, V) \tag{102}$$

where

$$T^h(x, V) = T_{ij}(x, V)dx^i \otimes dx^j \tag{103}$$

$$T^v(x, V) = T_{\mu\nu}(x, V)\delta y^\mu \otimes \delta y^\nu. \tag{104}$$

We fix up the horizontal part of T to refer to gravity and the vertical one to refer to electromagnetic phenomena associated to a primordial magnetic field B^μ. Thus, bearing in mind we can express the components of T as(ρ is the density and P the pressure of the cosmological fluid)

$$T_{ij}(x, V) = (\rho + P)V_iV_j - Pf_{ij}(x, V) \tag{105}$$

$$T_{\mu\nu}(x, V) = \frac{1}{2}B^2V_\mu V_\nu + \frac{1}{6}B^2h_{\mu\nu} + \Pi_{\mu\nu} \tag{106}$$

where $h_{\mu\nu}$ is the metric tensor of the vertical space, $B^2 = B^\mu B_\mu$ and

$$\Pi_{\mu\nu} = \frac{1}{3}B^2h_{\mu\nu} - B_\mu B_\nu. \tag{107}$$

$\Pi_{\mu\nu}$ is the anisotropic pressure.

$$R_{ij} - \frac{1}{2}(R + S)f_{ij} = kT_{ij} \tag{108}$$

describe the interaction of anisotropic geometry with the classical gravitational matter. The field equations on the vertical sub-bundle are dependented on the S-curvuture and written as

$$S_{\mu\nu} - \frac{1}{2}(R + S)h_{\mu\nu} = kT_{\mu\nu}. \tag{109}$$

The Einstein equations can relate the S-curvature of the vertical sub-space to the phenomenon of weak anisotropy created by the presence of the primordial magnetic field. The availability of energy–momentum tensors (105) and (106) implies the following expression for the Raychaudhuri equations (98) and (99) for an orthogonal congruence ($\tilde{\omega}_{ij} = \tilde{\omega}_{\mu\nu} = 0$)

$$\frac{d\tilde{\theta}_1}{d\tau} = -\frac{1}{3}\tilde{\theta}^2_{(1)} - \tilde{\sigma}^{ik}\tilde{\sigma}_{ik} - \frac{1}{2}k(\rho + 3P) \tag{110}$$

$$\frac{d\tilde{\theta}_2}{d\tau} = -\frac{1}{3}\tilde{\theta}^2_{(2)} - \tilde{\sigma}^{\mu\nu}\tilde{\sigma}_{\mu\nu} + k\left(B_\mu V^\mu\right)^2 \tag{111}$$

since

$$\begin{aligned} R_{ij}V^iV^j &= k\left(T_{ij}V^iV^j - \frac{1}{2}T^i_i\right) \\ S_{\mu\nu}V^\mu V^\nu &= k\left(T_{\mu\nu}V^\mu V^\nu - \frac{1}{2}T^\mu_\mu\right). \end{aligned} \tag{112}$$

In a gravitational theory on the Finslerian tangent bundle curvature effects can be considered as *tidal forces*. Possible modifications of them are ought to the non-linear character (internal interaction of tidal forces, dark tidal forces etc). For an anisotropic space–time with $\tilde{\sigma}_{ij} = \tilde{\sigma}_{\mu\nu} = \dot{\theta}^{(1)} = \dot{\theta}^{(2)} = 0$ and $\tilde{\omega}_{ij}$, $\tilde{\omega}_{\mu\nu}$ constant, the tidal fields are $R_{ij}V^iV^j$ and $S_{\mu\nu}V^\mu V^\nu$ are due to the vorticities. This play the role of vacuum energy.

The conservation laws for the energy–momentum tensor, are given by

$$T^{ij}_{;j} = 0, \qquad T^{\mu\nu}|_{\nu} = 0 \tag{113}$$

are based on the assumption of the P-curvature's vanishing $P^i_{jkh} = 0$.

Energy conditions

A direct consequence of the coservation laws (113) is the *dominant energy condition*(D.E.C) for both energy–momentum tensors which includes the *weak energy condition* (W.E.C)

$$T_{ij}(x, V)V^iV^j \geq 0, \qquad T_{\mu\nu}(x, V)V^{\mu}V^{\nu} \geq 0, \tag{114}$$

valid for all timelike vectors. The W.E.C. yield to the inequalities

$$k(\rho + 3P) \geq 0, \qquad (B_{\mu}V^{\mu})^2 \leq \frac{1}{2}B^2, \tag{115}$$

expressed with respect to the present matter and magnetic field. The horizontal energy–momentum tensor $T_{ij}(x, V)$ is responsible for gravitational-phenomena observed on the base space–time manifold M therefore the *strong energy condition*(SEC)

$$T_{ij}V^iV^j \geq \frac{1}{2}T^k_k \tag{116}$$

is valid and the field equation (108) lead to the inequality

$$R_{ij}V^iV^j \geq 0. \tag{117}$$

We remark that

$$S_{\mu\nu}V^{\mu}V^{\nu} = -\left(B_{\mu}V^{\mu}\right)^2 \leq 0 \tag{118}$$

which expresses the negative energy of the magnetic field due to the tension of magnetic lines to remain straight.

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