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General very special relativity in Finsler cosmology

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General very special relativity (GVSR) is the curved space-time of very special relativity (VSR) proposed by Cohen and Glashow. The geometry of general very special relativity possesses a line element of Finsler geometry introduced by Bogoslovsky. We calculate the Einstein field equations and derive a modified Friedmann-Robertson-Walker cosmology for an osculating Riemannian space. The Friedmann equation of motion leads to an explanation of the cosmological acceleration in terms of an alternative non-Lorentz invariant theory. A first order approach for a primordial-spurionic vector field introduced into the metric gives back an estimation of the energy evolution and inflation.

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I. INTRODUCTION

It is widely known that relativity violations are arising from breaking the Lorentz symmetry. Lorentz violations are a very wide research area traced back to Dirac in the early 1950s and applied in many aspects of modern physics since the question of Lorentz symmetry invariance was incorporated into the foundations of both general relativity and quantum field theory [1]. Plenty of different high energy theories face the challenge of Lorentz symmetry breaking; therefore the proposal of a different geometrical point of view from general relativity makes sense. The consideration of such a scenario implies a class of modified dispersion relations for elementary particles depending on both coordinates and momenta. In this case the geometry of space-time may be direction dependent generating a local anisotropy [2–4]. A possible candidate for a geometry which incorporates the anisotropies directly to the metric is Finsler geometry. A typical example of studying Lorentz violations within the framework of Finsler geometry is presented in [5].

An interesting case of Lorentz violation where the Finsler geometry turns up is the model of very special relativity (VSR) characterized by a reduced Lorentz symmetry [6]. The Lorentz violations are generated by a subgroup of the full Lorentz group, called $ISIM(2)$. The adoption of this theory is not in contrast to experimental constraints since it appears to be compatible with all current limits of local Lorentz and CPT invariance, confirming some new physics [6–8]. Some experimental analysis on the upper bounds of the Lorentz violation are described in [9].

The combination of Lorentz violations to gravitational phenomena is not compatible with the geometrical framework of Riemann geometry since general relativity is applicable only to low-energy descriptions of nature. The study of relativity violations together with gravitational phenomena requires a type of space-time geometry which allows local non-Lorentz invariance while preserving general coordinate invariance. A direct consequence of Lorentz violations is the production of local anisotropies and we expect any gravitational phenomenon to be affected by the breaking of classical local flatness (see, for example, [10,11]). In Finsler geometry all geometrical objects are direction dependent while preserving the general coordinate invariance [12]. Thus this special type of geometry is a possible choice for the investigation of geometrodynamics allowing Lorentz violations [13–17].

The deformation of the group $ISIM(2)$ leads to the construction of a Finslerian line element proposed by Bogoslovsky (see [18], and references therein). The whole set of Lorentz transformations are replaced by the deformed group of transformations $DISIM_b(2)$ which is a subgroup of the Weyl group. The line element $ds^2 = \eta_{ij}dx^i dx^j$ is no longer preserved under the $DISIM_b(2)$ transformations. The line element that is preserved under these transformations is the non-Riemannian [19]

$$ds = (\eta_{ij}dx^i dx^j)^{(1-b)/2}(n_k dx^k)^b. \tag{1.1}$$

The vector field n^μ is interpreted as a "spurion vector field" and it defines the direction of the "aetheral" motion's 4-velocity. The dependence of the metric function on the vector n^μ indicates the anisotropic character of space-time. The parameter b is dimensionless and is restricted by

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various experiments [19,20]. It is inserted into the mass tensor

$$m_{ij} = (1 - b)m(\delta_{ij} + bn_i n_j) \tag{1.2}$$

of every particle coming from a Lagrangian constructed in [18]. The canonical momenta satisfies the mass-shell condition $[\eta_{ij} = \mathrm{diag}(+1, -1, -1, -1)]$

$$\eta^{ij} p_i p_j = m^2(1 - b^2)\left(\frac{n^k p_k}{m(1 - b)}\right)^{2b/(1+b)}, \tag{1.3}$$

which upon quantization leads to the Klein-Gordon equation

$$\Box\phi + m^2(1 - b^2)\left(\frac{-in^\mu}{m(1 - b)}\partial_\mu\right)^{2b/(1+b)}\phi = 0. \tag{1.4}$$

The dispersion relation (1.3) implies that the relationship of energy and momentum is affected by the type of the metric geometry. The aether-drift experiments restrict b within the very tight limits $|b| < 10^{-10}$ [19] and the anisotropy of inertia implies $|b| < 10^{-26}$ [20]. The modified dispersion relation constructed by Gibbons *et al.* in [18] has also been reproduced in a Randers-type Finsler space [21].

We proceed by constructing the geometrical machinery of space-time using the Finslerian connection and curvature coming from an osculating Riemann metric [12]. The metric function of the Finsler space is the one prescribed by Bogoslovsky where the Minkowski metric tensor is substituted by the Friedmann-Robertson-Walker (FRW) metric defined in standard cosmology. The derivation of the gravitational field equations is similar to the one that appeared in [13]. The Friedmann equation of motion for a linearized spurion vector field parallel to the fluid flow lines leads us to a self-accelerated cosmological model. The kinematical equations of a scalar field are also considered providing an inflationary solution for the scale factor.

II. BOGOSLOVSKY'S METRIC APPLIED TO COSMOLOGY

The effects of local Lorentz violation are likely applicable on cosmological contexts, such as those involving the cosmological constant, dark matter, and dark energy. The homogeneous FRW cosmological solutions may acquire anisotropic corrections, leading to a realistic anisotropic cosmology complied to the observational data [10].

Bogoslovsky's metric may shed light on some problems of modern cosmology which are compatible to local anisotropies of the geometrical structures of space-time. We can construct a geometrical machinery of cosmology by introducing comoving coordinates to the metric function

$$F(x, y) = (\eta_{ij} y^i y^j)^{(1-b)/2}(n_k y^k)^b. \tag{2.1}$$

We replace the Minkowskian metric with the Robertson-Walker one

$$a_{\mu\nu} = \mathrm{diag}\left(1, -\frac{a^2(t)}{1 - kr^2}, -a^2(t)r^2, -a^2(t)r^2\sin^2\theta\right), \tag{2.2}$$

where t is the cosmic proper time, r, θ, and ϕ the comoving spherical coordinates, $k = 0, \pm 1$, and $a(t)$ the scale factor of the expanding volume. The new metric function

$$F(x, y) = (a_{\mu\nu} y^\mu y^\nu)^{(1-b)/2}(n_\rho y^\rho)^b \tag{2.3}$$

is a direct result of a coordinate linear transformation [22]

$$B_\mu = \frac{\partial x^i}{\partial x^\mu} B_i \tag{2.4}$$

and it directly determines the metric of the Finslerian space-time

$$f_{\mu\nu}(x, y) = \frac{1}{2}\frac{\partial F^2}{\partial y^\mu \partial y^\nu}(x, y). \tag{2.5}$$

This straightforward generalization for curved space-times and Machian gravitational theories is used [5] to give an explanation of local anisotropies in terms of geometrical phase transitions. The consideration of such a metric function embodies two types of geometries: the dynamics is described by the Finslerian metric produced by $F(x, y)$ while all the information about gravity is encoded to the FRW metric $a_{\mu\nu}$. A similar formulation is deduced by Bekenstein in [23] contemplating a different approach for Finsler geometry. The variables $y^\mu = \frac{dx^\mu}{dt}$ represent the 4-velocity components of the fluid flow lines, hence $y^\mu = (1, 0, 0, 0)$.

A null or timelike spurionic vector field?—The study of general very special relativity (GVSR) requires the existence of a null spurionic vector field. However, this preferential direction of ether is most naturally expected to be tangent to the flow lines of the cosmological fluid like every primordial vector field [24]. Thus, n^μ must be parallel to the velocity of the comoving observer, i.e.,

$$n^\mu = \lambda y^\mu. \tag{2.6}$$

Hence, the spurion vector field becomes of timelike character at a late time period of the universe with $|n^\mu| \ll 1$. Therefore, it is written in coordinate form

$$n^\mu = (n(t), 0, 0, 0) \tag{2.7}$$

with the time component very small. The timelike spurionic vector field does not essentially affect the mass-shell condition (1.3) since only quadratic terms of n^μ turn up at the contractions of canonical momentum $p_\mu = m\frac{\partial F}{\partial y^\mu}$.

The osculating space and the gravitational field equations.—All the geometrical quantities of Finsler geometry depend both on coordinates and velocity. However, we can study the geometrical properties of a Finsler space by restricting the vector field y^μ to belong to an individual tangent space for a given position coordinate. In such a case the velocity coordinates are functions of the position; therefore we can measure distances by using the metric [12]

$$g_{\mu\nu}(x) = f_{\mu\nu}(x, y(x)). \tag{2.8}$$

This method, known as the *osculating Riemannian* approach (for details see [12]), can be specialized for the tangent vector field $y(x)$ of the cosmological fluid flow lines.

We are interested in producing the Einstein field equations as in [13]. After calculating the connection and the curvature for the Riemannian osculating metric $g_{\mu\nu}(x)$ we are led to [17]

$$L_{\mu\nu} - \frac{1}{2} L g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2.9}$$

where all the quantities in (2.9) are functions of (x, y), $y \equiv y(x)$. The energy-momentum tensor for the signature $(+, -, -, -)$ is defined to be

$$T_{\mu\nu} = -P g_{\mu\nu} + (\mu + P) y_\mu y_\nu, \tag{2.10}$$

where P is the pressure and μ is the energy density of an ideal cosmic fluid.

The dispersion relation (1.3) is also modified after plugging the FRW metric and the vector n^μ into (1.1).

III. THE FRIEDMAN EQUATION FOR A LINEARIZED VECTOR FIELD

The calculation of the curvature and the Ricci tensor leads to the construction of the Friedmann equation of motion for a locally anisotropic universe. We can approximate the 0-component of the primordial-spurionic vector field n^μ at first order approach

$$n(t) \sim At + B. \tag{3.1}$$

The special form (3.1) is a Taylor type approximation

$$n(t) = n(t_0) + \dot{n}(t_0)(t - t_0) + O((t - t_0)^2), \tag{3.2}$$

where

$$A = \dot{n}(t_0), \qquad B = n(t_0) - t_0 \dot{n}(t_0). \tag{3.3}$$

Since all of the other components of the spurionic vector field vanish, only the diagonal elements of the metric and the Ricci tensor survive. Under the assumption of a weak Lorentz violation we can restrict our parameter A to be small enough

$$A = \dot{n}(t_0) \to 0 \tag{3.4}$$

considering an almost constant value of the field.

Connection and curvature.—By virtue of the metric (2.8), we are able to calculate the Christoffel symbols and the curvature (see Appendix A). The Ricci tensors $L_{\mu\nu}$ can be approximated for $b \to 0$, $A \to 0$, and this implies the following components:

$$\begin{aligned} L_{00} &= 3\frac{\ddot{a}}{a} + 3\frac{Ab}{B}\frac{\dot{a}}{a} + O(A^2), \\ L_{11} &= -\frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2} + \frac{5A}{B} b \frac{a\ddot{a}}{1 - kr^2} + O(A^2), \\ L_{22} &= -r^2(a\ddot{a} + 2\dot{a}^2 + 2k) - \frac{5A}{B} b r^2 a\ddot{a} + O(A^2), \\ L_{33} &= -r^2(a\ddot{a} + 2\dot{a}^2 + 2k)\sin^2\theta - \frac{5A}{B} b r^2 a\ddot{a}\sin^2\theta \\ &\quad + O(A^2). \end{aligned} \tag{3.5}$$

The extra terms appearing at the Ricci components are the dominant ones since they are multiplied by the parameter Ab/B, where B is necessarily small to enable the curvature additional terms to be measurable ($|n^\mu| \ll 1$). Following the usual procedure we can construct the equation of motion for the scale factor $a(t)$ [25]

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} + 2\frac{A}{B} b \frac{\dot{a}}{a} = \frac{8\pi G}{3}\left[\mu - 2\frac{A}{B} b P\left(t + \frac{B}{A}\ln B\right)\right]. \tag{3.6}$$

The (3.6) form of the Friedmann equation may not be well suited for late time acceleration since the pressure term is a sign of an early universe regime, where the linearization approach possibly breaks down. As the universe evolves we expect the anisotropic nature of space-time to be converted to a smoother structure. Therefore the linearized approach of the present model is more convenient for a matter-dominated phase.

The equation of motion for a matter-dominated universe.—Taking into account that there is no pressure in a matter-dominated universe we can obtain the following equation of motion:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} + 2\frac{A}{B} b \frac{\dot{a}}{a} = \frac{8\pi G \mu}{3}. \tag{3.7}$$

We emphasize the extra contribution generated by the geometrodynamical term $2\frac{A}{B} b \frac{\dot{a}}{a}$. If the sign of the parameter Ab/B is fixed up to be negative, the extra term at the equation of motion (3.7) will create a self-accelerating cosmology. Despite the fact that the additional term creates acceleration and might replace dark energy contributions, it cannot give an answer to the question why the vacuum does not gravitate? This difficulty gives rise to the complicated task of distinguishing modifications of curvature from dark energy [26]. However, the extra accelerating term can be contemplated as a relic left back by an earlier phase where the Finslerian geometrodynamics was characterized by a nonlinear nature [$n(t)$, $b(t)$ relatively large]. The same Friedmann equation is also produced for the DGP cosmology for a spatially flat space-time with a different continuity equation [27]. We also remark that the substitution $z_t = 2\frac{A}{B} b$ reveals a correspondence of the present model to the FRW cosmological model described in [13] constructed by a Randers-Finsler–type metric function.

Matter density and continuity equation.—The continuity equation for the cosmological fluid of the universe can be directly produced by the conservation law of the energy-momentum tensor $\nabla_\nu T^{\mu\nu} = 0$, where the covariant derivative comes from the connection of the osculating Riemann space-time. The zero component of the conservation law $\nabla_\nu T^{0\nu} = 0$ implies

$$\dot{\mu} + \Big(2A^0_{00} + \sum_{i=1}^{3} A^i_{0i}\Big)\mu + \dot{P}(1 - g^{00}) + P\Big[(1 - g^{00})\Big(2A^0_{00} + \sum_{i=1}^{3} A^i_{0i}\Big) + \sum_{i=1}^{3} g^{ii}A^0_{ii} - \dot{g}^{00}\Big] = 0. \quad (3.8)$$

If we make use of the equation of state $w = P/\mu$ and the connection components calculated in Appendix B, the approximation for $A \to 0$ leads to

$$\dot{\mu} + 3\frac{\dot{a}}{a}\mu + 5\frac{Ab}{B}\mu + \frac{w}{w + 2Ab/B \cdot t + 2b\ln B} \times \Phi[t, a, \dot{a}; b, A, B]\mu = 0, \quad (3.9)$$

where Φ is a function of time, the parameters, and the unknowns a, $\dot{a}$. In the case of a matter-dominated universe, where $w = 0$, the differential equation can be integrated and gives back the solution

$$\mu(t) \propto a^{-3}(t)\exp\Big(\frac{-5Ab}{B}t\Big) \quad (3.10)$$

which is in alliance to the one found in [13].

The Friedman equation in terms of Ω's.—The whole picture of the cosmological model can be effectively depicted by exploring the relation of the Ω parameters. Using the usual definitions of the Hubble parameter and the Ω parameters, we can rewrite (3.7) to the form

$$H^2 + \frac{k}{a^2} + 2\frac{Ab}{B}H = \frac{8\pi G\mu}{3} \quad (3.11)$$

which implies the equation

$$\Omega_M + \Omega_K + \Omega_X = 1, \quad (3.12)$$

where

$$\Omega_X = -2\frac{A}{BH}b \quad (3.13)$$

is the density parameter produced by the extra term of the Friedmann equation. The term Ω_X might give a significant contribution to the acceleration compared to the dark energy parameter $\Omega_\Lambda \simeq 0.7$.

Order of magnitude of unknown parameters.—Since the rhs of (3.11) is positive, we restrict $H^2 > |2\frac{Ab}{B}H|$. We can estimate the order of magnitude of the quantity $\frac{Ab}{B}$ in case it dominates the expansion over Λ, where

$$\frac{Ab}{B} \sim \frac{\Lambda c^2}{6H}. \quad (3.14)$$

Given a typical value of the Hubble parameter $H_0 \simeq 71\ \mathrm{km/s/Mpc} \sim 10^{-18}\ \mathrm{sec}^{-1}$ and the cosmological constant $\Lambda \sim 10^{-57}\ \mathrm{cm}^{-2}$ [24] we deduce

$$\frac{Ab}{B} \sim 10^{-19}\ \mathrm{sec}^{-1} \quad (3.15)$$

measured in Hubble units.

IV. ENERGY EVOLUTION

The calculation of the connection components coming from the osculating metric (2.8) can give us a picture of how the energy of a particle is affected by the universe expansion and the extra parameters introduced into the metric function (2.1).

Energy of a massless particle.—The 4-momentum of a massless particle is defined by $P^\mu = \frac{dx^\mu}{d\lambda}$, where $P^0 = E$ is the energy of the particle. The parameter λ is the evolution parameter of the particle's path described by the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + A^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0, \quad (4.1)$$

and the zero component of the geodesic equation yields [28]

$$E\frac{dE}{dt} = -A^0_{\alpha\beta}P^\alpha P^\beta = -\frac{Ab}{B}(E^2 - a_{ij}P^iP^j) + (1 - b)\frac{\dot{a}}{a}a_{ij}P^iP^j. \quad (4.2)$$

The particle is massless, $m = 0$, and thus the dispersion relation (1.3) is simplified to the usual form $E^2 = -a_{ij}P^iP^j$; hence (4.2) implies

$$\frac{1}{E}\frac{dE}{dt} - (b - 1)\frac{\dot{a}}{a} = -\frac{2Ab}{B} + O(A^2) \quad (4.3)$$

which can be integrated directly and gives back

$$E(t) \propto a^{b-1}(t)\exp\Big(-\frac{2Ab}{B}t\Big). \quad (4.4)$$

The solution (4.4) possesses an additional redshift effect due to the Lorentz violations inherited by the parameter b and the spurion vector field. The solution behaves as $E(t) \propto 1/a(t)$ if the extra terms at the equation of motion (3.7) are negligible.

Energy of a massive nonrelativistic particle.—We consider a massive nonrelativistic particle of mass m traveling on a geodesic of the space-time

$$\frac{d^2x^\mu}{d\tau^2} + A^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0. \quad (4.5)$$

The 4-momentum of the particle is defined in natural units

($c = 1$) as $P^\mu = m\frac{dx^\mu}{d\tau}$, where τ is the particle's proper time. The chain rule applied to the derivative $\frac{d}{d\tau}$ implies

$$\frac{d}{d\tau} = \frac{E}{m} \cdot \frac{d}{dt}. \tag{4.6}$$

In virtue of (1.3) the zero component of (4.5) leads to the differential equation for the energy

$$E \cdot \frac{dE}{dt} = -\frac{Ab}{B} E^2 \tag{4.7}$$

since $a_{ij}P^iP^j/m^2 \to 0$ for a massive nonrelativistic particle. The integration of the differential equation implies

$$E = E_0 \exp\left(-\frac{2Ab}{B} t\right). \tag{4.8}$$

We can calculate the effect of geometrodynamics by measuring the amount of time for a variation of energy $0.1\% E_0$ and find $t \sim 10^{15}$ sec, where we used the $\frac{Ab}{B}$ estimation from (3.15).

V. THE GENERALIZATION OF THE KLEIN-GORDON EQUATION

Since the geometry of space-time is determined by the process of osculating a Finslerian space to a Riemannian one, the metric is only position dependent. As long as the velocity is fixed up to be $y = y(x)$, a Riemannian metric is defined. Therefore we can apply the general covariance principle and construct a curved version of the Klein-Gordon equation (1.4)

$$\Box\phi + m^2(1 - b^2)\left(\frac{-in^\mu}{m(1-b)}\nabla_\mu\right)^{2b/(1+b)}\phi = 0. \tag{5.1}$$

The ∇_μ operator is the covariant derivative coming from the $A^\rho_{\mu\nu}$ connection (see Appendix B), and the box operator is

$$\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = g^{00}\ddot{\phi} - g^{\mu\nu}A^0_{\mu\nu}\dot{\phi} \tag{5.2}$$

and n^μ is the spurion defined in (2.7) and (3.1). We impose the scalar field $\phi = \phi(t)$ due to the assumption of homogeneity with respect to weak Lorentz violations. After expanding for small values of b, we end up with the following approximation:

$$\Box\phi = \left(1 - \frac{2Ab}{B}t\right)\ddot{\phi} + 3H\dot{\phi}\left(1 - \frac{2Ab}{B}t + \frac{2Ab}{3BH}\right) + O(A^2). \tag{5.3}$$

A small A and b approximation for the $m^2(1 - b^2) \times (\frac{-in^\mu}{m(1-b)}\nabla_\mu)^{2b/(1+b)}$ operator

$$m^2(1 - b^2)\left(\frac{-in^\mu}{m(1-b)}\nabla_\mu\right)^{2b/(1+b)}\phi = m^2\phi + 2m^2b(\hat{D}\phi) + \frac{2bA}{B}t \cdot m^2\phi + 2b\ln B \cdot m^2\phi + O(A^2) \tag{5.4}$$

leads to the kinematical equation of the scalar field

$$\ddot{\phi} + 3\left(H + \frac{2Ab}{3B}\right)\dot{\phi} + m^2\phi\left(1 + \frac{4Ab}{B}t + 2b\ln B\right) + O(A^2) = 0, \tag{5.5}$$

where $2m^2b\hat{D}\phi \to 0$ since $b \to 0$ is faster than the logarithmic term $\hat{D}\phi$. The time derivatives of the scalar field come from the covariant version of the box operator; therefore the potential of the scalar field is

$$V(\phi, t) = \frac{1}{2}m^2\phi^2\left(1 + 2\frac{A}{B}bt + O(A^2)\right). \tag{5.6}$$

We can eliminate H if we combine the Klein-Gordon equation with the Friedmann equation of motion producing a differential equation for the scalar field ϕ.

The energy-momentum tensor is expressed by [29]

$$T^\alpha_\beta = g^{\alpha\gamma}\phi_{,\gamma}\phi_{,\beta} - \left(\tfrac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi, t)\right)\delta^\alpha_\beta \tag{5.7}$$

and determines directly the energy density and the pressure

$$\mu = \tfrac{1}{2}g^{00}\dot{\phi}^2 + V, \qquad P = \tfrac{1}{2}g^{00}\dot{\phi}^2 - V. \tag{5.8}$$

Thus we can insert (5.8) into (3.6) and find the Friedmann equation of motion

$$H^2 + \frac{2Ab}{B}H = \frac{8}{3}\pi G\left[\frac{1}{2}\dot{\phi}^2(1 - 2b\ln B) + \frac{1}{2}m^2\phi^2\left(1 + 2b\ln B + 4\frac{Ab}{B}t\right) + O(A^2)\right]. \tag{5.9}$$

The elimination of H from (5.5) and (5.9) yields

$$\ddot{\phi} + \sqrt{12\pi G}(\dot{\phi}^2 + m^2\phi^2)^{1/2}\dot{\phi} + m^2\phi + \frac{Ab}{B} \cdot f(t, \phi, \dot{\phi}) + 2b\ln B \cdot g(t, \phi, \dot{\phi}) + O(A^2) = 0, \tag{5.10}$$

where

$$f(t, \phi, \dot{\phi}) = -\dot{\phi} + 4m^2t\phi + 2m^2t\phi^2\dot{\phi}\sqrt{12\pi G}(\dot{\phi}^2 + m^2\phi^2)^{-1/2} \tag{5.11}$$

and

$$g(t, \phi, \dot{\phi}) = m^2\phi - \tfrac{1}{2}\sqrt{12\pi G}(\dot{\phi}^2 + m^2\phi^2)^{-1/2} \times (\dot{\phi}^2 - m^2\phi^2)\dot{\phi} \tag{5.12}$$

are functions of ϕ and its first derivative (for details, see, e.g.,[29]).

The limit $|\dot{\phi}| \ll m\phi$.—In this special case the potential is much larger compared to the kinetic energy; thus we can derive the simplified differential equation

$$\ddot{\phi} + m^2\left(1 + 4\frac{Ab}{B}t + 2b\ln B\right)\phi = 0. \quad (5.13)$$

The solution of (5.13) is expressed with the aid of Airy functions [30]

$$\phi(t) = C_1 Ai(\zeta) + C_2 Bi(\zeta), \quad (5.14)$$

where

$$\zeta = \left(-2m\frac{B}{Ab}\right)^{2/3}\left(-\frac{1}{4} - \frac{Ab}{B}t - \frac{b}{2}\ln B\right). \quad (5.15)$$

The argument of the Airy function is fixed up as $\zeta \sim -\frac{1}{4} \times (-2m\frac{B}{Ab})^{2/3}$, given the small values of the parameter $\frac{Ab}{B}$; hence we can regard ζ to be independent of time. We can approximate the function $\phi(t)$ using the asymptotics of the Airy function after taking into consideration the negative sign of argument ζ [30]. The substitution of (5.14) to the Friedmann equation (5.9) yields

$$a(t) = \exp\left(-\frac{Ab}{B}t\right)\exp[h(t)], \quad (5.16)$$

where

$$h(t) = \frac{2}{3}C_3(\zeta)^{1/2}\left(1 + \frac{Ab}{B}t + 2b\ln B\right)^{3/2}. \quad (5.17)$$

Given the positive sign of $\ddot{a}(t)$ in (5.16), the solution secures an inflationary phase for a time interval where the high potential takes over the expansion.

VI. DISCUSSION

The essential result of our approach is the adoption of a Finslerian metric function applied to cosmology, coming from Cohen and Glashow's very special relativity [6]. The calculation of Einstein's field equations for an osculating Riemannian space-time gives back a Friedmann equation for a self-accelerating universe under the assumption that the sign of the extra parameter is negative. The estimation of the energy evolution implied by the modified geodesic equations may lead to experimental constraints for the VSR theory using observational data from the large scale structure. The specific limit for a massive nonrelativistic object implies a small variation of energy within a period of time close to the age of the universe restricting our calculations within the acceptable observational limits even in the case where the model wins totally over the dark energy.

The construction of the kinematical equation of a scalar field [18] with the aid of the Finslerian metric (2.8) and the Friedmann equation (3.7) can lead to a better understanding of the nature of the Finslerian gravitational field. Indeed, the Lorentz violations provide a modified potential for the curved Klein-Gordon equation (5.5) affecting the validity of the strong energy conditions. Given a large potential compared to the kinetic energy, the solution of the scale factor implies an inflating phase of the expansion depending on the GVSR assumptions inherited by the metric function (2.1). An interesting task for future work is the study of the model's early time behavior producing an inflationary solution without the aid of a scalar field, considering stronger effects of Lorentz symmetry breaking. A similar inflationary scenario has also been produced by gravitational mechanisms, as a direct result of Lorentz violations not depending on the vacuum's fluctuations and grand unified theories [31].

An explanation of acceleration lies on the fact that dark energy acts as a repulsive force introducing a cosmological constant at the Einstein field equations. In such a case the cosmological constant is a finely tuned ground state of a potential implying negative pressure at the equations of motion. A universe with pressure free matter can be self-accelerating under the restriction of a modified Ricci curvature which imposes an asymptotically de Sitter geometry of space-time. The machinery of osculating a Finslerian space to a Riemannian one leads at first order approach directly to an asymptotically de Sitter universe. However, the classification of the present model as a low curvature modification (e.g., ΛCDM, DGP) needs to be proven. This is a vital task since all such cosmological models reproduce Newtonian gravity locally [32].

A possible estimation of the spurionic vector field (from high energy physics or other methods), within the limits of the present cosmological model, can set forth an answer to the vital question about the small value of b posed by Gibbons *et al.* [18], connecting Lorentz violations to the *dark energy* problem. A further investigation of the present model taking into account the calculation of cosmological perturbations and the cosmic microwave background data may relate Lorentz violations to the problem of large angle anisotropies and inhomogeneities.

The introduction of Finsler geometry as a geometry of space-time opens up a new direction toward the study of geometric phase transitions. The concept of geometric phase transitions generated by Bogoslovsky's line element (1.1) has already been studied in [5] for the special case of a flat Finslerian space-time. An interesting generalization can be applied to a curved Finsler space for a better understanding of how Lorentz violations, with a varying $b = b(t)$, may evolve as the universe expands. Since Lorentz violations produce anisotropies, it is natural for them to "dilute" to thermal energy and a large amount of entropy [24]; therefore the special limit of the present model will be asymptotically recovered.

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APPENDIX A: THE METRIC $g_{\mu\nu}$

The definition of the Finsler metric function (2.1) implies the calculation of the space-time's metric. Assuming the comoving character of the spatial coordinates and the vector field tangent to the cosmic flow lines, the computation of the metric components of the osculating Riemann space-time are simplified to the form [$i, j = 1, 2, 3$ and $a_{\mu\nu}$ is the Robertson-Walker metric (2.2)]

$$\begin{aligned} g_{00} &= 1 + 2\ln(At + B)b + O(b^2), \\ g_{ij} &= a_{ij} + a_{ij}[2\ln(At + B) - 1]b + O(b^2), \end{aligned} \tag{A1}$$

which implies the connection $A^{\rho}_{\mu\nu} = g^{\rho\sigma}\frac{1}{2}\times(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma})$ and the curvature $L^{\mu}_{\nu\alpha\beta} = A^{\mu}_{\nu\beta,\alpha} - A^{\mu}_{\nu\alpha,\beta} + A^{\mu}_{\sigma\alpha}A^{\sigma}_{\nu\beta} - A^{\mu}_{\sigma\beta}A^{\sigma}_{\nu\alpha}$. The Ricci tensor components (3.5) are calculated for the limit $A \to 0$.

APPENDIX B: THE CONNECTION COMPONENTS $A^{\rho}_{\mu\nu}$

$$\begin{aligned} &A^0_{00} = \frac{Ab}{B} + O(A^2), \\ &A^1_{01} = A^2_{02} = A^3_{03} = \frac{\dot{a}}{a} + \frac{Ab}{B} + O(A^2), \\ &A^0_{ij} = -(1-b)\frac{\dot{a}}{a}a_{ij} - \frac{Ab}{B}a_{ij} + O(A^2), \\ &A^1_{11} = \frac{kr}{1-kr^2}, \qquad A^1_{22} = -r(1-kr^2), \\ &A^1_{33} = -r\sin^2\theta(1-kr^2), \qquad A^2_{12} = A^3_{13} = \frac{1}{r}, \\ &A^2_{33} = -\sin\theta\cos\theta, \qquad A^3_{23} = \cot\theta. \end{aligned} \tag{B1}$$

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