Notes on Frege's *Grundlagen §§55-76*

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Logical investigation of the fine structure of the concept of number

Three Fundamental Principles

1) Always to separate sharply the psychological from the logical, the subjective from the objective.
2) Never to ask the meaning of a word in isolation, but only in the context of a proposition.
3) Never to lose sight of the distinction between concept and object.

Anti-psychologism
Context Principle
Concepts are unsaturated; they are 'filled' with objects.

(Partial) Rejection of the Kantian synthetic a priori

GL§4

Frege rejects Aristotelian Logic (AL). Hence, he rejects the first Kantian criterion of analyticity. This criterion is sufficient for analyticity, but not necessary: a sentence can be analytic even if the concept of the predicate is *not* contained in the concept of the subject. "5+7=12" is analytic for Frege. For Frege the second Kantian criterion is enough for analyticity, but the second does not imply the first if AL is rejected.

Arithmetic is analytic, because arithmetic reduces to logic and logic is analytic (but N.B. this is the New Logic).
But for Frege, geometry is synthetic a priori.

	Analytic	**Synthetic**
A Priori	YES	YES
A Posteriori	NO	YES

Central notion: proof from primitive truths.

Frege-analyticity: a sentence is analytic if it is derived from general logical laws and definitions.
Frege-syntheticity: a sentence is synthetic if it is derived from non-logical truths.
Frege-a priority: a sentence is a priori if its proof depends only on general laws which need no and admit of no proof.
So, a sentence can be a priori without being analytic (e.g., geometrical truths).
Frege-a posteriority: a sentence is a posteriori if its proof depends only on assertions about particular objects. (Add?: if it depends on general laws, they should not be such that they need no and admit of no proof.)

Analyticity is something that gets transferred from the truths of logic and definitions, via a derivation, to another sentence that logically follows from them.
But what exactly gets transferred? What exactly is it that logical truths have that makes them analytic?
Three thoughts:
a) they satisfy the law of non-contradiction. But this is another logical law.

b) They are universally applicable ("universal scope"). Their domain is "everything thinkable".
Frege's hierarchical view of knowledge. GL§14
synthetic a posteriori truths (sciences)
synthetic a priori truths (geometry): spatial intuition is required (synthetic); they govern "all that is spatially intuitable" (a priori)
Both of these truths can be denied without contradiction. Hence they cannot be derived form logic alone (+ definitions).
NB. Non-Euclidean geometries.
analytic a priori truths (logic and arithmetic): universal scope.

c) they need no intuition; proof is enough. But the very general principles from which they derive are not provable. Still, their truth cannot be denied, so back to (a).

Arithmetic does not involve (pure) intuition

GL§§5-6

Rejection of Kantian intuition

The argument from large numbers

Zero.

Defence of Leibniz's insight: arithmetical truths are truths of reason.

Notion of proof.

Generation of numbers from first element (1) and the successor function.

But the gaps in Leibniz's proofs must be made explicit.

Arithmetical truths are not synthetic a posteriori

GL§§7-8

Against Mill's empiricism

"the spark of good sense" in Mill.

The status of Arithmetical laws

GL§§9-11

Arithmetical laws are not empirical generalisations

Arguments:

1. Addition is logical subordination
2. Arithmetical laws are not the product of enumerative induction.
   2a. There is no uniformity in numbers.
   2b. Numbers are outside space and time.
3. Enumerative induction requires the laws of arithmetic.

GL§§12-14

Arithmetical laws are not synthetic a priori truths.

Critique of intuition

The peculiarities of numbers make them unable to be intuited.

What numbers are not

GL§§18-45

Numbers are not properties of external things *GL§§21-23*

The concept of number is connected with the way in we regard things.

Numbers are not physical things *GL§§24-25*

There can be difference in number without any physical difference (e.g., *two* boots, *one* pair of boots).

Ergo, numbers do not supervene on physical things.

Numbers are not subjective entities *GL§§26-28*
They are not ideas, else there would be many ones.
(anti-psychologism)

Numbers are non-sensible, objective objects.

Numbers are not sets or pluralities *GL§§28-45*

Concepts

GL§§46-54
Concepts are objective.
Objects fall under concepts.
e.g., 'moon of Venus'.
Hierarchy of concepts.
Existence is a property of concepts. "Affirmation of existence is in fact nothing but the denial of number zero" *GL§53*
Hence, the ontological argument fails.

Frege's Fundamental Thought

GL§46 & *§55*
The content of a statement of number is an assertion about a concept.

Numerical Quantifiers

Frege *GL§55*
"The number 0 belongs to a concept, if the proposition that α does not fall under that concept is true universally, whatever α may be".

The number 0 belongs to concept F
$N(F)=0$ iff $\neg\exists xFx$.

"The number 1 belongs to a concept F, if the proposition that α does not fall under F is not true universally, whatever α may be, and if from the propositions

‘α falls under F’ and ‘β falls under F’

it follows universally that α and β are the same”.

The number 1 belongs to concept F

$N(F)=1$ iff $\exists x(Fx \wedge \forall y\ Fy \rightarrow y=x)$.

The number 2 belongs to concept F

$N(F) =2$ iff $\exists x \exists y(Fx \wedge Fy \rightarrow x \neq y \wedge (\forall z\ Fz \rightarrow z=x \vee z=y))$.

Generally,

“the number (n +1) belongs to a concept F, if there is an object α falling under F and such that the number n belongs to the concept ‘falling under F, but not α’”

For instance,

‘letter of the word ‘Mars’’ = 4 (n+1)

α=‘a’

‘letter of the word ‘Mars’ not identical with ‘a’’ = 3 (n).

Numerical Quantifiers abbreviated:

$\exists_0 xFx$ iff ----

$\exists_1 xFx$ iff ----

$\exists_2 xFx$ iff ----

Arithmetical equations can be expressed by numerical quantifiers:

1+3 =4

$(\exists_1 xFx \vee \exists_3 xQx \wedge (\neg \exists zPz \wedge Qz)) \rightarrow (\exists_4 wPw \vee Qw)$.

Numerical quantifiers secure the expressibility of the truths of arithmetic in purely logical notation.

But they are consistent with the claim that there are no numbers (or that numbers are not *sui generis* entities). A physicalist may well accept them and think that talk of numbers is talk of physical entities.

For instance,

(i) There is exactly one apple on the table.

$\exists x(\textit{Apple-on-the-table}x \wedge \forall y\ \textit{Apple-on-the-table}y \rightarrow y=x)$

(i) does not yet give us numbers as entities. Besides, (i) may be thought of as the proper understanding of

(ii) The number of apples on the table is one.

(ii) seems to commit us to *the number 1* but, if (ii) is understood as (i), the commitment no longer follows.

The Julius Caesar problem (*GL§56*)

'The number n belongs to the concept F' *defined as above* does not tell us what numbers are, since it does not exclude Julius Caesar being a number.

Besides, with the above definitions, we cannot prove the following:

If number α belongs to F and number β belongs to F, then $\alpha=\beta$.

Ergo, *no* conditions of identity for numbers.

And no uniqueness of numbers, since we cannot justify the 'the' (uniqueness) in the expression '*the* number that belongs to F'.

Using Frege's later terminology: We have fixed the sense of the expression 'the number n belongs to F', (for any n and F), but we have not yet fixed its *reference*: numbers as self-subsistent objects that can be recognised as the same again.

Numbers as objects

GL§57

Making clear the Fundamental Thought.

'the number 0 belongs to F'

The number 0 is predicated of a concept F. But 0 is only an element in the predicate (the number 0) and hence it is *not* a property of a concept.

The thought (I think) is this:

The schema is:

‘the number -- belongs to F’
‘the number –’ is an (unsaturated) concept. Hence, it takes objects as its arguments; hence numbers (*names of which fill the gap*) are objects.

Numerals are singular terms. Hence, true arithmetical propositions (like 1+1=2) commit us to numbers.
But Frege’s main concern is with the *application* of arithmetic to science.
Hence he goes for (ii) above being the prime form of identities that involve numbers in science.

(ii’) The number of satellites of earth is one.

Here the ‘is’ is the ‘is’ of identity: one is identical to the number of satellites of earth.
The thought: ‘one’ and ‘the number of satellites of earth’ have the same reference but different *senses*.

Numbers are abstract objects
GL§58-61
Senses are not ideas.
Words have meaning only in the context of a proposition. (Context Principle)
Reference via truth.
The meaning of a word is not a mental image or an idea.
Numbers are objective, abstract (not-sensible, not spatio-temporal) objects.

How are numbers, qua abstract objects, “given to us”?
GL§62
N.B. They are given to us neither in experience nor in (pure) intuition.
Context principle
Reference via truth.
Number-words: singular terms (their reference is *objects*).
No entity without identity. A criterion for deciding when two words refer to the *same* entity.
All this implies that the statements we should look into should be statements of *numerical identity*, viz.,

The number which belongs to the concept F is the same as the number which belongs to the concept G.
In light of the above, it should be clear how this statement is about identities of objects. (Recall *GL§57*)

Recognition judgements
Numbers are given to us via the truth of recognition judgements (numerical identities).
They are given to us as objects via singular terms that make up these true recognition judgements.
Important: by means of the context principle Frege denies that numbers can be given to us immediately in the sense in which a Platonist might say that we might have immediate non-sensible intuition of number 1, (that is in the sense in which reference precedes truth—thanks to Bill Demopoulos).

Hume's principle
GL§63
"When two numbers are so combined as that the one has always an unit answering to every unit of the other, we pronounce them equal".

The role of 1-1 correspondence
General concept of identity + concept of number ⇒ identity of numbers.

Abstraction Principles
GL§63-67

Line a is parallel to line b (a//b). This is an *identity statement*.
Introduction of the concept of direction:

(D) The direction of the line *a* is the same as the direction of the line *b* if, and only if, *a*//*b*.

Lines are given in intuition and yet directions (introduced as above) are abstract entities *not* given in intuition.
The concept of direction is discovered by a process of intellectual activity which takes its start from intuition.

The shape of the triangle ABC is the same as the shape of the triangle DEF if, and only if, ABC is similar to DEF.

Abstraction principles explain our capacity to refer to abstract objects.
'The direction of the line *a*' is a singular term; it refers to an object.
(D) enables us to identify this object as the same again (criterion of identity) under a different description ("guise"), e.g., 'The direction of the line *b*'.

BUT:
(D') The direction of the line *a* is q.

(D') cannot help us identify directions, except in the case in which q has the form 'the direction of the line *b*'.

To affirm statements like (D') we would first need a grasp of the concept of direction. But we do not have such an independent grasp.

What is the problem? Definitions (e.g., an explicit definition of direction of direction of the form: q is a direction, if there is a line *b* whose direction is q) do not say anything about objects; they only lay down the meaning of a symbol.
So with (D') we do not introduce directions as objects.

N=
GL§68

(N=) The number which belongs to the concept F is the same as the number which belongs to the concept G iff F 1-1 G.

Or, #F=#G iff F ≈ G

Note: the right hand-side does nto assert something that is based on intuition. Frege's key thought is that 1-1 correspondence is a logical relation.
Note: (N=) is a second-order principle, Number-concepts, unlike direction concepts, are second-order (concepts of concepts).

But the same problem as above arises here.
(N'=) The number which belongs to the concept F is q.

Frege's fundamental Thesis
(i) and (ii) above are equivalent and this explains how arithmetic (numbers) is applicable to reality.
$\exists_n$xFx ≡*n*=NxFX

But claims of the form *n*=NxFx can se settled only if n has the form NxGx.
The Julius Caesar problem (*GL§56*)
The definition "will not decide for us whether [Julius Caesar] is the same as the [number one]—if I may be forgiven an example which looks nonsensical".
Why not
The number which belongs to the concept 'moon of earth' = Julius Caesar?

What exactly is the Caesar problem?
If (N=) is all there is to the general concept of number, and even if we grant that numerals are singular terms and hence refer to objects, what kinds of objects they refer to is not thereby fixed.
Unless numbers are independently given by means other than (N=), there is no obstacle in taking (N=) to refer to the Roman Emperors.
Perhaps, this is not a problem in pure arithmetic. But it is a problem in applied arithmetic (and hence in science).
Recall that Frege wanted to show that numbers are unique objects. Recall that his rejection of the numerical quantifiers approach was based on the claim that it offers

no conditions of identity for numbers, and it does not explain the 'the' (uniqueness) in the expression '*the* number that belongs to F'.

Frege now faces a similar problem. For any arbitrary object, (N=) does not tell us whether it is or is not the number that belongs to a concept. Conversely, (N=) cannot tell us that 0, 1, 2, … (and only them) are *the* numbers.

Frege's Solution

GL§68-69

In the case of directions:

(P) The direction of line *a* is the extension of the concept 'parallel to line *a*'.

In the case of numbers:

(N) The number which belongs to the concept F is the extension of the concept 'equal to the concept F',

where 'equal' means equinumerous: two concepts F and G are equinumerous *iff* $F \approx G$.

Important point (GL§70-72): the notion of 1-1 correspondence does not presuppose the concept of number. It can be explained in logical terms (assuming second-order logic). A correspondence from F to G is 1-1 if an element x in F corresponds to an element y in G and no other element x' in F corresponds to y, and x does not correspond to any element y' other than y. More precisely, a correspondence from F to G is 1-1 if an element x in F corresponds to an element y in G and an element x' in F corresponds to y, then x = x', and if x also corresponds to an element y' then y = y'.

In the above definition, extensions can be thought of as sets: the number of Fs is thus the set of all concepts which are equinumerous with the concept F. Hence, number concepts are second-order concepts, viz. concepts of concepts. As Frege put it:

The same number belongs to the concept F as to the concept G iff the extension of the concept 'equal to the concept F' is identical with the extension of the concept 'equal to the concept G'.

Numbers are extensions of concepts of concepts. Hence, *numbers are objects*.

Explicit definitions of the concept of number

GL§72

This is a departure from the project associated with Hume's principle (based largely on the Caesar objection).

To remind you:

(N) The number which belongs to the concept F is the extension of the concept 'equinumerous to the concept F'.

Recall also that 'equinumerosity' is a logical concept.

Hence, '*n* is a number' means:

There exists a concept such that *n* is the number which belongs to it.

Frege now offers definitions of numbers based on his explicit definition of the number which belongs to concept F.

Derivation of Hume's Principle

GL§73

Given the explicit definition

(N) The number which belongs to the concept F is the extension of the concept 'equinumerous to the concept F'

Frege proves that (N=) can be derived from (N).

The extension of the concept 'equal to the concept F' is the same as the extension of the concept 'equal to the concept G' iff the concept F is equal to the concept G.

That is (substitute from (N))

The number which belongs to the concept F is the same as the number which belongs to the concept G iff the concept F is equal to the concept G. (N=).

This is extremely important. (N) refers to extensions while (N=) refers to concepts. The sole use of (N) in *GL* is to prove (N=). After that, Frege keeps talking about concepts.

So if (N=) can be defended independently of (N), then the paradoxes that it leads to can be avoided. This a core thought in the *neo-logicist* programme.

Explicit definitions of numbers

GL§74

0 is the number which belongs to the concept ‘not identical with itself’.

N_0 [x: x ≠x]

Definition of successor

GL§76

‘n follows in the series of natural numbers directly after m’ means:

there exists a concept F, and an object falling under it x, such that the number of which belongs to the concept F is n and the number which belongs to the concept ‘falling under F but not identical with x’ is m.

Claim: 1 is the successor of 0

Proof:

F: identical with 0

The number n that falls under F is 0. (n=0)

F’: falls under F but not identical with 0.

But F’ amounts to ‘identical with zero’, since only 0 falls under the concept ‘identical with 0 but not identical with 0’.

The number that belongs to F’ is m.

But: the number m which belongs to the concept ‘identical with zero’ follows in the series of natural numbers directly after 0(=n).

Hence, m=1.

A simpler example:

The number that belongs to the concept ‘is a letter of the word Mars’ is 4.

One object that falls under it is ‘a’.

How can we show that 4 is the successor of three?

The number that belongs to the concept ‘is a letter of the word Mars not identical with ‘a’’ is 3.

1 is the number which belongs to the concept ‘identical with zero’.
N_1 [x: x=the number of the concept N_0]

2 is the number which belongs to the concept ‘either identical with zero or with one’
N_2 [x: x=the number of the concept $N_1 \vee$ the number of the concept N_0]
and so forth.

0 is the set of all concepts under which nothing falls; 1, the set of all ‘singly instantiated’ concepts; 2, the set of all ‘doubly instantiated’ concepts; and so on.

The numbers are the objects 0,1,2,…
Associated with the sequence of concepts N_0, N_1, N_2,…

Frege’s Theorem
GL§§76

Second-order logic + HP (N=) $\Rightarrow$ PA (that every number has a successor).

Notes on Formalism

SP

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Frege against the Formalists

Some central thoughts

Grundgesetze der Arithmetik Vol. II §§86-137

Sign-formalism

Numerical signs are the essential constituents of the arithmetical theory.

Signs have representational capacity; their utility consists in the representational capacity §98

Use-mention distinction

Let a designate the smallest root of the equation (1).

Let 'a' designate the smallest root of the equation (1).

Let a be the smallest root of the equation (1).

Formalism conflates use and mention

A sign represents the same thing repeatedly.

What do the signs of the formalist represent? Shapes? But almost impossible to have exactly the same shape twice. Abstractions of actual signs? "Different sings cannot be made to coincide by abstraction, and to regard them as the same is simply to make a mistake" §99 Classes of similar sings? But classes are objects.

Frege's point cuts deep. The formalist cannot equate the sings arithmetic is about with concrete shapes or whatever. Perhaps, he then needs to treat them as abstract objects (e.g., concepts) see §100

Formalism cannot treat numerals as proper names, for there are many figures one. §100

Unmamed numbers

Game-formalism

Arithmetic is a rule-governed game with signs. Sings have no other content than that assigned to them by the rules of the game.

Cf. chess

This is consistent with there being numbers but in any case, they are not the concern of arithmetic.

Because of this it may explain the applicability of arithmetic to nature.

The status of rules

Do they license *inferences* from a to b? Inference proceeds from truths to truths.

Are they arbitrary? If yes, how can they be truth-preserving?

Are the rules definite? No. For each numeral, there must be one or more separate rules,

e.g., ‘1+1==2’ (that is, figure 1+1 may be replaced by figure 2)

‘2+1=3] (that is, figure 2+1 may replaced by figure 3)

An indefinite number of rules. §111

For Frege arithmetical propositions express thoughts; hence, they have truth-values and senses.

Application

No content, no application. “It is applicability alone that elevates arithmetic from a game to the rank of science” §91

Arithmetic has no special subject-matter. It is applicable to everything. (the same ratio can occur in many disparate domains.) Hence, its application cannot be accounted for by the individual sciences.

Mixed sentences: what kinds of rule govern them?

Formalism as a meta-thesis

A game is no theory. But there can be a theory of the game (with theorems etc.)

In the game itself, there are no theorems or proofs etc. It is only in the meta-theory that there are. Theorems about the rules of the game. So much of mathematical activity is really meta-mathematical. §93

But it seems that theorems of the theory have sense and meaning (they are about the game), while the expressions of the game have not.
The rules themselves play a double role. When the are part of the game, they have no sense (they are just strings of symbols); but when they are in the theory of the game they have sense, since they specify what moves are permitted.
For instance, an equation 'a+b=b+a' in the theory of the game expresses a rule, can be used in deductions etc. An equation 'a+b=b+*a'in* the game is meaningless. §108

Problems with Infinity

Hilbert's Programme
Mathematics has a specific subject matter. Hence disagreement with Frege (and Logicism) and agreement with Kant.
Perhaps, like Kant, maths is a body of synthetic (a priori?) truths. But it is not about the form of intuition, though intuition is central to it.
So what maths is about? What its subject-matter is?
Concrete extralogical objects, that is symbols. Numerical symbols in the case of Arithmetic. Not any symbols, however, but those that display a certain structure (which may be thought of as a *spatial* structure).

1, 11, 111, 1111, 11111, …

The numerals (the symbols) have no meaning in and of themselves (since they are concrete objects). Abbreviations can be introduced, e.g., 2 is 11, 3 is 111, etc.
Arithmetical equations can be true or false, but their truth or falsity is determined by the symbols (and the abbreviations).

2+3 = 3+2

This means equation of two symbols and reports (truly) an identity: 2+3 and 3+2 is the self-same numeral, viz., 11111.

Similarly, 3>2 reports (truly) that numeral three is longer than numeral two, i.e., 111 is longer than 11.

What is special about these truths is that they can be proved by finitary means. Generally, what is special about Arithmetic is that a certain portion of it can be done by finitist means. (Primitive recursive Arithmetic).

This means can be extended by means of further symbols, e.g.,
a+**b** =**b** +**a**

How can these formulas be understood? As schemata, which are true if their instances are true. (More on the status of these formulas in a bit).

But doesn't this create problems with infinity?

Hilbert's example:
The largest known prime number is
p= 170 141 183 460 469 231 731 687 303 715 884 105 727

One can prove (by finitary means) that
(A) there is at least one new prime number between p+1 and p!+1.

How can H. deal with this? By taking existential statements to be long disjunctions, i.e., either p+1 or p+2, or, …, or p!+1 is prime.

But (A) implies (B): there exists a prime number greater than p.
This (B) involves "a leap" into infinity. (B) is an infinite disjunction.

The same problem arises with negative universal statements: e.g., it's not the case that all numbers have a certain property.
a+1=1+**a**, for all **a**

Its denial would have to be an infinite conjunction.

So we face a dilemma: either accept infinities or revise logic (that is argue that statements with unrestricted quantifiers do not have determinate truth-values) (e.g., that for all **a, a**+1=1+**a**, is incapable of negation).

Hilbert wants to move between the horns by arguing how none of the two sides of the dilemma is compelling. His own alternative: the introduction of ideal elements. To preserve classical logic, we need to supplement finitary arithmetic with ideal elements. What are they?

Excursion into infinity.
Total infinities are illusions
No infinity in nature, since nothing is infinitely divisible.
No infinity in nature, since the universe is not infinite (though it may be unbounded).
But does infinity plays a justified role in our thinking?
Arithmetic and set theory

$1^2+2^2+\ldots n^2=1/6n(n+1)(2n+1)$
This already implies an infinity of instances (though the instances are decidable by finitary means).

Set theory: Cantor's paradise: transfinite numbers
1,2,3,4,… infinite set of order ω.
But then, ω+1, ω+2, …, ω+ω (=2ω)
2ω +1, 2ω +2, …, 2ω +ω (=3ω)
etc.

Different sizes of infinity
e.g.,
1, 2, 3, 4, …
real number [0,1]

Actual infinity vs potential infinity (which is not truly infinite).
Are we then driven out of Cantor's paradise?

Hilbert's suggestion: consider ideal points at infinity in geometry and their role in defining parallel lines. These are ideal (but useful) entities.
Similary, an *ideal number*.

So we need to supply finitary arithmetic we ideal elements. What are they? Formulas (ideal statements)
Consider
a+**b** =**b** +**a**
This is "an independent structure" which formalises the equation a+b =b +a. This last equation is already a formal structure (a schema) for getting equations like 1+2=2+1 etc. These last equations can be true or false (decidable by finitary means), but as we have seen they are true or false in virtue of the properties of the numerical symbols that appear in them. The schemata are neither true nor false (they are schemata) (they do not communicate thoughts, they do not signify anything).

Yet, they are concrete objects given in intuition. By means of these formulas we can prove other formulas, according to determinate rules (which are finitary). And we can derive from them formulas which do have meaning (like 1=2=2+1).

So mathematics is a stock of two kinds of formula: those concrete statements which refer to symbols and they can be true or false (this being dependent on the properties of the symbols) (like 1=1=2) and those formulas which are meaningless (yet, concrete) and which are the "ideal structure" of the mathematical theory. (We could say, I suppose that they are about ideal objects, but this might be misleading).

By means of these ideal statements (structure) we need not revise logic.

NB. The link with *instrumentalism*.

How is the mathematical theory built?
Formalisation in logic. (the logical calculus as a meta-mathematical calculus).

But: no reason to take logic to be anything meaningful or antecedently given. It's an ideal structure (structure of ideal statements) within which the notion of proof is formalised.

Proof becomes a formal procedure that proceeds according to rules.

The axioms on which proofs rest specify the relations among the formal concepts of the system (hence, they are meaningless).

So a mathematical proof is always a deduction schema of the form:

A

$A \rightarrow Q$

Q

So, in effect, all axioms become schemata (or disguised rules of inference)

e.g.,

$\forall x\ Ax \rightarrow Ab$

$\neg\forall x\ Ax \rightarrow \exists x\ \neg Ax$

$\neg\exists x\ Ax \rightarrow \forall x\ \neg Ax$

Or,

$a + 1 \neq 0$

The axiom of complete induction.

This is known as if-then-ism. (deductivism)

Implicit definitions.

"An axiom system is regarded not as a system of statements about a subject matter but as a system of conditions for what might be called a relational structure" Paul Bernays

Hilbets' scaffolding or schema for concepts and their relations.

How does maths apply to the world?

(i) 1+1=2

(ii) One apple and one apple makes two apples

How does maths apply to the world?

Using numerical quantifiers

$\exists_1 xFx \equiv \exists x(Fx \wedge \forall y\ Fy \rightarrow y=x)$

transform (ii) as

(ii') $\exists_1 xFx \vee \exists_1 xQx \wedge (\neg\exists zPz \wedge Qz)) \rightarrow (\exists_2 wPw \vee Qw)$

Use (ii') in deductions from other mathematical statements and draw conclusions.

These will be mathematical statements.

Then translate them back into statements like (ii).

But can there be a mathematics-free theory? Putnam's objection: science cannot be unmathematical.

Truth gives its place to consistency.

Consistency is required for showing the fruitfulness of the ideal structure. In fact, it suffices, as well.

A consistency proof is a conservativeness proof. The ideal structure is conservative over the original domain. (For otherwise, the extension would produce contradictions).

The importance of conservativeness: a means to justify the extension.

A math theory M should not just be self-consistent; it should also be consistent with any (consistent) scientific theory T. If this holds, then M + T does not allow the derivation of extra consequences that are not consequences of T.

M is conservative wrt T

Suppose (for *reductio*) it is not. That is M & T entail C. But T does not entail C.

Hence, T is consistent with not-C.

M is consistent with any scientific theory. Hence M is consistent with (T & not-C).

Then, M &T is consistent with not-C.

But then M & T does not entail C. (**absurd**).

How is consistency proved? Axioms + rules do not lead to $1 \neq 1$ as the last formula in a proof.

The motivation: certitude in mathematics. All maths should be as certain as elementary number theory.

Every mathematical problem is solvable.

Goedel’s incompleteness theorems.

Godel's First Incompleteness Theorem. Any adequate axiomatizable theory is incomplete. In particular the sentence "This sentence is not provable" is true but not provable in the theory.

By means of his Gödel numbering, he was able to produce a sentence which expressed a coded proposition to the effect:

‘This sentence is unprovable-in-Peano-Arithmetic.’

It follows, provided Peano Arithmetic is consistent, that if the Gödelian formula were to be provable, it would be true, contrary to what it itself claims. So it must be unprovable, and hence true.

Truth is not the same as provability. Truth outruns provability.

Godel's Second Incompleteness Theorem. In any consistent axiomatizable theory (axiomatizable means the axioms can be computably generated) which can encode sequences of numbers (and thus the syntactic notions of "formula", "sentence", "proof") the consistency of the system in not provable in the system.

No consistent theory can prove its own consistency.

Any system have to be more comprehensive than that envisaged by Hilbert.

Any consistency proof of a formal system requires *greater* mathematical resources than are available in the system, and so it certainly isn't possible to justify infinite sets finitistically.

Notes on What Numbers Could not Be, by Paul Benacerraf

Stathis Psillos

13/4/05

The target: the view that numbers are objects (with determinate identity conditions). Hence, the target is Platonism.

The key point: number-theory underdetermines what numbers are.
In set-theory: numbers: a progression of a certain type. Infinite ordered set N.
This account is enough for arithmetic, but it leaves entirely open what the elements of the set N are.

$\varnothing$,{$\varnothing$}, {{$\varnothing$}}, {{{$\varnothing$}}}…. (1)

$\varnothing$,{$\varnothing$}, {$\varnothing$, {$\varnothing$}}, {$\varnothing$,{$\varnothing$},{$\varnothing$, {$\varnothing$}}… (2)

N_0, N_1, N_2,… (3)

0 is the number which belongs to the concept 'not identical with itself'.
N_0 [x: x $\neq$x]
1 is the number which belongs to the concept 'identical with zero'.
N_1 [x: x=the number of the concept N_0]
2 is the number which belongs to the concept 'either identical with zero or with one'
N_2 [x: x=the number of the concept N_1 $\vee$ the number of the concept N_0]
and so forth.

So: does 1 belong to 2?
Yes according to (2) not according to (1).

So we are in a situation in which 3=b and 3=c, and yet b$\neq$c.
We have fixed the sense, but there is *no* way we can fix the reference.

N.B. This is a re-statement of the Caesar problem. But B. uses it to argue that it's wrong to think of numbers as objects in the first place. Frege, instead, used it to show that, assuming that numbers are objects, some analyses of numbers (these that do not meet the Caesar problem) are inadequate.

Couldn't it be that one of the (1) or (2) or (3) or …., is correct but we don't (cannot) know which it is?

B.'s answer: No way to tell which is the correct account. Can there exist a correct account if there is no way to tell which it is?

But is this argument excessively verificationist? Or question-begging against the Platonist? But it does pose a problem if Platonism implies that a correct account can be found.

B.'s conclusion: numbers are *not* sets.

Cf. Frege's programme

B.'s way out:

1. **Why the Caesar problem is not pressing?** Identity statements are meaningful only if the objects whose identity is sought are specified by some extra condition.

So: 3 is Julius Caesar is senseless.
No common category under which the two expressions are subsumed. But B. thinks of specific categories, e.g., number, Roman Emperor, piece of furniture etc.
Why couldn't we use the same general category, e.g., object or entity?

Cf. Locke on personal identity.

Cf. Frege
(N'=) The number which belongs to the concept F is q.
There is no problem if q= the number which belongs to the concept G.

Then why N= cannot give us numbers as *objects*? Even by B.'s standards, it seems that we can have numbers as objects in this case.
Of course, Frege thought that identity statements should be unqualified, hence (N'=). For if I cannot tell when q is *not* a number, I cannot tell what numbers are (since I have not specified them uniquely, that is, I have not excluded some objects from being numbers).

But is it meaningless to ask: is it the case that 3 is Julius Caesar?

That numbers are not sets does not imply that numbers are not objects.

2. B.'s further argument: numbers are not objects at all.

The negative thesis: any system of objects that forms a suitable progression is adequate for arithmetic.
Arithmetic is about a certain type of abstract structure. The structure is not specified by the kinds of objects that instantiate it but by the types of relations that hold between them.
The individuality of the objects that instantiate the abstract structure does not matter. What matters is the structure they jointly exhibit.

B.'s conclusion: No need to identify numbers with any objects at all.

But this is too quick. Cf. Shapiro
"places-are-offices"
"places-are-objects"

In any case, B. claims:
"For arithmetical purposes the properties of numbers which do not stem from the relations they bear to one another in virtue of being arranged in a progression are of no consequences whatsoever" (p.70).

But note the relativity of this argument. It does not follow that numbers have no other properties *simpliciter*.

It does not follow that the elements of a structure have no properties other than those relating them to other elements of the structure.
In fact, they do.

Structuralism is consistent with Platonism!

The positive thesis:
B. goes for a kind of game-formalism (number-terms are not genuinely singular terms). There are simply numerical symbols (word) and rules for manipulating them within a certain linguistic structure.

But a linguistic structure need not be an abstract structure. B. in effect takes it to be abstract because it equates it with the class of all linguistic structures of a certain sort, in which certain words function is the same ways.
e.g., ένα, δύο, τρία κλπ.
One, two, three etc.

And in any case, this abstract structure is not the same as the abstract structure that mathematicians study.
And how is truth accounted for in this context? To take arithmetic at face-value and to claim that it is true is to say that there are certain objects.
It could be truth-in-a-structure. But still, there must be objects in the structure (even if they are “places-as-objects”.

The argument:
(1) If PA is satisfiable by any set of objects, it is satisfiable by an infinite number of such sets.
(2) No reason to take one of these sets of objects as metaphysically privileged.
(3) Ergo, no unique set of objects that satisfies PA.

(4) Platonism implies uniqueness.
(5) Therefore, Platonism is wrong.

Possible challenge: Is (4) sound?

Should Platonists endorse uniqueness? But problems with semantics.

Possible challenge: Is (2) sound?

A thought: objects, **sui generis** objects, fundamental objects.

Fictionalism in Maths

Stathis Psillos 17/5/05

Four Positions

(iii) There is exactly one apple on the table.

(iv) The number of apples on the table is one.

(i) can be expressed by means of numerical quantifiers thus:

(i') $\exists$x(*Apple-on-the-table*x $\wedge$ $\forall$y *Apple-on-the-table*y $\rightarrow$ y=x)

(i') does not give us numbers as entities. It does not quantify over numbers. Besides, (i') may be thought of as the proper understanding of (ii). But (ii) does quantify over numbers. It commits us to *the number 1*.

Four possible stances:

(A)

(i) $\Leftrightarrow$ (ii)

(i) is true, hence (ii) is true. But (ii), taken at face value implies there are numbers. So there are numbers.

(A) can be taken as an ontologically inflationary position. It adds numbers to the world and shows how facts about them can be known. In essence, the Fregean approach. (Though Frege proceeded via the abstraction principle N=).

(B)

(ii) should not be taken at face value. (ii) really *means* (i). Or (ii) is reducible to (i). (i) is just a shorthand for (i). (i) is true, but if taken at face value, it does not commit to numbers. (ii) is *true*, but made true by non-numbered facts.

(B) is a reductive account. There are no numbers, but number talk is meaningful. It is made true by non-numbered facts.

(C)

(i) should not be taken at face value. (i) *really* means (ii). (i) is reducible to (ii). (i) is a shorthand for (ii). (ii) is true, so there are numbers. (i) is true too, but made true by numbered facts.

(C) is a reductive account too, but the reduction goes to the opposite direction than in (B). There are numbers. Number talk is meaningful and numerical statements are made true by numbered facts.

(D)

(ii) should be taken at face value. But (ii) is false, while (i) is true. Hence it is *not* the case that (i) ⇔ (ii). Since (ii) is false, there are no numbers.

(D) is *not* a reductive position. But it denies that there are numbers. (D) is akin to Field's fictionalism.

Fictionalism or instrumentalism about numbers

There are no numbers (or other mathematical entities), but mathematics is still useful. Mathematicalia are useful fictions.

Motivations

Nominalism

Two senses of nominalism: against universals and against abstract entities. But the two do not necessarily coincide.

Nominalism: Everything that exists is particular and concrete.

'particular' rules out universals

'concrete' rules out abstract objects.

Ontological problems with abstract objects.

In what sense do they exist if they make no causal difference?

Cf. also Benacerraf II.

Epistemological problems with abstract objects.

How can they be known?

Problems with causal theory of knowledge.

Can reliabilism help?

Field's answer: A Platonist, even if he assumes reliabilism, is forced to conclude that it's an *inexplicable coincidence* that mathematical beliefs are true. But this conclusion should undermine any putative justification conferred on those beliefs by our mathematical methods. It's not enough to have an explanation of the mathematical beliefs on the one hand, and an explanation of the mathematical facts on the other hand. What's needed for it not to be a coincidence is an explanation of the *correlatio*n. A genuine explanation of a correlation would presumably be a *counterfactual-supporting* one.

Bottom line: something wrong with the *abstract* nature of numbers.

Why take mathematics at face value? Uniform semantics. Cf. Benacerraf I.
Ergo: mathematics is taken at face value, but it is *false*; there are no mathematical entities.

What of indispensability arguments? Field says they are the only serious arguments for Platonism in maths, but they fail because mathematics is *dispensable*.

Cf. Craig's Theorem.

William Craig constructed a general method according to which given any first-order theory T and given any effectively specified sub-vocabulary O of T, one can construct another theory T' whose theorems are exactly those theorems of T which contain no constants other than those already in the sub-vocabulary O. What came to be known as Craig's Theorem is the following: for any scientific theory T, T is replaceable by another (axiomatisable) theory Craig(T), consisting of all and only the theorems of T which are formulated in terms of the observational vocabulary V_O. Craig requires that two conditions be satisfied: (1) The non-logical vocabulary of the original theory T is effectively partitioned into two mutually exclusive and exhaustive classes, one containing all and only theoretical terms, the other containing all and only observational ones. (2) The theory T is axiomatisable, and the class of proofs in T (that is the class of applications of a rule of inference with respect to the axioms of the theory and whatever has been previously inferred from them) is effectively defined. Then Craig shows how to construct these axioms of the new theory Craig(T). There will be an infinite set of axioms (no matter how simple the set of axioms of the original theory T is), but there is an **effective procedure** which specifies them. The new theory Craig(T) which replaces the original theory T is 'functionally equivalent' to T, in that all observational consequences of T also follow from Craig(T): the latter establishes all those deductive connections between observation sentences that the initial theory T establishes. So, for any V_O-sentence O_0, if T implies O_0 then Craig(T) implies O_0. This point was ceased upon by instrumentalists, who argued that theoretical commitments in science were dispensable: theoretical terms can be eliminated *en bloc*, without loss in the deductive connections between the observable consequences of the theory.

Field, against Craig's theorem, claims that theoretical terms are indispensable in science. Considerations of simplicity etc. **Attractive theory**.
But maths is dispensable.

How is this possible?
Conservativeness of mathematics

A mathematical theory M is conservative if, for any body of nominalistic assertions N and any particular nominalistic assertion Φ, then Φ is not a consequence of M + N unless it is a consequence of N.

The importance of conservativeness: a means to justify the extension of a theory.
A math theory M should not just be self-consistent; **it should also be consistent with *any* (consistent) scientific theory T**. If this holds, then M + T does not allow the derivation of extra consequences that are not consequences of T.
M is conservative wrt T
Suppose (for *reductio*) it is not. That is M & T entail C. But T does not entail C.
Hence, T is consistent with not-C.
M is consistent with any scientific theory. Hence M is consistent with (T & not-C).
Then, M &T is consistent with not-C.
But then M & T does not entail C. (**absurd**).

Intuitively, conservativeness captures the traditional idea that maths has no empirical content. Traditionally, this idea was captured by the thought that maths is a body of a priori and necessary truths (assertions that do not suffer truth-value revisions no matter what the empirical world is like). Fieldian conservativeness might be thought of something like this: necessity – truth.

Indeed, Conservativeness is equivalent with the following:
Let A be a nominalistic assertion. Then A is not a consequence of M unless A is a logical truth.
(This captures the claim that mathematics has no empirical consequences).

Conservativeness does not imply falsity, but explains the usefulness of mathematics. Maths is useful because it is a **conservative extension** of mathematics-free (that is nominalistic) scientific theories. It facilitates deductions. All that is required for this is that a mathematical theory M be consistent. Then, its conservativeness explains its usefulness. Mathematics is a body of *useful* fiction.

How is the falsity of maths grounded?
N.B. Conservativeness does not imply falsity. If a math theory M is consistent then it *could* be true. So what is the argument for falsity?

No direct argument. Indirect argument: the only reason we have for believing that mathematicalia exist, viz., the indispensability argument, is no good. Mathematics is *dispensable* to science. So Field does not offer a direct argument for nominalism. Rather, he undercuts the grounds for being Platonist.

Pure maths vs applied maths. The indispensability argument applies to applied maths (those math theories that are used in science). So, Field's argument, if correct, undercuts commitment to mathematicalia that play a role in science.
What about pure maths? This is already a problem for Platonists that base their views on the indispensability argument.
Field (like Hilbert before him) goes for mere consistency. Pure maths is a fictional game whose only rule is consistency. But consistency does *not* imply existence.

How do we get to dispensability?

Remember, Field accepts uniform semantics. If there are no math entities, then mixed statements (which employ math and physical vocabulary) will be false (even if there are the relevant physical entities). Hence, current physical theories are, strictly speaking, false (since they employ math language).

Field embarks on a nominalisation programme. This is important for two reasons. *First*, it shows that there are nominalised versions of physical theories and hence that these versions can be true. Better, if a fictionalist about maths says that a physical theory is true, he means that its nominalised version is true. *Second*, it shows how mathematics can be dispensed with in formulating physical theories.

Nominalisation: Take a physical theory T which uses mathematical vocabulary. Replace it with another physical theory T' which has exactly the same nominalistic consequences as T, but is mathematics-free.

Take any assertion A of a physical theory. Replace it with another assertion A* which is nominalistically acceptable, i.e., it does not quantify over mathematical entities.

A simple example:

(iii) One apple and one apple makes two apples

Using numerical quantifiers

$\exists_1 xFx \equiv \exists x(Fx \wedge \forall y\ Fy \rightarrow y=x)$

transform (iii) into

(iii') $\exists_1 xFx \vee \exists_1 xQx \wedge (\neg\exists zPz \wedge Qz)) \rightarrow (\exists_2 wPw \vee Qw)$.

How does maths facilitate deductions?

(iii) One apple and one apple makes two apples

(iv) 1+1=2

(iv) is the "abstract counterpart" of (iii).

Use (iii) in deductions with other mathematical statements (abstract counterparts of nominalistic statements) and draw conclusions. These will be mathematical statements. Then translate them back into nominalistic statements like (iii).

Say A1* and A2* are nominalistic premises.

Use their abstract counterparts M1, M2

M1&M2 imply M3.

Translate M3 into A*3.

Conservativeness entail that A1* & A2* imply A*3.

We have proved A*3 in N+M, hence, by conservativeness, it is provable in N alone.

Field's example

Let's introduce new numerical quantifiers. These are part of the language of theory N. Note that these quantifiers do not quantify over numbers.

Theory N also has some other non-mathematical vocabulary.

Let theory S contain arithmetic plus some set theory.

Now suppose N contains these sentences:

1. There are exactly twenty-one aardvarks. ($\exists_{21}$x A(x)
2. On each aardvark there are exactly three bugs
3. Each bug is on exactly one aardvark

Suppose we want to know whether the following sentence is a consequence of N:

4. There are exactly sixty-three bugs.

(4) *is* a consequence of 1-3. But proving this in the language of N is cumbersome. We can move more quickly. Consider the addition of S to N. We can then infer the following from 1-3:

1'. The cardinality of the set of aardvarks is 21

2'. All sets in the range of the function whose domain is the set of aardvarks, and which assigns to each entity in its domain the set of bugs on that entity, have cardinality 3

3'. The function mentioned in 2 is 1-1 and its range forms a partition of the set of all bugs

1'-3' are "abstract counterparts" of 1-3, respectively, in that the equivalences $1\leftrightarrow 1'$, $2\leftrightarrow 2'$, and $3\leftrightarrow 3'$ are provable in N+S.

We can now use basic set theory and arithmetic, including the mixed set theory that guarantees that there exists a set of all and only the aardvarks.

We can then prove the following in N+S:

(a) if all members of a partition of a set X have cardinality α, and the cardinality of the set of members of the partition is β, then the cardinality of X is $\alpha \cdot \beta$

(b) the range and domain of a 1-1 function have the same cardinality

(c) $3\cdot 21=63$

So from 1'-3' we can derive:

4'. The cardinality of the set of all bugs is 63.

But 4' follows from 4 (4' is 4's abstract counterpart). Since the mathematical theory S is a conservative extension of N, we can infer that 4 follows from the theory N (and in particular 1-3), without actually deriving 4 from 1-3.

The case of Hilbert's Geometry

Primitives: 'x is between y and z': means intuitively that x is a point on the line segment between y and z (inclusive)

'x, y are congruent to z, w': means intuitively that the distance between x and y is the same as the distance between z and w

The theory includes a number of axioms governing these primitives. For example, the following should be axioms (or consequences of axioms):

If y Bet xz then y Bet zx

If y Bet xz and z Bet yw then y Bet xw

xy Con xy

If y Bet xz and xy Con xz then y=z

Usually we develop geometry using the *numerical* notion of distance. But Hilbert showed that we can have this without using numbers in the axioms. For we can prove *representation* and *uniqueness* theorems.

Definition: a *legitimate distance function* is a two-place function, d, defined on points, such that:
(a) for any points x, y, z and w, xy Con zw iff d(x,y)=d(z,w)
(b) for any points x, y, and z, y Bet xz iff d(x,y)+d(y,z)=d(x,z)

Representation theorem: there exists a legitimate distance function

Uniqueness theorems: if d_1 and d_2 are legitimate distance functions then R(d_1, d_2), where in this particular case R would be "linear transformations" — i.e., "R(d_1, d_2)" would be replaced with "for some constant, c, for any x, y, $d_1(x,y)=c \cdot d_2(x,y)$". So we say "the distance function is unique up to linear transformation". This corresponds to the fact that a unit of measure is arbitrary.

More generally:
A fictionalist should construct theories that are at least as good as current theories in mathematical physics, but don't entail the existence of mathematicalia. This means
(I) Find a vocabulary for describing the physical structure of the world 'intrinsically', without reference to mathematical entities.
(II) Set up definitions, and proving *representation theorem*s. These show how a physical structure can be related to (represented by) a mathematical structure.
(III) Based on these representation theorems use the mathematical theory to draw conclusions etc.
(IV) State some simple laws using nominalistic vocabulary which have all the same nominalistic consequences as the mathematical physical theory, but don't entail the existence of mathematical entities.

Motivation: intrinsic vs extrinsic explanation
Physical (nominalistic) explanation are intrinsic e.g., electrons instinsically explain the silver-grey tracks.

Mathematical explanations are extrinsic. Math entities (if they exist) are extrinsic to the process to be explained (This is mainly because math entities, if they exist, are acausal.)
Methodological Principle I: Intrinsic explanations are to be preferred to extrinsic ones.
Methodological Principle II: Underlying every good extrinsic explanation there is an intrinsic explanation.

Representation theorems link physical facts with abstract counterparts. They then facilitate deductions, but given Conservativeness, the deductions can be case in purely intrinsic terms.
e.g., Newtonian mechanics.

Field's Extended Representation Theorem
For any model of a theory N with space-time S that uses comparative predicates but not numerical functors there are:
(a) a 1–1 spatio-temporal co-ordinate function $\Phi : S \rightarrow R^4$, which is unique up to generalised Galilean transformation,
(b) a mass density function $\rho : S \rightarrow R^+ \cup \{0\}$, which is unique up to a positive multiplicative transformation, and
(c) a gravitation potential function $\Psi : S \rightarrow R$, which is unique up to positive linear transformation,
all of which are structure preserving (in the sense that the comparative relations defined in terms of these functions coincide with the comparative relations used in N); moreover, the laws of Newtonian gravitational theory in their functorial form hold if Φ, ρ, and Ψ are taken as denotations of the relevant functors.

Rich Physical Ontology
Space-time points
Magnitudes are defined on space-time points

Objections to Nominalism
Technical
Shapiro I: Mathematics is not really conservative.
We can construct in nominalistic physics (in particular Nominalised Newtonian Mechanics, NNM) a Goedel-sentence g 'saying' that NNM is consistent. But g cannot

be proved within NNP (because of Goedel's Second Incompleteness Thm), though it can be proved within M + NNM. So M is not deductively conservative over NNM.

N.B. Field's nominalistic theories are second-order.

N.B. Field has shown that M is semantically conservative over nominalistic theories N, viz., if N+M $\models$ s, then N $\models$ s. But since second-order logic is not complete, we cannot infer that M is deductively conservative over N. In any case, it is not. Semantic conservativeness still holds, but given the lack of syntactic conservativeness, maths is no longer just a useful instrument for shortening deductions.

Field retreated to first-order formulations of nominalistic theories. But **Shapiro II**: within first-order logic, Field's representation theorems cannot be proved. Hence, Field can no longer claim that the usefulness of maths can be seen by means of representation theorems.

How is logical consequence defined? In terms of models. But models are abstract entities.
Field's reply: the Tarskian definition of logical consequence is not literally true. Logical consequence (or consistency, or necessity) cannot be defined: it is a "primitive" notion that we understand by learning rules of use.

Philosophical
Is Field's dispensability strategy generalisable? E.g., Quantum Mechanics
Is his preferred ontology (space-time points) nominalistic? What are regions of space-time points if not *sets* of space-time points?
Can we really distinguish the mathematical vocabulary from the physical vocabulary?

Field's more recent views: mathematical objects vs mathematical objectivity.
Claim: a fictionalist can account of mathematical objectivity. It is logical objectivity (which does not require mathematical objects). The objectivity of logical relations.
Mathematical objectivity does not require mathematical truth. (Putnam's model-theoretic argument—truth is too easy.)

Weakening fictionalism (Balaguer)
Nominalistic scientific realism
(NC) Empirical science has a purely nominalistic content that captures its "complete picture" of the physical world.
(COH) It is coherent and sensible to maintain that the nominalistic content of empirical science is true and the platonistic content of empirical science is fictional.

The argument rests heavily on the claim that mathematical entities, if they exist, are causally inert. So, there would be no empirical difference (no change in the world?) if they did not exist.

(A) "The physical system S is forty degrees Celcius".

Though (A) expresses a *mixed* fact, it does not express a *bottom-level* mixed fact (since the number 40, being causally inert, does not contribute causally to the temperature that S has.) The truth that (A) expresses supervenes on more basic facts that are *not* mixed. It supervenes on a purely physical fact and a purely mathematical fact. But this suggests that (A) has a nominalistic content "that captures its complete picture of S: that content is just that S holds its end of the '(A) bargain', that is, S does its part in making (A) true" (Balaguer, 133).

But is this right? Is the picture complete without the mathematical fact?

Balaguer claims that since physical facts and mathematical facts are independent of each other, it can be the case that there are physical facts of the sort needed to make an empirical statement true, but no mathematical facts.

Then, theories are strictly speaking false. Balaguer argues that capturing the nominalistic content of theories is enough, since we do not lose "any important part of out picture of the physical world" (134).
"The nominalistic content of a theory T is just that the physical world holds up its end of the 'T bargain', that is, does its part in making T true" (135).
Maths is no longer dispensable, but it is true that nominalistic facts would obtain and keep their end of the bargain, even if there were no mathematical facts.

In the end, fictionalists do not have to replace platonistic theories with nominalistic ones. They need to argue that when these theories are accepted, they commit us only to the truth of their nominalistic consequences.

Nominanilistically adequate theories.