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DETERMINISTIC APERIODIC TILE SETS

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Abstract

Wang tiles are square tiles with colored edges. We construct an aperiodic set of Wang tiles that is strongly deterministic in the sense that any two adjacent edges of a tile determine the tile uniquely. Consequently, the tiling group of this set is not hyperbolic and it acts discretely and co-compactly on a CAT(0) space.

1 Introduction

Eberlein in [E1] posed the problem of characterizing algebraically compact non-positively curved manifolds which do not admit a Riemmanian metric of strictly negative sectional curvature. More precisely he asked if the fundamental group of such manifolds always contains a free abelian subgroup of rank 2. Bangert and Schroeder [BaS] have shown that this is true under the additional assumption that the manifold is analytic.

Gromov remarks in [Gr1] that one can ask the same question in the more general context of CAT(0) spaces: Let X be a locally finite, metric polyhedron satisfying the CAT(0)-inequality and let G be a group acting co-compactly on X. Assume moreover that X is not hyperbolic in the sense of Gromov. Does then G contain a free abelian subgroup of rank 2?

Eberlein in [E2] showed that if M is a compact, non-positively curved manifold which does not admit a metric of strictly negative sectional curvature, then there is an isometric embedding $f : \mathbb{R}^2 \to \tilde{M}$. Gromov in [Gr2, sec. 4.2C] (see also [Br]) showed that this generalizes to metric polyhedra satisfying the CAT(0)-inequality and which are not hyperbolic.

Gromov in [Gr2, sec. 4.7.A], notices that there is an affinity between the question mentioned earlier and the existence of aperiodic sets of tiles. Wang in 1961 conjectured that if there is a tiling of the plane using a finite set of tiles then there is a periodic tiling of the plane for the same set of tiles. Periodic here means that $\mathbb{Z}^2$ acts freely by color preserving translations on this tiling. Berger in 1966 disproved this conjecture: He found a finite set of colored square tiles such that no tiling of the plane by translates of these tiles is periodic.

Although the similarity between these two problems is striking to our knowledge all examples of groups associated to tilings are not non-positively curved. In this paper we describe a set of square tiles whose associated complex satisfies the CAT(0)-inequality. Moreover the fundamental group of this complex is biautomatic. However our example does not give a counter-example for the question mentioned earlier as it contains a free abelian subgroup of rank 2.

CAT(0) complexes have been studied recently by a number of authors (see [GSho], [BBri1,2], [W], [BBuy], [H], [BuM]). Gersten and Short in [GSho] showed that the fundamental groups of 2-dimensional CAT(0) complexes satisfying small cancellation conditions (like the suare complexes that we will be considering here) are automatic. The theory of automatic and bi-automatic groups is, among other things, an attempt to define non-positively curved groups (generalizing hyperbolic groups) in the framework of combinatorial group theory. The abelian subgroup question is open in this context as well: Does a (bi)automatic group which is not hyperbolic contain a subgroup isomorphic to $\mathbb{Z}^2$?

Ballmann and Brin in [BBri2] show that the fundamental group of a 2-dimensional CAT(0) complex is either virtually $\mathbb{Z}^2$ or it contains a *non-abelian* free group of rank 2. They also show in [BBri1] that the fundamental groups of (3,6)-complexes (CAT(0)-complexes made out of hexagons) are either hyperbolic or contain subgroups isomorphic to $\mathbb{Z}^2$. Wise in [W] constructed examples of 2-dimensional CAT(0) square complexes whose fundamental group is not residually finite and Burger and Mozes (see [BuM]) constructed such complexes with simple fundamental group. Ballmann and Buyalo in [BBuy] show that there is a CAT(0) square complex whose fundamental group is hyperbolic which does not admit a metric of strictly negative curvature.

A set of Wang-tiles T is a finite set of squares with colored oriented edges placed on the plane with their edges horizontal or vertical. Conway (see [CL]) has associated to a tile set T a group, defined as follows:

Let $C(T)$ be the complex obtained by gluing the squares in T along the sides which have the same color respecting their orientation. If $C(T)$ is connected we define the tiling group of T, $G(T)$, to be the fundamental group of this complex. Note that if the plane can be tiled by T and $C(T)$ is not connected then the plane can be tiled by a proper subset of T, T_1, whose associated complex $C(T_1)$ is connected.

We call a set of Wang tiles strongly deterministic if for all colors C_1, C_2

there exists at most one tile with adjacent edges colored by C_1, C_2. This imposes interesting properties on valid tilings. It is easy to see that any valid tiling is now uniquely determined by tiles along any continuous curve that intersects every horizontal and vertical line of the plane. We say that the tiles along the curve *force* all other tiles. In the same way, any connected pattern forces the tiles on the smallest rectangle containing the pattern.

The following observation provides the connection between tilings and the question mentioned earlier.

Fact. *If T is an aperiodic and strongly deterministic set of Wang-tiles then the (metric) complex $C(T)$ satisfies (locally) the* CAT(0) *inequality. Moreover the fundamental group of this complex, $G(T)$, is not hyperbolic.*

To see this note that since T is strongly deterministic if v is a vertex of $C(T)$ every simple closed path in the link of v contains at least four edges. Therefore $C(T)$ is locally CAT(0). On the other hand, given a tiling of the plane by T there is an obvious way to map isometrically the tiled plane to the universal covering of $C(T)$. Therefore $\widetilde{C(T)}$ is not hyperbolic.

This fact shows that to find a tiling group which acts discretely and cocompactly on a CAT(0) space and which is not hyperbolic it is sufficient to find a strongly deterministic set of Wang-tiles that tile the plane only aperiodically. We do this in the rest of this paper.

2 Preliminaries

Wang-tiles are unit squares with colored edges. A *tile set* T is a finite collection of Wang-tiles, placed with their edges horizontal and vertical. A *tiling* is a mapping $f : \mathbb{Z}^2 \to T$ that assigns a Wang-tile at each integer lattice point of plane. Tiling f is *valid* at point $(x, y) \in \mathbb{Z}^2$ if the four edges of the tile in position (x, y) have the same color as the abutting edges of the adjacent tiles, i.e. if the upper edge of $f(x, y)$ has the same color as the lower edge of $f(x, y+1)$, the left edge of $f(x, y)$ has the same color as the right edge of $f(x-1, y)$, etc. Note that the Wang tiles may not be rotated. Tiling f is valid if it is valid at all points $(x, y) \in \mathbb{Z}^2$.

A tiling f is *periodic* with period $(a, b) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ iff $f(x, y) = f(x + a, y + b)$ for every $(x, y) \in \mathbb{Z}^2$. If there exists a periodic valid tiling with tiles of T, then there exists a *doubly periodic* valid tiling [R], i.e. a tiling f such that, for some $a, b > 0$, $f(x, y) = f(x + a, y) = f(x, y + b)$ for all $(x, y) \in \mathbb{Z}^2$. A tile set T is called *aperiodic* iff (i) there exists a valid tiling, and (ii) there does not exist any periodic valid tilings. The existance of aperiodic tile sets was first proved by Berger [Be] in 1966. He used his aperiodic tile set to prove that the *tiling problem* is undecidable: Given a tile set the tiling problem asks whether there exists any valid tilings of the plane. According to Berger's result there does not exist any algorithm that would solve the problem for all tile sets. A simplified proof was presented later by Robinson [R], based on another aperiodic tile set. Since then many other aperiodic tile sets have been constructed, see Chapter 7 of [GruSh]. The smallest known aperiodic set contains 13 tiles [Cu], [K2].

A tile set T is called *NW-deterministic* if there do not exist two different tiles in T that would have same colors on their upper and left edges, respectively. In other words, for all colors C_1 and C_2 there exists at most one tile in T whose upper and left edges have colors C_1 and C_2, respectively. It was shown in [K1] that there exist aperiodic NW-deterministic tile sets. Based on such a set it was shown that the tiling problem is undecidable even in the restricted class of NW-deterministic tile sets. SW-, NE- and SE-deterministic tile sets are defined analogously. In the present work it is shown that there exists aperiodic tile sets that are deterministic in all four directions simultaneously (Theorem 1). We call a tile set *4-way deterministic* if it is NW-, SW-, NE- and SE-deterministic.

An even stronger condition can be satisfied: There exist aperiodic tile sets in which for all colors C_1 and C_2 there exists at most one tile having two adjacent edges colored by C_1 and C_2 (Corollary 1).

Let S and T be two tile sets, and let $\varphi : S \to T$ be a function. We say that φ is a *tile homomorphism* if it respects colors, that is, $\varphi(s_1)$ and $\varphi(s_2)$ have identical colors on some edges if s_1 and s_2 have identical colors on the corresponding edges.

Let us apply φ to tilings by applying φ to all tiles separately:

$$\forall x, y \in \mathbb{Z} : \varphi(t)(x, y) = \varphi\big(t(x, y)\big) .$$

If φ is a homomorphism then it preserves valid tilings, that is, if f is a valid tiling using S then $\varphi(f)$ is a valid tiling using T.

Our general framework for constructing a 4-way deterministic aperiodic tile set is the following: Let T be an aperiodic tile set. We use Robinson's aperiodic tile set [R] (see Figure 1) as T. Consider a fixed valid tiling f using T. It is enough to construct a 4-way deterministic tile set S that can produce the same tiling f, in the sense that there exists a valid tiling g using S and a tile homomorphism $\varphi : S \to T$ that maps g into f. Then tile set

$$\big\{(s, t) \in S \times T \mid \varphi(s) = t\big\}$$

where adjacent tiles have to match in both S- and T-components is aperiodic and 4-way deterministic. It is 4-way deterministic because S is 4-way deterministic and the S-components uniquely determine the T-components of the tiles. It does not allow any valid periodic tiling because T does not allow any valid periodic tilings. And there is at least one valid tiling, obtained from g and f.

3 A Deterministic Aperiodic Tile Set

In this section we describe a detailed construction of an aperiodic 4-way deterministic tile set. Robinson's aperiodic tile set is shown in Figure 1. There are seven different tiles: one *cross* and six *arms*. The tiles may be rotated freely, which increases the number of tiles to 28. Instead of colored edges the tiles have incoming or outgoing arrows on the sides: The tiling is valid if each arrow meets an arrow with the same direction in the neighboring tile. The arrows are used in order to make tilings more readable – they can be naturally replaced by colors that code the directions and positions of the arrows.

Each tile contains *central arrows* in the middle of their edges and possibly some additional *side arrows*. A central arrow together with a side arrow is called a *double* arrow, a central arrow alone is a *single* arrow.

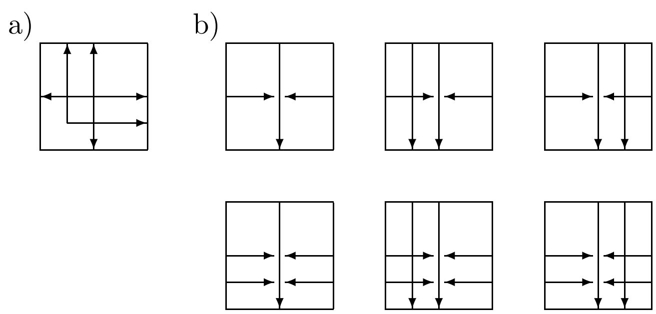

Figure 1: Robinson's seven basic tiles: a) a cross and b) arms.

The first tile containing outgoing arrows on all four edges is called a cross. The cross is said to face to the two directions of its double arrows. The cross shown in Figure 1 faces up and to the right.

The other tiles are called arms. Every arm contains a *principal* arrow: the central arrow running across the tile from one side to the opposite side. The arm is said to point to the direction of its principal arrow. The arm may also have a side arrow parallel to the principal arrow. The side arrow may be on either side of the principal arrow. Each arm also has two incoming arrows at right angles to the principal arrow. If the incoming arrows have side arrows then they must be toward the head of the principal arrow.

As in [R], a cross must be forced to occur in alternate columns in alternate rows. This is accomplished by adding a new component – a parity tile – to every basic tile. The four parity tiles are depicted in Figure 2. In every valid tiling of the plane the parity tiles alternate both horizontally and vertically.

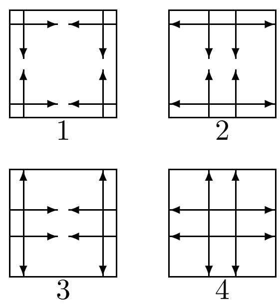

Figure 2: The parity tiles.

The parity tiles are attached to the basic tiles so that parity tile 1 in Figure 2 is attached only to the crosses. Parity tiles 2 and 3 are attached to vertical and horizontal arms, respectively. Finally, parity tile 4 may be attached to any of the basic tiles. Each basic tile has two possibilities for the parity tiles so that the total number of tiles becomes 56. The index 1,2,3 or 4 of the parity tile will be called simply the parity of the tile.

The set of 56 tiles described above is exactly the same that was used by Robinson in [R], so that his analysis of possible tilings can be used. For each positive integer n, four $(2^n - 1)$-squares are defined recursively. A cross with parity 1 is a 1-square. There are four 1-squares because there are four possible orientations of the cross.

For every $n \geq 2$ a $(2^n - 1)$-square consists of four $(2^{n-1} - 1)$-squares facing each other, separated by a cross and rows of arms leading radiately out from the center (see Figure 3). The cross in the center is called the *central cross* of the $(2^n - 1)$-square. There are four $(2^n - 1)$-squares because the orientation of the central cross is arbitrary. The $(2^n - 1)$-square is said

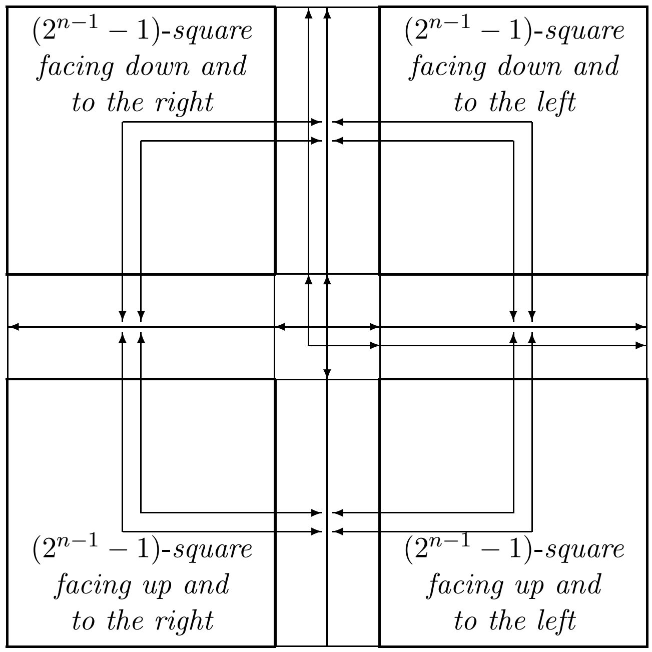

Figure 3: Constructing a $(2^n - 1)$-square facing up and to the right.

to face to the same directions as its central cross. For examples of 3- and 7-squares see [R] or [GruSh]. Because the tiling is valid inside (2^n-1)-squares for all n, there exist valid tilings of the infinite plane.

The following lemma was proved in [R]. It proves that Robinson's tile set is aperiodic.

LEMMA 1. *In any valid tiling of the plane using Robinson's tiles every* 1*-square (i.e. a cross with parity 1) belongs to a unique* 3*-square,* 7*-square,* 15*-square, etc. Consequently the tiling is aperiodic.* □

Let us see next how deterministic Robinson's tile set is. Assume that the arrows on two adjacent edges of a tile are known. The following three cases are possible.

1. On both edges the arrows point away from the tile: The tile has to be a cross. Its orientation is determined uniquely by the location of side arrows on the two known edges.
2. One edge contains incoming and one edge outgoing arrows: The tile is an arm directed towards the outgoing arrow. The position of side

arrows is determined uniquely by the side arrows on the two known edges.

3. On both edges the arrows point into the tile: Also in this case the tile has to be an arm. However there are two possibilities for its direction. The positions of side arrows are determined uniquely.

Case 3 shows that Robinson's tile set is not deterministic in any diagonal direction. We modify the tile set in such a way that the orientation of an arm is determined by its neighbors. It is sufficient to know whether the arm is horizontal or vertical.

The new tiles have *labeled arrows* on their edges. Each arrow has to meet an arrow with the same label and direction in the next tile. Each edge contains three arrows, all of which have the same direction. (We could as well use just one arrow whose label is a triple.) The arrow in the center is called a central arrow, the other two arrows are side arrows.

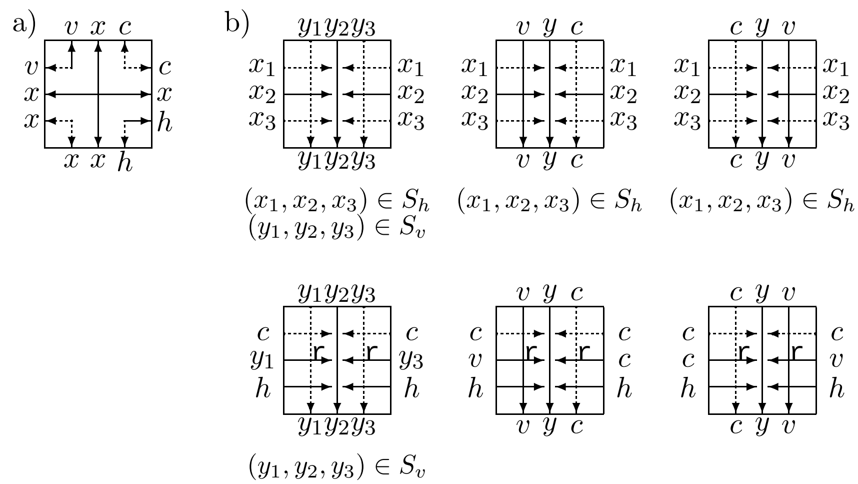

Figure 4: The seven types of modified tiles. Variables x and y denote any label in $\{v,h,c\}$. Labels (x_1,x_2,x_3) come from $S_h=\{(x,x,v),(v,x,x)\mid x=v,h \text{ or } c\}$, and labels (y_1,y_2,y_3) from $S_v=\{(y,y,h),(h,y,y)\mid y=v,h \text{ or } c\}$. When tiles are rotated by 90° labels h become v, and vice versa. Solid arrows correspond to the arrows on Robinson's tiles.

Labels h,v and c will be used (referring to horizontal arms, vertical arms and crosses, respectively). Available tiles are shown in Figure 4. Tile a) is a cross – the others are arms. The direction of an arm is the direction of its principal arrow, i.e. the central arrow that goes through the tile. The tiles may be rotated arbitrarily, but whenever a tile is turned 90° each label v has to be changed to h and vice versa. Parity tiles of Figure 2 will be attached in the same way as in the case of Robinson's tiles: Parity tiles $1, 2$ and 3 are combined with crosses, vertical arms and horizontal arms, respectively, and parity tile 4 may be combined with any tile.

In Figure 4 some arrows are drawn solid and some dashed. This indicates how the tiles relate to Robinson's tiles: If the dashed lines and labels are removed from a tile, the corresponding Robinson's tile is obtained. This is explained in more detail below.

Note that only some triples of labels are possible: On vertical arrows triples in

$$S_v = \{(x, x, h) \text{ and } (h, x, x) \mid x \text{ arbitrary } \}$$

correspond to single arrows, and triples in

$$D_v = \{(v, x, c) \text{ and } (c, x, v) \mid x \text{ arbitrary } \}$$

correspond to double arrows. Similarly for horizontal arrows, elements of

$$S_h = \{(x, x, v) \text{ and } (v, x, x) \mid x \text{ arbitrary } \}$$

represent single arrows, and

$$D_h = \{(h, x, c) \text{ and } (c, x, h) \mid x \text{ arbitrary } \}$$

represent double arrows. It is important to note that, in the case of arms having incoming double arrows at right angles to the principal arrow (i.e. the tiles on the lower row in Figure 4), the incoming central arrows have the same labels as the side arrows of the principal arrow. This is indicated in Figure 4 by small dots at the intersection points.

There exists a tile homomorphism from the new tile set into the Robinson's set. The homomorphism is obtained if each triple of labeled arrows on an edge are replaced by single and double arrows as follows:

- On vertical arrows elements of S_v represent single arrows. Combinations (v, x, c) (and (c, x, v)) belonging to D_v represent double arrows, where the side arrow is on the left side (right side, respectively) of the central arrow. (The side arrow is on the same side as label v.)
- On horizontal arrows elements of S_h represent single arrows, and combinations (h, x, c) and (c, x, h) of D_h represent double arrows, where the side arrow is on the same side as label h.

In Figure 4 the solid arrows indicate the homomorphic images of the tiles: the tiles are in the same order as their images in Figure 1.

The homomorphic preimages of $(2^n - 1)$-squares will be called simply $(2^n - 1)$-squares. Because there is a tile homomorphism to Robinson's aperiodic tile set, there are no valid periodic tilings by the new tile set. In fact, in every valid tiling each tile with parity 1 belongs to a unique (2^n-1)-square, for every $n \geq 2$ (see Lemma 1).

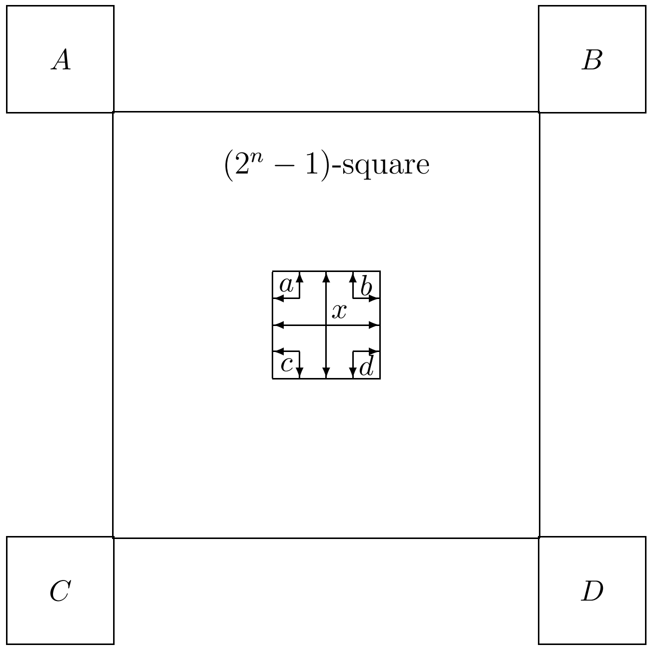

Figure 5: Interpretation of labels: a, b, c and d in the central cross of a (2^n-1)-square characterize tiles A, B, C and D outside the square, respectively.

The idea behind the labels is to code information about tiles outside the corners of a (2^n-1)-square in its central cross. Consider the (2^n-1)-square of Figure 5, and the four tiles denoted by A, B, C and D immediately outside its corners. The labels a, b, c and d of the side arrows in the central cross characterize tiles A, B, C and D, respectively. For example, $a = v$, $a = h$ or $a = c$ mean that tile A is a vertical arm, a horizontal arm or a cross, respectively. The label x of the central arrow characterizes the tile in the corner that is opposite the faces of the square. For example, $x = c$ on a cross facing up and to the right.

Keeping this interpretation of labels in mind, one can easily build arbitrarily large (2^n-1)-squares. As an example, Figure 6 contains the 7-square facing up and to the right. The labels of the central cross uniquely determine all other labels. Note especially how the arms in the middle of the rows of arms leading out from the central cross transmit information correctly to the quadrants: The arms are of the type represented on the lower row of Figure 4, so that the meeting central arrows have the same labels as the side arrows of the principal arrow.

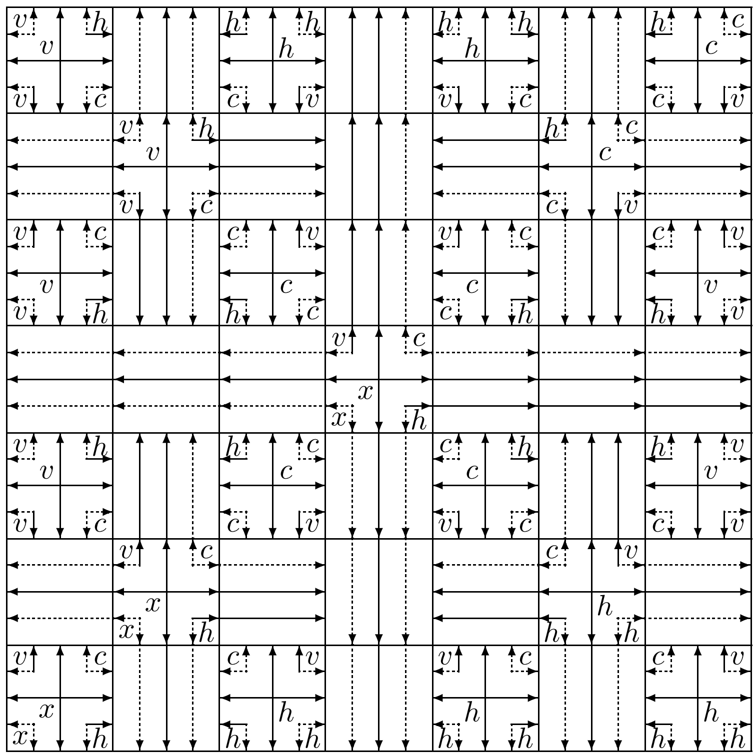

Figure 6: The 7-square facing up and to the right. Only the principal arrows and their side arrows of the arms are drawn. Labels are given in the crosses where the rows of arrows start. Label x is arbitrary. Arrows that are removed by the tile homomorphism into Robinson's tile set are drawn dashed, arrows that remain are drawn solid.

Finally a restriction is imposed that makes the tile set deterministic in all four diagonal directions. Consider a tile with parity 4, as in Figure 7. The tiles on its right and left sides are horizontal arms (since they have parity 3), and the tiles above and below are vertical arms (parity 2). Its diagonal neighbors are crosses (parity 1). It will be required that the labels of the closest meeting side arrows in the parity 2 and 3 tiles characterize the tile with parity 4. In Figure 7 this means that label x is v, h or c

if the parity 4 tile in the center is a vertical arm, a horizontal arm or a cross, respectively. This restriction is satisfied on $(2^n - 1)$-squares, since the crosses with parity 1 form 1-squares, and according to the interpretation above, the labels of their side arrows characterize correctly their diagonal neighbors, that is, tiles with parity 4.

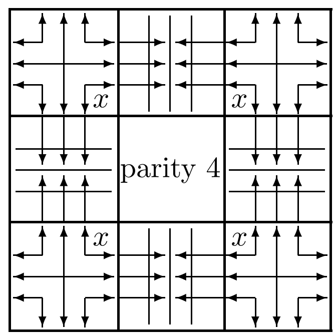

Figure 7: The restriction on tiles with parity 4: Label x has to characterize the tile in the center.

The restriction can be implemented by labeling arrows on the parity tiles using labels v, h and c, as shown in Figure 8. All arrows of the parity tile 4 are labeled by the same label. If the label is v (h or c, respectively) the tile may be combined only with a vertical arm (horizontal arm or cross, respectively). The incoming vertical arrows of the parity tile 2 are labeled but the horizontal arrows are unlabeled. The labels of the incoming arrows may be different on the upper and lower edges. The tile may be combined with vertical arms, in which the side arrows of the meeting arrow heads are labeled with the same labels used in the parity tile (see Figure 8). Symmetrically, the horizontal arrows of the parity tile 3 are labeled. The tile may be combined with horizontal arms, in which the side arrows have the same labels. The parity tile 1 remains unlabeled and it may be combined with any cross.

The tile set is now complete. Clearly the tiles can be made Wang-tiles having colored edges. The color of each edge consists of labels and directions of the arrows on the edge together with the position, possible label and the direction of the arrows on the parity component.

Let us count the number of different tiles. There are 87 tiles in Figure 4. This includes all possible choices of labels. The tiles may be rotated freely, and each tile has two alternatives for its parity component. This gives

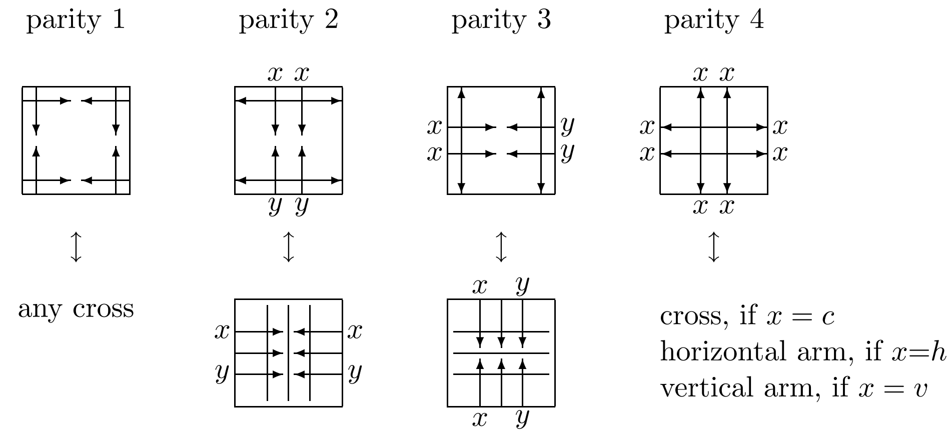

Figure 8: The labels on the parity tiles. Tiles that they may be combined with are shown below the parity tile.

altogether $87 \cdot 4 \cdot 2 = 696$ tiles. Not all tiles appear on $(2^n - 1)$-squares, so the set contains some unnecessary tiles.

Theorem 1. *The tile set above is 4-way deterministic, and aperiodic.*

Proof. There exist valid tilings, because the tiling property is satisfied on $(2^n - 1)$-squares for all n. None of the valid tilings is periodic, because there exists a tile homomorphism into Robinson's aperiodic tile set. Thus our tile set is aperiodic.

Let us prove that the set is deterministic in all four directions. Assume that the colors of two adjacent edges of a tile are known. Clearly the parity component is uniquely determined. (Any one edge alone determines it.)

Consider then the main component. If both edges have outgoing arrows the tile has to be a cross. Because $S_h \cap D_h = \emptyset$ and $S_v \cap D_v = \emptyset$, the orientation of the cross is uniquely determined by the labels on the two known sides. (The cross faces the directions where the labeling triples belong to D_h and D_v. The other two edges have triples belonging to S_h and S_v.) The variable x in Figure 4a is also fixed by the triples: it is the label of the central arrows.

Assume then that one edge has an outgoing arrow and the other known edge an incoming one. The tile is an arm directed in the direction of the outgoing arrow. Using the facts that $S_h \cap D_h = \emptyset$ and $S_v \cap D_v = \emptyset$ it is possible to find out the type of the arm among the six types of arms in Figure 4. (Top three tiles have horizontal labels from S_h, lower three from D_h. Two leftmost tiles have vertical labels from S_v, other four have vertical labels from D_v. The remaining ambiguity is resolved by the location of the side arrow: the side arrow is either on the left or on the right side of the principal arrow.) In all six types of tiles the unknown labels are determined uniquely by the two known edges.

Finally, assume that both edges have incoming arrows. The tile is an arm. If its parity is 2 it is vertical, if the parity is 3 it is horizontal. If the parity is 4 the labels on the parity tile tell whether the arm is horizontal or vertical (see the discussion above the theorem). Since the arm is directed away from one of the known edges, its direction is uniquely determined. The labels are determined in the same way as above.

In all cases the colors of two adjacent edges fix the tile, that is, the tile set is NW-, SW-, NE- and SE-deterministic. □

Consider the following stronger condition. We call a tile set *strongly deterministic* if for all colors C_1 and C_2 there exists at most one tile with adjacent edges colored by C_1 and C_2. The condition is stronger because it is not known which two edges have the given colors.

Corollary 1. *There exists an aperiodic, strongly deterministic tile set.*

Proof. In the construction we use an arbitrary aperiodic tile set T that is 4-way deterministic. According to Theorem 1 such tile sets exist. For each Wang-tile t in T we introduce 4 new tiles $t_{i,j}$, $i,j \in \{0,1\}$, and for each color C 4 colors $C^h_{i,j}$ for horizontal edges and 4 colors $C^v_{i,j}$ for vertical edges ($i,j \in \{0,1\}$). If the colors of the left, upper, right and lower edges of t are A, B, C and D, respectively, the edges of $t_{i,j}$ are colored by $A^v_{i,j}, B^h_{i,j}, C^v_{i+1,j}$ and $D^h_{i,j+1}$, respectively (see Figure 9). The calculation of indices is done modulo 2, i.e. $1+1=0$.

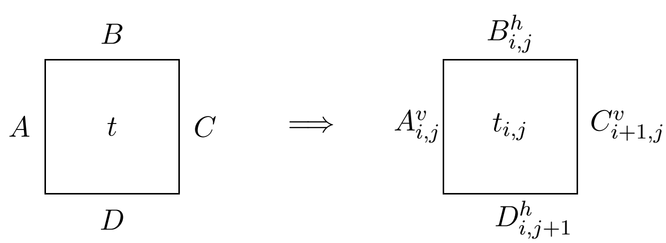

Figure 9: Modifying tiles to get a strongly deterministic tile set. Computation of indices is done modulo 2, and $i,j \in \{0,1\}$

The new tile set is aperiodic: A valid tiling is obtained from a valid tiling by T if tile t in position (x, y) is replaced by tile

$$t_{x \bmod 2, y \bmod 2}$$

for each $x, y \in \mathbb{Z}$. On the other hand, mapping each $t_{i,j}$ into t is a tile homomorphism into T, so that there are no valid periodic tilings with the new tiles.

Finally, let us show that the new tile set is strongly deterministic. Let $X^a_{i,j}$ and $Y^b_{k,l}$ be two arbitrary colors, where $a, b \in \{h, v\}$, $i, j, k, l \in \{0, 1\}$, and X and Y are colors used in T. Assume $t_{x,y}$ is a tile having two adjacent edges colored by $X^a_{i,j}$ and $Y^b_{k,l}$, respectively. Clearly a and b determine which one of the two colors appears on a horizontal edge and which one on a vertical edge. If $(i, j) = (k, l)$ then the colors must be on the left and upper edges of the tile. If $(k, l) = (i + 1, j)$ then the colors are used on upper and right edges, and if $(k, l) = (i, j + 1)$, then they are used on left and lower edges. (All computations are done modulo 2.) Finally, if $(k, l) = (i + 1, j + 1)$ then the colors must appear on the right and lower edges. In this way it is uniquely determined which two edges are colored by the given two colors.

Now the tile $t_{x,y}$ is uniquely determined: Since T is deterministic in all four diagonal directions, tile t is determined by colors X and Y, and indices x and y are determined by indices i and j of the color on one edge. $\square$

4 Final remarks

We note that the CAT(0)-complex we construct in this paper has similar properties to the complex used by Wise in [W] to produce examples of non-residually finite groups acting discetely and co-compactly on a CAT(0)-space (in Wise's terminology our complex contains many 'anti-tori'). One can obtain a CAT(0) complex with a non-residually finite fundamental group by gluing two copies of the complex constructed here along a closed geodesic.

As we pointed out in the introduction our complex contains tori. It is easy to see that in the complex associated to the Robinson tile set there is a torus: indeed the first two arms on the first line and the first two arms in the second line of Figure 1 are glued together in this complex and form a torus. This torus is not destroyed by the labelling we introduce in the rest of the paper, so in our complex there is also a torus. Since our complex is CAT(0) the fundamental group of the torus injects in the fundamental group of the complex.

In order to construct a CAT(0) complex whose fundamental group is not hyperbolic and does not contain $\mathbb{Z}^2$, it is reasonable to consider the following tiling problem: Let S be a finite set of tiles with colored oriented edges placed on the plane with their edges horizontal and vertical. We say that S is closed under reflection if for each tile in S the reflection of this tile along a horizontal or vertical line is also in S. We consider then the following tiling rule: A tiling of the plane using tiles from a set of tiles S which is closed under reflection, is valid if two adjacent tiles meet along an edge with the same color and orientation and two tiles that are the reflection of each other are never adjacent.

The question now is: Is there an aperiodic reflection-closed set of tiles? If there is one one can ask if there is a strongly deterministic such set of tiles. Such a set would give an example of a CAT(0) complex whose fundamental group is not hyperbolic and does not contain a subgroup isomorphic to $\mathbb{Z}^2$. This complex is obtained as follows: For each tile T in S there are 3 more tiles in S obtained by successively reflecting T. Call T and these 3 tiles equivalent. To construct the complex associated to S pick one tile from each equivalence class and glue these tiles as usual respecting the colors and orientation of sides.

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