CHEEGER CONSTANTS OF SURFACES AND ISOPERIMETRIC INEQUALITIES

PANOS PAPASOGLU

Abstract. We show that the if the isoperimetric profile of a bounded genus non-compact surfaces grows faster than $\sqrt{t}$, then it grows at least as fast as a linear function. This generalizes a result of Gromov for simply connected surfaces.

We study the isoperimetric problem in dimension 3. We show that if the filling volume function in dimension 2 is Euclidean, while in dimension 3 is sub-Euclidean and there is a g such that minimizers in dimension 3 have genus at most g, then the filling function in dimension 3 is 'almost' linear.

1. Introduction

If M is a riemannian manifold of dimension n one defines the Cheeger constant h of M by

$$h(M) = \inf_A \{\frac{vol_{n-1}(\partial A)}{vol_n(A)} : vol_n(A) \leq \frac{1}{2} vol_n(M)\}$$

where A ranges over all open subsets of M with smooth boundary. If M is a simplicial manifold one can define the Cheeger constant of M similarly.

As usual we call vol_2 area and vol_1 length. If M is a simplicial 2-manifold or a 2-manifold with a riemannian metric we denote by $A(M)$ the area of M. Similarly if p is a (simplicial or riemannian) path we denote by $l(p)$ the length of p.

It follows from work of Hersch [22] and Yang and Yau [39] (see also [24]) that there is a bound on the Cheeger constant of a closed surface that depends only on its area. So for example there is a constant c such that any riemannian manifold homeomorphic to the 2-sphere S, which has area 1, has $h(S) \leq c$. A similar result holds for graphs on bounded genus surfaces by work of Lipton and Tarjan [25] and Gilbert, Hutchinson and Tarjan [15]. In this paper we use these bounds of

1991 *Mathematics Subject Classification.* 53C20,53C23,20F65.

This work is co-funded by European Social Fund (75%) and National Resources (25%) (EPEAEK II) PYTHAGORAS.

Cheeger constants to study isoperimetric profiles of surfaces and higher isoperimetric inequalities.

To make the paper self-contained we give proofs of the Cheeger constant bounds in the first section. Our proofs have the advantage that they are more direct than the existing proofs as they rely only on Besicovitch theorem ([2]).

We state these bounds both in the simplicial and in the riemannian setting. The bounds in the simplicial case are applied in the last section to higher isoperimetric inequalities. We provide explicit bounds but the constants below are far from optimal.

Proposition 2.3 . *Let S be a riemannian manifold or a simplicial complex homeomorphic to the 2-sphere. Then the Cheeger constant, $h(S)$, of S satisfies the inequality:*

$$h(S) \leq \frac{16}{\sqrt{A(S)}}$$

where $A(S)$ is the area of S.

In general we have an upper bound that depends on the genus:

Proposition 2.6. *Let S be a closed orientable surface of genus $g \geq 1$ equipped either with a riemannian metric or with a simplicial complex structure. Let $A(S)$ be its (simplicial or riemannian) area. Then the Cheeger constant, $h(S)$, of S satisfies the inequality:*

$$h(S) \leq \frac{4 \cdot 10^3 \cdot g^2}{\sqrt{A(S)}}$$

One sees easily that the same bound applies to surfaces with boundary (just collapse the boundary curves to points to obtain a closed surface). One can get bounds for non-orientable surfaces too by passing to the orientable double cover.

If (M^n, g) is a riemannian manifold of infinite volume the isoperimetric profile function of M^n is a function $I_M : \mathbb{R}^+ \to \mathbb{R}^+$ defined by:

$$I_M(t) = \inf_{\Omega} \{vol_{n-1}(\partial\Omega) : \Omega \subset M^n,\, vol_n(\Omega) = t\}$$

where Ω ranges over all regions of M^n with smooth boundary. One can define similarly an isoperimetric profile function $I_M : \mathbb{N} \to \mathbb{N}$ for simplicial manifolds M^n.

Other functions related to the isoperimetric problem are the filling area, FA_0, and homological filling area, FA^h, functions of M that we define below. For more information on filling invariants and applications we refer the reader to the seminal paper of Gromov [17].

If p is a smooth contractible closed curve in M we define its filling area, $FillA_0(p)$, as follows: We consider all riemannian discs D such that there is a 1-lipschitz map $f : D \to X$ with $f|_{\partial D} = p$. We define $FillA_0(p)$ to be the infimum of the areas of this collection of disks. We define now the filling area function of M by:

$$FA_0(t) = \sup_p \{FillA_0(p) : l(p) \leq t\}$$

where p ranges over all smooth contractible closed curves of M and D over riemannian disks filling p.

More generally we can consider 1-cycles c (i.e. unions of closed curves) that can be filled by 2-cycles to define the homological filling area function (see sec. 2 for details).

Gromov ([18], ch. 6, see also [10], ch.6) showed the following:

Gromov's Theorem. *Let (M^n, g) be a simply connected riemannian manifold. Assume that there is some t_0 such that for all $t > t_0$, $FA_0(t) \leq \frac{1}{16\pi} t^2$. Then there is a constant K such that for all $t > t_0$, $FA_0(t) \leq Kt$.*

Gersten [14] observed that this theorem holds also for homological filling area FA^h (see also [18], 6.6E, 6.6F), while Olshanskii [27] gave an elementary proof of Gromov's theorem (see as well [8], [29], [12], for other proofs).

If the dimension of M is 2 then there is an obvious link between filling area and isoperimetric profile, so from Gromov's theorem we readily obtain the following:

Corollary. *Let (S, g) be a riemannian manifold homeomorphic to the plane. Assume that there is some t_0 such that for all $t > t_0$, $I_S(t) \geq 4\sqrt{\pi}\sqrt{t}$. Then there is a constant $\delta > 0$ such that for all $t > t_0$, $I_S(t) \geq \delta t$.*

We note that the isoperimetric problem for surfaces has been studied extensively (see [7], [16], [26], [32], [33], [35]).

We see that the 'gap' in the filling functions implies a 'gap' for the isoperimetric profiles of riemannian planes. It is reasonable to ask whether there are gaps in the isoperimetric profile of other surfaces. Although this does not hold in general we show that this is true for planes with holes or more generally surfaces of finite genus.

Theorem 3.5. *Let S be a plane with holes equipped either with a riemannian metric or with a simplicial complex structure. Assume that there is some $K > 0$ such that for all $t \in [K, 100K]$, $I_S(t) \geq 10^2\sqrt{t}$. Then there is a constant $\delta > 0$ such that for all $t > K$, $I_S(t) \geq \delta t$.*

One obtains as a corollary that the same holds for finite genus surfaces:

Corollary 3.6. *Let S be a non-compact surface of finite genus equipped either with a riemannian metric or with a simplicial complex structure. Assume that there is some $K > 0$ such that for all $t \in [K, 100K]$, $I_S(t) \geq 10^2\sqrt{t}$. Then there is a constant $\delta > 0$ such that for all $t > K$, $I_S(t) \geq \delta t$.*

It is an interesting question whether Gromov's theorem on filling area has an analogue for higher dimensional filling functions. The bounds of Cheeger constants of surfaces can be used to obtain some partial results in this direction. We will state our results in the convenient setting of simplicial complexes. We remark that if M is a compact riemannian manifold then the filling functions of its universal covering, $\tilde{M}$, are determined (up to some lipschitz constant) by $\pi_1(M)$ (see [13], theorems 10.3.3, 10.3.1 and [9]), so one can forget the riemannian metric and work with a triangulation and simplicial chains to calculate the filling functions of $\tilde{M}$.

To fix ideas when we refer to chains and cycles we mean always chains and cycles for simplicial homology with $\mathbb{Z}$ coefficients (in fact our results apply to $\mathbb{Z}_2$ coefficients as well). We denote by $H_n(X)$ the nth-homology group of the space X with $\mathbb{Z}$-coefficients.

Let X be a simplicial complex such that $H_k(X) = 0$. If

$$S = \sum n_i \sigma_i$$

is a (simplicial) k-chain we define the k-th volume of S by $vol_k(S) = \sum |n_i|$. If S is a k-cycle we define the filling volume of S by

$$Fillvol_{k+1}(S) = \inf\{vol_{k+1}T : \partial T = S\}$$

For $k \geq 1$ we define the $(k+1)$th-filling volume function, FV_{k+1}, of X by:

$$FV_{k+1}(n) = \sup\{Fillvol_{k+1}(S) : S \text{ is a } k\text{-cycle such that } vol_k(S) \leq n\}$$

If S is a k-cycle such that $vol_k(S) \leq n$ and $Fillvol_{k+1}(S) = FV_{k+1}(n)$ we say that S is a *minimizer* for $FV_{k+1}(n)$.

If S is a 2-cycle one can define the *genus* of S. Indeed S is represented by a map $f : \Sigma \to X$ where Σ is a closed surface and f is simplicial and 1-1 on open 2-simplices (see [21], sec.2.1, p.109). We define the genus of S to be the genus of the surface Σ.

We remark that $FA^h = FV_2$. As we noted earlier Gromov's theorem applies to FA^h as well so we have that if

$$\lim_{n\to\infty} \frac{FV_2(n)}{n^2} = 0$$

then there is some $K > 0$ such that $FV_2(n) \leq Kn$ for all $n \in \mathbb{N}$. In general we say that FV_k is euclidean if there is some $K > 0$ such that

$$\frac{1}{K} n^{\frac{k}{k-1}} \leq FV_k(n) \leq K n^{\frac{k}{k-1}}, \quad \forall n \in \mathbb{N}$$

and we say that FV_k is sub-euclidean if

$$\lim_{n\to\infty} \frac{FV_k(n)}{n^{\frac{k}{k-1}}} = 0$$

So by Gromov's theorem if FV_2 is sub-euclidean then it is linear. We note that a naive guess that if FV_3 is sub-euclidean then it is linear is contradicted by Pansu's theorem ([28]), FV_3 in Heisenberg's group grows like $n^{\frac{4}{3}}$. On the other hand Gromov conjectures ([18], sec. $6B_2$) that if X is a $Cat(0)$ space with a co-compact group action then sub-euclidean filling implies linear filling in any dimension. More generally it is believed that this is true for spaces satisfying a cone-type inequality (see [38]).

Another possible direction is to examine all filling functions simultaneously. Some specific conjecture is: If FV_2 is bounded by a quadratic function and FV_3 is sub-euclidean then FV_3 is bounded by a linear function. Of course one can state this conjecture in any dimension: If FV_i is euclidean for $i = 2, ..., k-1$ and FV_k is sub-euclidean then FV_k is linear. The following theorem is giving some evidence in favor of this conjecture.

Theorem 4.1. *Let X be a simplicial complex such that $H_1(X) = H_2(X) = 0$. Assume that the following hold:*

- *There is some $K > 0$ such that $FV_2(n) \leq Kn^2$ for all $n \in \mathbb{N}$.*
- $$\lim_{n\to\infty} \frac{FV_3(n)}{n^{\frac{3}{2}}} = 0$$
- *There is some $g \in \mathbb{N}$ such that if S is a minimizer $2-$cycle in X^2 then S is represented by a surface of genus at most g.*

Then for every $\epsilon > 0$ we have

$$\lim_{n\to\infty} \frac{FV_3(n)}{n^{1+\epsilon}} = 0$$

1.1. **Outline of the proofs.** The proofs of proposition 2.3 and proposition 2.6 are based on Besicovitch lemma (and more generally co-area inequalities). The idea can be grasped easier in the case of the sphere. We consider a minimal length simple closed curve p that subdivides the sphere S into two pieces S_1, S_2 such that $A(S_1), A(S_2)$ have both area bigger than $A(S)/4$. Let's say $A(S_1) \geq A(S_2)$. Now S_1 is a disk and subpaths of p of length $< l(p)/2$ are geodesic in S_1 because p is minimal. Applying Besicovitch lemma one sees that $l(p)$ is smaller than $4\sqrt{A(S)}$. This implies proposition 2.3.

To prove theorem 3.5 we show in fact that $I_S(t) \geq \frac{1}{\sqrt{K}} t$ for all $t > K$. To show this we argue by contradiction. We take a 1-cycle c of minimal filling area such that $FillA(c) > K$ and $l(c) < \frac{1}{\sqrt{K}} FillA(c)$. If $c = \partial R$ then we collapse the 'holes' of R to points to get a sphere S. Applying proposition 2.3 to S we find a 1-cycle γ in S such that $K \leq FillA(\gamma) \leq 4K$ which satisfies the inequality

$$FillA(\gamma) > \frac{l(\gamma)^2}{100}$$

This is somewhat tricky and the proof uses also Besicovitch lemma and exploits the convexity of x^2. Note though that intuitively the existence of such a curve is obvious since S is 'positively curved' at some points. Finally we lift γ back to R and we get a cycle with smaller filling area than c that has the same properties as c; this contradicts the minimality of c. It is an easy corollary that the theorem holds for finite genus surfaces in general.

The proof of theorem 4.1 is by contradiction. We assume that for some $\epsilon > 0$ we have

$$\lim_{n\to\infty} \frac{FV_3(n)}{n^{1+\epsilon}} = \infty$$

We take M 'big enough' and we consider a 2-cycle S which is a minimal area counterexample to $FV_3(n) \leq Mn^{1+\epsilon}$. We show that there is a $\delta = \delta(\epsilon, g, K) > 0$ such that $diam(S) > \delta\sqrt{A(S)}$. This is where we use the bounds of Cheeger constants. To simplify let's say that S is a sphere. By proposition 2.3 there is a simple closed curve p of length smaller than $4\sqrt{A(S)}$ on S which divides it into two pieces with comparable area. Now if $diam(S)$ is small the filling area of p is much smaller than $A(S)$. So one can subdivide S into two 2-cycles of area roughly between $A(S)/4$ and $3A(S)/4$. It follows by the convexity of $n^{1+\epsilon}$ that one of these two 2-cycles is also a counterexample to $FV_3(n) \leq Mn^{1+\epsilon}$, a contradiction. Given now that the diameter of S is big we take a minimal volume 3-cycle R filling S and we fix a point x on S. We consider 'balls' around x in R and using again the convexity of $n^{1+\epsilon}$ and an elementary inequality (lemma 4.5) we see that the 2-cycle, say S_1, given by the boundary of some of these 'balls' has filling volume of the order of $A(S_1)^{3/2}$ contradicting the hypothesis of the theorem.

2. Cheeger constants of surfaces

If M is a riemannian manifold of dimension n one defines the Cheeger constant h of M by

$$h(M) = \inf_A \{\frac{vol_{n-1}(\partial A)}{vol_n(A)} : vol_n(A) \leq \frac{1}{2} vol_n(M)\}$$

where A ranges over all open subsets of M with smooth boundary. If M is a simplicial manifold one can define the Cheeger constant of M similarly; now A runs over all simplicial submanifolds of M. To be more precise, we take A to be a union of closed n simplices and we define $\partial A = A \cap \overline{M - A}$. In the simplicial setting we define $vol_n(A)$ to be the number of n-simplices of A and $vol_{n-1}(\partial A)$ to be the number of $n-1$-simplices of ∂A.

We remark that this definition makes sense also if M is more generally an n dimensional simplicial complex. To make this definition coincide with the existing literature on graphs one should first take the barycentric subdivision and then calculate the Cheeger constant. However as here we are only concerned with surfaces we will not pass to barycentric subdivisions.

As usual we call vol_2 area and vol_1 length. If M is a simplicial 2-manifold or a 2-manifold with a riemannian metric we denote by $A(M)$ the area of M. Similarly if p is a (simplicial or riemannian) path we denote by $l(p)$ the length of p.

As we mentioned in the introduction it follows from work of Hersch [22] and Yang and Yau [39] (see also [24]) that there is a bound on the Cheeger constant of a closed surface that depends only on its area (see also [25] and [15] for similar results for graphs).

We give in this section an alternative approach to these results. We will treat first the simplicial case and then we will outline the argument in the riemannian case. In both cases our proof is based on Besicovitch lemma [2] (see also [20], sec. 4.28, p.252, this lemma is sometimes referred to as Almgren's lemma, [8], [1]):

Lemma 2.1. *Let D be a riemannian manifold homeomorphic to the disc and let $\gamma = \partial D$. Suppose γ is split in 4 subpaths, $\gamma = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$. Let $d_1 = d(\alpha_1, \alpha_3)$, $d_2 = d(\alpha_2, \alpha_4)$ Then*

$$A(D) \geq d_1 d_2$$

We introduce some notation:

If X is a simplicial complex and K is a subcomplex of X we denote by $star(K)$ the subcomplex of X consisting of all closed simplices intersecting K. We denote by $star_i(K)$ the subcomplex obtained by repeating the star operation i times. If v is a vertex of X we define the ball of radius n and center v, $B_v(n)$, by $B_v(n) = star_n(v)$.

We state below Besicovitch lemma ([8]) in the simplicial setting:

Lemma 2.2. *Let D be a simplicial disc and let $\gamma = \partial D$. Suppose γ is split in 4 subpaths, $\gamma = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$. Let $d_1 = d(\alpha_1, \alpha_3)$, $d_2 = d(\alpha_2, \alpha_4)$ Then*

$$A(D) \geq d_1 d_2$$

Proof. We consider $star(\alpha_1)$ and we remark that its boundary has at least $2d_2$ edges. Since each closed 2-simplex in $star(\alpha_1)$ intersects the boundary of $star(\alpha_1)$ at at most 2 edges we conclude that $A(star(\alpha_1)) \geq d_2$. Now we repeat d_1 times, i.e. we consider $star_i(\alpha_1)$ for $i = 1, 2, ..., d_1$ and we remark as before that

$$A(star_i(\alpha_1)) \geq d_2$$

It follows that $A(D) \geq d_1 d_2$. □

Remark 1. The same inequality applies for disks with a cell complex structure in which all cells are polygons with 2 or 3 sides (bigons or triangles). Indeed the proof above applies in this case too.

We start with the inequality for the sphere where the idea of the proof is more transparent.

Proposition 2.3. *Let S be a riemannian manifold or a simplicial complex homeomorphic to the 2-sphere. Then the Cheeger constant, $h(S)$, of S satisfies the inequality:*

$$h(S) \leq \frac{16}{\sqrt{A(S)}}$$

where $A(S)$ is the area of S.

Proof. We deal first with the simplicial case. Let p a closed curve on the 1-skeleton of S of minimal length dividing S on two regions which have both area bigger or equal to $\frac{A(S)}{4}$. Let's say $S - p = S_1 \sqcup S_2$ (where S_1, S_2 are open). Without loss of generality we assume that $A(S_1) \geq A(S_2)$. We remark now that there are no 'shortcuts' for p that are contained in S_1. More precisely if $a, b \in p$ and q is a path in S_1 joining a, b then $l(q)$ is at least as big as the length of the shortest subpath of p joining a, b. Indeed assume this is not the case. Let's say $p - \{a, b\} = p_1 \cup p_2$

with $l(p_1) \geq l(p_2)$. Without loss of generality we may assume that q intersects p only at a, b. Then $p_1 \cup q$ is a simple closed curve shorter than p which has the same properties as p, a contradiction. We note in particular that S_1 is connected.

We claim that

$$l(p) \leq 4\sqrt{A(S)}$$

This is clearly true if $l(p) < 4$.

Otherwise we subdivide p in 4 arcs $p = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$ such that $l(\alpha_i) \geq \frac{l(p)}{4} - 1$ for all i. Since there are no 'shortcuts' as we observed above

$$d(\alpha_1, \alpha_3) \geq \frac{l(p)}{4} - 1, \quad d(\alpha_2, \alpha_4) \geq \frac{l(p)}{4} - 1$$

Applying lemma 2.2 to S_1 we have that

$$A(S_1) \geq (l(p) - 1)^2/16 \Rightarrow 3A(S) \geq (l(p) - 1)^2/4 \Rightarrow l(p) \leq 4\sqrt{A(S)}$$

We conclude that

$$h(S) \leq \frac{4\sqrt{A(S)}}{A(S)/4} \leq \frac{16}{\sqrt{A(S)}}$$

We treat now the riemannian case. The argument is along the same lines. To sidestep the issue of existence and regularity of the minimal closed curve p we argue instead with ϵ-minimal curves. More precisely we consider the set U of all simple closed curves dividing S in two discs which have both area bigger or equal to $\frac{A(S)}{4}$. Let L be the infimum of the lengths of the curves in U. Given $\epsilon > 0$ we consider $p \in U$ with $l(p) > L - \epsilon$. Let's say that $S - p = S_1 \sqcup S_2$ and $A(S_1) \geq A(S_2)$. Then p does not have ϵ-shortcuts in S_1. That is if $q \subset S_1$ is a path joining $a, b \in p$ then $l(q) - \epsilon$ is smaller than the length of the shortest subpath of p joining a, b.

We subdivide now p in 4 arcs $p = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$ such that $l(\alpha_i) = l(p)/4$ for all i. Since there are no ϵ-'shortcuts' as we observed above

$$d(\alpha_1, \alpha_3) \geq \frac{l(p)}{4} - \epsilon, \quad d(\alpha_2, \alpha_4) \geq \frac{l(p)}{4} - \epsilon$$

Applying lemma 2.1 to S_1 we have that

$$A(S_1) \geq (l(p) - \epsilon)^2/16 \Rightarrow 12A(S) \geq (l(p) - \epsilon)^2 \Rightarrow l(p) \leq 2\sqrt{3A(S)} + \epsilon$$

It follows that

$$h(S) \leq \frac{2\sqrt{3A(S)} + \epsilon}{A(S)/4} \Rightarrow h(S) \leq \frac{8\sqrt{3}}{\sqrt{A(S)}}$$

Where the last inequality follows since the former inequality holds for every $\epsilon > 0$. We note that we obtain a slightly better constant in the riemannian case.

□

Remark 2. The same inequality for the Cheeger constant applies for spheres with a cell complex structure in which all cells are polygons with 2 or 3 sides (bigons or triangles). Indeed the proof above applies in this case too.

To treat the general case of compact surfaces we need some technical lemmas.

Definition . Let S be a compact surface with boundary. A simple arc p intersecting the boundary only at its endpoints is said to be parallel to the boundary if $S - p$ has a contractible component. Two disjoint simple arcs p_1, p_2 intersecting the boundary only at their endpoints are said to be parallel if $S - (p_1 \cup p_2)$ has a contractible component.

Lemma 2.4. *Let S be a surface of genus g with k boundary components. Then there are at most $k + 2g$ pairwise disjoint arcs on S with their endpoints on ∂S such that no arc is parallel to the boundary and no two arcs are parallel.*

Proof. We remark that $\pi_1(S)$ is a free group of rank $k + 2g$. A set of i arcs which are not parallel pairwise and are not parallel to the boundary induces a reduced graph of groups decomposition of $\pi_1(S)$ with i edges and trivial edge stabilizers. However the number of edges of any such decomposition can not exceed the rank of $\pi_1(S)$. □

Lemma 2.5. *Let S be a closed orientable surface of genus $g \geq 1$. If $U = \{p_1, ..., p_k\}$ is a set of pairwise disjoint simple closed curves on S such that no component of $S - U$ is contractible then $k \leq 2g$.*

Proof. Without loss of generality we may assume that U is maximal. Then if we pinch each curve to a point we obtain a space with fundamental group the free group of rank $2g$, F_{2g}. The set U induces a reduced graph of groups decomposition of F_{2g} with k edges and trivial edge stabilizers, so $k \leq 2g$.

□

Proposition 2.6. *Let S be a closed orientable surface of genus $g \geq 1$ equipped either with a riemannian metric or with a simplicial complex structure. Let $A(S)$ be its (simplicial or riemannian) area. Then the Cheeger constant, $h(S)$, of S satisfies the inequality:*

$$h(S) \leq \frac{4 \cdot 10^3 \cdot g^2}{\sqrt{A(S)}}$$

Proof. We treat first the simplicial case. The proof in the riemannian case follows the same lines, we outline at the end the changes which are needed in this case.

Let $U = \{p_1, ..., p_k\}$ be a set of closed curves on the 1-skeleton of S such that:

1. $S = S_1 \cup S_2$ with $\partial S_1 \cap \partial S_2 = S_1 \cap S_2 = U$, and $A(S_i) \geq A(S)/4$, $i = 1, 2$.
2. At most one curve p_i bounds a disk in S.
3. The sum of the lengths $L = l(p_1) + ... + l(p_k)$ is minimal among all sets of curves satisfying 1,2.

We claim that

$$L \leq 10^3 \cdot g^2 \cdot \sqrt{A(S)}$$

Suppose that this is not the case. By lemma 2.5, $k \leq 2g + 1$ (note that the p_i's are not necessarily disjoint but can be made disjoint by pushing them slightly inside S_1 or S_2). It follows that there is a curve $p_i \in U$ such that

$$l(p_i) \geq \frac{10^3 g^2 \sqrt{A(S)}}{2g + 1}$$

We set $n = [\sqrt{A(S)}] + 1$. Let's assume that $A(S_1) \geq A(S_2)$. We remark now that S_1 is connected. Indeed suppose S_1 is a disjoint union of two open sets, T_1, T_2. Let's say that $A(T_1) \leq A(T_2)$. We consider $T_1 \cup S_2$ and $S_1 - T_1$ and we remark that they are separated by a subset of U. This contradicts the minimality of U (property 3). It follows that $\partial S_1 = U$.

We pick now a vertex $v \in p_i$. We claim that S_1 is not contained in the ball of radius $3n$ and center v, $B_v(3n)$. Suppose not. We subdivide p_i at $4g + 1$ segments of length bigger than

$$\left[\frac{10^3 g^2 \sqrt{A(S)}}{(2g + 1)(4g + 1)}\right] \geq 100n$$

We consider geodesic arcs in S_1 from v to the endpoints of these segments. If some such arc is parallel to the boundary we can use it to 'cut away' a disc from S_1 and contradict the minimality of U (or in case the disc has area more than half of the area of S_1 we replace S_1 by the disc and contradict property 3).

Otherwise by perturbing these arcs slightly we may arrange so that they are disjoint. Since we have $4g + 1$ arcs by lemma 2.4 two of them are parallel. Using them we can cut away a disk from S_1 (or replace S_1 by a disc) which contradicts the minimality of U (property 3).

We consider now $D_r = B_v(r) \cap S_1$ for $n \leq r \leq 2n$. We remark that if the length of $\partial D_r \cap S_1$ is bigger than $2n$ for all r then $A(D_n) \geq n^2 \geq A(S)$, a contradiction.

So ∂D_r has length less than $2n$ for some r. On the other hand the length of $D_r \cap p_i$ is bigger than $2n$. So $S_1 \cap D_r$ and D_r have both boundary length smaller than the length of ∂S_1. So we can replace S_1 by whichever of the two has area bigger than $A(S_1)/2$. If this new domain has more than one boundary component that bounds a disc we just erase this component. We remark now that the boundary length of the new domain is smaller than than the boundary length of S_1 and this contradicts the minimality of U (property 3).

The same proof applies in the riemannian case with few changes. We define a set of closed curves $U = \{p_1, ..., p_k\}$ as before. Now we may additionally assume that the p_i are simple and disjoint. To insure this and avoid existence issues we assume that the sum of their lengths exceeds the minimal possible value by $\epsilon > 0$ (condition 3) among all curves that satisfy 1,2. As before we argue that there is some p_i such that

$$l(p_i) \geq \frac{10^3 g^2 \sqrt{A(S)}}{2g+1} - \epsilon$$

We argue as before and w consider $D_r = B_v(t) \cap S_1$ for $r \in [n, 2n]$. Now by the co-area formula if the length of $\partial D_r \cap S_1$ is bigger than $2n$ for almost all r then $A(D_n) \geq n^2 \geq A(S)$, which gives a contradiction as before. The rest of the proof applies verbatim to the riemannian case as well. □

Remark 3. We remark that there is no function of volume that gives an upper bound for the Cheeger constant of manifolds of dimension higher than 2. Indeed it's enough to prove this for the ball of dimension 3. We can obtain examples contradicting the existence of such a bound by considering sequences of expanders and thickening them.

3. Isoperimetric profiles of surfaces

If (M^n, g) is a riemannian manifold of infinite volume the isoperimetric profile function of M^n is a function $I_M : \mathbb{R}^+ \to \mathbb{R}^+$ defined by:

$$I_M(t) = \inf_{\Omega} \{vol_{n-1}(\partial\Omega) : \Omega \subset M^n,\ vol_n(\Omega) = t\}$$

where Ω ranges over all regions of M^n with smooth boundary. One can define similarly an isoperimetric profile function $I_M : \mathbb{N} \to \mathbb{N}$ for simplicial manifolds M^n. In this section we will study isoperimetric profiles and filling functions of surfaces (so vol_2 is area and vol_1 is length).

Other functions related to the isoperimetric problem are the filling area, FA_0, and homological filling area, FA^h, functions of M that we define now.

If p is a smooth contractible closed curve in M we define its filling area, $FillA_0(p)$, as follows: We consider all riemannian discs D such that there is a 1-lipschitz map $f : D \to X$ with $f|_{\partial D} = p$. We define $FillA_0(p)$ to be the infimum of the areas of this collection of disks. We define now the filling area function of M by:

$$FA_0(t) = \sup_p \{FillA_0(p) : l(p) \leq t\}$$

where p ranges over all smooth contractible closed curves of M.

More generally we can consider 1-cycles c (i.e. unions of closed curves) that can be filled by 2-cycles to define the homological filling area function. To define $FillA(c)$ we consider surfaces with boundary $(S, \partial S)$, equipped with a riemannian metric, such that there is a 1-lipschitz map $f : S \to X$ with $f|_{\partial S} = c$. We define then:

$$FA^h(t) = \sup_c \{FillA(c) : l(c) \leq t\}$$

where if $c = c_1 \sqcup ... \sqcup c_n$ with c_i closed curves, we define $l(c) = l(c_1) + ... + l(c_n)$. One defines FA_0 and FA^h similarly in the simplicial setting as well.

Gromov ([18], ch. 6, see also [10], ch.6) showed the following:

Gromov's Theorem. *Let (M^n, g) be a simply connected riemannian manifold. Assume that there is some t_0 such that for all $t > t_0$, $FA_0(t) \leq \frac{1}{16\pi} t^2$. Then there is a constant K such that for all $t > t_0$, $FA_0(t) \leq Kt$.*

We remark that Wenger ([37]) improved $\frac{1}{16\pi}$ to $\frac{1-\epsilon}{4\pi}$ (for any $\epsilon > 0$) which is optimal as the example of the euclidean plane shows. In fact Gromov's theorem applies more generally to 'reasonable' geodesic metric spaces where a notion of area can be defined (e.g. simplicial complexes). We note also that Gromov has shown a stronger ('effective') version than the one we state; it is enough in fact to have a subquadratic filling for a sufficiently big range of areas to conclude that the filling is linear.

In the case of surfaces the isoperimetric profile and the filling area functions are closely related. In fact FA_0 is linear for a space if and only if the space is Gromov hyperbolic (see [18]). On the other hand if a simply connected surface S, equipped with a riemannian metric, is not Gromov hyperbolic then for any t there is an embedded loop γ in S with $l(\gamma) > t$ such that $FillA_0(\gamma) > \frac{1}{16\pi} l(\gamma)^2$. If $t_1 = FillA_0(\gamma)$ we see that for any $t > 0$ there is some $t_1 > t$ such that $I_S(t_1) < 4\sqrt{\pi}\sqrt{t_1}$.

Theorem 3.1. *Let (S, g) be a simply connected riemannian surface. Assume that there is some t_0 such that for all $t > t_0$, $I_S(t) \geq 4\sqrt{\pi}\sqrt{t}$. Then there is a constant $\delta > 0$ such that for all $t > t_0$, $I_S(t) \geq \delta t$.*

We remark that in many cases FA_0 and FA^h are equal (e.g. this holds for the Euclidean and Hyperbolic plane).

This does not hold always however. We give now some examples to illustrate the relationship between the filling area functions and the isoperimetric profile. If $f(t), g(t)$ are functions we write $f(t) \sim g(t)$ if

$$\limsup_{t\to\infty} \frac{f(t)}{g(t)} < \infty, \quad \liminf_{t\to\infty} \frac{f(t)}{g(t)} > 0$$

Example 3.2. Let X be the punctured Euclidean plane $\mathbb{R}^2 - \mathbb{Z}^2$. Then $FA_0(t) \sim t$ (see [3], [31]) while $FA^h(t) \sim t^2$. The isoperimetric profile I_X is the inverse of FA^h so $I_X(t) \sim \sqrt{t}$.

Example 3.3. Let X be the cylinder $S^1 \times \mathbb{R}$ with the standard product metric. Then $FA_0(t) \sim t^2$. Indeed for any X, FA_0 is the same for X and for the universal covering $\tilde{X}$. Here $\tilde{X} = \mathbb{E}^2$ (the euclidean plane). On the other hand if s is the length of the $\mathbb{S}^1$ factor for any $t > 2s$ we have $FA^h(t) = \infty$. Similarly for the isoperimetric profile there is some t_0 such that for all $t > t_0$, $I_X(t) = 2s$.

Example 3.4. Let X_n be the space obtained by removed a ball of radius n from the hyperbolic plane $\mathbb{H}^2$. Let X be isometric to the hyperbolic plane. We fix a point $O \in X$ and we consider a sequence of points x_n such that $d(x_n, O) = 2^n$. For each n we remove the disk of radius n and center x_n from X and we glue along the boundary of the disk a copy of X_n. The space Y obtained has $FA_0(t) \sim FA^h(t) \sim I_Y(t) \sim t$. We remark that Y is not Gromov hyperbolic.

It is reasonable to ask whether Gromov's theorem extends to all surfaces. The answer is no in general but we can show that the theorem holds for surfaces of bounded genus (this applies for example to riemannian planes with infinitely many holes, compare [31]). We have the following:

Theorem 3.5. *Let S be a plane with holes equipped either with riemannian metric or with a simplicial complex structure. Assume that* *there is some* $K > 0$ *such that for all* $t \in [K, 100K]$, $I_S(t) \geq 10^2\sqrt{t}$. *Then there is a constant* $\delta > 0$ *such that for all* $t > K$, $I_S(t) \geq \delta t$.

Proof. We treat the simplicial case first. We will show that $I(t) \geq \frac{1}{\sqrt{K}}t$ for all $t > K$.

We argue by contradiction. So let c be a 1-cycle with minimal filling area and $FillA(c) > K$ such that $l(c) < \frac{1}{\sqrt{K}}FillA(c)$. Let's say that $c = \partial R$. We claim that $FillA(c) > 100K$. Indeed if $FillA(c) \leq 100K$ then $l(c) \geq 100\sqrt{FillA(c)}$ hence

$$\frac{1}{\sqrt{K}}FillA(c) > 100\sqrt{FillA(c)} \Rightarrow FillA(c) > 10^4K$$

By our minimality assumption R is connected, so R is a sphere with holes (possibly a disc). We collapse all holes to points and we obtain a sphere Σ will a cell complex structure in which all cells are either bigons or triangles.

In this way we obtain a map $f : R \to \Sigma$ which is 1-1 on open 2-simplices from R.

By Proposition 2.3 (and the remark following it) there is a simple closed curve p in Σ^1 such that $FillA(p) \geq A(\Sigma)/4$ and

$$l(p) \leq 4\sqrt{A(\Sigma)}$$

It follows that

$$FillA(p) \geq \frac{1}{64}l(p)^2$$

We consider now the set of curves q in Σ^1 with filling area $FillA(q) \geq K$ that satisfy $FillA(q) > \frac{1}{100}l(q)^2$. Clearly this set is not empty. Let γ in Σ^1 of minimal filling area with this property. We will show that $4K \geq FillA(\gamma) \geq K$. Assume this is not the case.

We subdivide γ in 4 arcs $\gamma = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$ such that $l(\alpha_i) \geq \frac{l(\gamma)}{4} - 1$ for all i.

We claim that

$$d(\alpha_1, \alpha_3) \geq \frac{l(\gamma)}{4}, \quad d(\alpha_2, \alpha_4) \geq \frac{l(\gamma)}{4}$$

We argue by contradiction. Assume that $d(\alpha_1, \alpha_3) < \frac{l(\gamma)}{4}$ and let w be a path from α_1 to α_3 of length $r \leq \frac{l(\gamma)}{4}$. Then, using w, we split γ into two curves γ_1, γ_2 such that $\gamma_1 \cap \gamma_2 = w$, $\gamma_1 \cup \gamma_2 = \gamma \cup w$. Let's say $l(\gamma_1) \geq l(\gamma_2)$. Then

$$l(\gamma_1) = \frac{l(\gamma)}{2} + a + r, \quad l(\gamma_2) = \frac{l(\gamma)}{2} - a + r$$

for some $a < l(\gamma)/4$. To simplify the notation we set $l = l(\gamma)$. Since we assume that $FillA(\gamma) > 2K$ we have that $FillA(\gamma_1) > K$, $FillA(\gamma_2) > K$, so

$$FillA(\gamma) \leq FillA(\gamma_1) + FillA(\gamma_2) \leq \frac{1}{100}[(\frac{l}{2}+a+r)^2+(\frac{l}{2}-a+r)^2] =$$

$$= \frac{1}{100}(\frac{l^2}{2}+2a^2+r^2+lr)$$

Since $a \leq \frac{l}{4}, r \leq \frac{l}{4}$ we have

$$\frac{l^2}{2}+2a^2+r^2+lr \leq \frac{l^2}{2}+\frac{l^2}{8}+\frac{l^2}{16}+\frac{l^2}{4} < l^2$$

which is a contradiction. We may now apply Lemma 2.2 to γ and conclude that $FillA(\gamma) \geq \frac{l(\gamma)^2}{16}$ which is again a contradiction. We conclude that $4K \geq FillA(\gamma) \geq K$.

We lift now γ via f to R. γ lifts to a set of arcs (or a single simple closed curve) that separate R into two 2-chains R_1, R_2. Let's denote this set of arcs by α. Let's say that $K \leq A(R_1) \leq 4K$. Then $\partial R_1 = c_1 \cup \alpha$ and $\partial R_2 = c_2 \cup \alpha$ with $c_1 \cup c_2 = c$. By our assumption on c we have

$$l(c_1)+l(c_2) \leq \frac{1}{\sqrt{K}}(A(R_1)+A(R_2)) \quad (1)$$

On the other hand since c is minimal with this property we have

$$l(c_1)+l(\gamma) \geq \frac{1}{\sqrt{K}}A(R_1) \quad (2)$$

$$l(c_2)+l(\gamma) \geq \frac{1}{\sqrt{K}}A(R_2) \quad (3)$$

By the way γ was defined we have

$$A(R_1) \geq \frac{1}{100}l(\gamma)^2$$

Since $A(R_1) \leq 4K$ we have $l(\gamma) \leq 20\sqrt{K}$. From the hypothesis of the theorem since $A(R_1) \in [K, 100K]$ we have

$$100A(R_1) \leq l(\gamma)+l(c_1) \Rightarrow l(c_1) \geq 80\sqrt{K}$$

Substituting in (1) we obtain

$$l(c_2) \leq \frac{1}{\sqrt{K}}(4K+A(R_2)) - 80\sqrt{K}$$

Therefore

$$l(c_2)+l(\gamma) \leq \frac{1}{\sqrt{K}}A(R_2) - 76\sqrt{K}$$

and from (3)

$$\frac{1}{\sqrt{K}}A(R_2) \leq \frac{1}{\sqrt{K}}A(R_2) - 76\sqrt{K}$$

which is a contradiction.

The proof in the riemannian case is identical. One has just to note that when we collapse the boundary curves to points we obtain a riemann metric with some singularities. The estimates for Cheeger constants apply however to this case as well. One can see this e.g. by approximating the singular metric by a non singular one or by noting that the proof we gave of the Cheeger constant bounds work also for singular metrics. □

Corollary 3.6. *Let S be a non-compact surface of finite genus equipped either with a riemannian metric or with a simplicial complex structure. Assume that there is some $K > 0$ such that for all $t \in [K, 100K]$, $I_S(t) \geq 10^2\sqrt{t}$. Then there is a constant $\delta > 0$ such that for all $t > K$, $I_S(t) \geq \delta t$.*

Proof. There is a finite set $\{p_1, ..., p_n\}$ of smooth, rectifiable, simple closed curves (or a finite set of simple closed curves lying in S^1 is the simplicial case) such that

$$S - \{p_1, ..., p_n\} = B \sqcup B_1 \sqcup .... \sqcup B_k$$

with B a surface of finite area and $B_1, ..., B_k$ planes with holes. From the previous theorem we have that there are $\delta_1, ..., \delta_k$ such that for all $t > K$ we have

$$I_{B_i}(t) \geq \delta_i t, \;\; i = 1, ..., k$$

Let $\delta' = \min\{\delta_1, ..., \delta_k\}$. We set $V = A(B)$ and $L = l(p_1) + ... + l(p_n)$.

If Ω is a domain in S with rectifiable boundary there is some $i \in \{1, ..., k\}$ such that

$$A(\Omega \cap B_i) \geq \frac{A(\Omega) - V}{k}$$

Let $c = \partial\Omega \cap B_i$. Since $\partial(\Omega \cap B_i) \subset c \cup \{p_1, ..., p_n\}$ we have that $l(\partial(\Omega \cap B_i)) \leq L + l(c)$. If

$$\frac{A(\Omega) - V}{k} > K \Leftrightarrow A(\Omega) > kK + V$$

we have

$$L + l(c) \geq \delta' \frac{A(\Omega) - V}{k} \Rightarrow l(c) \geq \delta' \frac{A(\Omega) - V}{k} - L$$

It follows that if

$$\delta'\frac{A(\Omega)}{k} \geq 2(\frac{\delta' V}{k} + L) \Leftrightarrow A(\Omega) \geq 2V + \frac{2kL}{\delta'}$$

we have

$$l(\partial\Omega) \geq l(c) \geq \delta'\frac{A(\Omega)}{2k}$$

We conclude that for all

$$t > \max(kK + V, 2V + \frac{2kL}{\delta'})$$

we have

$$I_S(t) \geq \frac{\delta'}{2k}t$$

We note further that if $\delta_1 = \inf\{I_S(t) : t > K\}$ and

$$\delta = \min(\frac{\delta_1}{K}, \frac{\delta'}{2k})$$

then for all $t > K$ we have

$$I_S(t) \geq \delta t$$

□

Remark 4. The previous theorem implies that if the filling area function is subquadratic for surfaces of finite genus then it is actually linear. In fact one may give a similar proof to another generalization of Gromov's theorem. Let X be either a riemannian manifold or a simplicial complex. Let c be a 1-cycle. If $c = \partial R$ for some 2-chain R then we define the genus of R to be the genus of the 2-cycle we obtain from R by collapsing c to a point. If c is a 1-cycle in X we define the g-filling area of c by

$$FillA_g(c) = \inf\{A(S) : S \text{ is a 2-chain of genus at most } g \text{ s.t. } \partial S = c\}$$

Note that with this definition $FillA_0$ is slightly more general than before as it applies to 1-cycles and not just closed curves but this does not affect what follows. We define now the g-filling area of X by

$$FA_g(t) = \sup_c\{FillA_g(c) : c = \partial R,\ R \text{ of genus} \leq g \text{ and } l(c) \leq t\}$$

With this notation Gromov's theorem says that if FA_0 is subquadratic then it is bounded by a linear function. In fact now we can generalize this for any g: For any g there is some $\epsilon_g > 0$ such that if for some t_0, $FA_g(t) \leq \epsilon_g t$ for all $t > t_0$ then there is some $K > 0$ such that $FA_g(t) \leq Kt$ for all $t > t_0$. The proof goes along the same lines as the proof of theorem 3.5. We argue by contradiction assuming that we have a minimal 1-cycle c that violates the linear isoperimetric inequality. We fill it by a minimal area 2-chain S of genus at most g. Then we collapse all boundary components of S to obtain a closed 2-cycle Σ of genus $\leq g$. Using proposition 2.6 we show that we can 'cut' a 2-chain from Σ with small boundary length and big area. We lift this back to S and we argue as in theorem 3.5 to contradict our assumption that c is minimal.

We note finally that if one considers the function FA^h rather than FA_g (i.e. if one does not impose any bound on g) then there is no such gap between subquadratic and linear.

4. Isoperimetric inequalities

In this section we will study the question whether Gromov's 'gap' theorem for FA_0 extends to the 3-dimensional filling function FV_3. The filling area function FA_0 is important for group theory since it is related to the word problem. In fact if G is a finitely presented group and X is its Cayley complex then G has a solvable word problem if and only if $FA_0(X)$ is bounded by a recursive function (X might not be a simplicial complex but one can pass to a simplicial subdivision to make sense of $FA_0(X)$). The question whether there are other 'gaps' for FA_0 apart from between n and n^2 for finitely presented groups was answered in the negative (see [34], [6], [4]). It is easy to see that one can produce simplicial (or riemannian) planes with FA_0 of the form, say, n^r, $r \in (2, \infty)$ and Grimaldi-Pansu ([16]) study the finer question of characterizing completely filling functions for riemannian planes.

Gromov ([19]) has given estimates and formulated conjectures for higher dimensional filling functions of nilpotent groups (see also [36] and [5] for interesting examples of higher filling functions of groups).

We note that as we move to higher dimensions we have two possible ways to define filling functions. We can either define them by considering fillings of (singular) spheres by balls or more generally one may consider filling of higher dimensional cycles (e.g. filling of orientable surfaces of genus $g \geq 0$ in dimension 3). Here we take the second option, apart from being easier to define technically it seems more natural. For example, as it is shown in [30] filling of 2-spheres in groups is always subrecursive in contrast to FA_0 which is not subrecursive for groups with unsolvable word problem. So examining filling only of 2-spheres seems quite restrictive.

We refer to the introduction for the definition of the terms in the theorem below. To simplify notation we denote $Fillvol_2(c)$ by $FillA(c)$ if c is a 1-cycle and $Fillvol_3(S)$ by $FillV(S)$ if S is a 2-cycle.

Theorem 4.1. *Let X be a simplicial complex such that $H_1(X) = H_2(X) = 0$. Assume that the following hold:*

- *There is some $K > 0$ such that $FV_2(n) \leq Kn^2$ for all $n \in \mathbb{N}$.*
- $$\lim_{n \to \infty} \frac{FV_3(n)}{n^{\frac{3}{2}}} = 0$$
- *There is some $g \in \mathbb{N}$ such that if S is a minimizer $2-cycle$ in X^2 then S is represented by a surface of genus at most g.*

Then for every $\epsilon > 0$ we have

$$\lim_{n \to \infty} \frac{FV_3(n)}{n^{1+\epsilon}} = 0$$

We don't know whether the third condition on the bound of the genus of the minimizers is in fact necessary. It is quite crucial however for our proof. We use it to deduce that the diameter of a minimizer S is of the order of $\sqrt{A(S)}$. This in turn is based on the upper bound of Cheeger constants in terms of genus. So our proof would work as well if we assumed that there is an upper bound for the Cheeger constant of minimizers of the form $\frac{c}{\sqrt{A(S)}}$ or if we assumed that there is a lower bound for the diameter of minimizers of the form $\delta\sqrt{A(S)}$. It would be interesting to remove the condition on minimizers even in the case that X is a non-positively manifold homeomorphic to $\mathbb{R}^3$. We remark that it is not known whether the isoperimetric profile of a non-positively curved manifold homeomorphic to $\mathbb{R}^n$ is dominated by the isoperimetric profile of the Euclidean space $\mathbb{E}^n$ (this is known however for $n \leq 4$, see [11], [23]).

We are going to prove a somewhat stronger statement that implies theorem 4.1:

Theorem 4.2. *Let X be a simplicial complex such that $H_1(X) = H_2(X) = 0$ Assume that the following hold:*

- *There is some $K > 0$ such that $FV_2(n) \leq Kn^2$ for all $n \in \mathbb{N}$.*
- *There is some $g \in \mathbb{N}$ such that if S is a minimizer $2-cycle$ in X^2 then S is represented by a surface of genus at most g.*

Then given $\epsilon > 0$ there is a constant $\alpha = \alpha(K, g) > 0$ such that the following holds: If there is an n_0 such that for all $n > n_0$, $FV_3(n) < \alpha n^{\frac{3}{2}}$, then

$$\lim_{n \to \infty} \frac{FV_3(n)}{n^{1+\epsilon}} = 0$$

Proof. In the course of the proof we will need to introduce some new constants; we will indicate the previous constants that the new constants depend on, e.g. for the new constant c we write $c = c(A, B)$ to indicate that c depends on the previous defined constants, A, B. It is possible always to give explicit estimates for the constants but we refrain from doing this as we don't find it instructive.

To show the theorem it is enough to show that for any $\epsilon > 0$ there is some $\beta = \beta(K, g) > 0$ with the following property: If there is an n_0 such that for all $n > n_0$, $FV_3(n) < \beta n^{\frac{3}{2}}$ then

$$\limsup_{n\to\infty} \frac{FV_3(n)}{n^{1+\epsilon}} \neq \infty$$

Indeed we can then take $\alpha(\epsilon) = \beta(\epsilon/2)$.

We argue by contradiction. The value of β will be specified in the course of the proof. So we assume that for some $\epsilon > 0$ the following holds:

For any $M > 0$ there is some 2-cycle S such that $FillV(S) > MA(S)^{1+\epsilon}$.

Lemma 4.3. *There is a $\delta = \delta(\epsilon, g, K) > 0$ such that for any $M > 0$ if S is a 2-cycle of minimal area such that $FillV(S) > MA(S)^{1+\epsilon}$ then $diam(S) > \delta\sqrt{A(S)}$.*

Proof. We set $n = \sqrt{A(S)}$. As we saw in the proof of proposition 2.6 there is a decomposition of S in two pieces S_1, S_2 such that:

- $S = S_1 \cup S_2$
- $S_1 \cup S_2 = \partial S_1 \cap \partial S_2$
- $A(S_i) \geq A(S)/4$, $i = 1, 2$
- $l(S_1 \cap S_2) \leq 10^3 g^2 n$
- $S_1 \cap S_2$ has at most $2g + 1$ components.

We claim now that if p is a closed curve on X^1 of diameter less than $\delta l(p)$ then $FillA(p) \leq 40K\delta l(p)^2$.

To see this subdivide p into $[1/\delta] + 1$ segments of length at most $\delta l(p) + 1$. Let $v_1, ..., v_r$ be the successive endpoints of these segments. We consider geodesic segments $[v_1, v_3], ..., [v_1, v_{r-1}]$ and we use them to break p into $r - 2$ loops each of which has length at most $3\delta l(p) + 3$. Since $FillA(p)$ is less or equal to the sum of the areas of these loops we have:

$$FillA(p) \leq K([1/\delta] - 1)(3\delta l(p) + 3)^2 \leq K\frac{1}{\delta}(6\delta l(p))^2 \leq 40K\delta l(p)^2$$

Assume now that $diam(S) \leq \delta n$. We will show that this leads to a contradiction if δ is too small.

Let $c = S_1 \cap S_2$. Since c has at most $2g+1$ components and $diam(c) \leq diam(S) \leq \delta n$ using the above estimate for a single simple closed curve we obtain for c:

$$FillA(c) \leq 40K(2g+1)\delta l(c)^2 \leq 40K(2g+1)\delta(10^3 g^2 n)^2$$

Let $\tilde{S}$ be a 2-cycle filling c with $A(\tilde{S}) \leq 40K(2g+1)\delta l(c)^2$. We break S into two 2-cycles using $\tilde{S}$: $\tilde{S}_1 = S_1 + \tilde{S}$ and $\tilde{S}_2 = S_2 - \tilde{S}$. We set $\delta' = 40K(2g+1)10^6 g^4 \delta$. So

$$A(\tilde{S}) \leq \delta' A(S)$$

If δ is sufficiently small $A(\tilde{S})$ is smaller than $A(S_1)$, $A(S_2)$. Using the minimality of S we have:

$$FillV(S) \leq FillV(\tilde{S}_1) + FillV(\tilde{S}_2) \leq MA(\tilde{S}_1)^{1+\epsilon} + MA(\tilde{S}_2)^{1+\epsilon}$$

Since $A(S_1)$ and $A(S_2)$ are bigger than $A(S)/4$ there is some $a \in [\frac{1}{4}, \frac{3}{4}]$ such that $A(S_1) = aA(S)$ and $A(S_2) = (1-a)A(S)$. We have

$$A(\tilde{S}_1) \leq aA(S) + \delta' A(S), \;\; A(\tilde{S}_2) \leq aA(S) + \delta' A(S)$$

Substituting above we have

$$FillV(S) \leq MA(S)^{1+\epsilon}[(a+\delta')^{1+\epsilon} + (1-a+\delta')^{1+\epsilon}]$$

Since the function $x^{1+\epsilon}$ is strictly convex $a^{1+\epsilon} + (1-a)^{1+\epsilon} < 1$ for all $a \in [1/4, 3/4]$. It follows that if δ' is small enough $(a+\delta')^{1+\epsilon} + (1-a+\delta')^{1+\epsilon} < 1$. Clearly one can give an explicit estimate for δ' in terms of ϵ.

Now if

$$\delta \leq \frac{\delta'}{40K(2g+1)10^6 g^4}$$

we have

$$FillV(S) \leq MA(S)^{1+\epsilon}$$

which is a contradiction.

□

We need a technical lemma:

Lemma 4.4. *Given $\epsilon > 0$ there is some $\lambda > 0$ such that for any $x \in (0, 1/2]$ the following inequality holds:*

$$(x + \lambda x)^{1+\epsilon} + (1 - x + \lambda x)^{1+\epsilon} < 1$$

Proof. We consider the function

$$f(x) = 1 - (x + \lambda x)^{1+\epsilon} - (1 - x + \lambda x)^{1+\epsilon}$$

We have

$$f'(x) = (1+\epsilon)[-(1+\lambda)^{1+\epsilon}x^{\epsilon} + (1-\lambda)(1-x+\lambda x)^{\epsilon}]$$

We remark now that there is a constant $c > 0$ such that if $\lambda < 1/2$ we have

$$-(1+\lambda)^{1+\epsilon}x^{\epsilon} + (1-\lambda)(1-x+\lambda x)^{\epsilon} > 0, \forall x \in [0, c]$$

Since $f(0) = 0$ we conclude that $f(x) > 0$ for $x \in [0, c]$, if $\lambda < 1/2$.

Now we remark that the function $x^{1+\epsilon}$ is strictly convex. It follows that $x^{1+\epsilon} + (1-x)^{1+\epsilon}$ restricted on the interval $[c, 1/2]$ is strictly smaller than 1. It follows that there is some $\lambda > 0$ such that

$$(x + \lambda x)^{1+\epsilon} + (1 - x + \lambda x)^{1+\epsilon} < 1$$

for all $x \in [c, 1/2]$. So there is some $1/2 > \lambda > 0$ such that for any $x \in (0, 1/2)$ we have

$$(x + \lambda x)^{1+\epsilon} + (1 - x + \lambda x)^{1+\epsilon} < 1$$

□

In what follows given $M > 0$ we consider a 2-cycle S of minimal area such that $FillV(S) > MA(S)^{1+\epsilon}$. Let R be a 3-chain such that $\partial R = S$ and $V(R) = FillV(S)$. We consider R as a subset of X. We fix a vertex $x \in S$ and we consider $B_i(x)$ in X. Let $R = \sum n_j \sigma_j$ with $n_j \in \mathbb{Z}$. We define R_i to be the chain:

$$R_i = \sum_{\sigma_k \in B_i(x)} n_k \sigma_k$$

We consider now all decompositions of ∂R_i as sum of two chains $\partial R_i = R_1 + R_2$. We consider the minimal value of $A(S - R_1)$ over all such decompositions. Let $\partial R_i = O_i + I_i$ be a decomposition of ∂R_i such that $A(S - O_i)$ attains this minimum.

With this notation we have the following lemma.

Lemma 4.5. *There is a $\lambda > 0$ such that for any $M > 0$ if S is a 2-cycle S of minimal area such that $FillV(S) \geq MA(S)^{1+\epsilon}$ then the following holds:*

$$A(I_i) \geq \lambda \min\{A(O_i), A(S - O_i)\}, \quad \forall i \in [\tfrac{\delta n}{4}, \tfrac{\delta n}{2}]$$

where δ is given by lemma 4.3 and we denote as before $n = \sqrt{A(S)}$.

Proof. Let λ be as in lemma 4.4. We argue by contradiction, ie we assume that the inequality of the lemma does not hold for some i. We consider the 2-cycles ∂R_i and $S - \partial R_i$. We remark that $A(\partial R_i)$ and $A(S - \partial R_i)$ are both smaller than $A(S)$. By our assumption on S we have the inequalities:

$$FillV(\partial R_i) < MA(\partial R_i)^{1+\epsilon}$$

$$FillV(S - \partial R_i) < MA(S - \partial R_i)^{1+\epsilon}$$

We also have

$$FillV(S) \leq FillV(\partial R_i) + FillV(S - \partial R_i)$$

Hence

$$FillV(S) < M[A(\partial R_i)^{1+\epsilon} + A(S - \partial R_i)^{1+\epsilon}] \qquad (1)$$

Now $A(\partial R_i) = A(O_i) + A(I_i)$ and $A(S - \partial R_i) = A(I_i) + A(S - O_i)$. Let

$$m = max\{A(O_i), A(S - O_i)\}$$

We set

$$x = \frac{m}{A(S)}$$

From inequality (1) and from our assumption we obtain:

$$FillV(S) < MA(S)^{1+\epsilon}[(x + \lambda x)^{1+\epsilon} + (1 - x + \lambda x)^{1+\epsilon}] < MA(S)^{1+\epsilon}$$

where the last inequality follows from lemma 4.4. This is clearly a contradiction.

□

Lemma 4.6. *Let $\mu > 0$, $a \in \mathbb{N}, a > 3$ and let $f(i) \in \mathbb{R}$ be a sequence such that $f(i) > \mu a$ for all $i \in \mathbb{N} \cap [a, 2a]$. Let*

$$F(s) = \sum_{i=a}^{s} f(i)$$

and assume that $F(2a) < \mu a^3$. Then for some r we have

$$F(r) > \frac{1}{8\sqrt{3\mu}} f(r)^{\frac{3}{2}}$$

Proof. We consider the sequence

$$g(i) = 3\mu i^2, \quad i = 1, 2, ..., a$$

We have $f(a) > g(1)$ and

$$\sum_{i=1}^{a} g(i) > \sum_{i=a}^{2a} f(i)$$

We consider the smallest r for which

$$f(r) < g(r - a + 1)$$

We have then

$$F(r) > F(r-1) = \sum_{i=a}^{r-1} f(i) > \sum_{i=1}^{r-a} g(i)$$

We set $s = r - a + 1$ and we have

$$\sum_{i=1}^{r-a} g(i) = 3\mu \frac{(s-1)s(2s-1)}{6}$$

On the other hand

$$f(r) < g(s) = 3\mu s^2$$

Since $s \geq 2$ we have

$$3\mu \frac{(s-1)s(2s-1)}{6} \geq \frac{1}{8\sqrt{3\mu}} (3\mu s^2)^{\frac{3}{2}} > \frac{1}{8\sqrt{3\mu}} f(r)^{\frac{3}{2}}$$

so

$$F(r) > \frac{1}{8\sqrt{3\mu}} f(r)^{\frac{3}{2}}$$

□

We take now $M = M(n_0)$ 'sufficiently big' and we consider a 2-cycle S of minimal area such that $FillV(S) > MA(S)^{1+\epsilon}$. We will explain how we choose M at the relevant point of the proof.

Let R be a 3-chain such that $\partial R = S$ and $V(R) = FillV(S)$. We set $n = \sqrt{A(S)}$.

Let δ be as in lemma 4.3. We fix a vertex $x \in S$ and we consider $B_i(x) \cap S$ and $B_i(x) \cap R$ for $\delta n/4 \leq i \leq \delta n/2$. We define R_i, O_i, I_i as above. We remark now that the following inequalities hold:

$$A(O_i) \geq 2i \geq \delta n/2$$

$$A(S - O_i) \geq \delta n/2$$

$$V(R_i - R_{i-1}) \geq A(I_i)/3 \geq \lambda\delta n/6 \quad (*)$$

where λ is the constant provided by lemma 4.5 and the last inequality follows from the same lemma.

We consider now the finite sequence

$$f(i) = A(I_i),\ i \in [\delta n/4, \delta n/2]$$

We may choose M so that $\delta n/4 > 3$. We may also assume that δ is such that $\delta n/4 \in \mathbb{N}$ (we just replace δ by δ' so that $\delta' n/4 = [\delta n/4]$).

We set $a = \delta n/4$ and we remark that $f(i) \geq \frac{\lambda}{2} a$ for all $i \in [\delta n/4, \delta n/2]$. We may choose β so that $3\beta < \frac{\lambda}{2}$. Since $V(S) < \beta n^3$ we have

and

$$\sum_{i=a}^{2a} f(i) \leq 3\beta n^3 < \frac{\lambda}{2} n^3$$

by our assumption and the inequality $(*)$.

Applying lemma 4.6 to f with $\mu = \frac{\lambda}{2}$ we conclude that there is some $r \in [\delta n/4, \delta n/2]$ such that

$$\sum_{i=a}^{r} f(i) > \frac{1}{8\sqrt{3\mu}} f(r)^{\frac{3}{2}}$$

We have also the inequality:

$$V(R_r) \geq \sum_{j=\delta n/4}^{r} f(j)/3 \geq \frac{1}{24\sqrt{3\mu}} f(r)^{\frac{3}{2}}$$

From lemma 4.5

$$f(r) \geq \lambda \min\{A(O_r), A(S - O_r)\}$$

We distinguish now two cases:

Case 1. $f(r) \geq \lambda A(O_r)$. In this case

$$A(\partial R_r) = A(O_r) + A(I_r) \leq (1 + \frac{1}{\lambda}) f(r)$$

while

$$V(R_r) \geq \frac{1}{24\sqrt{3\mu}} f(r)^{\frac{3}{2}}$$

So if

$$\beta \leq \min\left(\frac{1}{24\sqrt{3\mu}(1+\frac{1}{\lambda})^{\frac{3}{2}}}, \frac{\lambda}{2}\right)$$

and M is big enough so that $A(\partial R_r) > n_0$ we have

$$V(R_r) \geq \beta A(\partial R_r)^{\frac{3}{2}}$$

which contradicts our assumption.

Case 2. $f(r) < \lambda A(O_r)$. Then $f(r) \geq A(S - O_r)$ so $A(S - O_r) \leq A(O_r)$. In this case we pick $y \in S$ with $d(x, y) = diam S^1$ and we repeat the construction considering $B_y(i)$ instead of $B_x(i)$. We obtain a 3-chain as before which we denote $R'_{r'}$. $\partial R'_{r'} \cap S \subset A(S - O_r)$ we obtain a contradiction from $\partial R'_{r'}$ as in case 1.

□

Remark 5. The assumption that $FV_2(n)$ is bounded by a quadratic function does not play an essential role in the proof above. One may substitute this by $FV_2(n) \leq Kn^r$ for some $K > 0, r > 2$ and change the conclusion to: If

$$\lim_{n\to\infty} \frac{FV_3(n)}{n^{\frac{2r-1}{2r-2}}} = 0$$

then

$$\lim_{n\to\infty} \frac{FV_3(n)}{n^{1+\epsilon}} = 0$$

for any $\epsilon > 0$. This shows that there is some relationship between $FV_2(n)$ and $FV_3(n)$, always of course under the assumption of the bound on the genus of minimizers.

References

[1] F.J. Almgren, Jr, *An isoperimetric inequality*, Proc. Amer. Math. Soc. 15, 1964, p. 284–285.

[2] Besicovitch, A. S. *On two problems of Loewner*, J. London Math. Soc. 27, (1952). 141–144.

[3] M.Bonk, A.Eremenko, *Uniformly hyperbolic surfaces*, Indiana Univ. Math. J. 49 (2000), no. 1, 61–80

[4] N.Brady, M.R.Bridson, *There is only one gap in the isoperimetric spectrum*, Geom. Funct. Anal. 10 (2000), no. 5, 1053–1070

[5] N.Brady, M.R.Bridson, M.Forester, K.Shankar *Snowflake groups, Perron-Frobenius eigenvalues, and isoperimetric spectra*, preprint

[6] M.R.Bridson, *Fractional isoperimetric inequalities and subgroup distortion*, J. Amer. Math. Soc. 12 (1999), no. 4, 1103–1118.

[7] I.Benjamini, J.Cao, *A new isoperimetric theorem for surfaces of variable curvature*, Duke Math. J. 85, p. 359-396 (1996).

[8] B.H. Bowditch, *A short proof that a subquadratic isoperimetric inequality implies a linear one*, Michigan Math. J. 42 (1995), no. 1, 103–107.

[9] J.Burillo, J.Taback, *Equivalence of geometric and combinatorial Dehn functions*, New York J. Math. 8 (2002), 169–179

[10] M.Coornaert, T.Delzant, A. Papadopoulos,*Geometrie et theorie des groupes*, Lecture Notes in Mathematics, 1441. Springer-Verlag, Berlin, 1990. x+165 pp.

[11] C.B.Croke, *A sharp four-dimensional isoperimetric inequality.*, Comment. Math. Helv. 59 (1984), no. 2, 187–192

[12] C. Drutu, *Cones asymptotiques et invariants de quasi-isometrie pour des espaces metriques hyperboliques* Ann. Inst. Fourier (Grenoble) 51 (2001), no. 1, 81–97.
[13] D.B.A Epstein, J.W. Cannon, D.F. Holt, S.V.F. Levy, M.S. Paterson, W.P. Thurston,*Word processing in groups*, Jones and Bartlett Publishers, Boston, MA, 1992. xii+330 pp.
[14] S.M.Gersten, *Subgroups of word hyperbolic groups in dimension 2* , J. London Math. Soc. (2) 54 (1996), no. 2, 261–283.
[15] J.R.Gilbert, J.P. Hutchinson, R.E. Tarjan *A separator theorem for graphs of bounded genus* J. Algorithms 5 (1984) p.391-407.
[16] R. Grimaldi, P. Pansu, *Remplissage et surfaces de revolution*, J. Math. Pures Appl. (9) 82 (2003), no. 8, 1005–1046.
[17] M. Gromov, *Filling Riemannian manifolds* J. Differential Geom. 18 (1983), no. 1, 1–147.
[18] M. Gromov, *Hyperbolic groups*, Essays in group theory (S. M. Gersten, ed.), MSRI Publ. 8, Springer-Verlag, 1987 pp. 75-263.
[19] M.Gromov, *Asymptotic invariants of infinite groups* in 'Geometric group theory', (G.Niblo, M.Roller, Eds.), LMS Lecture Notes, vol. 182, Cambridge Univ. Press, 1993
[20] M.Gromov, *Metric structures for Riemannian and non-Riemannian spaces* With appendices by M. Katz, P. Pansu and S. Semmes, Progress in Mathematics, 152. Birkhauser Boston, Inc., Boston, MA, 1999. xx+585 pp.
[21] A.Hatcher, *Algebraic topology* , Cambridge University Press, Cambridge, 2002. xii+544 pp.
[22] J.Hersch, *Quatre proprietes isoperimetriques de membranes spheriques homogenes*, C.R. Acad. Sci. Paris **270** (1970) p. 1645-1648
[23] B.Kleiner, *An isoperimetric comparison theorem*, Invent. Math. 108 (1992), no. 1, 37–47.
[24] N. Korevaar, *Upper bounds for eigenvalues of conformal metrics*, J.Diff. Geom. **37** (1993) p.73-93
[25] R.J. Lipton, R.E.Tarjan *Applications of a planar separator theorem* SIAM J. Comput. 9 (1980), p.615-627
[26] F.Morgan, M. Hutchings, H.Howards, *The isoperimetric problem on surfaces of revolution of decreasing Gauss curvature*, Trans. Amer. Math. Soc. 352 (2000), no. 11, 4889–4909.
[27] A.Yu.Olshanski, *Hyperbolicity of groups with subquadratic isoperimetric inequality* Internat. J. Algebra Comput. 1 (1991), no. 3, 281–289.
[28] P.Pansu, *Une inégalité isopérimétrique pour le groupe d'Heisenberg*, C.R. Acad. Sci. Paris 295 (1982), p. 127-131.
[29] P.Papasoglu, *An algorithm detecting hyperbolicity* Geometric and computational perspectives on infinite groups p.193–200, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 25, AMS, Providence, RI, 1996
[30] P.Papasoglu, *Isodiametric and isoperimetric inequalities for complexes and groups*, J. London Math. Soc. (2) 62 (2000), no. 1, 97–106
[31] L.Polterovich, J-C. Sikorav, *A linear isoperimetric inequality for the punctured Euclidean plane*, preprint, arXive math.GR/0106216
[32] M. Ritoré, *The isoperimetric problem in complete surfaces of nonnegative curvature* J. Geom. Anal. 11 (2001), no. 3, 509–517

[33] M. Ritoré, *Constant geodesic curvature curves and isoperimetric domains in rotationally symmetric surfaces*, Comm. Anal. Geom. 9 (2001), no. 5, 1093–1138.

[34] M.V.Sapir, J-C.Birget, E.Rips, *Isoperimetric and isodiametric functions of groups*, Ann. of Math. (2) 156 (2002), no. 2,

[35] P.Topping, *Mean curvature flow and geometric inequalities*, J. Reine Angew. Math. 503 (1998), 47–61

[36] X. Wang, S.J. Pride, *Second order Dehn functions and HNN-extensions* J. Austral. Math. Soc. Ser. A 67 (1999), no. 2, 272–288.

[37] S.Wenger, *Gromov hyperbolic spaces and the sharp isoperimetric constant*, preprint, Arxiv math.DG/0609310

[38] S.Wenger, *Isoperimetric inequalities and the asymptotic rank of metric spaces*, preprint.

[39] P.Young, S.T.Yau, *Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds* , Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **7** (1980), p. 55-63.

E-mail address: panos@math.uoa.gr

Mathematics Department, University of Athens, Athens 157 84, Greece