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Inventiones mathematicae

Strongly geodesically automatic groups are hyperbolic

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Introduction

A finitely generated group G is called *hyperbolic* if for some $\delta > 0$ geodesic triangles in the Cayley graph of G are δ-thin. This means that for any triangle in the Cayley graph of G each side is in the δ-neighborhood of the union of the two other sides.

In 1984 Cannon in [C] showed the existence of a recursive structure on the set of all geodesics of a cocompact discrete group of isometries of hyperbolic space. His result immediately generalizes to all hyperbolic groups. The precise formulation of his result in the language of automatic groups is:

Theorem 1(Cannon). *If a group is hyperbolic then it is strongly geodesically automatic.*

The main result of this paper is that the converse is also true:

Theorem 2. *If a group is strongly geodesically automatic then it is hyperbolic.*

The proof of Theorem 2 is based on a simple geometric observation: If bigons in a graph are thin then triangles are also thin. In other words if there is an $\varepsilon > 0$ such that any two finite geodesics with common endpoints are in the ε neighborhood of each other then there is a $\delta > 0$ so that all geodesic triangles are δ-thin.

This of course gives an alternative definition of hyperbolic groups, namely a group is hyperbolic if for some $\varepsilon > 0$ bigons in its Cayley graph are ε-thin.

We note that this characterization of hyperbolicity is not valid for geodesic metric spaces in general as, for example, bigons are thin in the Euclidean

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plane but triangles are not. It is true however that if quasigeodesic bigons in a geodesic metric space are thin then the space is hyperbolic.

We apply theorem 2 to produce a partial algorithm to recognize hyperbolic groups: using work of [ECHLPT] we construct a partial algorithm which recognizes strongly geodesically automatic groups. Epstein, Short and Holt have suggested improvements to the algorithm that we present here and it is hoped that these improvements will make it computationally efficient. The first partial algorithm recognizing hyperbolic groups was given by Gromov [G]. Gromov's algorithm however was clearly impractical and it was impossible to apply it even in the simplest cases.

1. Thin bigons

In this section we prove that if bigons in a graph are "thin" then the graph is a (Gromov) hyperbolic space.

Definitions

A metric space X is called a *geodesic metric space* if for all points x, y in X there is an isometric map from the interval $[0, d(x, y)]$ to a path in X joining x and y. We denote the image of such an isometry by $[xy]$.

In what follows X is a geodesic metric space. We assume paths in X to be always parametrized with respect to arclength. If $\alpha : [0, l] \to X$ is a path we extend $\alpha : [0, \infty) \to X$ by defining $\alpha(t) = \alpha(l)$ for $t > l$. We consider connected graphs to be geodesic metric spaces by making each edge isometric to the unit interval or, if the edge's endpoints are equal, to the circle of length one. We define then the distance between two points to be the length of the shortest path joining them.

A geodesic metric space X is called *δ-hyperbolic* if for any geodesic triangle xyz in X and any point $w \in [xy]$ we have that $d(w, [xz] \cup [yz]) \leqq \delta$.

A group G given by a presentation $\langle S|R\rangle$ is called *δ-hyperbolic* if the Cayley graph $\Gamma(G|S)$ is δ-hyperbolic.

We will often omit the δ and we will call a δ-hyperbolic space or group, hyperbolic. We say that $e : \mathbb{R}^+ \to \mathbb{R}$ is a divergence function for X if for all $x \in X$, all $R \in \mathbb{R}^+$ and all geodesics $\gamma = [xy], \gamma' = [xz], e$ satisfies the following condition: If $r > 0, R + r \leqq \min(d(x, y), d(x, z)), d(\gamma(R), \gamma'(R)) \geqq e(0) > 0$ and α is a path in $\overline{X - B_x(R + r)}$ from $\gamma(R + r)$ to $\gamma'(R + r)$ then we have length $(\alpha) > e(r)$.

We say that geodesics *diverge* in X if there is a divergence function $e(r)$ such that $\lim_{r \to \infty} e(r) = \infty$.

We say that geodesics diverge exponentially in X if there is an exponential divergence function.

We need a result proved in [P]. For the reader's convenience we recall here the proof of this result.

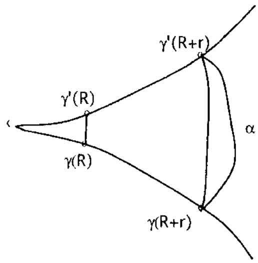

Fig. 1.

Theorem 1.1. *Let X be a geodesic metric space. If geodesics diverge in X then geodesics diverge exponentially in X.*

Proof. Let $f(r)$ be a divergence function for X such that $\lim_{r\to\infty} f(r) = \infty$. We define $e(r) = \inf\{\text{length}(\alpha) : \alpha$ is a path from $\gamma(R+r)$ to $\gamma'(R+r)$ in $\overline{X - B_x(R+r)}$ and $d(\gamma(R), \gamma'(R)) \geqq f(0)$ where γ, γ' are geodesics and $\gamma(0) = \gamma'(0) = x\}$ where the infimum is taken over all geodesics γ, γ' all $x \in X$ and all $R \in \mathbb{R}^+$. It is clear that $e(r)$ is a divergence function for X and that $\lim_{r\to\infty} e(r) = \infty$. ($e(r) \geqq f(r)$ for every r). Let

$$N' = \sup\{r : e(r) < 9e(0)\} .$$

Let $N = N' + 1 + 3e(0)$. Let

$$u = \sup\{t : e(t) < 4N + 2\} .$$

We will prove that if $r > u + N$ then $e(r) > \frac{3}{2}e(r - N)$.

Lemma 1.2. *Let γ, γ' be geodesics such that $\gamma(0) = \gamma'(0), d(\gamma(R), \gamma'(R)) > e(0)$. Then $d(\gamma(R+N), \gamma'(R+N)) \geqq 3e(0)$.*

Proof. Suppose that $d(\gamma(R+N), \gamma'(R+N)) < 3e(0)$. Let β' be a geodesic joining $\gamma(R+N), \gamma'(R+N)$. Consider the arc

$$\beta = [\gamma(R+N-3e(0))\gamma(R+N)] \cup \beta' \cup [\gamma'(R+N-3e(0))\gamma'(R+N)]$$

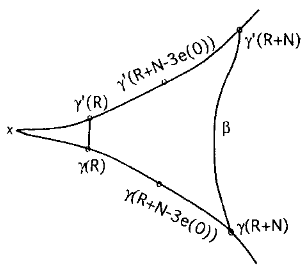

Fig. 2.

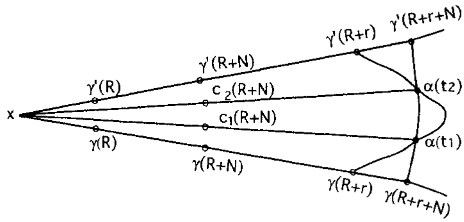

Fig. 3.

It is clear that

$$\beta \subset \overline{X - B_x(R + N - 3e(0))}$$

Also length $(\beta) = 6e(0) + \text{length}(\beta') < 9e(0)$. But length $(\beta) > e(N - 3e(0)) \geqq 9e(0)$ since $N - 3e(0) = N' + 1 > N'$. This is a contradiction and the lemma is proved.

Let $r > u + N$ and let γ, γ' be geodesics such that $\gamma(0) = \gamma'(0) = x, d(\gamma(R), \gamma'(R)) > e(0)$ and let α be an arc in $\overline{X - B_x(R+r)}$ such that $\alpha(0) = \gamma(R+r)$, $\alpha(M) = \gamma'(R+r)$ and $M = \text{length}(\alpha) \leqq e(r) + 1$.

Let

$$t_1 = \sup\{t : \alpha(t) \in B_x(R + r + N), t \leqq M/2\}$$

$$t_2 = \inf\{t : \alpha(t) \in B_x(R + r + N), t \geqq M/2\}$$

Let c_1, c_2 be geodesics from x to $\alpha(t_1), \alpha(t_2)$ respectively. Then

$$d(\gamma(R+N), c_1(R+N)) + d(c_1(R+N), c_2(R+N)) + d(c_2(R+N), \gamma'(R+N)) \geqq d(\gamma(R+N), \gamma'(R+N)) \geqq 3e(0)$$

so one of the three summands is greater than $e(0)$. Suppose that $d(c_1(R+N), c_2(R+N)) \geqq e(0)$. Then $\text{length}(\alpha|_{[t_1,t_2]}) \leqq \text{length}(\alpha) - 2N \leqq e(r) + 1 - 2N < e(r)$. So we have a path joining $\alpha(t_1), \alpha(t_2)$,contained in $\overline{X - B_x(R+r+N)}$, of length less than $e(r)$, which is impossible. We assume without restriction of generality that

$$d(\gamma(R+N), c_1(R+N)) \geqq e(0)$$

We consider now the path $\alpha' = \alpha|_{[0,t_1]} \cup [c_1(R+r), \alpha(t_1)]$. Then $\text{length}(\alpha') \leqq \frac{M}{2} + N \leqq (e(r)+1)/2 + N$. On the other hand $\text{length}(\alpha') \geqq e(r-N)$. So

$$e(r-N) \leqq \tfrac{1}{2}e(r) + N + \tfrac{1}{2} \Rightarrow e(r) \geqq 2(e(r-N) - N - \tfrac{1}{2})$$

But $e(r-N) \geqq 4N + 2 \Rightarrow e(r-N) - N - \frac{1}{2} \geqq \frac{3}{4}e(r-N)$, so

$$e(r) \geqq \tfrac{3}{2}e(r-N)$$

It is proved in [ABC] that if geodesics in a geodesic metric space diverge faster than linearly then the space is (Gromov) hyperbolic. So we have the following:

Corollary 1.3. *If geodesics diverge in a geodesic metric space then the space is hyperbolic.*

Definitions
A *bigon* in a graph Γ is a pair of geodesics γ, γ' with $\gamma(0) = \gamma'(0)$ and $\gamma(l) = \gamma'(l)$ where $l = \text{length}(\gamma)$. We say that bigons in a graph Γ are ε-*thin* if for any bigon γ, γ' with $\gamma(0) = \gamma'(0)$ and $\gamma(l) = \gamma'(l)$ we have $d(\gamma(t), \gamma'(t)) < \varepsilon$ for every $t, 0 < t < l$. We say that a bigon γ, γ' is M-*thick* if for some $t, d(\gamma(t), \gamma'(t)) \geqq M$

Theorem 1.4. *Let G be a finitely generated group and let $\Gamma = \Gamma(G|S)$ be the Cayley graph of G associated to a finite generating set S. If there is an ε such that bigons in Γ are ε-thin then G is hyperbolic.*

Proof. Assume that G is not hyperbolic. We will prove that for any $M \in \mathbb{N}$ there is a M-thick bigon. We define $f(r) = \inf\{d(\gamma(R+r), \gamma'(R+r))$ where γ, γ' are geodesics such that $\gamma(0) = \gamma'(0) = e$ and $d(\gamma(R), \gamma'(R)) \geqq 2M^2\}$ where the infimum is taken over all geodesics γ, γ' in Γ and all $R \in \mathbb{N}$ and e is the vertex corresponding to the identity in G.

It is obvious that if $\lim_{r\to\infty} f(r) = \infty$ then geodesics diverge in Γ which implies that G is hyperbolic.

We can therefore assume that $\liminf_{r\to\infty} f(r) = K < \infty$.

Note that since Γ is a graph, K is an integer and $f(r) = K$ for infinitely many values of r.

We distinguish now two cases:

Case 1. Suppose that there are two geodesics γ, γ' such that $\gamma(0) = \gamma'(0) = e$, $d(\gamma(R), \gamma'(R)) \geqq 2M^2$ and for some integers $t_1 < t_2$

$$d(\gamma(R+t_1), \gamma'(R+t_1)) \geqq Md(\gamma(R+t_2), \gamma'(R+t_2)) > 0$$

We show now how to construct an M-thick bigon in this case.

Let a be a geodesic arc such that $a(0) = \gamma(R+t_2)$, $a(l) = \gamma'(R+t_2)$.

Let $b_0 = \gamma|_{[0,R+t_2]}, b_l = \gamma'|_{[0,R+t_2]}$ and let b_i, $1 \leqq i \leqq l-1$ be geodesic arcs such that $b_i(0) = e$, $b_i(l_i) = a(i)$ where $l_i = \mathit{length}(b_i)$.

We note now that

$$Ml \leqq d(\gamma(R+t_1), \gamma'(R+t_1)) \leqq \sum_{i=0}^{l-1} d(b_i(R+t_1), b_{i+1}(R+t_1))$$

so at least one of the terms in the sum has to be greater or equal to M, i.e. $d(b_j(R+t_1), b_{j+1}(R+t_1)) \geqq M$ for some j, $0 \leqq j \leqq l-1$.

If $\text{length}(b_j) < \text{length}(b_{j+1})$ then we get an M-thick bigon defined by the geodesics b_{j+1} and b' where

$$b'(t) = b_j(t),\ 0 \leqq t \leqq l_j,\ b'(t) = a(j+t-l_j),\ l_j \leqq t \leqq l_j + 1$$

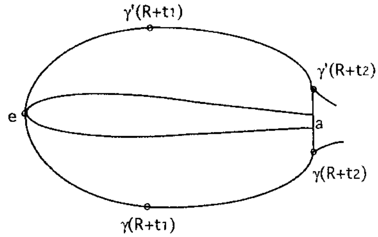

Fig. 4.

Similarly we define an M-thick bigon in case length(b_{j+1}) < length(b_j). If length(b_j) = length(b_{j+1}) then we get an M-thick bigon defined by the geodesics b' and b'' where

$$b'(t) = b_j(t), \quad 0 \leqq t \leqq l_j, b'(t) = a(j + t - l_j), \quad l_j \leqq t \leqq l_j + \tfrac{1}{2}$$
$$b''(t) = b_{j+1}(t), \quad 0 \leqq t \leqq l_{j+1},$$
$$b''(t) = a(j + 1 - t + l_{j+1}), \quad l_{j+1} \leqq t \leqq l_{j+1} + \tfrac{1}{2}$$

Case 2. We assume that no pair of geodesics γ, γ' as described in case 1 exist.

In this case there are pairs of geodesics which start from the same point, go far apart and then stay almost 'parallel'. To construct an M-thick bigon we have to use a pair of almost 'parallel' geodesics which are sufficiently long.

Let γ, γ' be geodesics with $\gamma(0) = \gamma'(0) = e$, $d(\gamma(R), \gamma'(R)) \geqq 2M^2$ and $d(\gamma(R+r), \gamma'(R+r)) = K$ for some $r \geqq (2^{KM} - 1)(KM)^2$. (Such geodesics exist because of our hypothesis that $\liminf_{r\to\infty} f(r) = K$).

We claim that

$$2M < d(\gamma(R+t), \gamma'(R+t)) < KM, \ 0 \leqq t \leqq r, \ t \in \mathbb{N}$$

Indeed if for some integer $t > 0$, $2M > d(\gamma(R+t), \gamma'(R+t))$ then

$$d(\gamma(R), \gamma'(R)) \geqq Md(\gamma(R+t), \gamma'(R+t))$$

and γ, γ' satisfy the hypothesis of case 1 with $t_1 = 0$, $t_2 = t$.

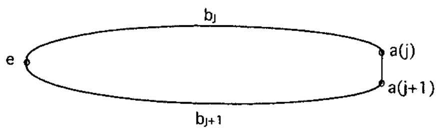

Fig. 5.

Similarly if for some integer $t \geqq 0$

$$d(\gamma(R+t), \gamma'(R+t)) \geqq KM$$

then

$$d(\gamma(R+t), \gamma'(R+t)) \geqq Md(\gamma(R+r), \gamma'(R+r))$$

and γ, γ' satisfy the hypothesis of case 1 with $t_1 = t$, $t_2 = r$.

Note that for integer t, $d(\gamma(R+t), \gamma'(R+t))$ is also an integer. We define a map:

$$g(t) = d(\gamma(R+t), \gamma'(R+t))$$

where t is an integer and $0 \leqq t \leqq r$. By the previous claim $g(t)$ is an integer and $2M \leqq g(t) \leqq KM$.

Since $r \geqq (2^{KM} - 1)(KM)^2$ there is a $J, 2M < J < KM$ such that the cardinality of $g^{-1}(J)$ is greater than $(2^{KM} - 1)(KM)$.

We will prove that there is a $J/2$-thick bigon. Since $J/2 \geqq M$ this will finish the proof of the theorem.

Call a point P on γ or on γ' a J-point if $P = \gamma(t)$ or $P = \gamma'(t)$ for some integer t and $d(\gamma(t), \gamma'(t)) = J$.

If P_1, P_2 are two J-points on $\gamma, P_1 = \gamma(t_1), P_2 = \gamma(t_2)$ we define their J distance, $d_J(P_1, P_2) = \mathrm{Card}\{t : \gamma(t) \text{ is a } J\text{-point and } t \in [t_1, t_2]\} - 1$

It is easy to see that d_J is a distance function. We define similarly the J distance between two points on γ' and we denote it also by d_J.

It is clear that $d_J(P_1, P_2) \leqq d(P_1, P_2)$.

Lemma 1.5. *Assume that there is not a $J/2$-thick bigon in Γ. Let $P = \gamma(t)$ be a J point. If $Q = \gamma'(t')$, $t' \geqq t$ is a J point with $d_J(Q, \gamma'(t)) = (2^n - 1)J$ then $d(P, Q) \leqq t' - t + J - n$.*

Proof. We will prove this by induction on n. It is obviously true for $n = 0$. Suppose that it is true for $n = m$. Let $P = \gamma(t)$, $Q = \gamma'(t')$, $t' > t$ be J points such that

$$d_J(Q, \gamma'(t)) = (2^{m+1} - 1)J$$

Let $Q_1 = \gamma'(t_1)$, $t_1 \geqq t$, $Q_2 = \gamma'(t_2), t_2 > t$ be such that

$$d_J(Q_1, \gamma'(t)) = (2^m - 1)J$$

$$d_J(Q_2, \gamma'(t)) = 2^m J$$

Let $P_1 = \gamma(t_1) P_2 = \gamma(t_2)$. Using the inductive hypothesis we have

$$d(P, Q_1) \leqq t_1 - t + J - m$$

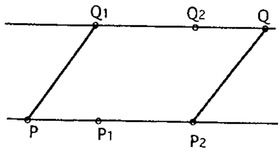

Fig. 6.

and

$$d(P_2,Q) \leqq t' - t_2 + J - m$$

If $d(P,Q_1) < t_1 - t + J - m$ then

$$\begin{aligned} d(P,Q) &\leqq d(P,Q_1) + d(Q_1,Q) \leqq t_1 - t + J - m - 1 + t' - t_1 \\ &= t' - t + J - (m+1) \end{aligned}$$

and the lemma is proved. Similarly we are done if $d(P_2,Q) < t' - t_2 + J - m$. We assume therefore that

$$d(P,Q_1) = t_1 - t + J - m$$

and

$$d(P_2,Q) = t' - t_2 + J - m$$

We conclude as before that $d(P,Q) \leqq t' - t + J - m$.

If $d(P,Q) < t' - t + J - m$ we are done (we remind the reader that $d(P,Q)$ is an integer). We assume therefore that $d(P,Q) = t' - t + J - m$. Consider the arcs $\alpha_1 = [PQ_1] \cup [Q_1Q]$ and $\alpha_2 = [PP_2] \cup [P_2Q]$. By our assumption α_1, α_2 define a bigon. We claim that α_1, α_2 is a $J/2$ thick bigon. More precisely we claim that $d(P_2, \alpha_1) \geqq J/2$.

Indeed, if $d(P_2,R) < J/2$ for some R on $[Q_1Q]$ then $d(R,Q_2) > J/2$. We have then

$$d(e,P_2) \leqq d(e,R) + d(R,P_2) < d(e,Q_2)$$

which is a contradiction since by hypothesis $d(e,P_2) = d(e,Q_2)$.

Similarly if $d(P_2,R) < J/2$ for some R on $[PQ_1]$ we have

$$d(P,R) > d(P,P_2) - J/2 \geqq d(P,P_1) + J/2$$

On the other hand

$$d(P,Q_1) \leqq d(P,P_1) + d(P_1,Q_1) = d(P,P_1) + J$$

Combining the previous two inequalities we get

$$d(R,Q_1) = d(P,Q_1) - d(P,R) < J/2$$

So

$$d(P_2,Q_1) \leqq d(P_2,R) + d(R,Q_1) < J$$

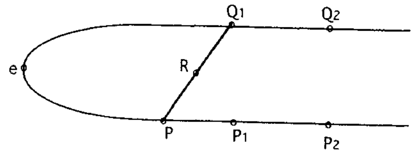

Fig. 7.

which implies that $d(e, P_2) < d(e, Q_2)$, a contradiction. This completes the proof of the lemma.

Since there are more than $(2^{KM} - 1)KM > (2^{J+1} - 1)J$ J-points on γ we conclude that there are J-points P, Q where

$$P = \gamma(t), Q = \gamma'(t'), t' > t \quad \text{with} \quad d_J(Q, \gamma'(t)) = (2^{J+1} - 1)J$$

We conclude using the previous lemma that

$$d(P, Q) \leqq t' - t + J - (J + 1) = t' - t - 1$$

This however contradicts the fact that γ' is a geodesic.

We arrived at this contradiction by assuming that there are no $J/2$-thick bigons in Γ. We conclude therefore that there is a $J/2$-thick bigon in Γ.

Remarks 1.6. 1) By refining the previous argument one can give an estimate of δ in terms of ε.
2) One can give a similar characterization of (Gromov) hyperbolic geodesic metric spaces. Namely, if for a space X there are $\varepsilon > 0, k > 0$ such that all $(1, k)$ quasigeodesic bigons are ε thin then X is hyperbolic (see [Po]).

2. Strongly geodesically automatic groups

Preliminaries. A $(q + 3)$-tuple $(S, M, M_0, M_1, \ldots, M_q)$ is called an *automatic structure* for a group G if the following hold:

i) $S = \{s_0, s_1, \ldots, s_q\}$ is an inverse closed set of generators for G, with $s_0 =$ identity.

ii) M is a finite state automaton with alphabet S such that the natural map p from the language accepted by M, denoted by $L(M)$, to G is onto; we denote $p(w)$ by $\overline{w}$.

iii) For $0 \leqq i \leqq q$ we have that M_i are finite state automata such that $(w_1, w_2) \in L(M_i)$ if and only if $\overline{w_1 s_i} = \overline{w_2}$ where w_1, w_2 both belong to $L(M)$.

Let S^* be the free monoid on S.

Any word $w \in S^*$ defines a path in the Cayley graph of G starting at the identity. We denote this path also by w and we denote by $w(t)$ the prefix of the word w of length t.

We denote by $|w|$ the length of the word w and by $|\overline{w}|$ the length of the shortest word on S representing $\overline{w}$.

A group G is called *automatic* if it can be equipped with an automatic structure. If the language of all shortest words in S representing elements of G is the language of an automatic structure for G we say that G is *strongly geodesically automatic*. We can order the words in S^* in the following way: First order the elements of S and then define for two words $v < w$ if and only if v is shorter than w or they are of equal length and in the first letter that they differ the letter of v is smaller than the letter of w.

A *shortlex* automatic structure for a group G is an automatic structure whose language consists of the smallest representatives in S^*, with respect to the order just defined, of elements of G. In this case we say that G is *shortlex* automatic. We recall some basic facts about automatic and hyperbolic groups:

Proposition 2.1. ([ECHLPT], *Lemma* 2.3.2). *Let G have an automatic structure $(S, M, M_0, M_1, \ldots, M_q)$. There is a constant k depending on the automatic structure such that if (w_1, w_2) is accepted by one of the M_i's then $|w_1(t)^{-1}w_2(t)| \leqq k$ for every t.*

Actually if a is equal to the maximum of the number of states of any of the automata M_i then $k = 2a - 1$ satisfies the above requirement.

We call k the *fellow traveller constant* of the automatic structure.

Proposition 2.2. ([ECHLPT], *Theorem* 3.4.5) *Let G be a δ-hyperbolic group and let S be any set of semigroup generators for G. Then the shortest words over S representing elements of G form a regular language, and this language is part of an automatic structure for G.*

Proposition 2.1 implies that if G is strongly geodesically automatic then bigons are k thin in the Cayley graph of $G, \Gamma(G|S)$.

Proposition 2.2 simply says that if a group is hyperbolic then it is strongly geodesically automatic. Note that if a group is strongly geodesically automatic then it is shortlex automatic. (see [ECHLPT], 2.5.2)

Combining these with theorem 1.2 we have

Corollary 2.3. *A group is hyperbolic if and only if it is strongly geodesically automatic.*

The algorithm

We will describe an algorithm which, given a presentation $\langle S|R\rangle$ of a group G will terminate if and only if G is strongly geodesically automatic.

We find first a shortlex automatic structure for G. An algorithm which does this is described in [ECHLPT], ch. 6.

The automatic structures for groups having as generating set S can be enumerated ([ECHLPT], ch. 5).

We carry out a number of procedures 'simultaneously'.

We list the automatic structures which have prefix closed languages containing the language of the shortlex automatic structure and having S as an alphabet.

We check for each one of them if the relators R are satisfied. For the ones for which R is satisfied we start checking if the rest of the defining relators of the group defined by the given automatic structure follow from R.

If this is so for an automatic structure we compute the fellow traveller constant k of the structure.

Let m be the number of states of the automaton generating the regular language of the automatic structure.

We check if the words in this language of length less than or equal to $m^2|S|^k + 1$ represent all the geodesic paths from the identity to elements of G in the ball of radius $m^2|S|^k + 1$ in the Cayley graph of G. (Note that we can construct the Cayley graph of G since we assume that we have found an automatic structure for G). If not we disregard this automatic structure and we continue the execution of the algorithm.

If G is strongly geodesically automatic the algorithm will terminate, i.e. for some automatic structure the words in the regular language of length less than or equal to $m^2|S|^k + 1$ represent all the geodesic paths from the identity to elements of G in the ball of radius $m^2|S|^k + 1$ in the Cayley graph of G.

Lemma 2.4. *The language of this structure consists of all the shortest words on S representing elements of G.*

Proof. We prove by induction on the length of the words that a word belongs to the language of the automatic structure if and only if it is shortest.

By hypothesis the assertion holds for all words of length less or equal to $m^2|S|^k + 1$. We assume that the assertion holds for all words of length less or equal to n where $n \geqq m^2|S|^k + 1$.

Let wx be a word in the language of length $n + 1$, where $x \in S$.

Suppose that wx is not shortest. Since we assume the language to be prefix closed we have that w belongs to the language. Since we assume that our language contains the shortlex language there is another accepted word w' with $|w'| < |wx|$ such that $\overline{wx} = \overline{w'}$.

The words w, w' are then accepted by some comparator automaton hence they fellow travel i.e. $|\overline{w(t)w'(t)^{-1}}| \leqq k$ for $t \leqq |w'|$. We denote by *state*$(w(t))$ the state of the word accepting automaton after it has read $w(t)$.

We note now that there are $|S|^k$ words of length less or equal to k and that the word accepting automaton has m states. Since the length of w is greater or equal to $m^2|S|^k+1$ we conclude that there are $t_1 < t_2$ such that $\overline{w(t_1)w'(t_1)^{-1}} = \overline{w(t_2)w'(t_2)^{-1}}$ and state$(w(t_1))$ = state$(w(t_2))$. Now if $wx = w(t_2)vx$ the word $w(t_1)vx$ is also in the regular language since state$(w(t_1))$ = state$(w(t_2))$. Also $|w(t_1)vx| \leqq n$.

On the other hand if $w' = w'(t_2)v'$ we have that

$$\overline{wx} = \overline{w'} \Rightarrow \overline{w(t_2)vx} = \overline{w'(t_2)v'} \Rightarrow \overline{w(t_1)vx} = \overline{w'(t_1)v'}$$

But $|w'(t_1)v'| < |w(t_1)vx|$ which is a contradiction.

Suppose now that there is a shortest word $wx, x \in S$ such that $|wx| = n+1$ which is not in the language. By the inductive hypothesis w is in the language and as before we have that there is a shortest word w' in the language such that $\overline{wx} = \overline{w'}$.

Now w, w' are accepted by some comparator automaton and we find as before $t_1 < t_2$ such that $state(w(t_1)) = state(w(t_2)), state(w'(t_1)) = state(w'(t_2))$ and $\overline{w(t_1)w'(t_1)^{-1}} = \overline{w(t_2)w'(t_2)^{-1}}$.

Now if $wx = w(t_2)vx$ the word $w(t_1)vx$ is not in the language since $state(w(t_1)) = state(w(t_2))$. Also $|w(t_1)vx| \leqq n$.

On the other hand if $w' = w'(t_2)v'$ the word $w'(t_1)v'$ is in the language since $state(w'(t_1)) = state(w'(t_2))$.

We have now

$$\overline{wx} = \overline{w'} \Rightarrow \overline{w(t_2)vx} = \overline{w'(t_2)v'} \Rightarrow \overline{w(t_1)vx} = \overline{w'(t_1)v'}$$

So $w(t_1)vx$ is a shortest word of length less than n which doesn't belong to the language and this is a contradiction.

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References

[ABC] Alonso, Brady, Cooper, Delzant, Ferlini, Lustig, Mihalik, Shapiro, Short: Group Theory from a Geometrical Viewpoint, E. Ghys, A. Haefliger, A. Verjovsky, eds, World Scientific, 1991

[C] J.W. Cannon: The Combinatorial structure of cocompact discrete hyperbolic groups. Geometriae Dedicata **16** (1984) 123–148

[ECHLPT] D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson, W. Thurston: Word Processing in Groups, Jones and Bartlett, 1992

[G] M. Gromov: Hyperbolic Groups. In: Essays in Group Theory, S.M. Gersten (ed.) MSRI series 8, Springer: Berlin Heidelberg New York, pp. 75–263

[P] P. Papasoglu: On the subquadratic isoperimetric inequality. Submitted for publication to the Proceedings of the Ohio Geometric Group Theory Conference, 1992

[Po] J. Pomroy, M.Sci. thesis, in preparation. Warwick University