GROUP SPLITTINGS AND ASYMPTOTIC TOPOLOGY

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Abstract. It is a consequence of the theorem of Stallings on groups with many ends that splittings over finite groups are preserved by quasi-isometries. In this paper we use asymptotic topology to show that group splittings are preserved by quasi-isometries in many cases. Roughly speaking we show that splittings are preserved under quasi-isometries when the vertex groups are fundamental groups of aspherical manifolds (or more generally 'coarse $PD(n)$-groups') and the edge groups are 'smaller' than the vertex groups.

§0. Introduction

The notion of quasi-isometry and the study of the relation of large-scale geometry of groups and algebraic properties has become predominant in group theory after the seminal papers of Gromov [G1,G2].

A classical theorem of Stallings ([St]) implies, that if G splits over a finite group and H is a group quasi-isometric to G then H also splits over a finite group.

Bowditch has shown recently ([Bo]) that splittings of hyperbolic groups over 2-ended groups are preserved by quasi-isometries.

By a theorem of Kapovich and Leeb [K-L] it follows that a group quasi-isometric to a non-geometric Haken 3-manifold splits over a group commensurable to a surface group. In this paper we show that group splittings are preserved by quasi-isometries in many cases. Our approach is based on asymptotic topology ('coarse topology') methods. Schwartz's asymptotic version of the Jordan separation theorem ([Sch,F-S]) is our main tool. In fact we will need a stronger version of Schwartz's theorem that was given recently by Kapovich and Kleiner ([K-K]) in the context of their work on coarse $PD(n)$-spaces and groups. We will therefore formulate our resluts in this more general setting.

We say that a group G is a coarse $PD(n)$ group if it acts discretely co-compactly on a coarse $PD(n)$-space. We say that G is a coarse $PD(n)$ group of dimension n if it is a coarse $PD(n)$-group that has an n-dimensional $K(G,1)$.

We note that examples of coarse $PD(n)$-groups are fundamental groups of closed aspherical n-manifolds.

Using the theory of coarse $PD(n)$-spaces we show that groups that are quasi-isometric to 'trees of coarse $PD(n)$-spaces' split. To pass from geometry to algebra we use a recent result of Niblo, Sageev, Scott and Swarup ([NSSS]).

We explain briefly the notation we use for graphs of groups. For more details see

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[Ba], [Se]. A graph of groups is given by the following data:
a) A finite graph Γ. Each edge $e \in \Gamma$ is oriented. We denote by $\partial_0 e$ the initial vertex of e and by $\partial_1 e$ the terminal vertex of e.
b)To each vertex $v \in \Gamma$ and each edge $e \in \Gamma$ there correspond groups A_v, A_e. If $v = \partial_0 e$ or $v = \partial_1 e$ we have monomorphisms (respectively) $i_0 : A_e \to A_v$ and $i_1 : A_e \to A_v$. We denote this collection of groups and morphisms by $\mathcal{A}$.
Using these data one defines the fundamental group $\pi_1(\Gamma, \mathcal{A})$. For the definition see [Se]. We recall that a graph of groups decomposition is called reduced if for any edge e which is not a loop neither $i_0 : A_e \to A_v$ nor $i_1 : A_e \to A_v$ is an epimorphism. With this notation we have the following:

Theorem 3.1. *Let G be a finitely generated group admitting a reduced graph of groups decomposition $G = \pi_1(\Gamma, \mathcal{A})$ such that all edge and vertex groups are coarse $PD(n)$ groups of dimension n. Suppose further that $(\Gamma, \mathcal{A})$ is not a loop with all edge to vertex maps isomorphisms and that it is not a graph of one edge with both edge to vertex maps having as image an index 2 subgroup of the vertex group. If H is quasi-isometric to G then H splits over a group that is quasi-isometric to an edge group of $(\Gamma, \mathcal{A})$.*

We obtain the following corollaries:

Corollary 3.2 ([F-M1,2])**.** *Let G be a solvable Baumslag-Solitar group. If H is a group quasi-isometric to G then H is commensurable to a solvable Baumslag-Solitar group.*

Corollary 3.3. *Let G be a finitely generated group admitting a reduced graph of groups decomposition $G = \pi_1(\Gamma, \mathcal{A})$ such that all edge and vertex groups are virtually $\mathbb{Z}^n$. Suppose further that $(\Gamma, \mathcal{A})$ is not a loop with all edge to vertex maps isomorphisms and that it is not a graph of one edge with both edge to vertex maps having as image an index 2 subgroup of the vertex group. If H is quasi-isometric to G then $H = \pi_1(\Delta, \mathcal{B})$ where all vertex and edge groups of Δ are virtually $\mathbb{Z}^n$.*

It turns out that splittings are invariant under quasi-isometries in the case that edge groups are 'smaller' than vertex groups:

Theorem 3.4. *Let G be a finitely generated group admitting a graph of groups decomposition $G = \pi_1(\Gamma, \mathcal{A})$ such that all vertex groups are coarse $PD(n)$-groups and all edge groups are dominated by coarse $PD(n-1)$ spaces. If H is quasi-isometric to G then H splits over some group quasi-isometric to an edge group of Γ.*

We say that a group G is dominated by a coarse $PD(n)$-space X if there is a uniform embedding $f : G \to X$ (for more details see sec. 3). Note that a subgroup of a coarse-$PD(n)$ group is dominated by a coarse $PD(n)$-space. So for example a free group is dominated by a coarse $PD(2)$-space.
The main geometric observation on which our results are based is that the groups in theorems 3.1, 3.4 are 'trees of spaces'. The simplest example of such groups are products $\mathbb{F}_k \times \mathbb{Z}^n$. To make the exposition easily accessible to readers not familiar with the geometry of graphs of groups and with 'coarse $PD(n)$-spaces' we treat this special case first in section 2. All 'asymptotic topology' arguments that we need are already present in this case. The link between algebra and geometry is provided by a result of Niblo-Sageev-Scott-Swarup ([NSSS]) generalizing the algebraic torus

theorem of Dunwoody-Swenson ([D-S]).
In section 3 we explain how to generalize these arguments to graphs of groups in which all edge and vertex groups are 'coarse $PD(n)$-groups of dimension n'. For this it suffices to understand the 'tree-like shape' of graphs of groups in general. The geometries of such groups have been described in several places (see [S-W], [Ep], [F-M1,2], [Wh]). We use similar arguments to treat the case of graphs of groups with vertex groups coarse $PD(n)$ groups and edge groups, groups that are 'dominated' by coarse $PD(n-1)$ groups.
In section 4 we discuss how the results of this paper (and Stallings' theorem) could be generalized and we ask some specific questions.
In the course of this work we found out that some of our results had been obtained earlier, independently, by Mosher, Sageev and Whyte. In particular they have shown a stronger version of theorem 3.1, Corollary 3.4 and some cases of theorem 3.4 ([MSW],[MSW1]). The main novelty (apart from the difference in the proofs) of this paper compared to [MSW] is that we improve $n-2$ to $n-1$ in theorem 3.4. So for example from our results it follows that if G, H are quasi-isometric groups and G is an amalgam of two aspherical 3-manifold groups along a surface group (or free group) then H also splits over a virtual surface (or virtually free) group. The work of [MSW] implies a similar result when G is an amalgam over $\mathbb{Z}$.
I would like to thank G.A. Swarup for many enlightening discussions on group splittings and B.Kleiner for explaining to me [K-K].

§1. Preliminaries

A (K,L)-*quasi-isometry* between two metric spaces X, Y is a map $f : X \to Y$ such that the following two properties are satisfied:
1) $\frac{1}{K}d(x,y) - L \leq d(f(x), f(y)) \leq Kd(x,y) + L$ for all $x, y \in X$.
2) For every $y \in Y$ there is an $x \in X$ such that $d(y, f(x)) \leq K$.
We will usually simply say quasi-isometry instead of (K,L)-quasi-isometry. Two metric spaces X, Y are called quasi-isometric if there is a quasi-isometry $f : X \to Y$.
A *geodesic metric space* is a metric space in which any two points x, y are joined by a path of length $d(x, y)$. In what follows we will be interested in graphs that we always turn into geodesic metric spaces by giving each edge length 1.
We recall now some notation and results from [K-K]. We refer the reader to this paper for more details.
Let X be a connected locally finite simplicial complex. The 1-skeleton X^1 is a graph and we turn it into a geodesic metric space as explained above. If $A \subset X^1$ the r-neighborhood of A is the set of points in X^1 at distance less or equal to r from A. More generally if K is a subcomplex of X we define the r-neighborhood of K, $N_r(K)$, to be the set of simplices intersecting $N_r(K^1)$.
The *diameter*, $diam(K)$, of K is by definition the diameter of K^1.
We call a map $f : X \to Y$ between metric spaces a *uniform embedding* (see [G2]) if the following two conditions are satisfied:
1) There are K, L such that for all $x, y \in X$ we have $d(f(x), f(y)) < K(d(x, y)) + L$.
2) For every $E > 0$ there is $D > 0$ such that $diam(A) < E \Rightarrow diam(f^{-1}A) < D$.
We say that the *distortion* of f is bounded by h ,where $h : \mathbb{R}^+ \to \mathbb{R}^+$, if for all $A \subset Y$, $diam(f^{-1}A) \leq h(diam(A))$.
If f satisfies only condition 1 above we say that f is a (K, L)-*lipschitz* map.
X is called *uniformly acyclic* if for each R_1 there is an R_2 such that for each

subcomplex $K \subset X$ with $diam(K) < R_1$ the inclusion $K \to N_{R_2}(K)$ induces zero on reduced homology groups.
If $K \subset X$ is a subcomplex of X and $R > 0$ we say that a component of $X - N_R(K)$ is *deep* if it is not contained in $N_{R_1}(K)$ for any $R_1 > 0$.
We say that K *coarsely separates* X if there is an $R > 0$ such that $X - N_R(K)$ has at least two deep components.
The appropriate context for the results in this paper seems to be that of 'coarse $PD(n)$ spaces'. We refer to [K-K] for a definition and an exposition of the theory of 'coarse $PD(n)$ spaces'. Important examples of 'coarse $PD(n)$ spaces' are uniformly acyclic PL-manifolds of bounded geometry.
We say that X is a 'coarse n-dimensional PL-manifold' if X is quasi-isometric to a uniformly acyclic n-dimensional PL-manifold of bounded geometry.
A reader not familiar with [K-K] can read this paper by replacing everywhere 'coarse $PD(n)$ space' by 'coarse n-dimensional PL-manifold'. The only result that we will use from [K-K] is the coarse Jordan theorem stated below (see theorem 7.7, footnote 11 and corollary 7.8 of [K-K]).

Proposition 1 (Coarse Jordan theorem). *Let X, X' be coarse $PD(n)$, $PD(n-1)$ spaces respectively, $Z \subset X'$ and let $g : Z \to X$ be a uniform embedding such that g is a (K, L) lipschitz map that is a uniform embedding with distortion bounded by f. Then*
1. If $Z = X'$ then there is an $R > 0$ such that $X - N_R(g(X'))$ has exactly 2 deep components.
2. There is an $N > 0$ such that every non-deep component of $X - g(Z)$ is contained in the N-neighborhood of $g(Z)$.
3. There is an $M > 0$ such that if $X - g(Z)$ has more than one deep component, X' is contained in the M-neighborhood of Z.
4. If $Z = X'$ then for every r, each point of $N_r(g(X'))$ lies within uniform distance from each of the deep components of $X - N_r(g(X'))$.
The constants M, N, R depend only on K, L, f, X, Y.

We note that the metric on Z in the above proposition is the metric induced by X'.
We say that $A \subset X$ *coarsely contains* B if for some R, $B \subset N_R(A)$. If A, B coarsely contain each other then we say that they are at finite distance from each other.

Let $f : X \to Y$ be a uniform embedding of a coarse $PD(n-1)$ space X to a coarse $PD(n)$ space Y. Let $K > 0$ be such that $Y - N_K(f(X))$ has two deep components. We will call a deep component of $Y - N_K(f(X))$ a *half coarse PD(n)-space* .
We call a group G a *coarse PD(n)-group* if it acts discretely co-compactly simplicially on a coarse $PD(n)$ space. We say that G is a coarse $PD(n)$ group of *dimension* n if it is a coarse $PD(n)$-group that has an n-dimensional $K(G, 1)$.
We call $(G; \{F_1, ..., F_n\})$ a *coarse PD(n)-pair* if :
1. G acts discretely simplicially on a coarse $PD(n)$ space X and there is a connected G-invariant subcomplex $K \subset X$ such that the stabilizer of each component of $X - K$ is conjugate to one of the $F_i's$.
2. The F_i's are coarse $PD(n-1)$ groups.

§2. The geometry of direct products of abelian and free groups

Theorem 2.1. *Let $G = \mathbb{F}_k \times \mathbb{Z}^n$ where $\mathbb{F}_k$ is the free group on $k > 1$ generators. If H is quasi-isometric to G then $H = \pi_1(\Delta, \mathcal{B})$ where all vertex and edge groups of Δ are virtually $\mathbb{Z}^n$.*

Proof. Let Γ_G be the Cayley graph of G with respect to the standard generators. Γ is quasi-isometric to $T_{2k} \times \mathbb{R}^n$ where T_{2k} is the homogeneous tree of degree $2k$. Clearly G acts discretely and co-compactly on $T_{2k} \times \mathbb{R}^n$.

Let's denote $T_{2k} \times \mathbb{R}^n$ by X and let $p : X \to T_{2k}$ be the natural projection from $X = T_{2k} \times \mathbb{R}^n$ to T_{2k}. With this notation we have:

Lemma 2.2. *Let $f : X \to X$ be a quasi-isometry. Then for any vertex v of T_{2k} there is a vertex $u \in T_{2k}$ such that $f(v \times \mathbb{R}^n)$ and $u \times \mathbb{R}^n$ are at finite distance from each other.*

Proof.

We note that if v is a vertex of T_{2k}, $v \times \mathbb{R}^n$ separates $T_{2k} \times \mathbb{R}^n$ in more than 2 deep components.
Moreover there are geodesic rays $r_1, r_2, r_3 : [0, \infty) \to X$ such that r_1, r_2, r_3 lie in distinct components of $X - p^{-1}(v)$ and $d(r_1(t), p^{-1}(v)) = t$.
Let $K > 0$ be such that $N_K(f(p^{-1}(v)))$ separates X in more than 2 deep components. Clearly for K sufficiently big $f(r_i)$, $i = 1, 2, 3$ are coarsely contained in distinct deep components of $X - N_K(f(p^{-1}(v)))$. We pick K so that this holds. Let's call C_i the deep component coarsely containing $f(r_i)$.
To simplify notation we set $S = N_K(f(p^{-1}(v)))$.
$f(r_i)$ is not necessarily connected. Let R_i be the path obtained by joining $f(r_i(n))$ to $f(r_i(n+1))$ by a geodesic path for all $n \in \mathbb{N}$. Clearly R_i is coarsely contained in C_i. If we parametrize R_i by arclength we have that $d(R_i(t), S)$ is a proper function from $[0, \infty)$ to $[0, \infty)$. Let $l_1 = p(R_1)$, $l_2 = p(R_2)$. We pick now a geodesic $l \subset T_{2k}$ such that $p^{-1}l \cap R_1$ and $p^{-1}l \cap R_2$ are both unbounded. We explain how to find such an l: If l_1, l_2 are finite then there are closed edges e_1, e_2 of T_{2k} such that $p^{-1}e_1 \cap R_1, p^{-1}e_2 \cap R_2$ are both unbounded, so we simply pick l to be any geodesic containing both these edges. If both l_1, l_2 are infinite, since they are connected they contain at least one geodesic ray each. We pick therefore l to be a line having an infinite intersection with both geodesic rays. If one of them is infinite and one finite we pick l similarly , requiring that it has an infinite intersection with a given ray and passes from a given edge.
We have therefore that S separates $p^{-1}l$. By part 3 of proposition 1 we conclude that there is a K_1 such that S is contained in the K_1 neighborhood of $p^{-1}l$.
We note that by parts 1,2 of proposition 1, $p^{-1}l \cap R_3$ is bounded. We can therefore find a geodesic ray $r \in T_{2k}$ intersecting l only at one point u, such that $p^{-1}r \cap R_3$ is unbounded. We have then $l = r_1' \cup r_2'$ and $r_1' \cap r_2' = u$ where r_1', r_2' are geodesic rays. As before we have that S separates both $p^{-1}(r \cup r_1')$ and $p^{-1}(r \cup r_2')$ so it lies in a finite neighborhood of both. We conclude that S lies in a finite neighborhood of all three: $p^{-1}(r \cup r_1')$, $p^{-1}(r \cup r_2')$ and $p^{-1}l$. Therefore S lies in a finite neighborhood of $p^{-1}u$.
We remark that if f is an (A, B)-quasi-isometry then the proof above shows that there is a C that depends only on A, B (and X) such that $f(v \times \mathbb{R}^n)$ and $u \times \mathbb{R}^n$ are in the C-neighborhood of each other. This is so because the constants in proposition 1 depend only on A, B. We will use this fact in the next lemma. ■

Lemma 2.3. *Let $G = \mathbb{F}_k \times \mathbb{Z}^n$ where $\mathbb{F}_k$ is the free group on $k > 1$ generators. If H is quasi-isometric to G then H splits over a group commensurable to $\mathbb{Z}^n$.*

Proof.

Let Γ_H be the Cayley graph of H. If $v \in T_{2k}$ is a vertex and $f : X \to \Gamma_H$ a quasi-isometry we have that $f(v \times \mathbb{R}^n)$ coarsely separates Γ_H to more than 2 deep components.
Let K be such that $\Gamma_H - N_K(f(v \times \mathbb{R}^n))$ has more than 2 deep components. We set $S = N_K(f(v \times \mathbb{R}^n))$. Note that $\Gamma_H - S$ has a finite number of deep components. We pick $M > 0$ such that the following holds:
For all $h \in H$ if $hS \cap S \neq \emptyset$ then $hS \subset N_M(S)$.
It follows from our remark at the end of the proof of lemma 2.2 that such an M exists.
We will show that there is a subgroup J of H quasi-isometric to S such that Γ_H/J has more than two ends.
We fix a vertex $e \in S$. We define an equivalence relation on the set of vertices $x \in S$:
Let $g_x \in H$ such that $g_x x = e$. Let $C_1, ..., C_k$ be the deep components of $\Gamma_H - S$ and let $D_1, ..., D_m$ be the deep components of $\Gamma_H - N_M(S)$.
We note that each D_i ,$i = 1, ..., m$ is contained in some C_j, $j = 1, ..., k$.
For each g_x and C_j we have that there are $i_1, ..., i_r$ such that $g_x C_j$ and $D_{i_1} \cup ... \cup D_{i_r}$ are at finite distance from each other. We use this to define a map $f_x : \{C_1, ..., C_k\} \to \mathcal{P}(\{D_1, ..., D_m\})$ where $f_x(C_j) = \{D_{i_1}, ..., D_{i_r}\}$ if and only if $g_x C_j$ and $D_{i_1} \cup ... \cup D_{i_r}$ are at finite distance from each other.
Let's write now $x \sim y$ for x, y vertices of S if $f_x = f_y$. Clearly $\sim$ is an equivalence relation with finitely many equivalence classes.
Let R be such that $B_R(e) \cap S$ contains all elements of the equivalence classes with finitely many elements and at least one element of each equivalence class with infinitely many elements.
For each vertex $y \in S$ we pick $t \in B_R(e) \cap S$, $t \sim y$ and we consider the group J generated by $\{g_y^{-1} g_t\}$. Clearly $S \subset J(B_R(e) \cap S)$. We claim that Je is contained in a neighborhood of S. Indeed for each $g \in J$ we have that gC_j and C_j are at finite distance from each other for all j. Let $A > 0$ be such that for any $v \in \Gamma_H$ we have that if $d(v, S) > A$ then v lies in a deep component of $\Gamma_H - S$. One sees easily that such an A exists by prop. 1, part 2.
If Je is not contained in any neighborhood of S then there is a $g \in J$ such that $gS \cap N_A(S) = \emptyset$. This follows again by lemma 2.2 and the remark at the end of the proof of lemma 2.2. Clearly then S intersects a single component of $\Gamma_H - gS$. Therefore there is some C_j such that all gC_j except one are contained in a single deep component of $\Gamma_H - S$. This however contradicts the fact that gC_j is at finite distance from C_j for all j.
We have therefore shown that J is quasi-isometric to S. Clearly Γ_H/J has more than two ends. By the algebraic torus theorem of Dunwoody-Swenson ([D-S]) we have that H splits over a group commensurable with $\mathbb{Z}^n$. In fact by proposition 3.1 of ([D-S]) we have that H splits over a group commensurable with J. To see this note that if T is an essential track corresponding to H as in lemma 2.3 of [D-S] then from lemma 2.2 it follows that no translate of T crosses T. ■

We return now to the proof of the theorem. We proceed inductively: We assume that we can write H as the fundamental group of a graph of groups $H = \pi_1(\Delta, \mathcal{B})$

where edge groups are commensurable with J and $(\Delta, \mathcal{B})$ can not be further refined. By this we mean that no vertex group B_v of Δ admits a graph of groups decomposition with edge groups commensurable to J and such that all edge groups B_e of edges adjacent to v are subgroups of vertex groups of the graph of groups decomposition of B_v.
By the accessibilty theorem of Bestvina-Feighn ([B-F]) this procedure terminates. Each vertex group of the graph of groups decomposition, say $H = \pi_1(\Delta, \mathcal{B})$, that we obtain in this way contains a finite index subgroup of J. Indeed if $|B_v : B_v \cap J| = \infty$ for some vertex group then $B_v \cap J$ coarsely separates the Cayley graph of B_v, so by applying the alebraic torus theorem once again we see that we can refine $(\Delta, \mathcal{B})$, a contradiction. We conclude therefore that all edge and vertex groups are commensurable to J which proves the theorem. ■

§3. Graphs of groups

To deal with splittings over coarse $PD(n)$ groups in general rather than $\mathbb{Z}^n$ we need a theorem of Niblo-Sageev-Scott-Swarup ([NSSS]) generalizing proposition 3.1 of [D-S]. We recall their notation and results:

Definitions. *Two sets P, Q are almost equal if their symmetric difference is finite. If G acts on the right on a set Z a subset P is almost invariant if Pg is almost equal to P for all $g \in G$.*
If G is a finitely generated group and J is a subgroup then a subset X of G is J-almost invariant if $gX = X$ for all $g \in J$ and $J\backslash X$ is an almost invariant subset of $J\backslash G$.
X is a non-trivial J-almost invariant subset of G if, in addition, $J\backslash X$ and $J\backslash(G - X)$ are both infinite.

If G splits over J there is a J-almost invariant subset X of G associated to the splitting in a natural way. If $G = A *_J B$ let X_A, X_B, X_J be complexes with $\pi_1(X_A) = A, \pi_1(X_B) = B, \pi_1(X_J) = J$ and let X_G be the complex obtained as usual by gluing $X_J \times I$ to $X_A \cup X_B$, so that $\pi_1(X_G) = G$. Let $\tilde{X}_G$ be the universal covering of X_G and let $p : \tilde{X}_G \to X_G$ be the covering projection. We note that each connected component of $p^{-1}(X_J \times (0,1))$ separates $\tilde{X}$ in two sets. Each of these two sets gives a J-almost invariant subset.
If X, Y are non-trivial J-almost invariant subsets of G we say that Y *crosses* X if all the four intersections $X \cap Y, X^* \cap Y, X \cap Y^*, X^* \cap Y^*$ (where $X^* = G - X, Y^* = G - Y$) project to infinite sets in $J\backslash G$. If X is a non-trivial J-almost invariant subset we say that the intersection number $i(J\backslash X, J\backslash X)$ is 0 if gX does not cross X for any $g \in G$. With this notation Niblo-Sageev-Scott-Swarup show the following ([NSSS], theorem 4.2):

Theorem. *Let G be a finitely generated group with a finitely generated subgroup J and a non-trivial J almost invariant subset X. If $\{g \in G : gX \, crosses \, X\}$ lies in $Comm_G(J)$, the commensurizer of J in G, then G splits over a subgroup commensurable with J.*

We note that from theorem 7.7 of [K-K] it follows that if $f : X \to Y$ is a uniform embedding of a coarse $PD(n)$-space X to a coarse $PD(n)$-space Y then Y is coarsely contained in $f(X)$. This implies that if $h : G \to H$ is a monomorphism between coarse $PD(n)$-groups then $|H : h(G)| < \infty$.

Theorem 3.1. *Let G be a finitely generated group admitting a graph of groups decomposition $G = \pi_1(\Gamma, \mathcal{A})$ such that all edge and vertex groups are coarse $PD(n)$ groups of dimension n. Suppose further that $(\Gamma, \mathcal{A})$ is not a loop with all edge to vertex maps isomorphisms and that it is not a graph of one edge with both edge to vertex maps having as image an index 2 subgroup of the vertex group. If H is quasi-isometric to G then H splits over a group that is quasi-isometric to an edge group of $(\Gamma, \mathcal{A})$.*

Proof. We recall from [S-W] the topological point of view on graphs of groups: To each vertex $v \in \Gamma$ and each edge $e \in \Gamma$ we associate finite simplicial complexes X_v, X_e such that $\pi_1(X_v) = A_v, \pi_1(X_e) = A_e$. Let I be the unit interval. We construct a complex X such that $\pi_1(X) = G$ by gluing the complexes X_v and $X_e \times I$ as follows: Let v be an endpoint of $e \in \Gamma$, say $v = \partial_0 e$. Then there is a monomorphism $i_0 : A_e \to A_v$. Let $f : X_e \to X_v$ be a simplicial map such that $f_* = i_0$. We identify then $(t, 0) \in X_e \times I$ to $f(t) \in X_v$.
Similarly we define an identification between $X_e \times \{1\}$ and X_v if $v = \partial_1 e$.
Doing all these identifications for the vertices $v \in \Gamma$ and the edges $e \in \Gamma$ we obtain a complex X such that $\pi_1(X) = G$. We metrize the 1-skeleton of $\tilde{X}$ as usual by giving each edge length 1. We note that with this metric $\tilde{X}$ is quasi-isometric to the Cayley graph of G.
We can obtain the Bass-Serre tree T associated to $\pi_1(\Gamma, \mathcal{A})$ from $\tilde{X}$, by collapsing each copy of $\tilde{X}_v \subset X$ to a vertex and each copy of $\tilde{X}_e \times I$ to I. This collapsing gives a G-equivariant map $p : \tilde{X} \to T$.
We note now that if $v \in T$ is a vertex then $p^{-1}v$ separates $\tilde{X}$ into more than 2 deep components. Moreover if l is an infinite geodesic in T $p^{-1}l$ is a coarse $PD(n+1)$-space (see theorem 11.13 of [K-K]). As in lemma 2.2 we have that if $h : \tilde{X} \to \tilde{X}$ is a quasi-isometry then for any vertex $v \in T$ there is a vertex $u \in T$ such that $h(p^{-1}(v))$ and $p^{-1}(u)$ are at finite distance from each other.

Let Γ_H be the Cayley graph of H. Arguing as in lemma 2.3 we show that there is a subgroup J of H quasi-isometric to an edge group of $(\Gamma, \mathcal{A})$ such that Γ_H/J has more than one end. As in lemma 2.3 we show that there is a connected subset S of Γ_H such that $\Gamma_H - S$ has more than 1 deep components and S/J is finite. Without loss of generality we can assume that for any $v \in S$, $Jv \subset S$. As in lemma 2.3 we see that we can find J as above such that, in addition, hC_i and C_i are at finite distance from each other for all deep components C_i of $\Gamma_H - S$. We fix now $v \in \Gamma_H$ and we identify H with the orbit Hv. The set $W = J(C_i \cap Hv)$ is clearly a non-trivial J almost invariant subset of H. If gW crosses W then gJ and J are at finite distance of each other. So the subgroups J and gJg^{-1} are at finite distance of each other. This implies that they are commensurable. Indeed gJg^{-1} is contained in finitely many cosets of J, say $Jx_1, ..., Jx_n$. We may assume that $x_1, ..., x_n$ lie in gJg^{-1}. It follows that

$$gJg^{-1} = (J \cap gJg^{-1})x_1 \cup ... \cup (J \cap gJg^{-1})x_n$$

Therefore J and gJg^{-1} are commensurable. Theorem 4.1 follows now from the result of Niblo-Sageev-Scott-Swarup quoted above. ■.

Corollary 3.2 ([F-M1,2])**.** *Let G be a solvable Baumslag-Solitar group. If H is a group quasi-isometric to G then H is commensurable to a solvable Baumslag-Solitar group.*

Proof. By theorem 3.1 H splits over a 2-ended group. Since H is amenable and is not virtually abelian, H can be written as a graph of groups with a single vertex and a single edge, such that the edge group is two ended and exactly one edge to vertex map is an isomorphism. If a is an element of the edge group generating an infinite normal subgroup of the edge group and t is the generator corresponding to the edge, then $tat^{-1} = a^k$ for some $k \in \mathbb{Z}$ and the solvable Baumslag-Solitar subgroup of H generated by $< t, a >$ is a subgroup of finite index.

Corollary 3.3. *Let G be a finitely generated group admitting a reduced graph of groups decomposition $G = \pi_1(\Gamma, \mathcal{A})$ such that all edge and vertex groups are virtually $\mathbb{Z}^n$. Suppose further that $(\Gamma, \mathcal{A})$ is not a loop with all edge to vertex maps isomorphisms and that it is not a graph of one edge with both edge to vertex maps having as image an index 2 subgroup of the vertex group. If H is quasi-isometric to G then $H = \pi_1(\Delta, \mathcal{B})$ where all vertex and edge groups of Δ are virtually $\mathbb{Z}^n$.*

Proof. By theorem 3.1 H splits over a subgroup quasi-isometric to $\mathbb{Z}^n$ and hence virtually $\mathbb{Z}^n$. We apply the same argument as in the proof of theorem 2.1 to conclude that $H = \pi_1(\Delta, \mathcal{B})$ where all vertex and edge groups of Δ are virtually $\mathbb{Z}^n$. ■

Definition. *A metric space Y is dominated by a coarse $PD(n)$-space if there is a uniform embedding $f : Y \to X$ where X is a coarse $PD(n)$-space. We say that a finitely generated group G is dominated by a coarse $PD(n)$-space if G equiped with the word metric is dominated by a coarse $PD(n)$ space.*

Some examples: A coarse $PD(n-k)$ group is dominated by a coarse $PD(n)$ space where $k = 0, ..., n-1$. A free group is dominated by a coarse $PD(2)$ space.

Theorem 3.4. *Let G be a finitely generated group admitting a graph of groups decomposition $G = \pi_1(\Gamma, \mathcal{A})$ such that all vertex groups are coarse $PD(n)$-groups and all edge groups are dominated by coarse $PD(n-1)$ spaces. If H is quasi-isometric to G then H splits over some group quasi-isometric to an edge group of Γ.*

Proof. We construct a complex X such that $\pi_1(X) = G$ as in theorem 3.1. We have as in 3.1 that there is a map $p : \tilde{X} \to T$ where T is the Bass-Serre tree of $(\Gamma, \mathcal{A})$. We note that $\tilde{X}$ is contained in a neighborhood of the 'vertex spaces of $\tilde{X}$': $p^{-1}(v)$ is a coarse $PD(n)$ space and $\tilde{X}$ is contained in the 1-neighborhood of $\bigcup p^{-1}(v)$ (where v ranges over vertices of T). Moreover if v, w are vertices of T adjacent to an edge e then for $t > 1$ $N_t(p^{-1}(v)) \cap N_t(p^{-1}(w))$ is a path connected subset of $\tilde{X}$ quasi-isometric to $p^{-1}e$ (here we consider this subset equipped with its path metric). In fact $p^{-1}e$ and $N_t(p^{-1}(v)) \cap N_t(p^{-1}(w))$ are contained in a finite neighborhood of each other.
Let $f : \tilde{X} \to \Gamma_H$ be a quasi-isometry from $\tilde{X}$ to the Cayley graph of H. If e is an edge of T then $f(p^{-1}e)$ coarsely separates Γ_H. Let e be an edge of T adjacent to the vertices v, w. Let R_0 be such that $f(p^{-1}e) \subset N_{R_0}(f(p^{-1}v)) \cap N_{R_0}(f(p^{-1}w))$. Note that we can pick R_0 uniformly for all $e \in T$. We distinguish now two cases:
Case 1: $p^{-1}e$ is not quasi-isometric to a coarse $PD(n-1)$-space.
Let r be such that $f(p^{-1}v), f(p^{-1}w)$ are coarsely contained in distinct components of $\Gamma_H - N_r(f(p^{-1}e))$. Let's call F_1, F_2 respectively these 2 components. Again we can pick r uniformly for all $e \in T$. We set $S = N_r(f(p^{-1}e))$, $C_1 = f(p^{-1}v)$, $C_2 = f(p^{-1}w)$. Let $R_1' > 2R_0$ be such that the following holds: Let u be a vertex of T, let $C = hf(p^{-1}u)$ (where $h \in H$) and let F be the component of $\Gamma_H - S$ coarsely

containing C. Then if $x \in C - F$ we have $d(x, S) < R_1'$. Note that the existence of an R_1' with this property follows from prop. 1, part 2.
Let $R_1 > R_1'$ be such that the following holds: If $x \in S$ then neither $B_x(R_1) \cap C_1$ nor $B_x(R_1) \cap C_2$ is contained in the R_1'-neighborhood of S.
Again we can pick R_1', R_1 uniformly for all $e, u \in T, h \in H$.
Case 2: $p^{-1}e$ is quasi-isometric to a coarse $PD(n-1)$-space.
We pick in this case too constants r, R_1', R_1 with similar properties:
Let r be such that $f(p^{-1}v)$, $f(p^{-1}w)$ are not contained coarsely in the same component of $\Gamma_H - N_r(f(p^{-1}e))$. We pick r so that $f(p^{-1}v), f(p^{-1}w)$ intersect each 2 deep components of $\Gamma_H - N_r(f(p^{-1}e))$ along a half coarse $PD(n)$ space. We set $S = N_r(f(p^{-1}e))$, $C_1 = f(p^{-1}v)$, $C_2 = f(p^{-1}w)$. Let $R_1 > 2R_0$ be such that the following hold: Suppose that u is a vertex of T, $C = hf(p^{-1}u)$ (where $h \in H$) and F_1, F_2 are distinct components of $\Gamma_H - S$ such that $C \cap F_1, C \cap F_2$ are coarse half $PD(n)$-spaces. Then if $x \in C - (F_1 \cup F_2)$ we have that $d(x, S) < R_1'$. We suppose further that the following holds: If F is any deep component of $\Gamma_H - S$ that intersects C_1 (or C_2) along a half coarse $PD(n)$-space then $B_x(R_1) \cap F \cap C_1$ is not contained in the R_1'-neighborhood of S (and similarly for C_2). Again we can pick r, R_1', R_1 uniformly for all $e, u \in T, h \in H$.
We note that there is an $R_2 > R_1$ such that the following hold:
a) for any $x \in S$ any two points in $B_x(R_1)$ that lie in the same deep component can be joined by a path lying in $B_x(R_2) - S$.
b) for any $v \in B_x(R_1)$ $d(v, S) = d(v, S \cap B_x(R_2))$.
We fix a vertex $v \in S$. For any vertex $x \in S$ we pick a $g_x \in H$ such that $g_x x = v$. We call the set of vertices in $B_v(R_2) \cap g_x S$ the *type* of x. We will show the following:

Lemma 3.5. *There is an $M > 0$ such that if x, y are of the same type then $g_x S$ and $g_y S$ lie in the M-neighborhood of each other.*

Proof. We note that two points in $B_v(R_1)$ lie in the same deep component of $\Gamma_H - g_x S$ if and only if they lie in the same deep component of $\Gamma_H - g_y S$.
We distinguish two cases:
Case 1: S is not quasi-isometric to a coarse $PD(n-1)$-space.
Let $C_1 = f(p^{-1}v), C_2 = f(p^{-1}w)$ where v, w are vertices adjacent to e. $g_x C_1, g_x C_2$ are coarsely contained in distinct components, say F_1, F_2 of $\Gamma_H - g_x S$. Let $c_1 \in B_v(R_1) \cap g_x C_1$ such that $d(c_1, g_x S) > R_1'$. Then c_1 lies in $g_x F_1$. Since x, y are of the same type $d(c_1, g_y S) > R_1'$ so c_1 lies in the deep component of $\Gamma_H - g_y S$ that coarsely contains $g_x C_1$. We pick similarly $c_2 \in B_v(R_1) \cap g_x C_2$. Since c_1, c_2 are not contained in the same component of $\Gamma_H - g_y S$ we have that $g_x C_1, g_x C_2$ are coarsely contained in distinct deep components of $\Gamma_H - g_y S$. We claim that $N_{R_0}(g_x C_1) \cap N_{R_0}(g_x C_2)$ is contained in the $R_1 + R_0$ neighborhood of $g_y S$ so $g_x S$ is coarsely contained in $g_y S$. Indeed let $a \in N_{R_0}(g_x C_1) \cap N_{R_0}(g_x C_2)$. Let $a_1 \in g_x C_1$, $a_2 \in g_x C_2$ with $d(a, a_1) \leq R_0$, $d(a, a_2) \leq R_0$. If a_1 (or a_2) is not contained in the deep component of $\Gamma_H - g_y S$ that contains $g_x C_1$ we have $d(a_1, g_y S) \leq R_1'$ so $d(a, g_y S) \leq R_0 + R_1'$. Otherwise we have that a_1, a_2 lie in distinct components of $\Gamma_H - g_y S$ and there is a path joining them of length less or equal to $2R_0$. This path intersects $g_y S$ so in this case a is at distance less than R_0 from $g_y S$. In the same way we see that $g_y S$ is coarsely contained in $g_x S$. Clearly if $M = R_1 + R_0$ we have that $g_x S$ and $g_y S$ lie in the M-neighborhood of each other. We note in particular that M does not depend on x, y.

Case 2: S is quasi-isometric to a coarse $PD(n-1)$-space.

Let $C_1 = f(p^{-1}v), C_2 = f(p^{-1}w)$ where v, w are vertices adjacent to e. g_xC_1, g_xC_2 are not contained coarsely in the same component of $\Gamma_H - g_xS$. Say g_xC_1, g_xC_2 intersect respectively the (distinct) deep components F_1, F_2 of $\Gamma_H - g_xS$ along half coarse $PD(n)$ spaces. Let $c_1 \in g_xC_1 \cap B_v(R_1)$ and $c_2 \in g_xC_2 \cap B_v(R_1)$ such that $d(c_1, g_xS) > R_1', d(c_2, g_xS) > R_1'$. As in case 1 we have that c_1, c_2 lie in distinct components of $\Gamma_H - g_yS$ and so $g_xC_1 \cap F_1, g_xC_2 \cap F_2$ are coarsely contained in distinct components of $\Gamma_H - g_yS$. We have then that $N_{R_0}(g_xC_1) \cap N_{R_0}(g_xC_2)$ is contained in the $R_1' + R_0$ neighborhood of g_yS so g_xS is coarsely contained in g_yS. In the same way we see that g_yS is coarsely contained in g_xS. So if $M = R_1' + R_0$ we have that that g_xS and g_yS lie in the M-neighborhood of each other. ■

We consider now the group J_1 generated by all elements $\{g_x^{-1}g_y\}$ where x, y are of the same type. We claim that it follows from lemma 3.5 that there is an $M_1 > 0$ such that for any $h \in J_1$, if $hS \cap S \neq \emptyset$ then hS is contained in the M_1-neighborhood of S.

Indeed we remark that there are finitely many types of vertices of S. Let $x_1, ..., x_n$ be vertices of S representing these types and let $g_{x_1}, ..., g_{x_n}$ be the corresponding group elements. There is an $M_0 > 0$ such that for any $i, j \in \{1, .., n\}$ if $g_{x_i}S$ and $g_{x_j}S$ are contained in a finite neighborhood of each other then they are contained in the M_0-neighborhood of each other. Consider now an $h \in J_1$ such that $hS \cap S \neq \emptyset$. Since every generator of J_1 maps S in a finite neighborhood of itself we have that hS and S are contained in a finite neighborhood of each other.

Similarly we see that for any $K > 0$ there is an $M_K > 0$ such that for any $h \in J_1$ if hS intersects the K-neighborhood of S then hS is contained in the M_K-neighborhood of S.

Let $a \in hS \cap S$. We have that g_ahS and g_aS are contained in a finite neighborhood of each other. Therefore they are contained in an $M_0 + 2M$ neighborhood of each other. Hence we can take $M_1 = M_0 + 2M$.

We fix $R > 0$ such that $S \subset J_1(B_v(R) \cap S)$. We distinguish two cases:

Case 1: There is an $M_2 > 0$ such that for any $h \in J_1$, hS is contained in the M_2-neighborhood of S. Then clearly Γ_H/J_1 has more than 1 end. Let $S' = J_1S$. Then $\Gamma_H - S'$ has finitely many deep components that coarsely contain S, say $F_1, ..., F_n$ and S'/J_1 is finite. Moreover J_1 acts on the set of deep components that coarsely contain S by permutations. Therefore there is a finite index subgroup of J_1 ,say J_1', such that F_1 is a non-trivial J_1'-almost invariant subset of H. If hF_1 crosses F_1 for some $h \in H$ then hF_1 and F_1 are at finite distance from each other. So the subgroups J_1' and $hJ_1'h^{-1}$ are commensurable. It follows by [NSSS] that H splits over a subgroup commensurable with J_1.

Case 2: Suppose now that no such $M_2 > 0$ exists. It follows that there is an $h_0 \in J_1$ such that h_0S does not intersect the R_1'-neighborhood of S. This implies that $\Gamma_H - (S \cup h_0S)$ has at least 3 deep components which coarsely contain S. We can now apply the same argument as in lemma 2.3 to conclude that H splits. We do this in detail here:

Let $A > M$ be such that $h_0S \subset N_A(S)$. We set $S' = N_A(S)$. $\Gamma_H - S'$ has at least 3 deep components that coarsely contain S.

For each $x \in S$ we pick $h_x \in J_1$ such that $h_xx \in B_v(R) \cap S$. If $F_1, ..., F_k$ are the deep components of $\Gamma_H - S$ that coarsely contain S and $D_1, ..., D_m$ the deep components of $\Gamma_H - S'$ that coarsely contain S, we have that for each j there are $D_{j_1}, ..., D_{j_r}$ such that h_xF_j and $D_{j_1} \cup ... \cup D_{j_r}$ are at finite distance from each other.

We use this to define a map $f_x : \{F_1, ..., F_k\} \to \mathcal{P}(\{D_1, ..., D_m\})$ where $f_x(F_j) = \{D_{i_1}, ..., D_{i_r}\}$ if and only if $h_x F_j$ and $D_{i_1} \cup ... \cup D_{i_r}$ are at finite distance from each other.
Let's write now $x \sim y$ for x, y vertices of S if $f_x = f_y$. Clearly $\sim$ is an equivalence relation with finitely many equivalence classes.
Let R_3 be such that $B_{R_3}(v) \cap S$ contains all elements of the equivalence classes with finitely many elements and at least one element of each equivalence class with infinitely many elements.
For each vertex $y \in S$ we pick $t \in B_R(v) \cap S$, $t \sim y$ and we consider the group J generated by $\{h_y^{-1} h_t\}$. Clearly $S \subset J(B_{R_3}(v))$. We note that for any $h \in J$ hF_i and F_i are in a finite neighborhood of each other for all i.
We claim that there is an $M_3 > 0$ such that for any $h \in J$, hS is contained in the M_3-neighborhood of S.
Indeed, if for some $h \in J$, hS does not intersect the R_1'-neighborhood of S then the following hold: hS lies in a single component of $\Gamma_H - S$, say F_i. Moreover at least 2 deep components of $\Gamma_H - S$ are coarsely contained in F_i. To see this recall that if S is not a coarse $PD(n-1)$-space there are 2 deep components of $\Gamma_H - S$, say F_1, F_2 containing, respectively, coarse $PD(n)$-spaces C_1, C_2, such that for any $x \in C_1$ (C_2) if $d(x, S) \geq R_1'$ then x lies in the deep component of $\Gamma_H - S$ that coarsely contains C_1 (C_2). This implies that hC_1, hC_2 are both contained in the same deep component of $\Gamma_H - S$, so hF_1, hF_2 are both contained in a finite neighborhood of a single deep component of $\Gamma_H - S$, a contradiction.
We argue similarly when S is a coarse $PD(n-1)$-space. It follows that there is a $M_3 > 0$ such that for any $h \in J$, hS is contained in the M_3-neighborhood of S. Therefore the set JS is at finite distance from S , JS/J is finite and $\Gamma_H - JS$ has more than one deep component. Using the criterion of [NSSS] as before we conclude that H splits over a group commensurable with J. ■

One can get a finer result when all edge groups are virtually $\mathbb{Z}^{n-1}$:

Theorem 3.6. *Let G be a finitely generated group admitting a graph of groups decomposition $G = \pi_1(\Gamma, \mathcal{A})$ such that all vertex groups are coarse $PD(n)$-groups and all edge groups are virtually $\mathbb{Z}^{n-1}$. If H is quasi-isometric to G then $H = \pi_1(\Delta, \mathcal{B})$ where all edge groups of Δ are virtually $\mathbb{Z}^{n-1}$ and all vertex groups H_v of Δ are either coarse $PD(n)$-groups or virtually $\mathbb{Z}^{n-1}$ or the pair $(H_v, \{H_{e_i}\})$ is a coarse $PD(n)$-pair, where H_{e_i} are the edge groups of the edges containing v.*

Proof. We construct a complex X such that $\pi_1(X) = G$ as in theorem 3.4. We have as in 3.4 that there is a map $p : \tilde{X} \to T$ where T is the Bass-Serre tree of $(\Gamma, \mathcal{A})$. By theorem 3.4 H splits over a group J commensurable to $\mathbb{Z}^{n-1}$. Moreover J lies at finite distance from $f(p^{-1}e)$ for some $e \in T$, where $f : \tilde{X} \to \Gamma_H$ is a quasi-isometry.
We proceed now inductively: We assume that we can write H as the fundamental group of a graph of groups $H = \pi_1(\Delta, \mathcal{B})$ where edge groups are commensurable with $\mathbb{Z}^{n-1}$ lie at finite distance from $f(p^{-1}e)$ for some $e \in T$ and $(\Delta, \mathcal{B})$ can not be further refined. By this we mean that no vertex group B_v of Δ admits a graph of groups decomposition with edge groups of the same type and such that all edge groups B_e of edges adjacent to v are subgroups of vertex groups of the graph of groups decomposition of B_v.
By the accessibilty theorem of Bestvina Freighn ([B-F]) this procedure terminates.
We note now that if H_v is a vertex group of $(\Delta, \mathcal{B})$ then $f^{-1}(H_v)$ is coarsely

contained in $p^{-1}u$ for some vertex $u \in T$. Indeed if not we could refine $(\Delta, \mathcal{B})$ further as in proposition 3.3.
If H_v is not a coarse $PD(n-1)$-group and it is not quasi-isometric to $p^{-1}u$ we have that H_v acts by quasi-isometries (via f) on the coarse $PD(n)$-space $p^{-1}u$ and it easily follows that the pair $(H_v, \{H_{e_i}\})$ is a coarse $PD(n)$-pair, where H_{e_i} are the edge groups of the edges containing v. ■

§4. Questions

It is reasonable to wonder whether theorems 3.1 and 3.4 can be subsumed under a theorem posing no restriction on edge groups. More precisely we have the following:

Question 1. *Let G be a finitely generated group admitting a graph of groups decomposition $G = \pi_1(\Gamma, \mathcal{A})$ such that all vertex groups are coarse $PD(n)$-groups. Is it true that if H is quasi-isometric to G then H splits over some group quasi-isometric to an edge group of Γ?*

The main motivation of this paper was to generalize Stallings' theorem on groups with infinitely many ends to splittings over groups that are not necessarily finite. This has been achieved to a large extend for splittings over virtually cyclic groups (see [Bo], [P]). However generalizing Stallings' theorem for splittings over any group poses serious difficulties. For example even in the virtually cyclic case we have that surface groups do split over $\mathbb{Z}$ but triangle groups that are quasi-isometric to them don't. In general there are examples of groups that are quasi-isometric to groups that split but which do not split themselves and even do not virtually split (that is none of their finite index subgroups splits). Moreover it is a hard problem to determine whether a group virtually splits, it is not known even for 3-manifold hyperbolic groups.
On the other hand looking closer at Stallings' theorem one realizes that it splits in 3 cases: Groups with 2 ends, virtually free groups and groups with more than 2 ends that are not virtually free groups. Or to rephrase it in asymptotic topology terminology, the first two cases correspond to groups of asymptotic dimension 1 which are coarsely separated by subsets of asymptotic dimension 0 (compact sets) and the third case corresponds to groups of asymptotic dimension ≥ 2 which are coarsely separated by subsets of asymptotic dimension 0 (compact sets). The first two cases can be considered as 'exceptional' as they belong to only 2 quasi-isometry (in fact commensurability) classes. It seems that these two cases are the harder to generalize. On the other hand the third case (the 'codimension 2' case) might be easier to deal with. More precisely we have the following:

Question 2. *Let G be a finitely generated group of asymptotic dimension $\geq n$. Suppose that a uniformly embedded subset S of asymptotic dimension $\leq n-2$ coarsely separates the Cayley graph of G. Is it true then that G splits?*

See [G2] for a definition of asymptotic dimension.

References

[Ba] H.Bass, *Covering theory for graphs of groups*, J. Pure Appl. Algebra 89, No. 1-2, pp. 3-47 (1993).

[B-F] M.Bestvina,M. Feighn, *Bounding the complexity of simplicial group actions on trees*, Invent. math. 103, p. 449-469 (1991).

[Bo] B.H. Bowditch, *Cut points and canonical splittings of hyperbolic groups*, Acta Math. 180, No.2, pp.145-186 (1998).

[D-S] M. Dunwoody, E. Swenson, *The algebraic torus theorem*, Invent. Math. 140, no. 3, 605–637 (2000).

[Ep] D. Epstein, *Word Processing in Groups*, Jones and Bartlett, Boston, 1992.

[F-S] B. Farb, R.Schwartz, *The large scale geometry of Hilbert modular groups*, J. Differential Geom., 44 (3), p. 435-478 (1996).

[F-M1] B. Farb, L. Mosher, *A rigidity theorem for the solvable Baumslag-Solitar groups*, Invent. Math. 131, no. 2, p. 419-451 (1998).

[F-M2] B. Farb, L. Mosher, *Quasi-isometric rigidity for the solvable Baumslag-Solitar groups II*, Invent. Math. 137, no. 3, p. 613-649 (1999).

[G1] M.Gromov, *Hyperbolic groups*, in 'Essays in group theory', S.M. Gersten (Ed.), MSRI publ. vol. 8, Springer Verlag, NY 1987, pp. 75-263.

[G2] M.Gromov, *Asymptotic invariants of infinite groups*, in 'Geometric group theory', (G.Niblo, M.Roller, Eds.), LMS Lecture Notes, vol. 182, Cambridge Univ. Press (1993).

[K-K] M. Kapovich, B. Kleiner, *Coarse Alexander duality and duality groups*, J. Differential Geom. 69, no. 2, 279–352 (2005).

[K-L] M. Kapovich, B. Leeb, *Quasi-Isometries preserve the geometric decomposition of Haken manifolds*, Invent. Math. 128, no.2, p.393-416 (1997).

[MSW] L. Mosher, M. Sageev, K.Whyte, *Quasi-actions on trees*, Research announcement, ArXiv, May 22, 2000.

[MSW1] L. Mosher, M. Sageev, K.Whyte, *Quasi-actions on trees I, Bounded valence*, Quasi-actions on trees. I. Bounded valence. Ann. of Math. (2) 158, no. 1, 115–164. (2003).

[NSSS] G.Niblo, M.Sageev, P.Scott, G.A.Swarup, *Minimal cubings*, Int. J. Alg. Comp., vol. 15, no. 2, p.343-366 (2005).

[P] P.Papasoglu, *Quasi-Isometry invariance of group splittings*, Annals of Math., vol. 161, no.2, p.759-830 (2005).

[Se] J-P.Serre, *Trees*, Springer Verlag, N.Y., 1980.

[Sch] R. Schwartz, *Quasi-isometric rigidity and Diophantine approximation*, Acta Math. 177 (1) p. 75-112 (1996).

[S-W] P. Scott, T. Wall, *Topological methods in group theory*, in Homological Group Theory, Proc. Symp. Durham 1977, LMS Lect. Note Ser. 36, pp.137-203 (1979).

[St] J.R. Stallings, *On torsion-free groups with infinitely many ends*, Ann. of Math. 88, p. 312-334 (1968).

[Wh] K. Whyte, *The quasi-isometry types of the higher Baumslag-Solitar groups*, GAFA v.11, p.1327-1343 (2002).

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