SIMPLY CONNECTED HOMOGENEOUS CONTINUA ARE NOT SEPARATED BY ARCS

MYRTO KALLIPOLITI AND PANOS PAPASOGLU

ABSTRACT. We show that locally connected, simply connected homogeneous continua are not separated by arcs. We ask several questions about homogeneous continua which are inspired by analogous questions in geometric group theory.

1. Introduction

In this paper we prove a theorem about homogeneous continua inspired by a result about finitely presented groups ([13]).

Theorem 1. *Let X be a locally connected, simply connected, homogeneous continuum. Then no arc separates X.*

We recall that an arc in X is the image of a 1-1 continuous map $\alpha : [0,1] \to X$. We say that an arc α separates X if $X \setminus \alpha$ has at least two connected components. We say that X is simply connected if it is path connected and every continuous map $f : S^1 = \partial D^2 \to X$ can be extended to a continuous $\bar{f} : D^2 \to X$ (where D^2 is the 2-disc and S^1 its boundary circle). The proof of Theorem 1 relies on Alexander's lemma for the plane (see [12]) and our generalization of this lemma to all simply connected spaces (see sec. 2).

There is a family air between continua theory and group theory. This became apparent after Gromov's theory of hyperbolic groups ([6]). Gromov defines a boundary for a hyperbolic group which is a continuum on which the group acts by a 'convergence action'. The classic 'cyclic elements' decomposition theory of Whyburn was extended recently by Bowditch ([3]) in this context and it gave deep results in group theory. 'Asymptotic topology', introduced by Gromov ([7]) and developed further by Dranishnikov ([4]), shows that the analogy goes beyond the realm of hyperbolic groups. The 'philosophy' of this is that topological questions that make sense for continua can be translated to

Date: November 25, 2006.
1991 *Mathematics Subject Classification.* 54F15.
Research co-funded by European Social Fund and National Resources EPEAEK II-Pythagoras.

'asymptotic topology' questions which make sense for groups (see [4] for a dictionary between topology and asymptotic topology).

One wonders whether Theorem 1 holds in fact for all locally connected, homogeneous continua of dimension bigger than 1:

Question 1. *Let X be a locally connected, homogeneous continuum. Is it true that if no Cantor set separates X then no arc separates X?*

We note that by a result of Krupski ([10]), if a Cantor set separates a homogeneous continuum X then $\dim X = 1$, while if an arc separates X then $\dim X \leq 2$. So the question above just asks if Theorem 1 applies to all locally connected homogeneous continua of dimension 2. Krupski and Patkowska ([11]) have shown that a similar property (the disjoint arcs property) holds for all locally connected homogeneous continua of dimension bigger than 1 which are not 2-manifolds.

Question 1 makes sense also for boundaries of hyperbolic groups. In fact a similar question can be formulated for finitely generated groups too (see [13]).

Not much is known about locally connected, simply connected homogeneous continua. One motivation to study them is the analogy with finitely presented groups. Another reason is that one could hope for a classification of such continua in dimension 2:

Question 2. *Are the 2-sphere and the universal Menger compactum of dimension 2 the only locally connected, simply connected, homogeneous continua of dimension 2?*

We recall that S^1 and the universal Menger curve are the only locally connected homogeneous continua of dimension 1 ([1]). Prajs ([15], question 2) asks whether S^2 is the only simply connected homogeneous continuum of dimension 2 that embeds in $\mathbb{R}^3$.

A related question about locally connected, simply connected continua that makes sense also for finitely presented groups is the following:

Question 3. *Let X be a locally connected, simply connected homogeneous continuum which is not a single point. Does X contain a disc?*

We remark that in the group theoretic setting the answer is affirmative for hyperbolic groups ([2]). By a result of Prajs ([14]) a positive answer to this would imply that S^2 is the only simply connected homogeneous continuum of dimension 2 that embeds in $\mathbb{R}^3$.

We refer to Prajs' list of problems ([15]) for more questions on homogeneous continua.

2. Preliminaries

Definition 1. *Let X be a metric space. A path p is a continuous map $p : [0,1] \to X$. A simple path or an arc α, is a continuous and $1-1$ map $\alpha : [0,1] \to X$. We will identify an arc with its image.*

For a path p we denote by ∂p the set of its endpoints, i.e. $\partial p = \{p(0), p(1)\}$.

An arc α separates X if $X \setminus \alpha$ has at least two connected components. If $x, y \in X$ we say that an arc α separates x from y if α separates X and x, y belong to distinct components of $X \setminus \alpha$.

Definition 2. *Let α be an arc of X. On α we define an order $<_\alpha$ as follows: If $x = \alpha(x')$, $y = \alpha(y')$ then $x <_\alpha y$ if and only if $x' < y'$.*

We denote by $[x,y]_\alpha$ the set of all $t \in \alpha$ such that $x \leq t \leq y$. Similarly we define $(x,y)_\alpha$, $[x,y)_\alpha$ and $(x,y]_\alpha$. When there is no ambiguity we write $[x,y]$ instead of $[x,y]_\alpha$ and $x < y$ instead of $x <_\alpha y$. Finally, if $t \in [0,1]$ we denote by $x + t$ the point $\alpha(x' + t)$ (where $x = \alpha(x')$).

We recall Alexander's lemma from plane topology (see Theorem 9.2, p.112 of [12]).

Alexander's Lemma (for the plane). *Let K_1, K_2 be closed sets on the plane such that either $K_1 \cap K_2 = \emptyset$ or $K_1 \cap K_2$ is connected and at least one of K_1, K_2 is bounded. Let $x, y \in \mathbb{R}^2 \setminus (K_1 \cup K_2)$. If there is a path joining x, y in $\mathbb{R}^2 \setminus K_1$ and a path joining x, y in $\mathbb{R}^2 \setminus K_2$ then there is a path joining x, y in $\mathbb{R}^2 \setminus (K_1 \cup K_2)$.*

It is easy to see that Alexander's lemma also holds for the closed disc D^2 in the case that $K_1 \cap K_2 = \emptyset$. In fact this implies that this lemma holds in general for every simply connected space. In particular we have the following:

Alexander's Lemma (for simply connected spaces). *Let X be a simply connected space, K_1, K_2 disjoint closed subsets of X and let $x, y \in X \setminus (K_1 \cup K_2)$. If there is a path joining x, y in $X \setminus K_1$ and a path joining x, y in $X \setminus K_2$ then there is a path joining x, y in $X \setminus (K_1 \cup K_2)$.*

Proof. Let p_1, p_2 be paths joining x, y such that $p_1 \cap K_1 = p_2 \cap K_2 = \emptyset$. We consider the closed path $p_1 \cup p_2$ and let $f : S^1 = \partial D^2 \to X$ be a parametrization of this path. Since X is simply connected, f can be extended to a map $F : D^2 \to X$. Then $F^{-1}(K_1)$ and $F^{-1}(K_2)$ are disjoint, closed subsets of D^2. Clearly, neither $F^{-1}(K_1)$ nor $F^{-1}(K_2)$ separates x, y, therefore, using Alexander's lemma for the closed disc, we have that there is a path p that joins x, y without meeting $F^{-1}(K_2) \cup F^{-1}(K_2)$. This implies that $F(p)$ is a path from x to y that does not meet $K_1 \cup K_2$. $\square$

For the rest of this paper we assume that X is a simply connected, locally connected continuum.

Lemma 1. *Let O be a connected open subset of X, K be a connected component of ∂O and let $x, y \in O$ such that $d(x, K) < \varepsilon$ and $d(y, K) < \varepsilon$. Then there is a path p in O connecting x to y such that p is contained in the ε-neighborhood of ∂O.*

Proof. Let U be the union of the open balls $B_t(\varepsilon)$ with center $t \in \partial O$ and radius ε. Let V be the connected component of U containing K. Clearly $x, y \in V$ so there is a path in X joining them that does not intersect ∂V. On the other hand $x, y \in O$ so there is a path in X joining them that does not intersect ∂O.

Since $\partial O \cap \partial V = \emptyset$ and $\partial O, \partial V$ are closed, applying Alexander's lemma for the simply connected space X, we have that p is a path lying in X joining x, y that intersects neither ∂O nor ∂V. Clearly p is contained in O and lies in the ε-neighborhood of ∂O. $\square$

Lemma 2. *Let α be an arc that separates X and let C be a connected component of $X \smallsetminus \alpha$. Then $\overline{C}$ is simply connected and ∂C is connected.*

Proof. Let $f : S^1 = \partial D^2 \to \overline{C}$. We will show that this map can be extended to a map $\widehat{f} : D^2 \to \overline{C}$.

X is simply connected, so there is a map $F : D^2 \to X$ such that $F|_{S^1} = f$. Furthermore, $X \smallsetminus \overline{C}$ is an open set, therefore $\partial F^{-1}(X \smallsetminus \overline{C}) \cap F^{-1}(X \smallsetminus \overline{C}) = \emptyset$ and since F is a continuous extension of f it follows that $F(\partial F^{-1}(X \smallsetminus \overline{C})) \subset \alpha$ (where by $\partial F^{-1}(X \smallsetminus \overline{C})$ we denote the boundary of $F^{-1}(X \smallsetminus \overline{C})$ in X).

Let $f' : \partial F^{-1}(X \smallsetminus \overline{C}) \to \alpha$ be the restriction of F in $\partial F^{-1}(X \smallsetminus \overline{C})$. Then, applying Tietze's extension theorem, we obtain an extension for f':

$$F' : \overline{F^{-1}(X \smallsetminus \overline{C})} \to \alpha.$$

Finally we define $\widehat{f} : D^2 \to \overline{C}$ as follows:

$$\widehat{f}(x) = \begin{cases} f(x) & \text{if } x \in \partial D^2, \\ F(x) & \text{if } x \in D^2 \setminus F^{-1}(X \setminus \overline{C}), \\ F'(x) & \text{if } x \in F^{-1}(X \setminus \overline{C}). \end{cases}$$

This shows that $\overline{C}$ is simply connected.

Suppose now that $S = \partial C$ is not connected and let p be a path that joins two different components of S, such that if a, b are the endpoints of p, then $(p \setminus \{a, b\}) \cap S = \emptyset$. Let $x \in p \setminus \{a, b\}$ and $y \in (a, b)_\alpha \setminus S$. (Figure 1).

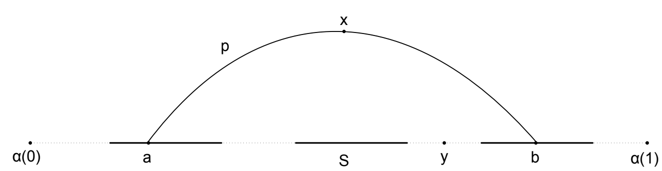

FIGURE 1

We set $K_1 = [\alpha(0), y]_\alpha \cap S$ and $K_2 = S \setminus K_1$. It is clear that K_1, K_2 are disjoint closed subsets of X and that neither K_1 nor K_2 separates x from y. Then Alexander's lemma (for the simply connected space X) implies that there is a path joining x, y in $X \setminus (K_1 \cup K_2) = X \setminus S$, a contradiction. □

Lemma 3. *Let α be an arc that separates X, $x, y \in \alpha$ and $\varepsilon > 0$ with $\varepsilon < d(x, y)$. Then for every connected component C of $X \setminus \alpha$ such that $x, y \in \partial C$ there are points $x', y' \in C$ with $d(x, x'), d(y, y') < \varepsilon$ and a path $p \in C$ that joins x', y' and is contained in the ε-neighborhood of $[x, y]_\alpha$.*

Proof. Let $B_{\frac{\varepsilon}{2}}(x)$ and $B_\varepsilon(x)$ be balls of center x and radius $\frac{\varepsilon}{2}$ and ε respectively. We consider the connected components of $\alpha \setminus \overset{\circ}{B}_{\frac{\varepsilon}{2}}(x)$ and we restrict to those that are not contained in $B_\varepsilon(x)$ (here we denote by $\overset{\circ}{B}_{\frac{\varepsilon}{2}}(x)$ the open ball). It is clear that there are finitely many such components, so we denote them by $I_1, I_2, \ldots, I_n$. Let $\delta_1 < \min\{d(I_i, I_j)\}$ for every $i, j = 1, 2, \ldots, n,\ i \neq j$. Similarly, let $J_1, J_2, \ldots, J_m$ be the connected components of $\alpha \setminus \overset{\circ}{B}_{\frac{\varepsilon}{2}}(y)$ that are not contained in $B_\varepsilon(y)$ and let $\delta_2 < \min\{d(J_i, J_j)\}$ for every $i, j = 1, 2, \ldots, m,\ i \neq j$. Let $\delta' < \min\{\delta_1, \delta_2, \frac{\varepsilon}{2}\}$.

From Lemma 2, we have that $\overline{C}$ is simply connected, therefore Lemma 1, for $\delta = \frac{\delta'}{4}$, implies that there is a path $q \in C$ that joins a point of $B_\varepsilon(x)$ with a point of $B_\varepsilon(y)$ and lies in the δ-neighborhood of α (Figure 2). We will show that there is a subpath of q that has the required properties.

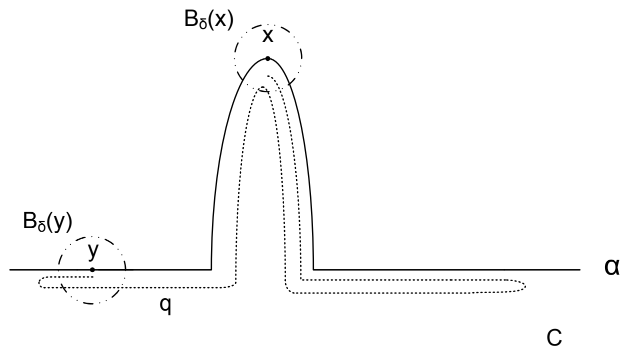

FIGURE 2

We assume that none of the $I_i, J_j,\ i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$ contain $[x, y]_\alpha$, since otherwise we are done. Thus, without loss of generality, let I_1 be the connected component of $\alpha \smallsetminus \overset{\circ}{B}_{\frac{\varepsilon}{2}}(x)$ that is contained in $[x, y]_\alpha$. We denote by $N_\delta([x, y]_\alpha)$ the open δ-neighborhood of $[x, y]_\alpha$. Suppose that there is a connected component $I = [a, b]_q$ of $q \smallsetminus N_\delta([x, y]_\alpha)$ with $a \in B_\delta(x)$ and $b \notin B_\delta(x)$. Then there is an $r > 0$ such that $(b - r, b)_q \notin N_\delta(\alpha)$. Indeed, if not then for every $r > 0$ there is an $I_i \neq I_1$ such that $(b - r, b)_q \in N_\delta(I_i)$. Thus $d(I_i, b) \leq \delta$. But $d(I_1, I_i) \leq d(I_i, b) + d(b, I_1) \leq \delta + \delta = 2\delta = \frac{\delta'}{2} < \delta'$, a contradiction. So there is an $r > 0$ such that for every $i \neq 1$ we have $(b - r, b)_q \notin N_\delta(I_i)$, therefore $(b - r, b)_q \notin N_\delta(\alpha)$, which is not possible. This contradiction proves the lemma. $\square$

3. PROOF OF THEOREM 1

We will prove the theorem by contradiction.

Remark: Since X is locally connected and compact, it follows that every open connected subset of X is path connected (see [9], Thm. 3.15, p.116). In particular the closure of every component of $X \smallsetminus \alpha$ is path connected.

Definition 3. *Let α_1, α_2 be arcs that separate X. We say that α_1 crosses α_2 at $x \in (\alpha_1 \setminus \partial\alpha_1) \cap (\alpha_2 \setminus \partial\alpha_2)$ if for any neighborhood of x in α_2, $(x-\varepsilon, x+\varepsilon)_{\alpha_2}$, there are $a, b \in (x-\varepsilon, x+\varepsilon)_{\alpha_2}$ separated by α_1. More generally, if $[x_1, x_2]$ is a connected component of $\alpha_1 \cap \alpha_2$, which is contained in $(\alpha_1 \setminus \partial\alpha_1) \cap (\alpha_2 \setminus \partial\alpha_2)$, we say that α_1 crosses α_2 at $[x_1, x_2]$ if for any neighborhood of $[x_1, x_2]$ in α_2, $(x_1-\varepsilon, x_2+\varepsilon)_{\alpha_2}$, there are $a, b \in (x_1-\varepsilon, x_2+\varepsilon)_{\alpha_2}$ separated by α_1. In this case, the endpoints x_1, x_2 are also called cross points of α_1, α_2.*

If $I_1 \subset \alpha_1$, $I_2 \subset \alpha_2$ are intervals of α_1, α_2 containing x in their interior, we say that I_1, I_2 cross at x. Similarly we define what it means for two intervals to cross at a common subarc. We call x (respectively $[x_1, x_2]$) a cross-point (respectively cross-interval) of α_1, α_2. We say that I_1, I_2 cross if they cross at some point x or at some interval $[x_1, x_2]$.

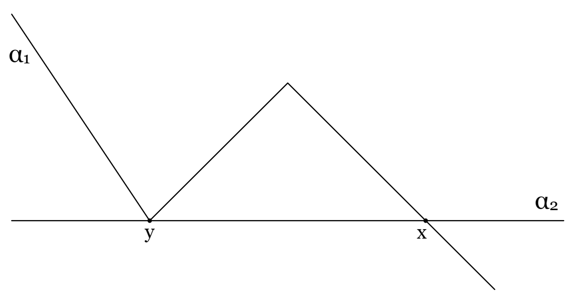

FIGURE 3

For example in Figure 3, x is a cross point of α_1, α_2, while y is an intersection point of α_1, α_2 which is not a cross point.

Lemma 4. *There is an arc that separates X in exactly two components.*

Proof. Suppose that this is not the case, so let α be an arc that separates X in more than two components. Since X has no cut points there are two connected components of $X \setminus \alpha$, say C_1, C_2, such that $\beta = \partial C_1 \cap \partial C_2 \neq \emptyset$ is a subarc (which is not a point) of α. Clearly β separates X. To simplify notation we denote by C_1, C_2 the components of $X \setminus \beta$ that satisfy $\partial C_1 = \partial C_2 = \beta$. Let C_3 be another component of $X \setminus \beta$. By Lemma 2 we have that ∂C_3 is connected, so $\partial C_3 = \gamma$ is a subarc of β, which separates X.

Lemma 4.1. *γ cannot be crossed by any other separating arc of X.*

Proof. Suppose that there is an arc γ' that separates X and crosses γ at t. Then there are $x, y \in \gamma,\ x < t < y$ that are separated by γ'. Let $Y = \overline{C}_1 \cup \overline{C}_2$. Since $\overline{C}_3$ is path connected, it follows that $\gamma' \not\subset Y$. We denote by C_x the connected component of $Y \setminus \gamma'$ that contains x. As in Lemma 2, we may show that Y is simply connected. We show then that Alexander's lemma for Y implies that ∂C_x has a connected component that separates x, y in Y.

This can be achieved as follows: Let C_y be the connected component of $Y \setminus \overline{C}_x$ that contains y. It is clear that $\partial C_y \subseteq \partial C_x$. Then no proper closed subset of ∂C_y separates x from y. Indeed, suppose that there is a closed $K \subset \partial C_y$ that separates x from y and let $z \in \partial C_y \setminus K$. Let U be an open neighborhood of z such that $U \cap K = \emptyset$. It is obvious that U intersects every component of $Y \setminus \partial C_y$, therefore, there are paths $q_1, q_2 \in Y \setminus \partial C_y$ that join x, y with points $x', y' \in U$ respectively. However U is path connected and since $U \cap K = \emptyset$, it follows that there is also a path $q \in U$ that joins x' with y'. Thus $p_1 \cup q \cup p_2$ is a path joining x, y without meeting K, a contradiction. Therefore, $I = \partial C_y$ is connected and separates x from y.

We note now that I does not cross γ. Indeed, suppose that there is an $a \in I \setminus \partial I$ in which γ' crosses γ. Let $V \subset X$ be sufficiently small neighborhood of a such that $(\gamma' \setminus I) \cap V = \emptyset$. We denote by J the connected component of $\gamma \cap V$ that contains the point a. Then we can pick points $x', y' \in J$ with $x' < a < y'$ in γ that are separated by γ'. Let $N_{x'}, N_{y'}$ be connected neighborhoods of x' and y' respectively, such that $N_{x'}, N_{y'} \subset V$. Applying now Lemma 3 for the component C_3 and for $\varepsilon < \min\{\mathrm{diam}(N_{x'} \cap C_3), \mathrm{diam}(N_{y'} \cap C_3)\}$, we have that every point of $N_{x'} \cap C_3$ can be joined with every point of $N_{y'} \cap C_3$ by a path in C_3 which lies in the ε-neighborhood of $[x', y']_\gamma$.

Let $t \in C_3 \cap N_{x'},\ s \in C_3 \cap N_{y'}$ and let q be a path that joins t and s as above. We note now that $N_{x'}$ and $N_{y'}$ are path connected, so there are paths $q_1 \in N_{x'}$ and $q_2 \in N_{y'}$ joining the endpoints of q with the points x' and y' respectively. Clearly then the path $p = q_1 \cup q \cup q_2$ joins x' with y' without meeting γ'. This is however impossible, since x' and y' are separated by γ'. Therefore, I does not cross γ. Thus $\gamma \setminus I$ is contained in a single component of $X \setminus I$, a contradiction. $\square$

We return now to the proof of Lemma 4: Let G be the group of homeomorphisms of X. For every $f \in G$ we have that $f(\gamma)$ separates X and from the previous lemma it follows that $f(\gamma)$ does not cross γ. Let $S = G \cdot \gamma$. Clearly S is uncountable. Let Q be a countable dense set of X. We define a map $R : S \to Q \times Q \times Q$ as follows: Let $p \in S$ and U_1, U_2, U_3 be three connected components of $X \setminus p$. For every U_i

we pick an $r_i \in Q$ and we associate $p \in S$ the triple (r_1, r_2, r_3). We remark that R is $1-1$ map, which is a contradiction. This completes the proof of Lemma 4. $\square$

Let γ be an arc that separates X in exactly two components C_1, C_2 with $\partial C_1 = \partial C_2 = \gamma$. We denote by G the group of homeomorphisms of X. Let $S = G \cdot \gamma$. It is clear that S is uncountable and that every $\alpha \in S$ also separates X in exactly two components U_1, U_2 such that $\partial U_1 = \partial U_2 = \alpha$. For an $\alpha \in S$ we will denote these two components by α^+ and α^- (Figure 4).

Henceforth we will consider only arcs in S.

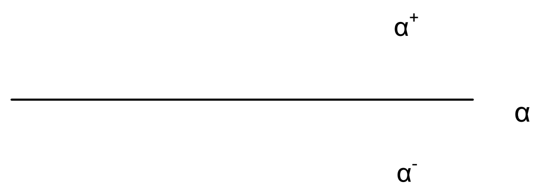

FIGURE 4

Lemma 5. *Let $\alpha_1, \alpha_2 \in S$ such that α_1 crosses α_2 at x (or at $[x_1, x_2]$). Then α_2 crosses α_1 at x (or at $[x_1, x_2]$).*

Proof. Suppose that there are $\alpha_1, \alpha_2 \in S$ such that α_1 crosses α_2 at x but α_2 does not cross α_1 at x. Then there is an interval $I \subset \alpha_1$ containing x at its interior that lies in the closure of one of the components of $X \setminus \alpha_2$, say $\overline{\alpha_2^+}$. Clearly then we have that $I \cap \alpha_2^- = \emptyset$.

Let $V \subset X$ be sufficiently small neighborhood of x such that $(\alpha_1 \setminus I) \cap V = \emptyset$. We denote by J the connected component of $\alpha_2 \cap V$ that contains the point x. We pick two points $a, b \in J$ with $a < x < b$ in α_2 which are separated by α_1 and let N_a, N_b be connected neighborhoods of a and b respectively such that $N_a, N_b \subset V$. As in proof of Lemma 4.1, for $\varepsilon < \min\{\operatorname{diam}(N_a \cap \alpha_2^-), \operatorname{diam}(N_b \cap \alpha_2^-)\}$, we can find a path p that joins a with b without meeting α_1, a contradiction. We argue similarly if α_1 crosses α_2 at an interval $[x_1, x_2]$.

$\square$

We recall now a version of Effros' Theorem ([5], [8] p. 561):

Theorem 2. *For every $\varepsilon > 0$ and $x \in X$ the set $W(x, \varepsilon)$ of $y \in X$ such that there is a homeomorphism $h : X \to X$ with $h(x) = y$ and $d(h(t), t) < \varepsilon$ for all $t \in X$, is open.*

Lemma 6. *There are arcs $\alpha = [a_1, a_2]$ and $\beta = [b_1, b_2]$ in S, such that $b_1 \in (a_1, a_2)_\alpha$ and if A is the connected component of $\alpha \cap \beta$ that contains b_1, then $a_1, a_2 \notin A$.*

Proof. We will need the following:

Lemma 6.1. *Let $\alpha \in S$. Then there is an arc $\beta \in S$ that crosses α.*

Proof. Let $\alpha, \gamma \in S,\ c \in \partial\gamma,\ a \in \alpha \setminus \partial\alpha$ and $g \in G$ such that $gc = a$. By the definition of S it is not possible that $g\gamma \subset \alpha$, since α separates X in exactly two connected components. Assume now that α does not cross $g\gamma$. We denote by A the connected component of $\alpha \cap g\gamma$ that contains a and let $\partial\alpha = \{a_1, a_2\}$.

We distinguish two cases: Suppose that $a_1, a_2 \notin A$. Let $z \in g\gamma$ such that $(z, gc)_{g\gamma}$ lies in the closure of one of the components of $X \setminus \alpha$, say α^+. Let $z' \in (z, gc)_{g\gamma} \setminus \alpha$ and $\varepsilon > 0$ such that $B_\varepsilon(z') \subset \alpha^+$. By Theorem 2 there is a $\delta > 0$ such that $B_\delta(a) \subset W(a, \varepsilon)$. Let $y \in B_\delta(a) \cap \alpha^-$ (Figure 5).

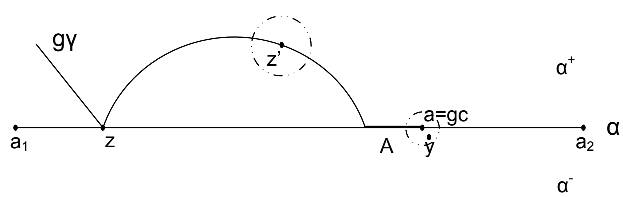

FIGURE 5

Then there is a homeomorphism, $h \in G$, with $h(a) = y$ and $d(t, h(t)) < \varepsilon$ for every $t \in X$. We consider the arc $\beta = h(g\gamma)$. Then clearly β crosses α, since $h(z') \in \alpha^+$ and $h(a) \in \alpha^-$.

Suppose now that $a_2 \in A$. We consider the homeomorphism $h \in G$ of the previous case. If $a_2 \notin h(A)$, then clearly Lemma 6.1 is proved. So let $a_2 \in h(A)$ and $\varepsilon' < \min\{\varepsilon, \frac{1}{2}d(\alpha, h(gc))\}$. As before, by theorem 2, there is a $\delta' > 0$ such that $B_{\delta'}(a_2) \subset W(a_2, \varepsilon')$. Let $y' \in B_{\delta'}(a_2) \cap \alpha^+$ (Figure 6). Then there is an $h' \in G$ with $h'(a_2) = y'$ and $d(t, h'(t)) < \varepsilon'$ for every $t \in X$. It is obvious now that α crosses $h'(\beta)$.

□

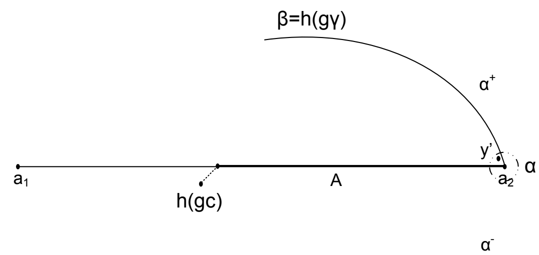

FIGURE 6

Let $\alpha = [a_1, a_2] \in S$. By Lemma 6.1 there is an arc $\beta = [b_1, b_2]$ that crosses α at $x \in \alpha \cap \beta$. Without loss of generality, suppose that x is the endpoint of a cross interval I of α, β. Let $\gamma \in S,\ c \in \partial\gamma$ and $g \in G$ such that $gc = x$. We denote by A the connected component of $g\gamma \cap \alpha$ that contains c and similarly by B the component of $g\gamma \cap \beta$ that contains c. Clearly if $a_1, a_2 \notin A$, then Lemma 6 is proved. Otherwise, we note that if A contains one of the endpoints of α, then $b_1, b_2 \notin B$, since I is a cross interval of α, β. So in this case, the required arcs are $g\gamma$ and β. $\square$

We return to the proof of Theorem 1.

Let $\alpha = [a_1, a_2], \beta = [b_1, b_2] \in S$ be paths as in Lemma 6, that is $b_1 \in (a_1, a_2)_\alpha$ and if A is the connected component of $\alpha \cap \beta$ that contains b_1, then $a_1, a_2 \notin A$. Let $t_1, t_2 \in \alpha \setminus \beta$ such that $b_1 \in (t_1, t_2)_\alpha$ and let p_1, p_2 be paths joining t_1, t_2 in $\overline{\alpha^+}$ and $\overline{\alpha^-}$ respectively (the points t_i exist since $a_1, a_2 \notin A$). We pick p_i such that $p_i \cap \alpha$ has exactly two connected components neither of which intersects β (this can be achieved using Lemma 3 for $\varepsilon < \frac{1}{2}\min\{d(t_1, \partial A_1), d(t_2, \partial A_2)\}$, where A_i is the connected component of $\alpha \setminus \beta$ that contains t_i and ∂A_i is its boundary in α).

Let $\varepsilon > 0$ with $\varepsilon < \frac{1}{2}d(A, p_1 \cup p_2)$. As in proof of Lemma 6.1, using Theorem 2, we can find a homeomorphism $h \in G$ such that $h(\beta)$ crosses α at $x \in \alpha \cap h(\beta)$, with $d(A, x) < \varepsilon$. Then we remark that $x \in (t_1, t_2)_\alpha$ and that the subarc of $h\beta$ with endpoints x and hb_1 does not intersect $p_1 \cup p_2$.

We pick now points $s \in (t_1, x)_\alpha \setminus (p_1 \cup p_2)$ and $t \in (x, t_2)_\alpha \setminus (p_1 \cup p_2)$ which are separated by $h\beta$ so that they satisfy the following: If y is a cross point of α and $h\beta$, lying in $[s, t]_\alpha$, then the subarc $[y, hb_1]_{h\beta}$

does not intersect the paths p_1, p_2. Such points exist by definition of x (Figure 7).

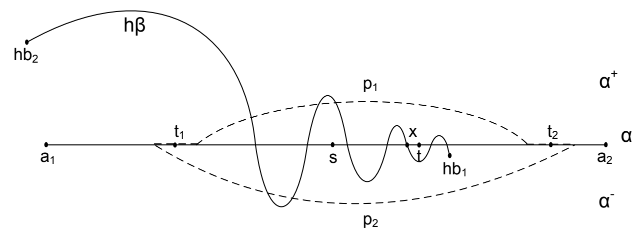

FIGURE 7

We consider now the closed paths $p_1 \cup [t_1, t_2]_\alpha$ and $p_2 \cup [t_1, t_2]_\alpha$. Let D_1, D_2 be discs and let $f_1 : D_1 \to \overline{\alpha^+}$, $f_2 : D_2 \to \overline{\alpha^-}$ be maps so that $f_1(\partial D_1) = p_1 \cup [t_1, t_2]_\alpha$, $f_2(\partial D_2) = p_2 \cup [t_1, t_2]_\alpha$ (such maps exist, since $\overline{\alpha^+}$ and $\overline{\alpha^-}$ are simply connected by Lemma 2).

We 'glue' D_1, D_2 along $[t_1, t_2]_\alpha$ and we obtain a disc D and a map $f : D \to X$ with $f(\partial D) = p_1 \cup p_2$. More precisely, we consider the disc $D = D_1 \sqcup D_2 / \sim$, where $\sim$ is defined as follows: $x_1 \sim x_2$ if and only if $x_1 \in \partial D_1,\ x_2 \in \partial D_2$ and $f_1(x_1) = f_2(x_2)$. Finally, we define $f : D \to X$ as:

$$f(t) = \begin{cases} f_1(t), & \text{if } t \in D_1, \\ f_2(t), & \text{if } t \in D_2. \end{cases}$$

By abuse of notation we identify points of $[t_1, t_2]_\alpha$ in D with their image under f. We note that the interior, say U, of D is homeomorphic to $\mathbb{R}^2$ and since t, s are separated by $h\beta$ in X, it follows by Alexander's lemma that t, s are separated in U by a connected component of $f^{-1}(h\beta) \cap U$. We call this component K (Figure 8).

Clearly $f(K)$ is a subarc of $h\beta$ that contains cross points or cross intervals of $h\beta$ with $[s, t]_\alpha$. Let c be such a cross point. Then we can write $f(K)$ as $f(K) = I_1 \cup I_2$, where I_i, $i = 1, 2$, are (connected) subarcs of $h\beta$, such that $I_1 \cap I_2 = c$. Furthermore, at least one of I_1, I_2 does not intersect $p_1 \cup p_2$ (this is by our choice of h and c). It follows that at least one of $f^{-1}(I_1) \cap U, f^{-1}(I_2) \cap U$ is compact.

We set $I_1' = I_1 \setminus c$, $I_2' = I_2 \setminus c$. We will define two sets K_1, K_2 such that the following are satisfied: K_1, K_2 are closed subsets of U that contain $f^{-1}(I_1')$ and $f^{-1}(I_2')$ respectively, $K_1 \cap K_2$ is connected contained in $f^{-1}(c)$ and $K_1 \cup K_2 = K$.

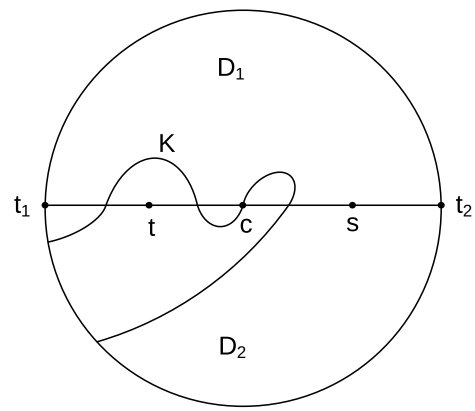

FIGURE 8

We consider the connected components of $f^{-1}(c) \cap K$. We remark that there is exactly one component of $f^{-1}(c) \cap K$, say C, that intersects both D_1 and D_2.

Let now C_1 be a connected component of $f^{-1}(c) \cap K$ different from C and suppose that $C_1 \subset D_1$. We consider the closure of K, $\overline{K}$, in $D_1 \cup D_2$. $\overline{K}$ is connected thus the closure of the component of $\overline{K} \cap D_1$ containing C_1 intersects ∂D_1. Indeed, we consider the set $\overline{K} \cap (D_1 - [t_1, t_2]_\alpha)$ as an open subset of the continuum $\overline{K}$. Let K' be the component of $\overline{K} \cap (D_1 - [t_1, t_2]_\alpha)$ that contains C_1. We recall that if U is an open subset of a continuum and C is a component of U then the frontier of U contains a limit point of C (Theorem 2.16, p.47 of [9]). It follows that the closure of K' intersects $[t_1, t_2]_\alpha$.

Therefore, we have that $f(K') \subset \overline{\alpha^+}$ so $f(K') \subset I_1$ or $f(K') \subset I_2$. We remark that if $f(K') \subset I_1$ then a non trivial interval of I_1 containing c lies in $\overline{\alpha^+}$.

We have a similar conclusion if $f(K') \subset I_2$. Therefore if a connected component of $f^{-1}(c) \cap K$ different from C lying in D_1 intersects the closure of both $f^{-1}(I_1'), f^{-1}(I_2')$ we have that an open interval of I_1 around c lies in $\overline{\alpha^+}$. This is impossible since c is a cross point. We argue similarly for connected components of $f^{-1}(c) \cap K$ contained in D_2.

We conclude that the union of the components of $f^{-1}(c) \cap K$ which lie in D_1, intersect exactly one of $f^{-1}(I_1'), f^{-1}(I_2')$. Clearly the same is true for the union of the components of $f^{-1}(c) \cap K$ contained in D_2. In particular exactly one of the following two holds:

1. If C_1 is a connected component of $f^{-1}(c) \cap K$ different from C lying in D_1 then the component of $D_1 \cap K$ containing C_1 intersects

$f^{-1}(I'_1)$, while if C_1 lies in D_2 the component of $D_2 \cap K$ containing C_1 intersects $f^{-1}(I'_2)$.

2. If C_1 is a connected component of $f^{-1}(c) \cap K$ different from C lying in D_1 then the component of $D_1 \cap K$ containing C_1 intersects $f^{-1}(I'_2)$, while if C_1 lies in D_2 the component of $D_2 \cap K$ containing C_1 intersects $f^{-1}(I'_1)$.

Assume that we are in the first case. Then we define K_1 to be the union of the components of $f^{-1}(c) \cap K$ intersecting D_1 together with $f^{-1}(I'_1)$. We define K_2 to be the union of the components of $f^{-1}(c) \cap K$ intersecting D_2 together with $f^{-1}(I'_2)$. It is clear that K_1, K_2 are closed and that $K_1 \cap K_2 = C$, $K_1 \cup K_2 = K$. Since K is connected, K_1, K_2 are connected too. We define K_1, K_2 similarly in the second case.

We note now that at least one of K_1, K_2 is compact subset of U, thus bounded in U. Since K separates s, t and K_1, K_2 are closed subsets of D, applying Alexander's lemma for the plane we have that at least one of K_1, K_2 separates s, t in U.

It follows that either $f^{-1}(I_1)$ or $f^{-1}(I_2)$ separates s, t. We remark that the same argument holds in the case c is replaced by a cross interval J: We have then that $I = I_1 \cup I_2$ with $I_1 \cap I_2 = J$ and as before either $f^{-1}(I_1)$ or $f^{-1}(I_2)$ separate s, t in U. Now we can continue subdividing intervals along cross points (cross intervals) that lie in $[s, t]_\alpha$ as follows: Let's say that $f^{-1}(I_1)$ separates s, t. We have that there is a connected component of $f^{-1}(I_1)$, say M, that separates them. We note that $f(M)$ is a subinterval of I_1 and if there is a cross point (or cross interval) of $[t, s]_\alpha, h\beta$ contained in its interior, we repeat the previous procedure replacing K by M. If not we have a contradiction. Therefore, either s, t are separated in U by the inverse image of an interval $f(K)$ of $h\beta$ which does not contain in its interior any cross point of $h\beta, \alpha$ lying in $[s, t]_\alpha$, or by iterating this procedure we conclude that the inverse images under f of intervals of $h\beta$ of arbitrarily small diameter separate s from t in U. It is clear that both are impossible, so the theorem is proven.

References

[1] R.D. Anderson, *One-dimensional continuous curves and a homogeneity theorem* Ann. of Math. (2) 68, p.1-16 (1958).

[2] M. Bonk, B. Kleiner, *Quasi-hyperbolic planes in hyperbolic groups* Proc. Amer. Math. Soc. 133, no. 9, p.2491-2494 (2005).

[3] B.H. Bowditch, *Cut points and canonical splittings of hyperbolic groups*, Acta Math. 180, No.2, p.145-186 (1998).

[4] A. Dranishnikov, *Asymptotic topology*, Russian Math.Surveys 55, No 6, p.71-116 (2000).

[5] E.G. Effros, *Transformation groups and C^*-algebras* Ann. of Math. (2) 81, p. 38-55 (1965).
[6] M. Gromov, *Hyperbolic groups*, Essays in group theory (S. M. Gersten, ed.), MSRI Publ. 8, Springer-Verlag, p. 75-263 (1987).
[7] M. Gromov, *Asymptotic invariants of infinite groups*, 'Geometric group theory', (G.Niblo, M.Roller, Eds.), LMS Lecture Notes, vol. 182, Cambridge Univ. Press (1993).
[8] C.L. Hagopian, *No homogeneous tree-like continuum contains an arc* Proc. Amer. Math. Soc. 88, no. 3, p.560-564 (1983).
[9] J.G. Hocking, G.S. Young, *Topology*, Dover (1961).
[10] P. Krupski, *Homogeneity and Cantor manifolds* Proc. Amer. Math. Soc. 109, no. 4, p.1135-1142 (1990).
[11] P. Krupski, H. Patkowska, *Menger curves in Peano continua* Colloq. Math. 70, no. 1, p.79-86 (1996).
[12] M.H.A. Newman, *Elements of the topology of plane sets of points*, Cambridge University Press (1951).
[13] P. Papasoglu, *Quasi-isometry invariance of group splittings*, Annals of Math. 161, pp. 759-830 (2005).
[14] J.R. Prajs, *Homogeneous continua in Euclidean $(n+1)$-space which contain an n-cube are n-manifolds* Trans. Amer. Math. Soc. 318, no. 1, p.143-148 (1990).
[15] J.R. Prajs, *Thirty open problems in the theory of homogeneous contiunua*, in Problems from Topology Proceedings, Ed. by E. Pearl vi+216 pages, Topology Atlas (2003).

E-mail address, Myrto Kallipoliti: `mirtok@math.uoa.gr`
E-mail address, Panos Papasoglu: `panos@math.uoa.gr`

(Myrto Kallipoliti) Mathematics Department, University of Athens, Athens 157 84, Greece

(Panos Papasoglu) Mathematics Department, University of Athens, Athens 157 84, Greece