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Homogeneous Trees are Bilipschitz Equivalent

P. PAPASOGLU
Mathematics Institute, University of Warwick, Coventry CV4-7AL, England
e-mail address: pp@maths.warwick.ac.uk

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Abstract. We prove that any two locally finite homogeneous trees with valency greater than 3 are bilipschitz equivalent. This implies that the quotient $h^1(G)/h^k(G)$, where $h^k(G)$ is the kth L_2-Betti number of G, is not a quasi-isometry invariant.

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0. Introduction

In geometric group theory one studies groups using the geometric properties of spaces on which these groups act. It is a classical result that if a finitely generated group acts properly, by isometries, on a proper geodesic metric space, with compact quotient, then the group is quasi-isometric to the space. Here the metric on the group is the word metric corresponding to a finite generating set. In this setting one tries on the one hand to extract algebraic information (e.g. subgroup theorems, solvability of the word problem, etc.) about the group from the large-scale geometry of the space on which the group acts and on the other hand one tries to understand the large-scale geometric invariants of the group as these are interesting for their own sake.

Of course when two groups are quasi-isometric one cannot distinguish them by their large-scale geometry but one can conclude that their large-scale geometric properties (isoperimetric inequalities, etc.) are the same. So one would like to know when two groups are quasi-isometric.

It is not known if quasi-isometry is preserved under free products, i.e. it is not known if, when G_1 is quasi-isometric to G_2 and H_1 is quasi-isometric to H_2, $G_1 * H_1$ is quasi-isometric to $G_2 * H_2$ (we assume here that all the groups are infinite). This would follow easily if one knew that any two quasi-isometric groups were in fact bilipschitz equivalent.

In this paper we prove that any two homogeneous trees are bilipschitz equivalent. This implies that any two finitely generated non-abelian free groups are bilipschitz equivalent. This question has been raised by Gromov in [Gr] in connection with L_2-cohomology. It is known that vanishing of the L_2-Betti numbers is a quasi-isometry invariant. As the L_2-Betti numbers behave multiplicatively with respect to subgroups of finite index it is natural to ask if the quotient of two L_2-Betti numbers is invariant under quasi-isometry. Gromov in [Gr] observes that a negative answer would follow if one knew that any two non-abelian free groups were bilipschitz equivalent.

1. Notation

In this paper by a tree we mean what is usually taken to be the set of vertices of the tree. So we have the following:

DEFINITION. A *tree* is a metric space X which satisfies the following two conditions:

(1) $\forall x, y \in X$, there is a finite sequence $x_1, \ldots, x_n$ such that $x_1 = x$, $x_n = y$, $d(x_i, x_{i+1}) = 1$ and $\Sigma_{i=1}^{n-1} d(x_i, x_{i+1}) = d(x, y)$.
(2) If $x_1, \ldots, x_n$ is a finite sequence in X such that $x_1 = x_n$ and $d(x_i, x_{i+1}) = 1$, $\forall i$, then $x_i = x_{i+2}$ for some i, $1 \leq i < n$.

In what follows X will be a tree. We think of X as a 'rooted tree', that is we pick a point $e \in X$ which we call the root of the tree.

A finite sequence $x_1, \ldots, x_n$ in X with the properties $x_i \neq x_{i+1}$ and $d(x_i, X_{i+1}) = 1$, $\forall i$, will be called a path from x_1 to x_n. As it is determined by x_1, x_n it will be denoted by $[x_1, x_n]$.

For $x \in X$ we define the star of x, $\mathrm{St}(x) = \{y \in X \mid d(x, y) = 1\}$.

We define: $|x| = d(e, x)$ for $x \in X$.

We define the n-ball and the n-sphere in X:

$$B_n = \{x \in X \mid |x| \leq n\},\ S_n = \{x \in X \mid |x| = n\}.$$

We define $p : X - \{e\} \to X$ by $p(x) = \mathrm{St}(x) \cap [e, x]$.

We define $N(x) = p^{-1}px$.

We write $x \wedge y = z$ if $[e, x] \cap [e, y] = [e, z]$.

Note that $d(x, y) = d(x, x \wedge y) + d(x \wedge y, y)$.

The homogeneous tree of valency k, denoted by X_k, is the tree for which $\mathrm{Card}(\mathrm{St}(x)) = k$ for every $x \in X_k$.

Two metric spaces M, N are bilipschitz equivalent if there is a one-to-one and onto map $f : M \to N$ and a constant $C > 0$ such that:

$$\forall x, y \in M,\ \frac{1}{C} d(x, y) \leq d(f(x), f(y)) \leq C d(x, y).$$

2. Homogeneous Trees are Bilipschitz Equivalent

We will prove that the homogeneous tree X_k is bilipschitz equivalent to the homogeneous tree X_{k+1}, for $k \geq 4$. It is obvious that bilipschitz equivalence is an equivalence relation so it follows that any two homogeneous trees X_k, X_n are bilipschitz equivalent, where $k, n \geq 4$.

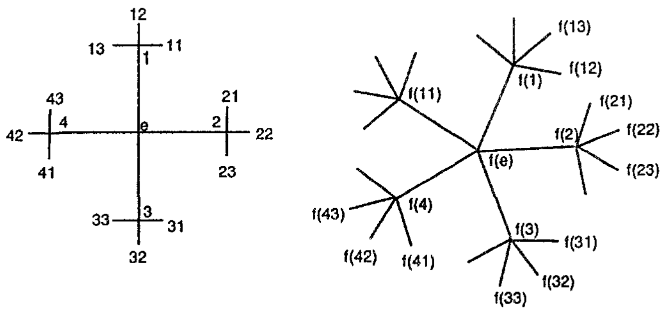

Fig. 1.

The idea of the proof is to define a one-to-one map which will take the n-sphere of X_k to the n-sphere of X_{k+1}. This cannot be done because the n-sphere of X_{k+1} has more points than the n-sphere of X_k so we define instead the map from $S_n \cup S_{n+1}$ to the sphere of X_{k+1}.

It is a good idea for the reader to draw pictures of the trees and try to understand the map geometrically while reading the formal proof.

THEOREM. *The trees X_k, X_{k+1} are bilipschitz equivalent for $k \geq 4$.*

Proof. We will denote by S_n, B_n the n-sphere and n-ball of X_k. Let e, e' be the roots of X_k, X_{k+1}. We will define inductively a one-to-one and onto map $f : X_k \to X_{k+1}$. We define: $f(e) = e'$ and since $\mathrm{Card}(p^{-1}(e)) < \mathrm{Card}(p^{-1}(e'))$ we extend $f : p^{-1}(e) \to p^{-1}(e')$ so that f is one to one.

We define f on S_2 as follows: We pick some $x_1 \in S_2$ and define $f(x_1) = p^{-1}(e') - f(S_1)$. We extend f on the rest of S_2 in an one to one way so that for every $x \in S_2$, $x \neq x_1$, $pf(x) = f(px)$. This is clearly possible. In Figure 1 we show one way to define $f : X_4 \to X_5$ on $S_2 \subset X_4$.

We define now f inductively so that for every $x \in S_n$, $n \geq 2$ either (1) or (2) holds:

(1) $pf(x) = f(px)$ and $\mathrm{Card}(f(N(x)) \cap N(f(x))$ is either $k-1$ or $k-2$,
(2) $pf(x) = pf(px) = f(ppx)$.

We remark that by our definition of f on S_2, for every $x \neq x_1$ (1) holds while for x_1 (2) holds.

Assume that we have defined f on the ball or radius n so that the above condition is satisfied for all $x \in B_n$. We will extend f on B_{n+1}, i.e. we will define f on S_{n+1} so that the above condition is satisfied.

$S_{n+1} = Y_1 \cup Y_2$ where

$$Y_1 = \{x \in S_{n+1} : px \text{ satisfies } (1)\}$$
$$Y_2 = \{x \in S_{n+1} : px \text{ satisfies } (2)\}$$

We extend f on Y_2 as follows: Y_2 is a disjoint union of sets $N(x)$, $x \in Y_2$. We define f from $N(x)$ to $p^{-1}f(px)$ in a one-to-one way. This is clearly possible as

$\mathrm{Card}(N(x)) = k - 1$ and $\mathrm{Card}(p^{-1}f(px)) = k$. It follows that every $x \in Y_2$ will satisfy condition (1). Y_1 is a disjoint union of sets of the form:

$$V(a) = \{x \in Y_1 : ppx = a\}.$$

We will define f on each of these sets.

$V(a)$ is a disjoint union of sets $N(x)$, $x \in V(a)$.

We define f on $V(a)$:

We observe that $pV(a)$ is equal to the set of vertices in $p^{-1}a$ which satisfy (1). If x satisfies (1) then $y \in N(x)$ satisfies (1) if and only if $f(y) \in f(N(x))$. Hence the set of vertices in $p^{-1}a$ satisfying (1) has cardinality $k-1$ or $k-2$. We conclude that $\mathrm{Card}(pV(a)) = k - 1$ or $\mathrm{Card}(pV(a)) = k - 2$.

Assume $\mathrm{Card}(pV(a)) = k - 2$.

Let $\{y_1, y_2\} = p^{-1}f(a) - f(pV(a))$.

Let $x_1, x_2 \in V(a)$ be such that $px_1 \neq px_2$. Such x_1, x_2 exist by our hypothesis that $k \geq 4$.

We define $f(x_1) = y_1$, $f(x_2) = y_2$. We extend f on $V(a) - \{x_1, x_2\}$ by mapping each of the sets $N(x) - \{x_1, x_2\}$ to $p^{-1}f(px)$ in a one-to-one way. This is clearly possible as $\mathrm{Card}(N(x) - \{x_1, x_2\}) = k-1$ or $\mathrm{Card}(N(x) - \{x_1, x_2\}) = k - 2$ (note that at most one of x_1, x_2 is in $N(x)$) and $\mathrm{Card}(p^{-1}f(px)) = k$.

We define f in the same way if $\mathrm{Card}(pV(a)) = k - 1$.

It is clear by this definition that every $x \in Y_1$ will satisfy either condition (1) or (2). This completes the definition of f. It is clear that f is one-to-one by definition. One can verify easily that

$$f(B_n) \supset \{x \in X_{k+1} | d(x, f(B_{n-2})) \leq 1\}$$

which implies that f is onto. It is quite easy to see that f is a bilipschitz equivalence. In what follows we give a formal proof of this.

LEMMA 1. *$d(f(x), f(y)) \leq 2d(x, y)$ for every $x, y \in X_k$.*

Proof. We will prove this by induction on $d(x, y)$. Suppose that $d(x, y) = 1$. We can assume without restriction of generality that $x = py$. Then by the definition of f either $pf(y) = f(py)$ or $pf(y) = pf(py)$. In the first case $d(f(x), f(y)) = 1$ and in the second $d(f(x), f(y)) = 2$ so the lemma is true when $d(x, y) = 1$.

Suppose now that the lemma is true when $d(x, y) \leq n$. Let x, y be such that $d(x, y) = n+1$ and let z be such that $d(x, z) = 1$ and $d(x, y) = d(x, z) + d(z, y)$. We have then

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f(z)) + d(f(z), f(y)) \leq 2d(x, z) + 2d(z, y) \\ &= 2d(x, y). \end{aligned}$$

LEMMA 2. *If $y \in [e, x]$ then $d(f(x), f(y)) > \frac{1}{3}d(x, y)$.*

Proof. If $d(x, y) = 1$ the assertion is obviously true as f is one-to-one. We assume therefore that $d(x, y) > 1$.

Let x_1, x_2, x_3 be consecutive vertices in $[y, x]$ (i.e. $|x_1| = |x_2| - 1 = |x_3| - 2$). By the definition of f, $|f(x_2)| = |f(x_1)|$ or $|f(x_2)| = |f(x_1)| + 1$. If $|f(x_1)| = |f(x_2)|$ then by the definition of f, $|f(x_3)| = |f(x_2)| + 1$.

It follows that

$$|f(x)| > |f(y)| + \frac{d(x, y) - 1}{2} \Rightarrow d(f(x), f(y)) > \tfrac{1}{3}d(x, y).$$

LEMMA 3. $d(f(x), f(y)) > \frac{1}{3}d(x, y)$.

Proof. Let $x \wedge y = z$. We assume that $z \neq x, y$ otherwise we are done by Lemma 2. We claim that $f(x) \wedge f(y) = f(z)$ or $f(x) \wedge f(y) = pf(z)$. Indeed if $x_1 = [z, x] \cap p^{-1}(z)$ and $y_1 = [z, y] \cap p^{-1}(z)$ then one sees by the definition of f that $f(x_1) \wedge f(y_1) = f(x) \wedge f(y)$ and $f(x_1) \wedge f(y_1) = f(z)$ or $f(x_1) \wedge f(y_1) = pf(z)$. As $z, pz \in [e, x]$ and $d(x, z) < d(x, pz)$, applying Lemma 2 we have

$$d(f(x), f(x) \wedge f(y)) \geq d(f(x), f(z)) \geq \tfrac{1}{3}d(x, z).$$

Similarly

$$d(f(y), f(x) \wedge f(y)) \geq \tfrac{1}{3}d(y, z).$$

Therefore

$$d(f(x), f(y)) \geq \tfrac{1}{3}d(x, y).$$

From Lemmas 1 and 3 it follows that f is a bilipschitz equivalence.

Remark. X_3 is also bilipschitz equivalent to X_4. One has to define the map f a bit more carefully in this case. In fact if $k > 3$ and X, Y are trees such that for every $x \in X \cup Y$ one has $3 \leq \mathrm{Card}(\mathrm{St}(x)) \leq k$ then X, Y are bilipschitz equivalent.

COROLLARY 1. *Any two free groups $\mathbb{F}_r, \mathbb{F}_n$ $(r, n \geq 2)$ are bilipschitz equivalent.*

COROLLARY 2. *The quotients $h^1(G)/h^k(G)$ of L_2-Betti numbers are not invariant under quasi-isometry.*

Proof. Let $\mathbb{F}_r, \mathbb{F}_n$ $(r, n \geq 2)$ be free groups and let H be any group. Then $\mathbb{F}_r * H$ is quasi-isometric to $\mathbb{F}_n * H$.

Indeed, any element g of $\mathbb{F}_r * H$ has a unique reduced normal form $g = a_1c_1 \ldots a_m c_m$ where $a_i \in \mathbb{F}_r$, $c_i \in H$ $(1 \leq i \leq m)$ and $a_i \neq e$ for $i > 1$, $c_i \neq e$ for $i < m$.

If $f : \mathbb{F}_r \to \mathbb{F}_n$ is a bilipschitz equivalence we define $h : \mathbb{F}_k * H \to \mathbb{F}_n * H$ by

$$h(g) = f(a_1)c_1 \ldots f(a_m)c_m.$$

It is clear that h is a bilipschitz equivalence hence, in particular, a quasi-isometry. If we pick H now so that $h^k(H) \neq 0$ we have $h^1(\mathbb{F}_r * H) = k + h_1(H)$, $h^1(\mathbb{F}_n * H) = n + h_1(H)$ while $h^k(\mathbb{F}_r * H) = h^k(\mathbb{F}_r * H) = h^k(H)$. So $h^1(\mathbb{F}_r * H)/h^k(\mathbb{F}_r * H) \neq h^1(\mathbb{F}_n * H)/h^k(\mathbb{F}_n * H)$.

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Reference

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