A general formula in Additive Number Theory

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Abstract

Classical Additive Number Theory in \( \mathbb{Z} \) investigates the existence of a finite integer \( \theta \) such that for a given infinite sequence of increasing non-negative integers \( a_\nu = f(\nu), \nu = 1, 2, \ldots \) the Diophantine equation
\[
f(x_1) + f(x_2) + \ldots + f(x_\theta) = n
\]
has solutions \( x_i \geq 1 \) for any \( n \geq 0 \), and determines, if possible, their number, asymptotically or otherwise (Goldbach, Waring, Hilbert, Hardy-Littlewood, Vinogradov, Erdös, \ldots). Our approach is different.

We denote by \( A(n, N, \theta) = A(n, \theta) \) the number of solutions in integers \( x_i \geq 0 \) of the Diophantine system:
\[
a_1 x_1 + a_2 x_2 + \ldots + a_N x_N = n \quad (1)
x_1 + x_2 + \ldots + x_N = \theta \quad (1')
\]

\( A(n, \theta) \) expresses in how many ways \( n \) is a sum of \( \theta \) integers taken from the set \( \{a_1, \ldots, a_N\} \).

The generating function for the \( A(n, \theta) \), \( n = 0, 1, \ldots, \theta = 0, 1, \ldots \), is:
\[
\sum_{n=0}^{\infty} A(n, \theta) x^n y^\theta = \prod_{\nu=1}^{N} \frac{1}{1 - x^{a_\nu} y} ,
\]
as seen by expanding all the \( (1 - x^{a_\nu} y)^{-1} \) formally in power series and, after multiplication, collecting terms with equal exponents in \( x \) and \( y \), respectively.

Considering \( x \) and \( y \) as complex variables we have by Cauchy’s theorem for the coefficients of power series of several complex variables:
\[
A(n, \theta) = \frac{1}{(2\pi i)^2} \oint_{c_1} \oint_{c_2} \frac{1}{x^{n+1} y^{\theta+1}} \prod_{\nu=1}^{N} \frac{1}{1 - x^{a_\nu} y} \, dx \, dy ,
\]
where the integrals are taken over the circles \( |x| = c_1 < 1, |y| = c_2 < 1 \), respectively.

The partial fraction expansion with regard to \( y \) of \( \prod_{\nu=1}^{N} (1 - x^{a_\nu} y)^{-1} \) gives
\[
\prod_{\nu=1}^{N} \frac{1}{1 - x^{a_\nu} y} = \sum_{\nu=1}^{N} \frac{1}{y^{N-1} L'(x^{a_\nu})(1 - x^{a_\nu} y)} ,
\]

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where $L'(t)$ is the derivative of $L(t) = \prod_{v=1}^{N}(t - x^{a_v})$.

Inserting in (2) and expanding $(1 - x^{a_v}y)^{-1}$ in power series of $y$ within the circle $c_2$ we get successively:

$$A(n, \theta) = \frac{1}{(2\pi i)^2} \int_{c_1} \int_{c_2} \frac{1}{x^{n+1}y^{\theta+1}} \left\{ \sum_{\nu=1}^{N} \frac{1}{y^{N-1} L'(x^{a_\nu})(1 - x^{a_\nu}y)} \right\} dx dy$$

$$= \frac{1}{2\pi i} \int_{c_1} \frac{1}{x^{n+1}} \left\{ \frac{1}{2\pi i} \int_{c_2} \frac{1}{y^{\theta+N} \sum_{\nu=1}^{N} L'(x^{a_\nu})(1 - x^{a_\nu}y)} dy \right\} dx$$

$$= \frac{1}{2\pi i} \int_{c_1} \frac{1}{x^{n+1}} \left\{ \sum_{\nu=1}^{N} \frac{1}{2\pi i} \int_{c_2} \frac{1}{y^{\theta+N} L'(x^{a_\nu})(1 + x^{a_\nu}y + \ldots)} dy \right\} dx$$

Integrating over $|y| = c_2$ we obtain by Cauchy’s theorem:

$$A(n, \theta) = \frac{1}{2\pi i} \int_{c_1} \frac{1}{x^{n+1}} \left\{ \sum_{\nu=1}^{N} \frac{x^{a_\nu(\theta+N-1)}}{L'(x^{a_\nu})} \right\} dx$$

$$= \sum_{\nu=1}^{N} \frac{1}{2\pi i} \int_{c_1} \frac{1}{x^{n+1-a_\nu(\theta+N-1)} L'(x^{a_\nu})} dx. \quad (3)$$

In order to expand $1/L'(x^{a_\nu})$ in a power series of $x$ we transform $L'(x^{a_\nu})$ as follows:

$$L'(x^{a_\nu}) = (x^{a_\nu} - x^a) \ldots (x^{a_\nu} - x^{a_{\nu-1}})(x^{a_\nu} - x^{a_{\nu+1}}) \ldots (x^{a_\nu} - x^{a_N})$$

$$= x^{a_1 + \ldots + a_{\nu-1}}(x^{a_{\nu-1} - a_1} - 1) \ldots (x^{a_{\nu-1} - a_{\nu-2}} - 1)x^{a_{\nu-1}}(N-\nu)(1 - x^{a_{\nu+1} - a_{\nu}}) \ldots (1 - x^{a_N - a_{\nu}})$$

$$= (-1)^{\nu-1} x^{a_1 + \ldots + a_{\nu-1} + a_{\nu-1}(N-\nu)}(1 - x^{a_{\nu-1} - a_1}) \ldots (1 - x^{a_{\nu-1} - a_{\nu-2}})(1 - x^{a_{\nu+1} - a_{\nu}}) \ldots (1 - x^{a_N - a_{\nu}}).$$

Inserting in (3) for $L'(x^{a_\nu})$ their above expressions and after calculations in the exponents, we obtain:

$$A(n, \theta) = \sum_{\nu=1}^{N} \frac{(-1)^{\nu-1}}{2\pi i} \int_{c_1} \frac{1}{x^{n+1-a_\nu(\theta+N-1) + a_\nu}} P_\nu(x) dx \quad (4)$$

where

$$P_\nu(x) = \frac{1}{(1 - x^{a_{\nu-1} - a_1}) \ldots (1 - x^{a_{\nu-1} - a_{\nu-2}})(1 - x^{a_{\nu+1} - a_{\nu}}) \ldots (1 - x^{a_N - a_{\nu}})}.$$  

Since all exponents $a_\nu - a_\mu$ are positive, we can expand for $|x| \leq c_1$ the factors $P_\nu(x)$ in power series of $x$:

$$P_\nu(x) = \sum_{\lambda=0}^{\infty} B_\nu(\lambda)x^\lambda$$

where $B_\nu(\lambda)$ are respectively the number of non-negative integer solutions of the linear Diophantine equations

$$(a_\nu - a_1)x_1 + \ldots + (a_\nu - a_{\nu-1})x_{\nu-1} + (a_{\nu+1} - a_\nu)x_\nu + \ldots + (a_N - a_{N-1})x_{N-1} = \lambda .$$
Substituting in (4) the $P_\nu(x)$ by their respective power series and using again Caushy’s theorem we finally arrive at:

$$A(n, \theta) = \sum_{\nu=1}^{N} (-1)^{\nu-1}B_\nu(s_\nu) ,$$

(5)

with $s_\nu = n + \sum_{i=1}^{\nu} a_i - (\nu + \theta) a_\nu$ and $B_\nu(s_\nu)$ respectively, the number of solutions of each of the following linear Diophantine equations

$$(a_\nu - a_1)x_1 + \ldots + (a_\nu - a_{\nu-1})x_{\nu-1} + (a_{\nu+1} - a_\nu)x_\nu + \ldots + (a_N - a_{\nu-1})x_{N-1} = s_\nu$$

(6)

$\nu = 1, \ldots, N$.

This formula reduces the investigation of the number of solutions of the initial system to that of the number of solutions of $N$ linear Diophantine equations involving the difference sets (positive) $\{a_\nu - a_\mu\}$ and the numbers $s_\nu$.

Geometrically speaking this means that the number of $Gitterpunkte$ in the intersection of the two $N$-dimensional planes (1) and (1’) in the positive quadrant $x_i \geq 0$ is equal to the alternate sum of the number of $Gitterpunkte$ of the $(N-1)$-dimensional planes (6) in the same quadrant.

As standard examples, theorems and conjectures we may cite $a_\nu = (\nu - 1)^2$, $\theta = 4$ (Lagrange), $a_\nu = (\nu - 1)^k$ (Waring), $a_\nu = \nu$-th prime, $\theta = 2$, $n$ even (Goldbach), $a_\nu = \nu^p$, $\theta = 2$, $n = n_1^p$, $p \geq 3$ (Fermat).

Obviously in order to attack the problem for a given sequence we have to take into account that $N$ is linked to $n$ by a function $N(n)$ (ex.g. for Lagrange $N(n) = \lfloor n^{\frac{1}{2}} \rfloor$). This complicates the matter but still an ad hoc suggestion would be to approximate the $B_\nu(s_\nu)$ as follows

$$B_\nu(s_\nu) \sim \frac{s_\nu^{N-2}}{(N-2)! (a_\nu - a_1) \cdots (a_N - a_\nu)} ,$$

which is valid for $n \to \infty$ but fixed $N$ (Polya-Szegö, Aufgaben und Lehrsätze aus der Analysis I: loosing no generality the differences $a_\nu - a_\mu$ can be assumed free of common divisors > 1).

Summing over $\nu$ (we write $N$ for short of $N(n)$) and reverting again to the polynomials $L(t)$, written now as $L_N(t)$, we obtain from (5)

$$\frac{1}{(N-2)!} \sum_{\nu=1}^{N} \frac{s_\nu^{N-2}}{L_N(a_\nu)} ,$$

(7)

as a plausible heuristic estimate of $A(n, \theta)$ for $n \to \infty$.

The behaviour of the expressions involving $n$ and $\theta$: 

$$\frac{n + \sum_{i=1}^{\nu} a_i - (\nu + \theta)a_\nu}{|a_\nu - a_\mu|} \quad \text{The cutting points of the planes (6) with the coordinate axes},$$

$$\frac{n + \sum_{i=1}^{\nu} a_i - (\nu + \theta)a_\nu}{\sqrt{\sum_{i=1}^{N} (a_\nu - a_\mu)^2}} \quad \text{The distances of the planes (6) from the origin},$$
would play, we believe, a decisive role in any such attempt.

As to its form the sum (7) bears a striking resemblance to the sums

\[ \sum_{\nu=1}^{N} \frac{a'_{\nu}}{L_{N}(a_{\nu})} \]

encountered in Lagrange interpolation with \( a_{\nu} \) replaced by \( s_{\nu} \) in the numerator. As known these expressions considered as functions of the exponent \( t \) are equal to:

\[
\begin{cases} 
(\frac{-1}{N-1}) \sum_{i=-t-1}^{a_{1} \cdots a_{N}} \frac{1}{a_{1}^{i_{1}} \cdots a_{N}^{i_{N}}} & \text{for } t \leq -1 \\
0 & \text{for } 0 \leq t \leq N - 2 \\
1 & \text{for } t = N - 1 \\
\sum_{i=t-N+1}^{a_{1}^{i_{1}} \cdots a_{N}^{i_{N}}} & \text{for } N \leq t 
\end{cases}
\]

Above facts may prove, eventually, useful in further developments.