Stochastic Forcing of Perturbation Variance in Unbounded Shear and Deformation Flows

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ABSTRACT

The problem of growth of small perturbations in fluid flow and the related problem of maintenance of perturbation variance has traditionally been studied by appeal to exponential modal instability of the flow. In the event that a flow supports an exponentially growing modal solution, the initially unbounded growth of the mode is taken as more or less compelling evidence for eventual flow breakdown. However, atmospheric flows are characterized by large thermally forced background rates of strain and are subject to perturbations that are not infinitesimal in amplitude. Under these circumstances there is an alternative mechanism for growth and maintenance of perturbation variance: amplification in a straining flow of stochastically forced perturbations in the absence of exponential instabilities. From this viewpoint the flow is regarded as a driven amplifier rather than as an unstable oscillator. We explore this mechanism using as examples unbounded constant shear and pure deformation flow for which closed-form solutions are available and neither of which supports a nonsingular mode. With diffusive dissipation we find that amplification of isotropic band-limited stochastic driving is unbounded for the case of pure deformation and bounded by a threefold increase at large shear for the case of a linear velocity profile. A phenomenological model of the contribution of linear and nonlinear damped modes to the maintenance of variance results in variance levels increasing linearly with shear. We conclude that amplification of stochastic forcing in a straining field can maintain a variance field substantially more energetic than that resulting from the same forcing in the absence of a background straining flow. Our results further indicate that existence of linear and nonlinear damped modes is important in maintaining high levels of variance by the mechanism of stochastic excitation.

1. Introduction

Production and maintenance of variance at small amplitude in both barotropic and baroclinic flows is commonly explained by appeal to exponential temporal modal instability of the undisturbed background flow. Initially infinitesimal disturbances are supposed by this explanation to grow exponentially, eventually reaching finite amplitude. This explanation assumes that the perturbation field is of such small amplitude that asymptotic dominance by the unstable mode is achieved. This, often implicit, assumption justifies ignoring the generally greater instantaneous growth rates of a large subset of "optimal perturbations" (Farrell 1988, 1989, 1990; Lacarra and Talagrand 1988). This assumption is questionable for many physical systems of interest and must fail for flows such as unbounded constant shear and pure deformation that do not support exponentially growing solutions. An alternative viewpoint is that the background flow is not significantly unstable in the sense of supporting exponentially growing temporal modes with rapid growth rates but rather a large strain rate allows the flow to serve as an amplifier of the inevitable field of perturbations ubiquitous in geophysical flows.

Consider a barotropic and/or baroclinic jet maintained by thermal or other strong internal forcing. In such a flow the forcing is assumed to maintain a background rate of strain field that is only modified by the perturbations. Certainly the initial growth of disturbances in such a flow can be studied using linear theory and, if amplitudes remain sufficiently small, the maintenance of the perturbation field can also be studied using linear theory. In this work we examine the circumstances under which a forced/dissipative system with strong rates of strain but without exponential instability behaves as an amplifier of perturbations. We use as model problems linearized stochastically forced barotropic constant shear and pure deformation flows with Ekman, diffusive, and high-order diffusive dissipation. To further elucidate the physics, we investigate the effect of nonisotropic forcing and of occlusion, making use of phenomenological models of these processes. An important advantage of these model prob-

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lems is the existence of complete, closed-form solutions (Kelvin 1887; Yamagata 1976; Farrell 1982; Tung 1983; Boyd 1983; Shepherd 1985; Craik and Criminale 1986). Making use of these solutions, the two dimensional stochastic excitation problem can be reduced to a quadrature from which the basic physics can be more easily explored.

While we have chosen the barotropic model because of its relatively simple interpretation, we note with Charney (1971) that the baroclinic quasigeostrophic equations are essentially isomorphic to the barotropic equations except for boundary terms and the identification of variables.

2. The plane wave solution

Consider the linearized barotropic vorticity equation governing the evolution of disturbances to a flow $U = -\Psi_y$, $V = \Psi_x$, where U is the background velocity in the x direction and V the background velocity in the y direction:

$$\nabla^2\psi_t + J(\Psi, \nabla^2\psi) = \nu\nabla^2\nabla^2\psi, \qquad (2.1)$$

where $\nabla^2 \equiv \partial_x^2 + \partial_y^2$, $J(\phi_1, \phi_2) \equiv \partial_x\phi_1\partial_y\phi_2 - \partial_y\phi_1\partial_x\phi_2$, and the x, y velocity is related to the perturbation streamfunction, ψ, by $(u, v) = (-\psi_y, \psi_x)$.

When the coefficient of viscosity, ν, assumes values appropriate to molecular diffusion, (2.1) is the two-dimensional linearized Navier–Stokes equation. In atmospheric applications a $\nu\nabla^2\nabla^2\psi$ viscous force can be assumed to model the effects of eddy mixing, and ν is then an eddy viscosity. While we will in the sequel concentrate on this form of dissipation, it is of interest to consider also the effects of more general dissipations of the form $(-1)^n\nabla^{2n}\psi$, which we will refer as dissipations of order n. (The ∇^{2n} operator is to be interpreted as the rotationally invariant operator ∇^2 compounded n times so that, in Fourier space, waves with the same total wavenumber will be subjected to equal dissipation.) The case $n = 1$ models Ekman dissipation, while values of $n > 2$ are often employed in numerical simulations. We will explicitly treat the $n = 2$ case below, and show that the analysis carries over to any dissipation order n so that we need only quote and selectively derive results for $n > 2$.

Consider a wave perturbation to an unbounded constant shear or deformation flow governed by (2.1). The constant background shear flow has streamfunction $\Psi = -\alpha y^2/2$, and kinematics require that perturbation wavenumbers vary with time as $k(t) = k_0$, $l(t) = l_0 - \alpha k_0 t$, where (k_0, l_0) is the initial wavenumber. The pure deformation flow corresponds to $\Psi = \alpha xy$; $k(t) = k_0 e^{\alpha t}$, $l(t) = l_0 e^{-\alpha t}$. For these cases it can be verified that (2.1) has the plane wave solution

$$\psi(x, y, t) = C_{k_0,l_0}(t)\cos(k(t)x + l(t)y + \phi_{k_0,l_0}), \qquad (2.2)$$

where

$$C_{k_0,l_0}(t) \equiv A_{k_0,l_0}\frac{k_0^2 + l_0^2}{k(t)^2 + l(t)^2} \times \exp\left\{-\nu\int_0^t (k(\tau)^2 + l(\tau)^2)d\tau\right\}. \qquad (2.3)$$

Here all quantities are real, ϕ_{k_0,l_0} is the initial phase, and A_{k_0,l_0} is the initial wave amplitude. The exponential term describes the effect of dissipation, which is proportional to the time-mean-square wavenumber, which in turn depends on the shear.

Each of these nonseparable waves, which together constitute an orthogonal basis, is, in isolation, a nonlinear solution of the barotropic vorticity equation (Kelvin 1887). This happy circumstance unfortunately does not carry over to a superposition of these solutions, and as a consequence, our analysis will be restricted to the linear problem for which the solution with any initial condition can be produced by Fourier synthesis of these plane waves, taking for definiteness $k_0 > 0$ and l_0 of either sign.

The energy density of a plane wave is defined as the mean energy per unit mass per unit area. For a single plane wave it is given by

$$\bar{E}_{k_0,l_0}(t) = \frac{\overline{u_{k_0,l_0}^2 + v_{k_0,l_0}^2}}{2} = \frac{B_{k_0,l_0}^2(t)}{2}\lim_{L\to\infty}\frac{1}{L^2} \times \int_0^L\int_0^L \cos^2(k(t)x + l(t)y)dxdy = \frac{B_{k_0,l_0}^2(t)}{4}, \qquad (2.4)$$

where

$$B_{k_0,l_0}^2(t) \equiv [k(t)^2 + l(t)^2]C_{k_0,l_0}^2(t). \qquad (2.5)$$

It is useful at this point to introduce the vector wavenumbers $\mathbf{k} = (k(t), l(t))$, $\mathbf{k}_0 = (k_0, l_0)$. The total streamfunction can then be written as

$$\psi = \sum_{\mathbf{k}_0} C_{\mathbf{k}_0}(t)\cos(\mathbf{k}\cdot\mathbf{x} + \phi_{\mathbf{k}_0}). \qquad (2.6)$$

Taking account of contributions from all plane waves of the form (2.2), the total energy density is

$$\bar{E}(t) = \frac{1}{2}\sum_{\mathbf{k}_0,\mathbf{k}_0'} C_{\mathbf{k}_0}(t)C_{\mathbf{k}_0'}(t)\mathbf{k}_0\cdot\mathbf{k}_0'\lim_{L\to\infty}\frac{1}{L^2} \times \int_0^L\int_0^L \cos(\mathbf{k}\cdot\mathbf{x} + \phi_{\mathbf{k}_0}) \times \cos(\mathbf{k}'\cdot\mathbf{x} + \phi_{\mathbf{k}_0'})dxdy. \qquad (2.7)$$

The only contribution to this integral arises when $\mathbf{k}_0' = \mathbf{k}_0$, and thus,

$$\bar{E}(t) = \frac{1}{4} \sum_{k_0, l_0} A^2_{k_0, l_0} \frac{(k_0^2 + l_0^2)^2}{k(t)^2 + l(t)^2} \times \exp\left\{-2\nu \int_0^t (k(\tau)^2 + l(\tau)^2) d\tau\right\}. \quad (2.8)$$

3. Stochastic dynamics

The stochastic forcing is assumed uniform and uncorrelated in space. Each Fourier component is equally excited in the chosen measure by this forcing. If the Fourier components are forced at time zero, the ensemble average of the energy density at a later time, t, is the sum of the space averages of the Fourier components at t:

$$\langle E(t) \rangle = \sum_{k_0, l_0} \bar{E}_{k_0, l_0}(t) = \bar{E}(t), \quad (3.1)$$

where the bracket indicates the ensemble average, and (2.8) has been used.

We also assume that the forcing is stationary and uncorrelated in time. Up to now we have calculated the average energy density at time t after the excitation of the Fourier components. The ensemble average of the energy density when the components are excited discretely in time is the sum over the excitations of the average energy density:

$$\langle E_{k_0, l_0} \rangle = \sum_{t_i} \bar{E}_{k_0, l_0}(t_i). \quad (3.2)$$

Hence, the ensemble average over all components of the energy density is

$$\langle E \rangle = \sum_{t_i} \bar{E}(t_i) = \sum_{t_i} \sum_{k_0, l_0} \bar{E}_{k_0, l_0}(t_i). \quad (3.3)$$

It should be noted that we can force the flow in a variety of ways. We can, for instance, impart to each component of the system on average either unit energy or unit enstrophy per unit time and unit area. In the sequel we will be mainly concerned with forcing that imparts unit energy per unit time and unit area.

4. Stochastic dynamics of unbounded constant shear flows

a. The case of diffusive dissipation

Consider now the unbounded constant shear flow for which

$$k(t) = k_0, \quad l(t) = l_0 - \alpha k_0 t. \quad (4.1)$$

The component (k_0, l_0) when excited with energy density $\dot{\epsilon}\delta t$ has amplitude:

$$A_{k_0, l_0} = 2\left(\frac{\dot{\epsilon}\delta t}{k_0^2 + l_0^2}\right)^{1/2}. \quad (4.2)$$

The energy density at a later time is given by (2.8):

$$\bar{E}_{k_0, l_0}(t) = \dot{\epsilon}\delta t \frac{k_0^2 + l_0^2}{k_0^2 + (l_0 - \alpha k_0 t)^2} \times \exp\left\{-2\nu \int_0^t (k_0^2 + (l_0 - \alpha k_0 \tau)^2) d\tau\right\}. \quad (4.3)$$

Note that waves excited with $l_0 > 0$ grow for $t \leqslant l_0/\alpha k_0$, as the crosswind wavenumber $l(t)$ is diminished to zero. After this time the wave decays. The sector $l_0 > 0$ will be referred to as the favorable sector. The energy of waves excited with $l_0 < 0$ decays for all time, and this sector will be referred to as the unfavorable sector. If it were not for the dependence of dissipation on the mean-square wavenumber, the ensemble average of the energy density would be constant and independent of the shear. (We will see an explicit example of such a case for Ekman damping in the next section.) In the presence of shear, the mean-square wavenumber, and with it the dissipation, is reduced during the period of the energy growth, creating the possibility for an ensemble-mean energy density higher than in the case of a flow with no shear.

Let the initial total wavenumber be denoted by $r_0 = (k_0^2 + l_0^2)^{1/2}$, where $k_0 = r_0 \cos\theta$ and $l_0 = r_0 \sin\theta$. The average energy density due to the excitation of a spectrum of waves with total wavenumber r_0 in the annular region between R_u and R_l is

$$\bar{E}(t) = \frac{2\dot{\epsilon}\delta t}{\pi(R_u^2 - R_l^2)} \times \int_{R_l}^{R_u} r_0 dr_0 \int_{-\infty}^{\infty} d\sigma \frac{e^{-2\nu r_0^2 g}}{1 + (\sigma - \alpha t)^2}, \quad (4.4)$$

where $\sigma = \tan\theta$, and

$$g \equiv t + \frac{\alpha^2 t^3/3 - \alpha t^2 \sigma}{\sigma^2 + 1} \quad (4.5)$$

is the cumulative dissipation factor.

In the case of no shear, $\alpha = 0$, the energy density can be found immediately:

$$\bar{E}_{\alpha=0}(t) = \frac{\dot{\epsilon}\delta t(e^{-2\nu t R_l^2} - e^{-2\nu t R_u^2})}{2\nu t(R_u^2 - R_l^2)}. \quad (4.6)$$

Note that at time zero the energy density is equal, as expected, to the energy input:

$$\bar{E}_{\alpha=0}(0) = \dot{\epsilon}\delta t. \quad (4.7)$$

To calculate the ensemble average energy density for stochastic excitation uncorrelated in time we proceed to the limit $\delta t \to 0$, $\dot{\epsilon} = O(1)$ and replace the summation in (3.3) with an integral. The integral (4.6) is tabulated so that the ensemble average of the energy density in the case of no shear can be written:

$$\langle E \rangle_{\alpha=0} = \int_0^{\infty} \bar{E}(t) dt = \frac{\dot{\epsilon}}{2\nu(R_u^2 - R_l^2)} \ln\left(\frac{R_u^2}{R_l^2}\right). \quad (4.8)$$

The ensemble-average wave energy density is proportional to the strength of the driving and inversely proportional to the coefficient of dissipation. The energy density vanishes in the limit $R_u \to \infty$ because the rapid increase of dissipation with wavenumber strongly damps short waves; it diverges as $R_l \to 0$ because the long waves are so weakly dissipated that energy accumulates and renders stationary statistics impossible. Because of divergences of this kind it is preferable to retain band-limited excitation $R_l < r_0 < R_u$.

To isolate the functional dependence of the dissipation in the general case with $\alpha \neq 0$, we nondimensionalize distance by L and time by the viscous time scale $\tilde{t} = (\nu/L^2)t$, so that $\tilde{r}_0 = Lr_0$, $\tilde{R}_l = LR_l$, $\tilde{R}_u = LR_u$, and the nondimensional shear becomes $\tilde{\alpha} = (L^2/\nu)\alpha$. In the terrestrial atmosphere $L \approx 10^6$ m, and typical dimensional shears are $\alpha = 10^{-5}$ s^{-1}. For an eddy viscosity of $\nu = 10^7$ m^2 s^{-1}, the nondimensional shear is $\tilde{\alpha} = 1$, while if we take a smaller eddy viscosity $\nu = 10^5$ m^2 s^{-1}, we have a nondimensional shear $\tilde{\alpha} = 100$.

The ensemble-average energy density becomes, in nondimensional variables, after dropping the tildes and performing the r_0 integration:

$$\langle E \rangle_\alpha = \frac{\dot{\epsilon} L^2}{\nu \pi (R_u^2 - R_l^2)} \int_0^\infty dt \times \int_{-\infty}^{\infty} \frac{e^{-2gR_l^2} - e^{-2gR_u^2}}{2g(1 + (\sigma - \alpha t)^2)} d\sigma, \quad (4.9)$$

where g, the cumulative dissipation factor, is given in (4.5). [We note that the presence of dissipation assures validity of the interchange of the order of integration in (4.9).]

The energy density maintained by stochastic forcing is proportional to the product of the strength of the driving, $\dot{\epsilon}$, and the viscous time scale L^2/ν. Finding the dependence on shear and spatial frequency bandpass of the forcing requires evaluation of the integral

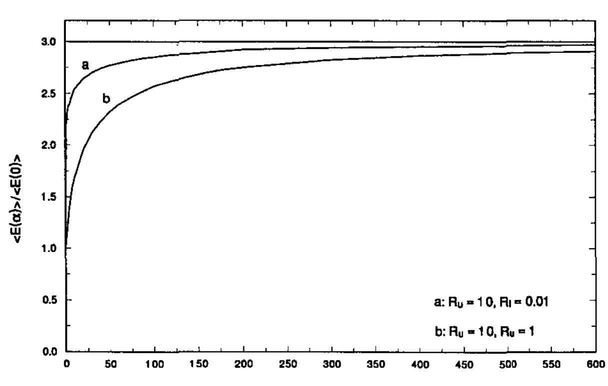

FIG. 1. The ratio of the ensemble-average perturbation energy density at shear α to the ensemble average perturbation density at zero shear as a function of shear for a constant shear flow: (a) bandpass $0.01 < r_0 < 10$; (b) bandpass $1 < r_0 < 10$. The asymptotic limit of large shear is indicated.

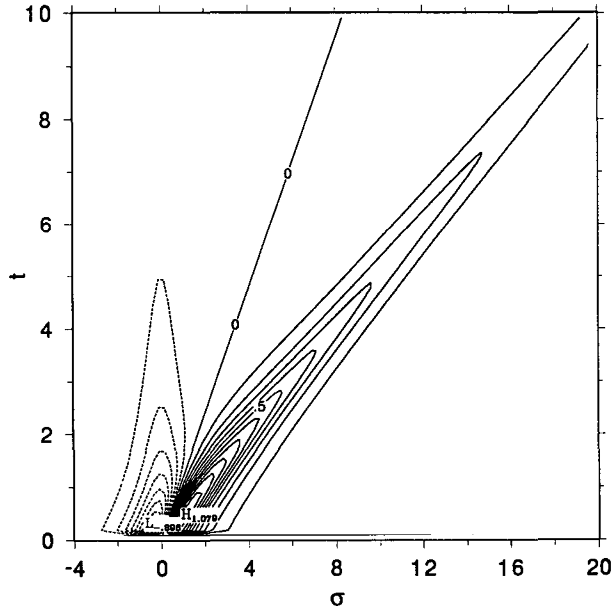

FIG. 2. Contribution of dynamics to the stochastic energy integral (4.9). Plotted are contours of the difference between the integrand and the integrand with the dynamics factor suppressed, that is, with the factor $1 + (\sigma - \alpha t)^2$ replaced by $1 + \sigma^2$. The bandpass is $0.01 < r_0 < 10$, and $\alpha = 2$.

in (4.9). Results of a numerical evaluation are shown in Fig. 1 for two cases of band limiting.

We can isolate the role of dynamics in (4.9) by suppressing the dynamic amplitude factor in the integrand while retaining the viscous dissipation. This is done by replacing $1 + (\sigma - \alpha t)^2$ in the denominator of the integrand with $1 + \sigma^2$. The difference between these two integrands highlights the contribution of dynamics to the stochastic energy integral. A plot of this difference as a function of σ and t for nondimensional shear, $\alpha = 2$, is shown in Fig. 2. Note that for negative σ, corresponding to waves adversely oriented with respect to the shear, dynamics results in a decreased contribution to the integral. However, this effect is more than compensated by the relatively larger region of positive values resulting from waves favorably oriented with respect to the shear. It is clear from this figure that the major contribution to the integral in (4.9) arises from the locus of favorably oriented waves along the line $\sigma = \alpha t$.

It turns out that the energy density can be calculated simply in the case of infinite shear. To evaluate (4.9) as $\alpha \to \infty$ we make the transformation:

$$\xi = \sigma - \alpha t, \quad t = t. \quad (4.10)$$

In the transformed variables (4.9) becomes

$$\langle E \rangle_\alpha = \frac{\dot{\epsilon} L^2}{\nu \pi (R_u^2 - R_l^2)} \int_0^\infty dt \times \int_{-\infty}^{\infty} \frac{e^{-2g'R_l^2} - e^{-2g'R_u^2}}{2g'(1 + \xi^2)} d\xi, \quad (4.11)$$

with

$$g' \equiv t - \frac{2\alpha^2 t^3/3 + \alpha t^2 \xi}{(\xi + \alpha t)^2 + 1}. \qquad (4.12)$$

The transformed integrand $I(\alpha, \xi, t)$ of (4.11) converges uniformly as $\alpha \to \infty$:

$$\lim_{\alpha \to \infty} I(\alpha, \xi, t) = 3\,\frac{e^{-(2/3)R_l^3 t} - e^{-(2/3)R_u^2 t}}{2t(1+\xi^2)}. \qquad (4.13)$$

The existence of the integrals and the additional circumstance of their uniform convergence enable us to write

$$\lim_{\alpha \to \infty} \langle E \rangle_\alpha = \frac{3\dot{\epsilon} L^2}{\nu\pi(R_u^2 - R_l^2)} \int_0^\infty dt \times \int_{-\infty}^{\infty} \frac{e^{-(2/3)R_l^3 t} - e^{-(2/3)R_u^2 t}}{2t(1+y^2)}\, dy. \qquad (4.14)$$

This integral is of the same form as in the case of no shear. It can also be integrated in closed form, yielding, in dimensional variables:

$$\langle E \rangle_{\alpha \to \infty} = \frac{3\dot{\epsilon}}{2\nu(R_u^2 - R_l^2)} \ln\left(\frac{R_u^2}{R_l^2}\right). \qquad (4.15)$$

Remarkably, the average energy density in infinite shear is three times the average energy density with zero shear (4.8). As can be seen from Fig. 1, the average energy density monotonically increases with shear as this asymptote is approached. The increase of the mean energy with increasing shear is due to the reduced dissipation of the waves that grow initially, that is, those with $l_0 > 0$. Such waves belong to what we have termed the favorable sector. If uncorrelated forcing were limited to exciting only waves in the unfavorable sector then the ensemble-average energy density would approach zero as the shear increased. If, instead, it were limited to exciting waves in the favorable sector then asymptotically as $\alpha \to \infty$ the energy density would be six times the ensemble average density with zero shear. This can be seen in Fig. 4.

Variation of energy density with shear and viscosity arises from two causes. The first is growth due to the extraction of background flow energy by downgradient Reynolds stresses. The second is the accumulation of excitation energy. In the case of no shear the second mechanism can be responsible for maintaining high levels of energy density in the presence of sufficiently small dissipation, as can be seen from (4.8). In the presence of shear both mechanisms operate. The accumulation mechanism arises from the effective dissipation over the evolution of the wave, which from (4.4) is

$$\bar{E}_{\rm diss} = \frac{2\dot{\epsilon}\,\delta t}{\pi(R_u^2 - R_l^2)} \int_{R_l}^{R_u} r_0\, dr_0 \int_{-\infty}^{\infty} \frac{e^{-2\nu r_0^2 g}}{1+\sigma^2}\, d\sigma. \qquad (4.16)$$

The difference $\bar{E} - \bar{E}_{\rm diss}$ is the contribution of dynamics to the maintenance of perturbation energy density. The relative importance of these mechanisms varies with shear, as can be seen in Fig. 3. It should be noted that as $\alpha \to \infty$, $\bar{E}_{\rm diss} \to 0$, and the maintenance of the energy density can be attributed solely to dynamical growth (necessarily favorably coincident with reduced diffusive dissipation, both occurring where the total wavenumber is near its minimum).

The divergence of $\langle E \rangle$, for all α as $\nu \to 0$, as seen in (4.11), is consistent with the fact that the energy of the waves does not decay in this limit but instead accumulates at ever smaller wavenumbers (Shepherd 1985; Farrell 1987). This lack of decay causes the $\nu \to 0$ limit to be nonstationary for energy density. Perhaps more interesting than the nonstationarity of the limit is the implication that arbitrary energy density can be maintained by stochastic excitation in the presence of sufficiently small dissipation.

Now consider the case of enstrophy driving. When excited with enstrophy $\dot{\zeta}\delta t$, the component (k_0, l_0) has amplitude:

$$A_{k_0,l_0} = 2\,\frac{(\dot{\zeta}\delta t)^{1/2}}{k_0^2 + l_0^2}. \qquad (4.17)$$

The energy density at a later time, given by (2.8), is

$$\bar{E}_{k_0,l_0}(t) = \frac{\dot{\zeta}\delta t}{(k_0^2 + (l_0 - \alpha k_0 t)^2)} \times \exp\left\{-2\nu \int_0^t (k_0^2 + (l_0 - \alpha k_0 \tau)^2)\, d\tau\right\}. \qquad (4.18)$$

Following the same steps that reduced (4.4) we obtain in the case of no shear ($\alpha = 0$) and enstrophy forcing, in dimensional variables:

$$\langle E \rangle_{\alpha=0} = \frac{\dot{\zeta}}{2\nu R_u^2 R_l^2}. \qquad (4.19)$$

The ensemble-average wave energy is proportional to the strength of the driving and inversely proportional to the dissipation. The energy density vanishes as

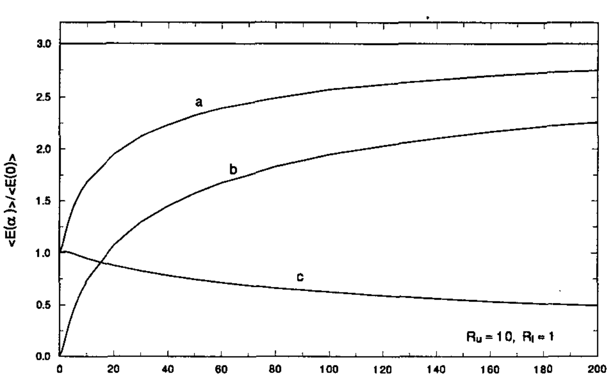

FIG. 3. Relative contribution to (a) the total average energy from (b) dynamics and (c) perturbation energy accumulation, as a function of shear for a constant shear flow. The bandpass is $1 < r_0 < 10$. The asymptotic limit of large shear is indicated.

$R_u \to \infty$, and the infrared divergence as $R_l \to 0$ is stronger than in the case of energy driving. This can be understood by noticing that the low-wavenumber waves are more strongly excited under enstrophy than under energy driving; in fact the ratio of the amplitudes is $1/r_0$, as can be seen by a comparison of (4.17) and (4.2).

The same procedure used to obtain asymptotic results for energy driving can be applied to enstrophy driving. Again we find that in the limit $\alpha \to \infty$ the energy density is three times that found in the zero shear case, in dimensional variables:

$$\langle E \rangle_{a\to\infty} = \frac{3\dot{\zeta}}{2\nu R_u^2 R_l^2}. \tag{4.20}$$

Apart from differences in detail, enstrophy driving produces results qualitatively similar to those obtained with energy driving.

b. General dissipation

For a general dissipation of order n, the viscous force in the rhs of (2.1) becomes $(-1)^n \nu \nabla^{2n} \psi$, and the foregoing procedure can be followed to yield the counterpart of (4.3) for the energy density:

$$\bar{E}_{k_0,l_0}(t) = \dot{\epsilon}\delta t \frac{k_0^2 + l_0^2}{k_0^2 + (l_0 - \alpha k_0 t)^2} \exp\left\{-2\nu \times \int_0^t (k_0^2 + (l_0 - \alpha k_0 \tau)^2)^{n-1} d\tau\right\}. \tag{4.21}$$

We note that the dissipation will now depend on the time average of the $n - 1$ moment of the instantaneous total wavenumber. The dissipation is independent of the wavenumber when $n = 1$, which corresponds to Ekman damping.

We nondimensionalize time by L^{2n-2}/ν, and shear by the reciprocal of the time scale. Transforming coordinates from (k_0, l_0) to (r_0, σ), we get for the ensemble-average energy density for energy driving:

$$\langle E \rangle_\alpha = \frac{2\dot{\epsilon} L^{2n-2}}{\nu\pi(R_u^2 - R_l^2)} \int_{R_l}^{R_u} r_0 dr_0 \times \int_0^\infty dt \int_{-\infty}^{\infty} \frac{e^{-2r_0^{2n-2} g^n}}{1 + (\sigma - \alpha t)^2} d\sigma, \tag{4.22}$$

where the variables are nondimensional and

$$g^n \equiv \int_0^t d\tau \left[\frac{1 + (\sigma - \alpha\tau)^2}{1 + \sigma^2}\right]^{n-1}. \tag{4.23}$$

After some algebra, we obtain the asymptotic energy density:

$$\langle E \rangle_{a\to\infty} = (2n - 1)\langle E \rangle_{\alpha=0}, \tag{4.24}$$

where $\langle E \rangle_{\alpha=0}$ is the ensemble average density in the absence of shear. It is interesting to note that the ensemble average energy density increases with the order of the dissipation. This linear result suggests that numerical simulations using high-order dissipation may lead to inherently different statistics characterized by stronger variance fields for higher-dissipation orders even though ν is adjusted so that waves with characteristic length scale L are equally damped. Also, note that in the case of Ekman damping, $n = 1$, the asymptotic value of the ensemble-average energy is the same as that in a flow with no shear and is in fact constant for all α.

c. Influence of forcing distribution

In order to explore the influence of forcing distribution it is instructive to treat explicitly the case of Ekman damping. The contribution from waves in the favored sector, $\sigma > 0$, $\langle E^f(\alpha) \rangle_{n=1}$ in nondimensional variables [refer to (4.22)] is

$$\langle E^f(\alpha) \rangle_{n=1} = \frac{\dot{\epsilon}}{\pi\nu} \int_0^\infty e^{-2t} dt \int_0^\infty \frac{d\sigma}{1 + (\sigma - \alpha t)^2} = \frac{\dot{\epsilon}}{\nu}\left[\frac{1}{4} + f(\alpha)\right], \tag{4.25}$$

where

$$f(\alpha) = \frac{1}{\pi} \int_0^\infty dt e^{-2t} \tan^{-1}(\alpha t) \tag{4.26}$$

is a monotonically increasing function (for $\alpha > 0$), $f(0) = 0$, and $\lim_{\alpha\to\infty} f(\alpha) = 1/4$. We note that if we were to force only the waves belonging to the favored sector the ensemble-average energy for large shears would asymptote to a value that is twice the energy density for a flow with no shear.

The contribution from the unfavorable sector $\sigma < 0$, $\langle E^u(\alpha) \rangle_{n=1}$ in nondimensional variables is

$$\langle E^u(\alpha) \rangle_{n=1} = \frac{\dot{\epsilon}}{\nu}\left[\frac{1}{4} - f(\alpha)\right]. \tag{4.27}$$

The energy contribution from the unfavorable sector monotonically decreases to zero with increasing shear. The ensemble-average energy density is plotted in Fig. 4 for Ekman damping and diffusive dissipation. The total ensemble average is the sum of the energy of the contributions from the two sectors and due to the cancellation of the $f(\alpha)$ term in (4.25) and (4.27) is $\dot{\epsilon}/2\nu$. This example shows succinctly that the increase of ensemble-average energy density with shear for dissipations of order $n > 1$ results from the coincidence of the reduction of the dissipation of the waves excited in the favored sector with the growth phase of their life cycle.

d. Statistical equilibrium with occluding waves

The cases we have treated up to this point lead to, at most, only a modest increase in the ensemble-average energy density with shear and to saturation of the en-

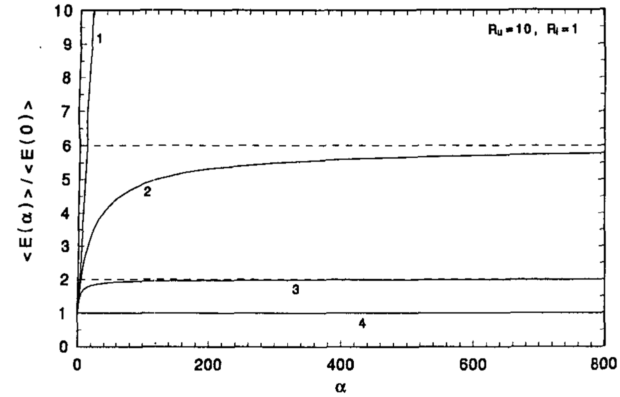

FIG. 4. The ratio of the ensemble-average perturbation energy density at shear α to the ensemble-average perturbation density at zero shear as a function of shear α for forcing of the waves in the favorable sector. Curve 1 is for Ekman damping and occlusion of the growing waves. Curve 2 is for diffusive dissipation ($n = 2$) and anisotropic forcing of waves with $\sigma > 0$. Curve 3 is the same as 2 but for Ekman damping ($n = 1$). Curve 4 shows the constant average energy density maintained by stochastic isotropic forcing in the case of Ekman damping. The bandpass is $1 < r_0 < 10$.

semble energy density for large shears. The saturation is due to the symmetry between growth and decay of each excited wave in the favored sector ($\sigma > 0$), while the asymmetry between the growth and decay phases of the waves introduced by dissipation leads to a modest increase of the energy with shear. If we suppress the kinematically induced decay of each growing wave, that is, suppress dynamic cyclolysis, the symmetry between the growing and decaying phases of the favored waves is broken with the result that the ensemble average energy density increases without bound as shear increases. This asymmetry between the growing and decaying phase of the waves models a crucial aspect of the physics of developing disturbances in the atmosphere. Growing disturbances that excite a linear mode acquire a repository for the energy obtained from their interaction with the mean flow although the mode itself may be damped (Farrell 1989). Alternatively, growth may continue until the waves reach an amplitude that leads them to form a long-lived nonlinear mode, often referred to as an occluded state. These processes can be included in the frame of this model by demanding that some of the growing waves after reaching their maximum amplitude subsequently dissipate without further kinematic deformation. These waves we will refer to as occluded. We model this process by allowing decaying disturbances ($\sigma \leqslant 0$) to continue their decay through both dynamic and diffusive mechanisms while growing disturbances ($\sigma > 0$), when they reach their maximum amplitude where $l = 0$ at $t = \sigma/\alpha$, transit with probability p to a second state in which these occluded waves continue to diffusively damp but without further kinematic deformation.

The ensemble-average energy density of the first state for the waves excited with total wavenumber r_0 is

$$E_l = \int_{-\infty}^{0} d\sigma \int_0^\infty dt \bar{E}_{r_0,\sigma}(t) + (1-p)\int_0^\infty d\sigma \int_0^\infty dt \times \bar{E}_{r_0,\sigma}(t) + p\int_0^\infty d\sigma \int_0^{\sigma/\alpha} dt \bar{E}_{r_0,\sigma}(t), \quad (4.28)$$

where $\bar{E}_{r_0,\sigma}(t)$, the average energy density at time t of a wave excited with total initial wavenumber r_0 and orientation σ is given by (4.3) for a dissipation of order $n = 2$.

The average energy of a disturbance in the second state, for dissipation of order $n = 2$, is

$$E_2(t) = \bar{E}_{r_0,\sigma}\left(\frac{\sigma}{\alpha}\right) \exp\left\{-2\nu r_0^2 \frac{t - \sigma/\alpha}{1 + \sigma^2}\right\}, \quad (4.29)$$

and the ensemble-average energy density of the system in the second state is

$$E_2 = p \int_0^\infty d\sigma \int_{\sigma/\alpha}^\infty dt E_2(t). \quad (4.30)$$

The ensemble average of the total system comprising both states is $\langle E_{oc}\rangle_\alpha = E_1 + E_2$.

The details of a calculation for the case in which the growing waves occlude with probability $p = 1$ is relegated to appendix A. We find that for dissipation of order n the ensemble average density grows with the shear and approaches, for large shears, an ensemble-average energy density proportional to α^{2n-1}. We plot the growth of the average energy for the case of occluded waves with Ekman damping, $n = 1$, in Fig. 4.

5. Stochastic dynamics of deformation flows

In the case of the unbounded deformation flow $\Psi = \alpha xy$, for which

$$k(t) = k_0 e^{\alpha t}, \quad l(t) = l_0 e^{-\alpha t}, \quad (5.1)$$

and energy driving we get at time, t, as in (4.5):

$$\bar{E}(t) = \frac{2\dot{\epsilon}\delta t}{\pi(R_u^2 - R_l^2)} \int_{R_l}^{R_u} r_0 dr_0 \times \int_{-\infty}^{\infty} \exp\left\{-\frac{\nu r_0^2}{\alpha} \frac{e^{2\alpha t} - 1 + \sigma^2(1 - e^{-2\alpha t})}{\sigma^2 + 1}\right\} \Big/ (e^{2\alpha t} + \sigma^2 e^{-2\alpha t}) d\sigma, \quad (5.2)$$

where we have used the same nondimensional variables as in section 4. To investigate the integrability in time of (5.2) we make the transformation, $\eta = \sigma e^{-2\alpha t}$, consistent with the observation that the dominant contribution to this integral comes from $\sigma \rightarrow \infty$. Performing the r_0 integration we obtain, in nondimensional variables:

$$\bar{E}(t) = \frac{\dot{\epsilon}\delta t L^2}{\nu\pi(R_u^2 - R_l^2)} \int_{-\infty}^{\infty} \frac{e^{-\lambda R_l^2} - e^{-\lambda R_u^2}}{\lambda(1 + \eta^2)} d\eta, \quad (5.3)$$

where

$$\lambda = \frac{1}{\alpha} \frac{e^{2\alpha t} - 1 + \eta^2 e^{4\alpha t}(1 - e^{-2\alpha t})}{\eta^2 e^{4\alpha t} + 1}. \quad (5.4)$$

We note that $\lim_{t\to\infty}\lambda = 1/\alpha$. At large t (5.3) behaves as

$$\bar{E}(t) = \frac{\dot{\epsilon}\delta t L^2 \alpha}{\nu} \frac{e^{-R_l^2/\alpha} - e^{-R_l^2/\alpha}}{R_u^2 - R_l^2}, \qquad (5.5)$$

a constant. Therefore, the time integral of $\bar{E}(t)$ diverges. Note that (5.3) does not have a time integral for any nonzero domain of integration in η. However, (5.2) has a time integral as long as the σ integration is restricted to finite limits.

The case of no background flow ($\alpha = 0$) is immediate. Integration of (5.2) can be done explicitly, and the result is the same as in the case of no shear (4.8).

Divergence of the time integral of (5.2), for any nonzero α, indicates failure of energy driving to maintain a statistically stationary state. A similar failure to attain a statistically stationary state is characteristic of inviscid flows where accumulation of perturbation energy is not limited by dissipation. In deformation flows the effect of dissipation is opposed by rapid growth of those disturbances with wave-front normals in the direction of diffluence. If these waves are suppressed, (5.3) leads to a statistically steady state. The resulting energy density as a function of shear is shown in Fig. 5 for the cases with components having initial wave-front–normal directions within 0.5 degrees, and 0.02 degrees from the direction of diffluence suppressed ($|\sigma| < 100$, and $|\sigma| < 4000$, respectively). The energy density increases as the domain of integration expands to include more of these rapidly growing waves, yielding, in the limit, a nonstationary energy density.

To investigate further the dynamics of deformation flows we suppress the singular direction and attain a steady state. We can then, as in the previous section, calculate the contribution to the energy density due to the accumulation of excitation energy. We define, as in (4.16), $\bar{E}_{\text{dissip}}$, and calculate the difference $\bar{E} - \bar{E}_{\text{dissip}}$.

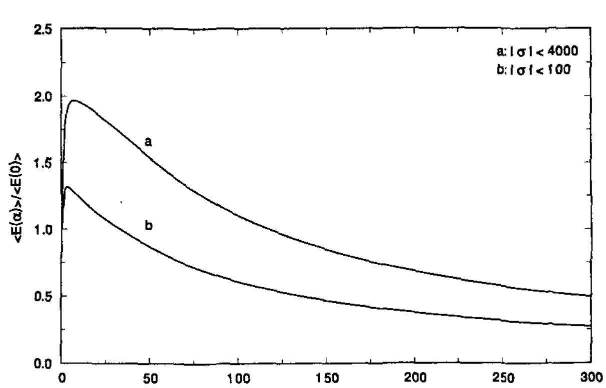

FIG. 5. The ensemble-average perturbation energy density at α normalized by the ensemble-average perturbation density at zero α as a function of α for deformation flows. The wave-front–normal directions are limited by (a) $|\sigma| < 4000$; (b) $|\sigma| < 100$. The bandpass is $1 < r_0 < 10$.

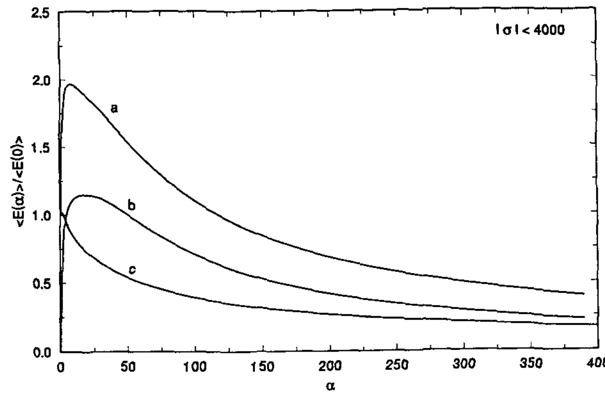

FIG. 6. Relative contribution to (a) the total average energy from (b) dynamics and (c) perturbation energy accumulation, as a function of α for a deformation flow with wave-front–normal directions limited by $|\sigma| < 4000$. The bandpass is $1 < r_0 < 10$.

The relative importance of dynamics and accumulation of energy to the energy density is shown in Fig. 6. In this band-limited deformation flow the energy density decays as $\alpha \to \infty$.

6. Energetics

We can form an energy equation by multiplying (2.1) by ψ and taking ensemble averages. If stationary statistics have been attained, as in the case of a constant shear flow, there will be a balance between the average energy input by the driving $\dot{\epsilon}$, the average dissipation, which for $n = 2$ is $D \equiv \langle \nu|\nabla^2\psi|^2 \rangle$, and the average energy lost to the mean flow $F \equiv \langle \alpha\overline{(uv)} \rangle$. We thus have

$$\dot{\epsilon} = \langle \nu|\nabla^2\psi|^2 \rangle + \langle \alpha\overline{(uv)} \rangle. \qquad (6.1)$$

When $F > 0$ the average perturbation energy input is greater than the average dissipation, and the resulting upgradient Reynolds stress tends to augment the mean shear. When $F < 0$ the wave field tends instead to reduce the mean shear. In the case of no shear, for which $F = 0$, there is balance between driving and dissipation. It should be remarked that the unbounded constant shear and deformation problems have on average no Reynolds stress divergence that would be associated with local wave–mean flow interaction. The mean-flow interaction occurs, in these model problems, at their infinitely removed boundaries. However, this peculiarity is not crucial to the linear problem in which alteration of the mean flow by the perturbation fields is ignored.

For the case of energy driving of a constant shear flow we find, in nondimensional variables, for $n = 2$ diffusion, that the ensemble-average dissipation is proportional to the ensemble average of the enstrophy and is given by

$$\langle \nu |\nabla^2\psi|^2\rangle = \frac{4\dot\epsilon}{\pi(R_u^2-R_l^2)}\int_{R_l}^{R_u} r_0^3\, dr_0 \times \int_0^\infty dt \int_{-\infty}^{\infty} d\sigma\, \frac{e^{-2r_0^2 g}}{1+\sigma^2}, \quad (6.2)$$

where g, the cumulative dissipation factor, is given by (4.5). A similar expression can be derived for the deformation flow. It is easy to check that the dissipation for $\alpha = 0$ is equal to $\dot\epsilon$ as expected from the energy equation (6.1).

The interaction with the mean flow, F, for a constant shear takes the form in nondimensional variables:

$$\alpha\langle\overline{uv}\rangle = -\frac{2\dot\epsilon L^2}{\pi(R_u^2-R_l^2)}\int_{R_l}^{R_u} r_0\, dr_0 \int_{-\infty}^{\infty} d\sigma \times \int_0^\infty dt\, \frac{\sigma-\alpha t}{(1+(\sigma-\alpha t)^2)^2}\, e^{-2r_0^2 g}. \quad (6.3)$$

A similar expression can be derived from the deformation flow. We note that for growing waves ($\sigma > 0$ and $0 < t < \sigma/\alpha$) the fluxes are downgradient, consistent with the growth of the wave during this phase. During the decaying phase the flux is upgradient.

Dissipation, D, and the average energy lost to the mean flow, F, as a function of shear, are shown in Fig. 7 for the constant shear flow and in Fig. 8 for the band-limited deformation flow. In the limit $\alpha \to 0$ perturbation energy input is balanced by dissipation in both cases. In the case of constant shear flow there is a slight overall downgradient flux for small shear, while for deformation flow the downgradient fluxes are substantial and extend to high shears. In both cases, the limit $\alpha \to \infty$ is characterized by upgradient Reynolds stress, indicating driving of the mean flow by the perturbations. This can be understood by observing that for large shears the dynamics associated with wave perturbations giving appreciable fluxes are nearly inviscid and nearly preserve enstrophy. This is due to the fact that the time scale for shearing is of the order of $1/\alpha$ while the dissipation rate $\dot E/E$ proceeds for $n = 2$ at a constant rate. As a result for very large shears the energy input is not dissipated rapidly enough by diffusion to prevent its transfer by upgradient Reynolds stress to the mean flow as the wave is sheared over. Appreciable downgradient Reynolds stress, indicative of net energy transfer from the background flow to the perturbation field, are seen in Fig. 8 for moderate values of deformation.

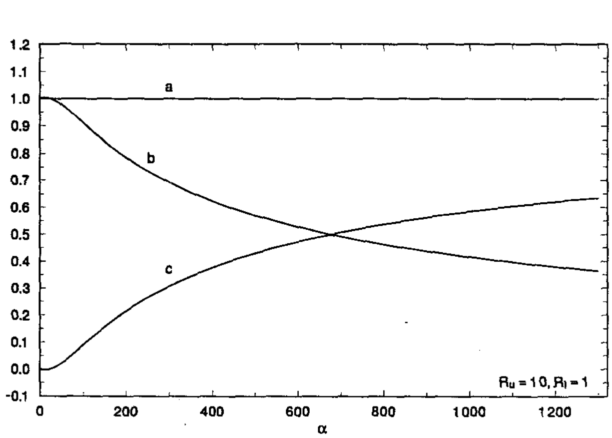

FIG. 7. Contributions to the energy balance in a constant shear flow. (a) Average perturbation energy input $\dot\epsilon$; (b) ensemble-average perturbation energy dissipation D; (c) average energy lost, through Reynolds stress, to the mean flow F; as a function of shear. The bandpass is $1 < r_0 < 10$.

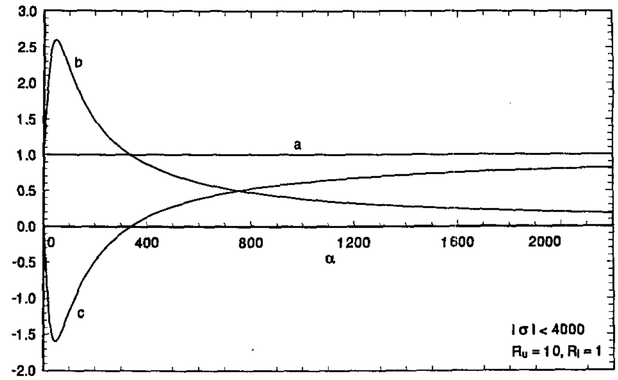

FIG. 8. Contribution to the energy balance in a deformation flow. (a) Average perturbation energy input $\dot\epsilon$; (b) ensemble-average perturbation energy dissipation D; (c) average energy lost, through Reynolds stress, to the mean flow F; as a function of α for a deformation flow with wave-front-normal directions limited by $|\sigma| < 4000$. The bandpass is $1 < r_0 < 10$.

There is no necessary connection between the magnitude of the average downgradient Reynolds stress and the average level of perturbation energy density. To see this, consider a small inviscid perturbation that grows for a long time, extracting energy by downgradient Reynolds stress, and then slowly decays, in so doing eventually producing an equal average upgradient stress. The total stress over the wave life cycle is zero, but the mean energy over the life cycle can be very large.

If we were to force only waves in the favored sector in a constant shear flow then the fluxes of the growing waves could lead to an average downgradient flux. Such a case is shown in Fig. 9, where the flux is downgradient for small and moderate shears. However, as the shear α becomes large, and the model dynamics essentially inviscid, the fluxes become upgradient and the waves tend to drive the mean flow.

General dissipation, $n > 2$, can be easily formulated, and the results are qualitatively similar to the $n = 2$ case. The case of Ekman friction, $n = 1$, leads to zero fluxes for all shears. In this case the dissipation is proportional to the energy, a quantity that, unlike the enstrophy, is not conserved in the nearly inviscid dynamics at large shear. However, if we force only waves in the favored sector, the symmetry is broken, and we obtain upgradient fluxes. These cases are shown in Fig. 9.

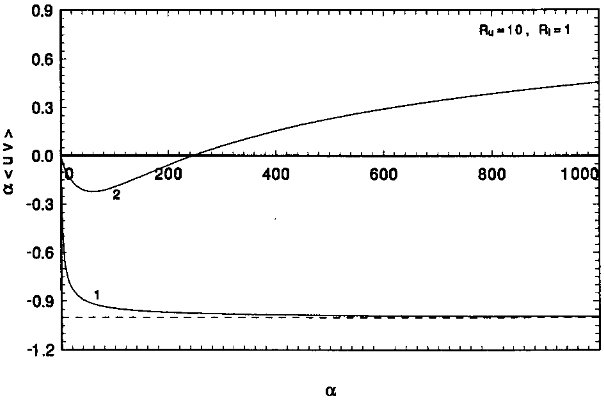

FIG. 9. Ensemble-average energy lost to the mean flow F as a function of shear α. Curve 1 is for Ekman damping and anisotropic forcing of the favorable sector ($\sigma > 0$). Curve is for diffusive dissipation ($n = 2$) and forcing only in the favorable sector. The bandpass is $1 < r_0 < 10$.

Finally, we consider the case of occlusion in a constant shear flow for general dissipation of order n. In such a case it is not obvious that the two states of the system considered together will yield an energy budget of the form (2.1). Detailed calculation shows that an energy budget of this form is valid also for the case of occlusion with the appropriate form of the dissipation substituted in (2.1). Calculation of the flux is relegated in appendix A, where it is shown that the flux eventually becomes constant as the shear increases. This behavior of the flux is consistent with the rapid growth of average perturbation energy as a function of shear found for occluded waves.

7. Discussion

The source of perturbation variance can be either exponential temporal modal instability or stochastic excitation of perturbations that maintain variance as an amplifier of perturbations rather than as an unstable oscillator. This second mechanism has been examined in this work, making use of closed-form solutions for viscous free shear and deformation flows, of which neither supports exponential temporal modal instability.

We find that mean perturbation energy in pure deformation flow increases without bound under stochastic excitation for any viscosity. In constant shear and diffusive dissipation the mean energy is limited to an increase of a factor of 3 over that arising from the same forcing of the unsheared fluid. These examples demonstrate the possibility of maintaining wave variance by amplification of perturbations. We expect that this mechanism is greatly enhanced in geophysical flows by the presence of moderately damped modes that serve to intercept the dynamic and essentially kinematic decay of shear waves so that variance accumulates. We explored this possibility using a phenomenological model of the occlusion process and found rapid increase in variance with shear in the presence of modeled occlusion. Unfortunately, it is difficult to rigorously extend the methods used here to jet flows where near-neutral modes exist because the simplicity of the Kelvin solutions is then lost (Craik and Criminale 1986). For this reason, an alternative numerical method of solution must be used (Farrell and Ioannou 1992).

At some high level of variance nonlinear interactions will become important, perhaps providing the feedback perturbations required to transform the amplifier into a self-excited oscillator. While the importance of nonlinearity in fully developed turbulence is undeniable, results obtained by exploiting linear theory are at least suggestive of directions for further study of mechanisms operating at higher variance levels.

Stochastic excitation both maintains the perturbation wave field and helps maintain the background flow. All perturbations that grow in free shear flow must eventually decay because there are no neutral modes. As this eventual decay takes place the associated upgradient Reynolds stress enhances the background flow. However, if the growth is large before this decay ensues, a variance field may be maintained at high average energy.

Observations of variance in the maximum shear region of the storm tracks typically reveal a factor of approximately 3 increase in average perturbation energy over that found in regions of minimum shear (Blackmon et al. 1977; James and Anderson 1984). In addition, observations of variance in the highest-shear regions reveal evidence of saturation of variance with increasing shear (Nakamura 1992). Comparison with the idealized results of this work suggests that maintenance of variance by stochastic forcing may provide an explanation for these behaviors, but confirmation must await development of more realistic models of the stochastically driven atmosphere.

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APPENDIX A

Variance and Energetics of Occluded Waves

The excited waves with $\sigma > 0$ (equivalently $l_0 > 0$) will reach maximum amplitude at $t_v = \sigma/\alpha$, at which time their crosswind wavenumber is instantaneously zero; that is, $\sigma - \alpha t_v = 0$, after which they will decay. We can model the process of occlusion by requiring that a portion of the growing waves ceases shearing when they reach maximum amplitude, and that they retain a zero crosswind wavenumber thereafter.

For $(-1)^n \nu \nabla^{2n}$ dissipation the steady-state energy density, $\langle E_{oc} \rangle_\alpha$, for which all the growing waves occlude, is given by

$$\langle E_{oc}\rangle_\alpha = E_1 + E_2, \quad \text{(A.1)}$$

where E_1, the energy due to all unoccluded waves, is

$$E_1 = \frac{\dot\epsilon L^{2n-2} r_0 dr_0}{\pi\nu}\left(\int_{-\infty}^{0} d\sigma \int_0^\infty dt + \int_0^\infty d\sigma \int_0^{\sigma/\alpha} dt\right) \times \frac{\exp(-2g^n r_0^{2n-2})}{1+(\sigma-\alpha t)^2}, \quad \text{(A.2)}$$

and E_2, the contribution to the mean energy by those waves that have occluded, is

$$E_2 = \frac{\dot\epsilon L^{2n-2} r_0 dr_0}{\pi\nu}\int_0^\infty d\sigma \int_{\sigma/\alpha}^\infty dt \exp(-2g_{oc}^n r_0^{2n-2}). \quad \text{(A.3)}$$

The variables are nondimensional (we have dropped the tildes). The cumulative dissipation factor g^n is given in (4.23), and the cumulative dissipation factor for the occluded waves, for which $t > \sigma/\alpha$, is

$$g_{oc}^n = g^n\left[\frac{\sigma}{\alpha}\right] + \frac{t-\sigma/\alpha}{(1+\sigma^2)^{n-1}}, \quad \text{(A.4)}$$

$$g^n\left[\frac{\sigma}{\alpha}\right] = \frac{\int_0^{\sigma/\alpha} d\tau[1+(\sigma-\alpha\tau)^2]^{n-1}}{(1+\sigma^2)^{n-1}}. \quad \text{(A.5)}$$

In (A.2) E_1 is certainly bounded by $(2n-1)\langle E\rangle_{\alpha=0}$, while E_2, which gives the contribution to the mean energy by the waves that have occluded, can be reduced to

$$E_2 = \frac{\dot\epsilon L^{2n-2} dr_0}{2\pi\nu r_0^{2n-2}}\int_0^\infty d\sigma(1+\sigma^2)^{n-1} \times \exp\left[-2r_0^{2n-2} g^n\left[\frac{\sigma}{\alpha}\right]\right]. \quad \text{(A.6)}$$

We now estimate the cumulative dissipation factor of the waves before they occlude. We have

$$g^n\left[\frac{\sigma}{\alpha}\right] = \frac{\sigma}{\alpha}\sum_{k=0}^{k=n-1} C_k^{n-1}\frac{\sigma^{2k}}{2k+1}\Big/\sum_{k=0}^{k=n-1} C_k^{n-1}\sigma^{2k}$$
$$= \frac{\sigma}{\alpha}\left[1 - \sum_{k=0}^{k=n-1}\frac{2k}{2k+1}C_k^{n-1}\sigma^{2k}\Big/\sum_{k=0}^{k=n-1} C_k^{n-1}\sigma^{2k}\right], \quad \text{(A.7)}$$

where C_k^m is the binomial coefficient. We note that the sequence $\{2k/(2k+1)\}$ is increasing, and as $\sigma \geqslant 0$ we can bound (A.7):

$$\frac{\sigma}{\alpha}\frac{1}{2n-1} < g^n\left[\frac{\sigma}{\alpha}\right] \leqslant \frac{\sigma}{\alpha}. \quad \text{(A.8)}$$

The contribution to the steady energy from the occluded waves is thus bounded

$$\frac{\dot\epsilon L^{2n-2} dr_0}{2\pi\nu r_0^{2n-2}}\int_0^\infty d\sigma(1+\sigma^2)^{n-1}\exp\left[-2r_0^{2n-2}\frac{\sigma}{\alpha}\right]$$
$$< E_2 < \frac{\dot\epsilon L^{2n-2} dr_0}{2\pi\nu r_0^{2n-2}}\int_0^\infty d\sigma(1+\sigma^2)^{n-1} \times \exp\left[-2r_0^{2n-2}\frac{\sigma}{(2n-1)\alpha}\right]. \quad \text{(A.9)}$$

The integrals in (A.9) can be reduced to obtain

$$\frac{\dot\epsilon L^{2n-2} dr_0}{2\pi\nu r_0^{2n-2}}\sum_{k=0}^{n-1} C_k^{n-1}\frac{(2k+1)!\alpha^{2k+1}}{(2r_0^{2n-2})^{2k+1}}$$
$$< E_2 < \frac{\dot\epsilon L^{2n-2} dr_0}{2\pi\nu r_0^{2n-2}}\sum_{k=0}^{n-1} C_k^{n-1} \times \frac{(2k+1)!(2n+1)^{2k+1}\alpha^{2k+1}}{(2r_0^{2n-2})^{2k+1}}. \quad \text{(A.10)}$$

We have thus proved that when all the growing waves occlude the steady-state energy for a ∇^{2n} dissipation increases with the shear as α^{2n-1}, for large shears. For Ekman damping, $n = 1$, in the case of occlusion the steady-state energy increases asymptotically linearly with α. For eddy diffusion, $n = 2$, the energy increases as α^3.

The energy equation (6.1) remains valid for cases in which the waves occlude (it can be proved by explicit calculation of the contributing terms) and we estimate the dependence on the shear of the interaction term $F \equiv \alpha\langle\overline{uv}\rangle$. For the case of occlusion the dominant contribution to F is proportional to

$$\lim_{\alpha\to\infty} F \approx -\int_0^\infty d\sigma \exp\left\{-2r_0^{2n-2} g^n\left[\frac{\sigma}{\alpha}\right]\right\}. \quad \text{(A.11)}$$

The fluxes are downgradient, and for large shears, using the bounds given by (A.8), we find that $\alpha\langle\overline{uv}\rangle$ grows linearly with the shear, implying that the momentum flux, $\langle\overline{uv}\rangle$, asymptotes to a constant for large shears.

As an example we give the results for the case of Ekman damping; that is, $n = 1$. The steady-state energy, in nondimensional variables, for the case in which all growing waves occlude is

$$\langle E_{oc}\rangle_\alpha = \frac{\dot\epsilon}{\pi\nu}\left(\frac{\pi}{2} + \frac{\alpha}{4} - \int_0^\infty dt e^{-2t}\tan^{-1}(\alpha t)\right). \quad \text{(A.12)}$$

The mean energy grows linearly with the shear in this case. Note that $\int_0^\infty dt e^{-2t}\tan^{-1}(\alpha t)$ monotonically increases with the shear from 0 to an asymptotic value of $\pi/4$ as $\alpha \to \infty$. The associated fluxes are downgradient, and the interaction term grows asymptotically linearly with the shear:

$$\alpha\langle\overline{uv}\rangle_{oc} = \frac{2\dot\epsilon}{\pi}\int_0^\infty e^{-2t}\tan^{-1}(\alpha t)dt - \frac{\alpha}{4}. \quad \text{(A.13)}$$

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