Perturbation Growth and Structure in Time-Dependent Flows

BRIAN F. FARRELL

Department for Earth and Planetary Sciences, Harvard University, Cambridge, Massachusetts

PETROS J. IOANNOU

Section of Astrophysics, Astronomy, and Mechanics, Department of Physics, University of Athens, Athens, Greece

ABSTRACT

Asymptotic linear stability of time-dependent flows is examined by extending to nonautonomous systems methods of nonnormal analysis that were recently developed for studying the stability of autonomous systems. In the case of either an autonomous or a nonautonomous operator, singular value decomposition (SVD) analysis of the propagator leads to identification of a complete set of optimal perturbations ordered according to the extent of growth over a chosen time interval as measured in a chosen inner product generated norm. The long-time asymptotic structure in the case of an autonomous operator is the norm-independent, most rapidly growing normal mode while in the case of the nonautonomous operator it is the first Lyapunov vector that grows at the norm independent mean rate of the first Lyapunov exponent. While information about the first normal mode such as its structure, energetics, vorticity budget, and growth rate are easily accessible through eigenanalysis of the dynamical operator, analogous information about the first Lyapunov vector is less easily obtained. In this work the stability of time-dependent deterministic and stochastic dynamical operators is examined in order to obtain a better understanding of the asymptotic stability of time-dependent systems and the nature of the first Lyapunov vector. Among the results are a mechanistic physical understanding of the time-dependent instability process, necessary conditions on the time dependence of an operator in order for destabilization to occur, understanding of why the Rayleigh theorem does not constrain the stability of time-dependent flows, the dependence of the first Lyapunov exponent on quantities characterizing the dynamical system, and identification of dynamical processes determining the time-dependent structure of the first Lyapunov vector.

1. Introduction

Linear stability theory addresses a set of problems in the dynamics of physical systems that include the origin, energetics, structure, and growth to finite amplitude of perturbations, and the conceptually distinct problem of error growth, which involves the rate of divergence of initially nearby trajectories in state space. In the context of the midlatitude atmosphere these address, respectively, the cyclogenesis and the predictability problems. While traditional stability analysis confined attention to determining the $t \rightarrow \infty$ asymptotic of perturbations to autonomous dynamical operators, more recent generalizations of stability theory have examined the more physically relevant finite-time stability of both autonomous and nonautonomous systems (Farrell and Ioannou 1996a, 1996b). In the case of an autonomous op-

erator the temporal asymptotic perturbation or error growth occurs at the rate of the most unstable eigenvalue of the linear dynamical operator and takes the form of the associated most unstable normal mode. The analogous structure and growth rate in the case of a nonautonomous operator are given by the time-dependent first Lyapunov vector and first Lyapunov exponent, respectively (Oseledec 1968). Asymptotic structure and growth rate are readily obtained for autonomous systems through eigenanalysis of the dynamical operator and well-known theorems constrain autonomous system stability (Rayleigh 1880). Analogous results are not available for nonautonomous systems and integral bounds on growth in time-dependent systems are relatively loose (Farrell and Ioannou 1996b, hereafter F&I). Moreover, while eigenanalyses of canonical autonomous systems provide familiar examples of growth rate and structures for time-independent systems, growth rates and structures of Lyapunov vectors are not commonly available and intuition concerning such issues as the Lyapunov structure, its variability with time, and the nature of the energetics producing Lyapunov growth lacks ex-

Corresponding author address: Dr. Brian F. Farrell, Department of Earth and Planetary Sciences, Harvard University, Pierce Hall, 29 Oxford St., Cambridge, MA 02138.
E-mail: farrell@io.harvard.edu

ample. Given that all physical systems are to a greater or lesser extent time dependent, and that the atmosphere in particular is highly time dependent, it is of more than strictly theoretical interest to better understand the nature of asymptotic stability of nonautonomous systems. Indeed, forecast accuracy is likely to be ultimately limited by asymptotic error growth, which is governed by the time-dependent tangent linear system linearized about the forecast trajectory until very near the end of the period of error growth when nonlinear effects become important. The approach of an initial perturbation to this asymptotic limit is of interest as is characterizing the universal structure assumed by the disturbance in this limit, which is the Lyapunov vector. (Hereafter, the terms Lyapunov vector and Lyapunov exponent refer to the first Lyapunov vector and the associated first Lyapunov exponent.)

We begin by examining destabilization of a barotropic model in which the flow and effective β are allowed to be time dependent. The results obtained are then interpreted from the point of view of nonnormal dynamics using simple time-dependent model systems.

2. The time-dependent stability problem

Consider perturbations to a time varying zonally homogeneous barotropic flow, $U(y, t)$, in a channel with y the northward direction and x the zonal direction. Choosing the inverse of the mean shear and the channel width as characteristic time- and space scales, the nondimensional barotropic vorticity equation for the meridionally and temporally varying component of the streamfunction $\Psi(x, y, t) = \psi(y, t)e^{ikx}$ is given by

$$\frac{\partial \nabla^2 \psi}{\partial t} = -ikU(y, t)\nabla^2 \psi - ik\left(\beta - \frac{d^2 U(y, t)}{dy^2}\right)\psi$$
$$- (-1)^n R_n \nabla^{2(n+1)}\psi, \qquad (1)$$

where k is the zonal wavenumber. The operator ∇^{2n} is defined as

$$\nabla^{2n} \equiv \left(\frac{d^2}{dy^2} - k^2\right)^n. \qquad (2)$$

The nondimensional dissipation constant R_n is chosen appropriately for dissipation of order n; for $n = 0$ the dissipation models Ekman damping of the equivalent barotropic atmosphere with time constant $1/R_0$; for $n = 1$ the dissipation constant R_1 is an inverse Reynolds number. The boundary conditions at the channel walls are $\psi(\pm 1, t) = 0$ and for $n = 1$ the nonslip condition $\psi_y(\pm 1, t) = 0$ is also imposed. The assumption of zonal homogeneity is severe and will be relaxed in future investigations, but for now the simplicity of this zonally homogeneous flow will be exploited to facilitate the exposition.

The perturbation barotropic vorticity equation can be interpreted as governing perturbations to a time-depen-

dent flow in which the time dependence is produced by external forcing such as a barotropic stratospheric flow forced by tropospheric planetary waves, or as the evolution equation of errors on a time-dependent zonal trajectory $U(y, t)$ obtained as the free solution of the nonlinear barotropic equation.

The barotropic equation is discretized using central differences on N grid points so that (1) takes the matrix form:

$$\frac{d\psi}{dt} = \mathbf{A}(t)\psi, \qquad (3)$$

where ψ is the column vector of streamfunction values at the discretization points, and $\mathbf{A}$ is the discretized dynamical operator matrix

$$\mathbf{A} = \nabla^{-2}\left(-ikU(y, t)\nabla^2 - ik\left(\beta - \frac{d^2 U(y, t)}{dy^2}\right)\right.$$
$$\left. - (-1)^n R_n \nabla^{2(n+1)}\right), \qquad (4)$$

in which the discretized operator ∇^{-2} has been rendered invertible by imposition of the boundary conditions. Convergence of the discrete approximate barotropic Eq. (3) to its continuous counterpart (1) for the examples considered was verified by doubling resolution.

Solution of (3) is expressed in terms of the finite time propagator:

$$\mathbf{\Phi}(t) = \lim_{\tau \to 0} \prod_{\nu=1}^{m} e^{\mathbf{A}(\nu\tau)\tau}, \qquad (5)$$

obtained by m advances of the state of the system by the infinitesimal propagators $e^{A(\nu\tau)\tau}$ at times $\nu\tau$, where m and τ satisfy the relation $t = m\tau$.

Asymptotic stability of (3) is determined by the Lyapunov exponent

$$\lambda = \lim_{t \to \infty} \frac{\log(\|\mathbf{\Phi}(t)\|)}{t}, \qquad (6)$$

which exists for all of the time-dependent flows to be considered. When the Lyapunov exponent is positive the flow is asymptotically unstable.

It is well known that $\mathbf{A}(t)$ can have a stable spectrum at each instant and yet the time-dependent system (3) can be asymptotically unstable; this is the regime in which the midlatitude atmosphere typically operates. To understand the mechanism of this destabilization, we select for analysis operators that have stable spectrum at each time instant.

The Lyapunov exponent for time-independent flows reduces to the maximum over the real parts of the spectrum of the operator $\mathbf{A}$ and the definition (6) is a natural extension of this notion of instability to time-dependent operators. In the case of time-independent flows, asymptotic stability can be assessed by examining only the spectrum of $\mathbf{A}$ without taking account of the structure

of the associated eigenvectors, but in time-dependent flows it is necessary to take account also of the non-normality of **A**, which is associated with the structure of the eigenvectors. It was shown in F&I that if the operator **A**(t) has a stable spectrum at each time instant, then for the time-dependent flow to be asymptotically unstable (i.e., $\lambda > 0$) **A**(t) must be nonnormal in all inner products and in addition exhibit transient growth in all inner products. At first sight it is surprising that the Lyapunov exponent is independent of the norm considering that the nonnormality of an operator depends on the norm (cf. appendix A). The reason that nonnormality in all inner products produces the same asymptotic growth stems from the fundamental fact that time-dependent operators with noncommuting instantaneous realizations cannot be simultaneously rendered normal by choosing a time-independent inner product (while this can easily be done for autonomous operators or time-dependent but serially commuting operators). For this reason nonautonomous operators that do not serially commute and therefore do not have the same eigenvectors are fundamentally and irreducibly nonnormal and we will show that instability of time-dependent flows with stable mean operators is a consequence of this requisite nonnormality of the operators.

3. Necessary conditions for existence of positive Lyapunov exponents in flows with stable mean operators

a. The time-dependent operator must be nonnormal and its time realizations must not commute

We begin analysis of the fundamental role of non-normality in destabilizing time-dependent operators by decomposing the time-dependent operator into a time mean and deviation operator:

$$\mathbf{A}(t) = \overline{\mathbf{A}} + \mathbf{N}(t), \qquad (7)$$

in which

$$\overline{\mathbf{A}} = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbf{A}(s) \, ds. \qquad (8)$$

In this decomposition $\overline{\mathbf{A}}$ is the autonomous mean operator and **N** the time-dependent deviation from the mean. We seek generic results independent of the specific time dependence and so require the deviation operator (matrix) **N**(t) to be a stochastic operator (matrix) with zero mean ($\overline{N_{ij}} = 0$) and bounded variance ($\overline{N_{ij}^2}$ bounded for all i, j). We choose the operator **N**(t) to be a sum of red noise processes:

$$\mathbf{N}(t) = \sum_{i=1}^{m} \epsilon_i(t) \mathbf{B}_i, \qquad (9)$$

where ϵ_i is a red noise process and $\mathbf{B}_i$ a time-independent matrix. The number of independent processes m is at most n^2 where n is the dimension of **N**. As a concrete example consider a Numerical Weather Prediction

(NWP) tangent linear model; in that case the mean operator together with the structure and temporal correlation of an orthogonal expansion of deviations from the mean operator can be used to determine the model's $\overline{\mathbf{A}}$ and **N**(t), respectively.

Consider first the case in which the commutator[1] between the mean operator and the deviation operator vanishes for all time: $[\overline{\mathbf{A}}, \mathbf{N}(t)] = 0$. In that case the propagator can be written:

$$\mathbf{\Phi}(t) = e^{\overline{\mathbf{A}}t + \int_0^t ds \mathbf{N}(s)} \approx e^{\overline{\mathbf{A}}t + O(t^{1/2})} \qquad (10)$$

given that asymptotically each entry of $\int_0^t ds \, \mathbf{N}(s)$ grows at most as $t^{1/2}$. This shows that when the commutator $[\mathbf{A}(t_1), \mathbf{A}(t_2)]$ of the operator matrix vanishes for all times t_1, t_2 the asymptotic stability of the flow is determined by the stability of the mean operator $\overline{\mathbf{A}}$.

Consider, for example, the barotropic vorticity equation on an f plane with Rayleigh friction of timescale $1/R_0$ and a mean flow consisting of a temporally varying constant shear, that is, $U(y, t) = \alpha y + \epsilon(t) y$ with $\overline{\epsilon} = 0$ and $\overline{\epsilon^2}$ bounded. The governing equation for the vorticity $\zeta = \nabla^2 \psi$ from (1) is

$$\frac{\partial \zeta}{\partial t} = -\alpha y \frac{\partial \zeta}{\partial x} - R_0 \zeta - \epsilon(t) y \frac{\partial \zeta}{\partial x}. \qquad (11)$$

We are free to determine the asymptotic stability of (11) in the vorticity variable as the choice of variable does not affect asymptotic stability properties. For this case and with the usual centered difference approximations to the derivative operator, the commutator $[\overline{\mathbf{A}}, \mathbf{N}]$ vanishes:

$$[\overline{\mathbf{A}}, \mathbf{N}] = \left[-\alpha y \frac{\partial}{\partial x} - R_0, -\epsilon(t) y \frac{\partial}{\partial x} \right] = 0, \quad (12)$$

implying that time dependence of a meridionally uniform shear cannot affect the stability of the mean flow.[2] This argument also applies to a baroclinic flow on an f-plane with time-dependent shear and this explains the stability of the Eady model with temporally varying shear (Hart 1971). Note that inclusion of either β or any dissipation other than Rayleigh friction or Ekman damping (which in this problem is equivalent to Rayleigh friction) will lead to noncommuting operators in (12) in which case no statement can then be made about the stability of the time-dependent flow from the above argument.

Consider now a temporally varying zonal velocity profile, $u(y, t)$, in an atmosphere with Rayleigh damping

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[1] The commutator of two linear operators **A**, **B** is defined as the operator $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$.

[2] Equation (12) implies that the eigenmodes of the mean operator and of the deviation operator are the same and therefore there is a variable (the projection on the modes) in which the modes are δ functions and the operators therefore commute and destabilization cannot occur.

and no vorticity gradient. Although this flow requires cancelation of the flow curvature term U_{yy} in the effective β term, $\beta - U_{yy}$, by a temporally and spatially varying $\beta(y, t)$, it is instructive as an example. The equation for perturbation vorticity is

$$\frac{\partial \zeta}{\partial t} = -\overline{U}(y)\frac{\partial \zeta}{\partial x} - R_0 \zeta - u(y, t)\frac{\partial \zeta}{\partial x}. \qquad (13)$$

The commutator

$$[\overline{\mathbf{A}}, \mathbf{N}] = \left[-\overline{U}(y)\frac{\partial}{\partial x} - R_0, \ -u(y, t)\frac{\partial}{\partial x}\right] = 0, \quad (14)$$

and consequently such an atmospheric state cannot be destabilized by velocity fluctuations of any magnitude.

Similar results can be obtained for the stability of a uniform flow $U(y, t) = U_0$ with time varying β. The equation for the vorticity takes the form:

$$\frac{\partial \zeta}{\partial t} = -U_0\frac{\partial \zeta}{\partial x} - \beta(t)\nabla^{-2}\frac{\partial \zeta}{\partial x} - (-1)^n R_n \nabla^{2n}\zeta, \quad (15)$$

and the commutator also vanishes for this case:

$$[\overline{\mathbf{A}}, \mathbf{N}] = \left[-U_0\frac{\partial}{\partial x} - (-1)^n R_n \nabla^{2n}, \ -\beta(t)\nabla^{-2}\frac{\partial}{\partial x}\right]$$
$$= 0, \qquad (16)$$

as ∇^{-2} commutes with $\partial/\partial x$ and ∇^{2n}. This proves that time variation of a meridionally uniform β cannot destabilize a constant mean flow.[3]

We conclude that destabilization of a stable mean operator by a time-dependent deviation operator requires that the operators not commute.

b. Effect of breaking the commutativity between the mean and deviation operator on the structure of the Lyapunov vector

The Lyapunov vector is the time-dependent structure to which all initial perturbations to a linear time-dependent system converge after sufficient time (cf. F&I for its properties).

We begin analysis of Lyapunov vector structure by considering a diagonalized 3×3 dynamical system with time-mean operator:

$$\overline{\mathbf{A}} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}, \qquad (17)$$

with each of the eigenvalues (λ, μ, ν) assumed to have

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[3] Inclusion in (1) of temporal variation in β with the form $\beta(t)$ in addition to variation in shear of the form $U(y, t) = (\alpha + \epsilon(t))y$ can be shown not to produce instability using simple integral arguments although the mean and deviation matrices do not commute in this case. However, a term of the form $\beta(y, t)$ may be destabilizing.

negative real part. We wish to find those deviation matrices of the form (9) generated by a single noise process, which will destabilize (17). Noise matrices $\mathbf{N}$ that independently excite the modes of the time-mean operator are of the form

$$\mathbf{N} = \epsilon(t)\mathbf{B} = \epsilon(t)\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \qquad (18)$$

but for $\mathbf{N}$ of form (18) and any choice of α, β, and γ, the commutator $[\overline{\mathbf{A}}, \mathbf{N}]$ vanishes and we know that the system is stable. The mean operator $\overline{\mathbf{A}}$ cannot be destabilized by a time-varying deviation $\mathbf{N}(t)$ unless this deviation operator mixes the eigenmodes, that is, does not commute with the mean operator. We wish to know if there are other requirements on the structure of a noise matrix in order that it destabilize a mean operator. Consider the deviation operator produced by the noise matrix $\mathbf{B}$ in (18) with $\beta = \gamma = 0$ to which has been applied a rotation $\mathbf{R}$ about the direction of the third eigenvector giving a deviation operator $\mathbf{N} = \epsilon(t)\mathbf{RB}$, which has the effect of mixing the first two eigenvectors. Although the commutator $[\overline{\mathbf{A}}, \mathbf{N}] \neq 0$ it can be shown that $\overline{\mathbf{A}}$ is not destabilized for any magnitude of ϵ. The reason is that although $\mathbf{N}$ mixes the first two modes, this mixing is one way; that is, the first eigenvector is rotated into the direction of the second but not vice versa. We are thus led to conclude that destabilization of a stable operator requires that the deviation operator mix at least two modes in such a manner that each projects on the other. A minimum such deviation operator for destabilization is produced by the noise matrix $\mathbf{RB}$ where $\mathbf{R}$ is a rotation with respect to the third eigenvector and $\mathbf{B}$ the noise matrix in (18) with α and β nonzero and $\gamma = 0$. In this case the system is unstable for $\overline{\epsilon^2}$ sufficiently large and the structure of the Lyapunov vector results from mixing of the first two modes. These arguments are general and any set of stable modes however damped can be destabilized by mode mixing in this way with noise of sufficient magnitude. Moreover, the Lyapunov vector that results must be a superposition of at least two eigenmodes of the mean operator $\overline{\mathbf{A}}$.

We turn now to the role of nonnormality of the mean operator. Consider a general matrix $\overline{\mathbf{A}}$; any such matrix can be diagonalized by transformation to normal coordinates. The general time-varying system with one noise process,

$$\frac{d\psi}{dt} = \overline{\mathbf{A}}\psi + \epsilon(t)\mathbf{B}\psi, \qquad (19)$$

in the variable $y = \mathbf{U}^{-1}\psi$, with $\mathbf{U}$ the eigenvector matrix of $\overline{\mathbf{A}}$, becomes

$$\frac{dy}{dt} = \overline{\mathbf{\Lambda}}y + \epsilon(t)(\mathbf{U}^{-1}\mathbf{BU})y, \qquad (20)$$

in which $\overline{\mathbf{\Lambda}}$ is the diagonal matrix of the eigenvalues of the time-mean operator. This is the system we in-
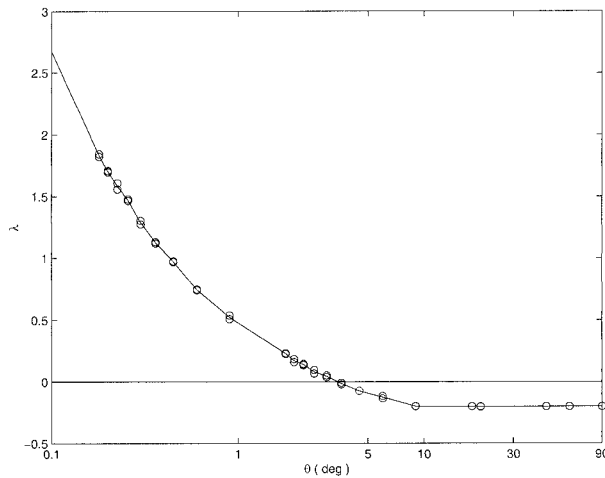
FIG. 1. The Lyapunov exponent as a function of the angle (in degrees) between the two nonorthogonal eigenvectors, which measure the nonnormality of the mean operator in the 3×3 example (21). The least damped eigenvector eigenvalue is -0.3 and the two nonnormal eigenvectors have eigenvalues $-0.5 \pm i0.1$. The noise process is red with variance 0.2 and a decorrelation time of one nondimensional time unit.

vestigated in the previous paragraphs. But in (20) it is crucial that the original deviation matrix is transformed to a deviation matrix in which the effective magnitude of the noise process ϵ has been increased by as much as the condition number $\|\mathbf{U}^{-1}\| \|\mathbf{U}\|$ of the eigenvector matrix of the time-mean operator because $\|\mathbf{U}^{-1}\mathbf{B}\mathbf{U}\| \leq \|\mathbf{U}^{-1}\| \|\mathbf{B}\| \|\mathbf{U}\|$, where $\| \cdot \|$ denotes the Euclidean norm. It follows that a highly nonnormal mean operator $\overline{\mathbf{A}}$, which is characterized by very large condition number, can be destabilized by very little noise provided that the deviation matrix mixes at least two of the eigenmodes of the nonnormal subspace of $\overline{\mathbf{A}}$.

As an example consider the 3×3 system with $\overline{\mathbf{\Lambda}}$ given by (17) and assume that this system was produced by diagonalization of a time-mean matrix having eigenvectors the columns of

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \cos\theta \\ 0 & 0 & \sin\theta \end{pmatrix}. \tag{21}$$

The degree of nonnormality induced by the nonorthogonal subspace is controlled by the angle θ between the two nonorthogonal eigenvectors. Consider a noise matrix $\mathbf{N} = \mathbf{R}_{\pi/4}\mathbf{B}$, where $\mathbf{B}$ is given by (18) with $\alpha = 0$ and $\beta = \gamma = 1$ and $\mathbf{R}_{\pi/4}$ is a rotation about the first eigenvector by an angle $\pi/4$. This noise matrix mixes the two nonorthogonal eigenvectors and as argued above for fixed magnitude noise process $\epsilon(t)$ the degree of instability is expected to increase with the degree of nonnormality of the mean operator $\overline{\mathbf{A}}$ as measured by θ. Indeed this is verified in Fig. 1, which shows the Lyapunov exponent as a function of θ.

c. On Rayleigh's theorem in time-dependent flows

For inviscid zonally homogeneous time-independent flows, Rayleigh's theorem provides a necessary condition for the existence of exponential instability, which is rooted in momentum conservation. We inquire whether a similar theorem constrains the possibility of positive Lyapunov exponents in inviscid time-dependent zonally homogeneous flows.

Consider the momentum stress divergence $M = d[uv]/dy$, where the bracket denotes the zonal average of u, v, which are, respectively, the zonal and meridional velocity. If the boundary conditions require vanishing of the perturbation momentum stress at $y = \pm 1$, then the perturbation cannot alter the integrated momentum of the mean flow because

$$\int_{-1}^{1} M \, dy = 0. \tag{22}$$

The tangent linear vorticity equation

$$\frac{\partial \zeta}{\partial t} = -U(y, t)\frac{\partial \zeta}{\partial x} - Q_y(y, t)v + \mathcal{D}, \tag{23}$$

implies the enstrophy tendency equation

$$\frac{\partial [\zeta^2/2]}{\partial t} = -Q_y(y, t)[v\zeta] + [\zeta\mathcal{D}], \tag{24}$$

where $\mathcal{D}$ denotes the dissipation, and $Q_y(y, t)$ the time-dependent vorticity gradient of the background flow. Integrating over the channel width, we obtain in the absence of dissipation ($\mathcal{D} = 0$) the following integral constraint at every instant:

$$\frac{1}{2} \int_{-1}^{1} \frac{\dfrac{\partial [\zeta^2]}{\partial t}}{Q_y(y, t)} \, dy = \int_{-1}^{1} [v\zeta] \, dy$$

$$= -\int_{-1}^{1} \frac{d[uv]}{dy} \, dy = 0, \tag{25}$$

which necessarily requires that in the time mean

$$\overline{\frac{1}{2} \int_{-1}^{1} \frac{\partial [\zeta^2]/\partial t}{Q_y(y, t)} \, dy} = 0. \tag{26}$$

A positive Lyapunov exponent also implies that at each latitude

$$\overline{\left[\frac{\partial \zeta^2}{\partial t}\right]} \geq 0. \tag{27}$$

If the vorticity gradient Q_y is time independent, we obtain from (26) as a necessary condition for instability that it change sign (Rayleigh 1880). However, this result does not generalize to time-dependent flows because episodic decrease in $[\zeta^2]$ can coincide with small values of Q_y so as to maintain constraint (26) while $\overline{\partial [\zeta^2]/\partial t} > 0$. The possibility of satisfying the integral constraint
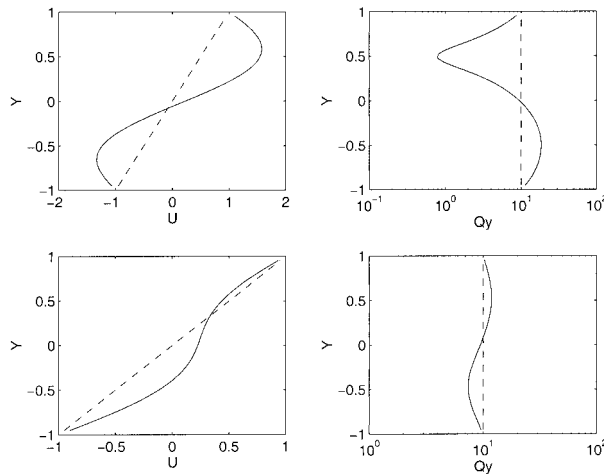
FIG. 2. (left panels) The mean flow velocity as a function of latitude for the Rayleigh stable example. (right panels) The associated mean vorticity gradient with $\beta = 10$.
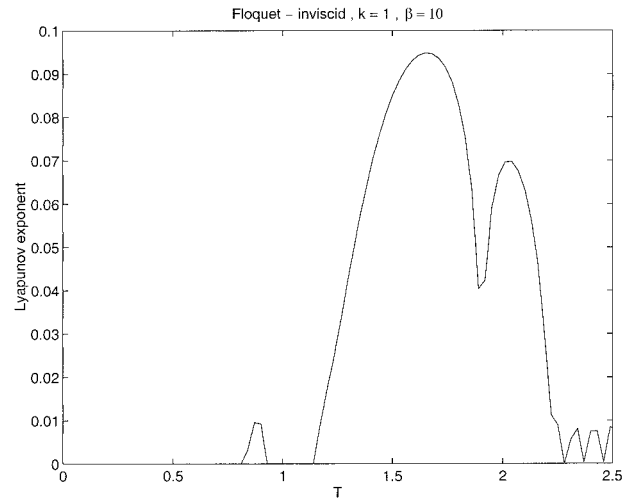

FIG. 3. The Lyapunov exponent as a function of the switching period T for the Rayleigh stable example, consisting of the velocity profiles in Fig. 2. The time-dependent flow results from periodic switching every T time units of the flows shown in Fig. 2. The case shown is for zonal wavenumber $k = 1$ and $\beta = 10$.

(26) in time-dependent flows while maintaining the positivity of $Q_y(y, t)$ is illustrated with a simple schematic example in appendix B; however, in order to verify that the Rayleigh theorem does not constrain the stability of dynamical systems arising from time-dependent flows it suffices to consider a time periodic flow produced by periodic discontinuous alteration between two mean velocity profiles. We choose nondimensional $\beta = 10$ in order to ensure that the associated mean vorticity gradient is of one sign. The two velocity profiles and the associated mean vorticity gradients are shown in Fig. 2. Each flow state is maintained for a time interval T. The Lyapunov exponent is readily calculated using Floquet analysis. The time-dependent flow is unstable and the growth rate as a function of the switching period T is shown in Fig. 3 for zonal wavenumber $k = 1$.

A situation in which (26) does require stability of a time-dependent flow is that of time-varying meridionally uniform shear flows, $U(y, t) = \alpha(t) y$, with time-varying $\beta(t)$ of one sign. For this case the constraint (25) becomes

$$\frac{1}{\beta(t)} \frac{\partial}{\partial t} \int_{-1}^{1} [\zeta^2] dy = 0, \qquad (28)$$

which implies constant $\int_{-1}^{1} [\zeta^2] \, dy$ from which it follows that perturbations cannot grow in vorticity and therefore cannot have positive Lyapunov exponent, as discussed in section 3a.

4. The mechanism producing instability in time-dependent flows with stable mean operators

We will show that temporal variation of **A** in (3) with sufficient amplitude leads generically to asymptotic instability (by generically we mean unless very specific conditions are met by **A**). We isolate this generic insta-

bility of time-dependent flows and understand how it is essentially different from asymptotic instability of autonomous dynamical systems by requiring again that at each instant **A** have neutral or damped spectrum. It is necessary for such a neutral or damped operator to be nonnormal for even instantaneous perturbation growth to occur. However, while it is well known that nonnormality can lead to episodic growth, it is another matter to sustain these instances of growth so as to produce asymptotic instability. Key to understanding the mechanism of this generic asymptotic instability in time-dependent operators is the observation that if the instantaneously evaluated operators do not commute with each other then there is no single metric in which the time-dependent operator is normal at all times. The instability results from concatenating the finite growth achieved by the optimal vectors of the instantaneous nonnormal operator while avoiding through time dependence the decay that would eventually occur if any of the instantaneous nonnormal but asymptotically stable operators were to persist indefinitely.

This process of destabilization by time dependence can be understood conceptually through an analysis method proposed by Zel'dovich et al. (1984) to explain the mean exponential increase in the length of material lines embedded in a random divergenceless flow. The analysis is simplified if the continuous operator **A**(t) is approximated by a piecewise constant sequence of operators, **A**$_i$, each of which represents the continuous operator for a finite time interval τ. The initial state ψ_0 evolves to the following state at time $n\tau$:

$$\psi_n = \left(\prod_{i=1}^{n} e^{\mathbf{A}_i \tau} \right) \psi_0. \qquad (29)$$

Passing to the limit of long time the Lyapunov exponent given in (6) can be expressed as

$$
\lambda = \lim_{n \to \infty} \ln \left(\frac{\|e^{A_n \tau} \psi_{n-1}\|}{\|\psi_{n-1}\|} \cdots \frac{\|e^{A_1 \tau} \psi_0\|}{\|\psi_0\|} \right) \Big/ n\tau
$$

$$
\equiv \lim_{n \to \infty} \frac{\sum_{i=1}^{n} \ln G_i}{n\tau}, \tag{30}
$$

where each element of the product is the incremental growth of the Lyapunov vector magnitude, G_i, over the interval τ:

$$
G_i = \frac{\|e^{A_i \tau} \psi_{i-1}\|}{\|\psi_{i-1}\|}. \tag{31}
$$

Consequently, the time average of the logarithm of the individual growths G_i approaches the Lyapunov exponent in (6) as $n\tau \to \infty$. The total perturbation growth over $n\tau$ can be expressed alternatively in terms of the projection of the state vector on the optimal (or right singular) vectors of the time-varying propagator:

$$
G_i = \sqrt{\sum_{k=1}^{n} \alpha_k^2 \sigma_k^2}, \tag{32}
$$

where the α_k are the projection coefficients of the unit vector lying in the direction of the state vector on the optimal (or right singular) vectors of the incremental propagator $e^{A_i \tau}$ and the σ_k are the associated optimal growths (singular values).

Consider a nondissipative operator that preserves volume in state space so that $\Pi_{k=1}^{n} \sigma_k = 1$. If we assume that the state vectors are uniformly distributed over the state space hypersphere, then the Lyapunov exponent would be the averaged growth rate over the hypersphere:

$$
\lambda = \left\langle \frac{\ln G}{\tau} \right\rangle = \frac{1}{\tau |\Sigma|} \int_{\Sigma} d\Sigma \, \ln \left(\sqrt{\sum_{i=1}^{n} \alpha_i^2 \sigma_i^2} \right), \tag{33}
$$

where Σ is the surface of the sphere $\Sigma_{i=1}^{n} \alpha_i^2 = 1$ and $|\Sigma|$ its area. It can be shown that for all volume-preserving distributions of σ_i (so long as at least one $\sigma_i > 1$), the Lyapunov exponent $\lambda > 0$, so that the dynamical system is asymptotically unstable despite the fact that at each time step the system is neutral (appendix C). The fundamental reason for this generic instability can be traced to the convexity of the logarithmic function. It is remarkable that growth is inevitable even when the state vector projects uniformly on the optimal vectors of the instantaneous operator. However, in practice for atmospheric applications, expression (33) gives significantly more accurate estimates for the Lyapunov exponent if the observed statistical distribution of the state vector on the optimal vectors is taken into account by assigning the α_i's in (33) their observed statistical properties of variance and temporal correlation.

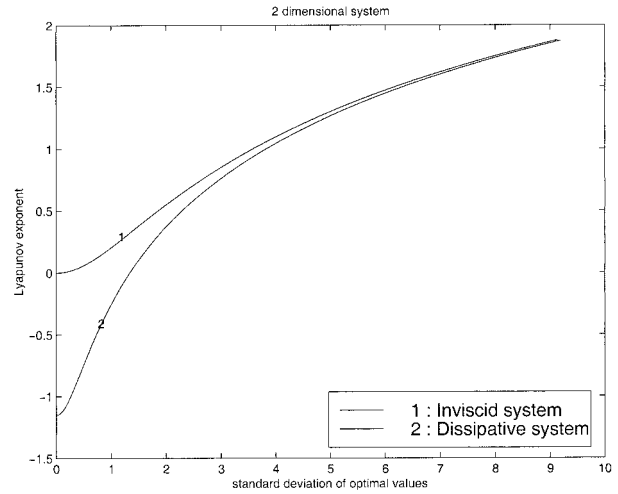The magnitude of growth calculated using (33) can

FIG. 4. The Lyapunov exponent as a function of the standard deviation of the optimal growths for a two-dimensional dissipationless (curve 1) and a representative dissipative system (curve 2).

be shown to depend primarily on the variance of the incremental growths over the characteristic interval τ, which can be conveniently measured by the standard deviation (std) of the optimal growths about their mean, which for nondissipative systems is unity. For nondissipative systems if the std of the optimal growths about unity is zero, the Lyapunov exponent is also zero, but as we have seen, for any other value of the std of the optimal growths of the incremental propagator the Lyapunov exponent is positive. As the variance increases the Lyapunov exponent approaches asymptotically the logarithm of the std of the optimal growths of the incremental propagators.

We illustrate the above by Monte Carlo evaluation of (33). The dependence of the Lyapunov exponent on the logarithm of the std of the optimal growths can be seen in Fig. 4 an example with $n = 2$ degrees of freedom, and in Fig. 5 an example with $n = 20$ degrees of freedom. An example with dissipation is also included in these graphs. While for dissipative systems arbitrarily small optimal growth std's do not necessarily result in positive Lyapunov exponent, even for dissipative systems the exponent soon asymptotes to that obtained in the inviscid case as can be seen in the example in Fig. 4, demonstrating that this generic mechanism leading to destabilization of time-dependent systems is robust.

5. A Floquet approximation to the Lyapunov exponent

A simple model atmospheric system provides an example of the above results. Consider perturbations of a time-varying flow $U(y, t)$ governed by the barotropic operator (4) restricted for simplicity to be dissipationless. Consider random flow states chosen discontinuously every time interval τ and obtain a further simplification by assuming that the sequence of these ran-
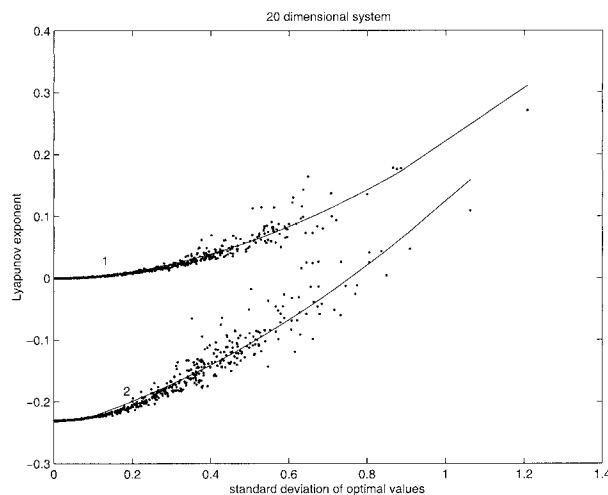
FIG. 5. The Lyapunov exponent as a function of the standard deviation of the optimal growths of a 20D dissipationless (curve 1) and a representative dissipative system (curve 2). The spread in the values is due to the fact that the optimal growths in a high-dimensional system are not characterized completely by their standard deviation. The continuous line provide a quadratic fit to the Lyapunov exponents.

dom but piecewise steady flow states is repeated with period $T = n\tau$. The assumption of periodicity allows us to easily obtain an approximation to the Lyapunov exponent by Floquet analysis. Because of the periodicity the Lyapunov exponent is given by eigenanalysis of the propagator, Φ_T, that advances the state of the system a time interval equal to the period, T, of the time-dependent flow. The Lyapunov exponent is given by

$$\lambda = \frac{1}{T} \log(|\lambda_p|), \tag{34}$$

where λ_p is the eigenvalue of the propagator with greatest absolute value.

It is an assumption easily verified that as the period $n\tau$ is increased the growth rate of the first eigenmode of the Floquet approximation approaches the growth rate of the Lyapunov vector of the aperiodic flow, that is, the growth rate obtained in the case the states never repeat. We consider recurrence intervals $n = 2, 10, 50, 100$ with each flow state constrained to be neutral by imposition of a large nondimensional $\beta = 10$, which renders the mean vorticity gradient one-signed for the velocity states chosen, assuring stability of the individual states by the Rayleigh theorem. We have verified that the Lyapunov exponent for n large enough approaches the Lyapunov exponent of an aperiodic system with the same fluctuation statistics of $U(y, t)$ and the approach to this limit is instructive. In the two-flow state case ($n = 2$) (upper left Fig. 6), we obtain islands of instability that are characteristic of parametric instability of time-dependent systems with strictly periodic variation of parameters. An example of such a system is the harmonic oscillator with periodically varying re-

storing force, and analysis of this system reveals the islands of instability of the familiar Mathieu equation. The islands of instability in our example gradually blend into a continuum as the number of states increases ultimately producing the universal instability for all τ on $[0, \infty]$ (cf. the lower-right graph of Fig. 6 for $n = 100$). It can be verified that the Lyapunov exponent vanishes on approach to both limits of this interval $\tau \to 0$ and $\tau \to \infty$ and that the maximum Lyapunov exponent occurs at an intermediate value of τ.

The asymptotic behavior of the Lyapunov exponent as $\tau \to 0$ and $\tau \to \infty$ found in the above example is generally valid. It is shown in appendix D that for a fixed number of independent piecewise constant realizations if fluctuations of the operator are bounded, uncorrelated, and rapid, then in the limit $\tau \to 0$ the Lyapunov exponent vanishes. In this limit the stability properties of the time-dependent system approach the stability properties of the mean autonomous system. In the large τ limit λ approaches the mean of the decay rates of the least-damped modes of the incremental operators. Given that for an operator that has a stable mean and stable realizations at each time the Lyapunov exponent vanishes for $\tau \to 0$ and $\tau \to \infty$, we anticipate that if destabilized by noise the Lyapunov exponent reaches a maximum at an intermediate τ and that this τ for which the maximum Lyapunov exponent is obtained depends on the timescale of transient growth. However, if the instantaneous operators are not restricted to be stable, then the Lyapunov exponent does not necessarily vanish as $\tau \to \infty$ but rather asymptotes to the average of the maximum growth rates of the instantaneous states and there may not be an intermediate maximum.

In order to sharpen the correspondence between our general analysis of instability of time-dependent operators and the specific instability of the barotropic atmospheric model, we need to more closely relate the parameters in the analysis to the model. While the atmospheric flow does not evolve in piecewise constant steps, nevertheless the time variation of the flow state is characterized by a finite decorrelation time of the order of a few days, which we take to be the τ appropriate for our analysis. In our example we assume a timescale of jet vacillation and a mean state of the jet as well as the spatial spectrum of variance about the mean state. Such a model consists of a flow state decomposed into a mean part and a time-dependent stochastic part, which is adequately modeled for our purposes as a red noise process with the observed spatial structure, variance, and decorrelation time.

6. Error growth in a barotropic atmosphere

As a simple model of the above form, consider evolution of errors in a zonally homogeneous barotropic channel with constant nondimensional mean shear flow, $\overline{U} = y$, on which are superimposed time-dependent wind
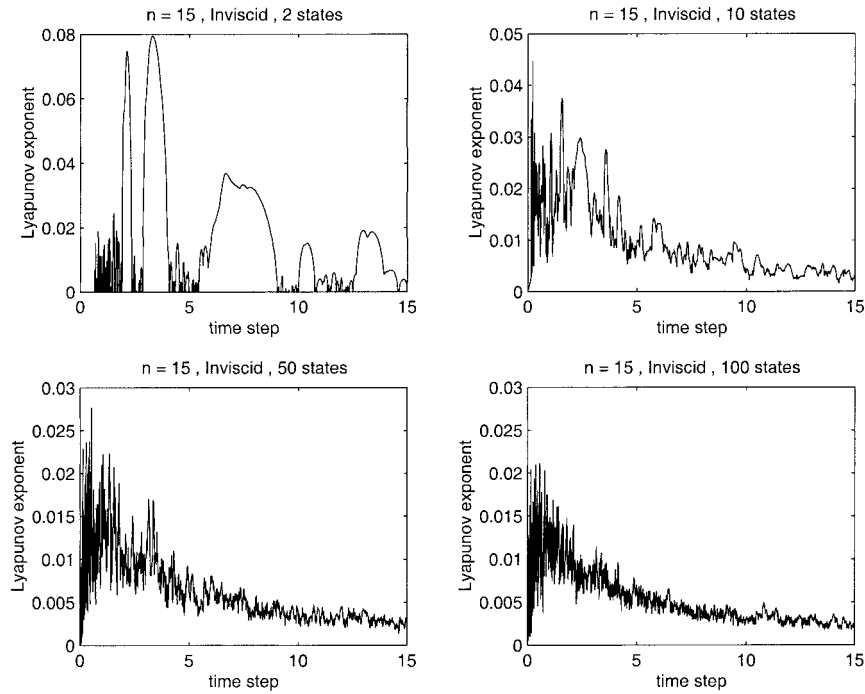
FIG. 6. The Lyapunov exponent as a function of the time step τ in an inviscid barotropic flow consisting of a periodic recurrence of (upper-left corner) two-flow states, (upper-right corner) ten-flow states, (lower-left corner) 50 flow states, and (lower-right corner) 100 states. Although each state is neutrally stable and obeys the Rayleigh criterion for stability, the resulting time-dependent system is unstable, approaching neutral stability in the limit $\tau \to 0$ and $\tau \to \infty$.

components produced by modulation of the six gravest latitudinal harmonics

$$U_f(y, t) = \sum_{m=1}^{3} \epsilon_m(t) \cos\left(\frac{2m - 1}{2}\pi y\right)$$

$$+ \sum_{m=1}^{3} \epsilon_m(t) \sin m\pi y, \qquad (35)$$

where $\epsilon(t)$ are red noise processes, with identical variance and decorrelation time, T_c, and zero mean. The magnitude of the velocity fluctuation is characterized by its nondimensional rms average. Typical velocity profile realizations for rms velocity fluctuations of magnitude 0.3 nondimensionalized by the mean shear and the half-channel width are shown in Fig. 7.

Consider the viscously damped barotropic vorticity Eq. (1), made nondimensional by spatial scale of the half-channel width, L, and timescale of the inverse shear $1/\alpha$. In these variables the Reynolds number is Re $= \alpha/\nu$ where ν is the coefficient of viscosity, the nondimensional magnitude of the noise process is related to its dimensional magnitude, $\tilde{\epsilon}$, by $\tilde{\epsilon}/\alpha L$, and the nondimensional β parameter is related to its dimensional value, $\tilde{\beta}$, by $\tilde{\beta}L/\alpha$. Note that in this nondimensionalization as the dimensional value of the shear $\alpha \to \infty$, the nondimensional value of the rms fluctuations $\epsilon \to 0$ while

Re $\to \infty$. In the calculations that follow we do not require that each realization be stable.

The nondimensional barotropic operator (4) can be decomposed into a mean part $\overline{\mathbf{A}}$, which is the barotropic operator governing evolution of perturbations on the Couette profile $\overline{U} = y$,
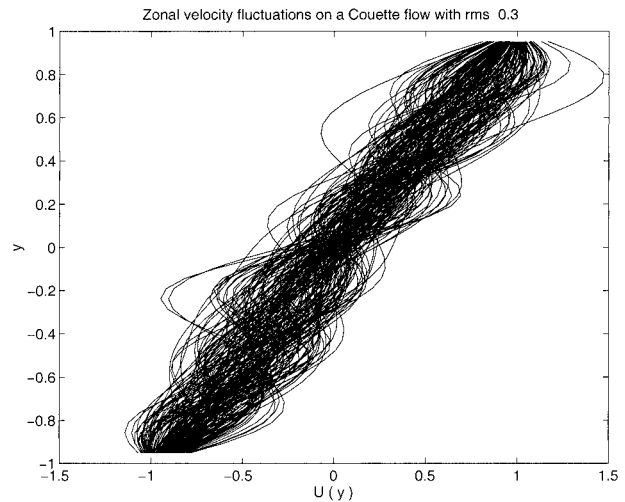

FIG. 7. The velocity profile that results from modulation of the five gravest harmonics in the barotropic flow example. The rms velocity fluctuation is 0.3.
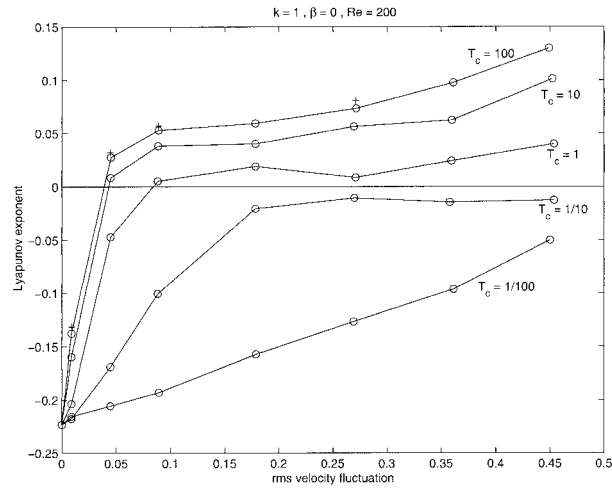
FIG. 8. The Lyapunov exponent as a function of rms velocity fluctuation for different decorrelation times of the noise process. The decorrelation times are marked on the curves. The Reynolds number is Re = 200 and the zonal wavenumber $k = 1$ and $\beta = 0$. For $T_c = 100$ the Lyapunov exponent has approached its asymptotic $\tau \to \infty$ limit, which is equal to the mean over the maximum real parts of the eigenvalues of the instantaneous operators (marked with crosses).

$$\overline{\mathbf{A}} = \nabla^{-2}\left(-iky\nabla^2 - ik\beta + \frac{1}{\mathrm{Re}}\nabla^4\right), \qquad (36)$$

and a deviation part,

$$\mathbf{N} = \nabla^{-2}\left(-ikU_f(y, t)\nabla^2 + ik\frac{d^2U_f(y, t)}{dy^2}\right). \quad (37)$$

Mean and deviation operators do not commute with each other and furthermore the deviation operator mixes the modes of $\overline{\mathbf{A}}$. Consequently, we expect the stable mean flow to be destabilized by introduction of velocity fluctuations of sufficient magnitude.

Forward integration yields the Lyapunov exponent. We have verified that the same Lyapunov exponent is obtained if the flow is approximated by a sequence of steady states each with time duration τ equal to the decorrelation time of the red noise process. The dependence of the Lyapunov exponent on the rms value of the velocity fluctuations for various decorrelation times of the noise, T_c, is shown in Fig. 8 for zonal wavenumber $k = 1$, Re = 200, and $\beta = 0$. As the decorrelation time decreases the Lyapunov exponent approaches the decay rate of the stable mean operator (36).[4] As the decorrelation time increases the Lyapunov exponent asymptotes to the average of the maximum

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[4] It should be noted that the limit $T_c \to 0$ is not the white noise limit. The white noise limit requires also that the fluctuations increased in amplitude as $T_c^{-1/2}$ so that as $T_c \to 0$ the noise approaches unit delta correlation in time. This white noise limit implies divergent velocity fluctuations that are unphysical. In the white noise limit the Lyapunov exponent does not vanish, as shown in F&I.
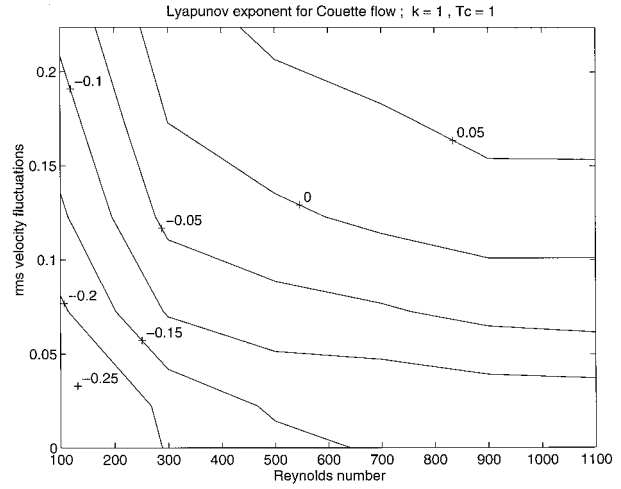

FIG. 9. The Lyapunov exponent as a function of Reynolds number and rms fluctuation about a mean Couette velocity profile. The case is for zonal wavenumber $k = 1$, $\beta = 0$, and noise decorrelation time $T_c = 1$.

growth rates of the instantaneous realizations. Because the individual flows have not been restricted to be stable as $T_c \to \infty$, the Lyapunov exponent approaches a positive limit. This contrasts with the $T_c \to \infty$ limit shown in Fig. 6 in which case the individual realizations were chosen to be stable and therefore the Lyapunov exponent tended to zero. The relevant regime for atmospheric examples is the case in which each realization is stable.

A contour plot of the resulting Lyapunov exponent as a function of nondimensional rms velocity fluctuations and Reynolds number is shown in Fig. 9 for zonal wavenumber $k = 1$ and a red noise process with decorrelation time $T_c = 1$. As expected the growth increases with increasing Reynolds number, and with increasing velocity fluctuations.

We expect that as β increases the Lyapunov exponent will decrease because the nonnormality of the operator is suppressed by β. The typical dependence is shown in Fig. 10, which shows the Lyapunov exponent as a function of β for Re = 200, wavenumber $k = 1$, rms velocity fluctuations 0.4, and decorrelation time $T_c = 1$. We also expect the Lyapunov exponent to be suppressed at high zonal wavenumbers again because nonnormality is suppressed at large k. We verify this by considering the flow damped with Rayleigh friction (Ekman damping) to avoid confusing this effect with the decrease of the Lyapunov exponent with wavenumber arising due to the wavenumber dependence of viscous damping. Figure 11 shows the typical dependence of Lyapunov exponent on zonal wavenumber.

In general, if the mean operator $\overline{\mathbf{A}}$ is unstable, then in the limit $T_c \to 0$ the Lyapunov exponent and structure approach the growth rate and structure of the most unstable mode of $\overline{\mathbf{A}}$. In the limit $T_c \to \infty$, the growth rate becomes the average of the growth rates of the individual realizations and the structure assumes the form

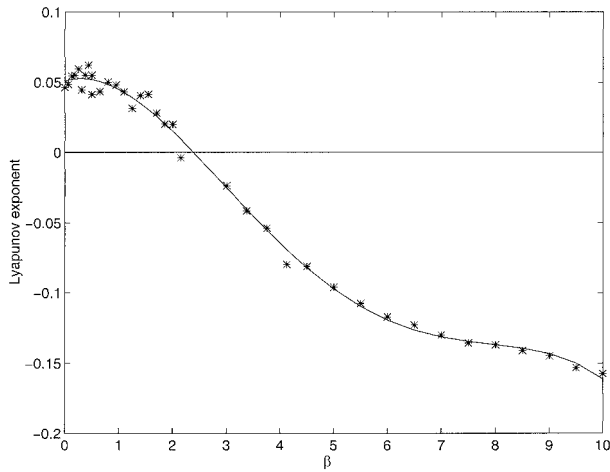FIG. 10. The Lyapunov exponent as a function of β. The case is for rms velocity fluctuation 0.4 about a constant shear flow, Reynolds number Re = 200, zonal wavenumber $k = 1$, and noise decorrelation time $T_c = 1$.
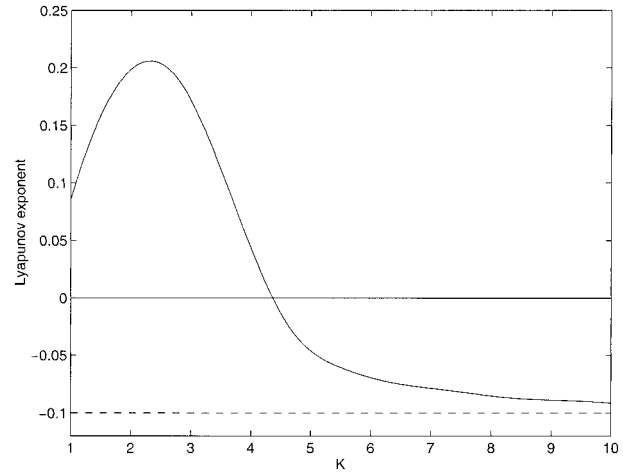

FIG. 11. The Lyapunov exponent as a function of zonal wavenumber k. The case is for rms velocity fluctuation 0.4 about a constant shear flow, $\beta = 0$, and noise decorrelation time, $T_c = 1$. The dissipation here is Ekman damping with coefficient $R_1 = 0.1$.

of the most unstable mode of the operator at a given time. For intermediate T_c assuming that the modes of $\overline{\mathbf{A}}$ are mixed, the structure becomes a mixture of modes and the time-dependent operator may be stabilized if the highly decaying modes are preferentially mixed; alternatively the nonnormal subspace of the time-dependent operator may dominate the Lyapunov exponent producing a first Lyapunov exponent and structure unrelated to the most unstable eigenmode of the mean operator; more likely, however, is for the unstable eigenmode to be part of the mixed and nonnormal subspace producing the growth. If now, as in the case at hand, the mean operator is stable the structure of the Lyapunov vector in the $T_c \to 0$ and $T_c \to \infty$ limit is as in the case of unstable $\overline{\mathbf{A}}$: for $T_c \to 0$ it is the structure of the least damped mode, for $T_c \to \infty$ it assumes the form of the maximally growing (alternatively least damped) structure of each flow realization. This brings up the question of how to most economically characterize the structure of the Lyapunov vector for intermediate T_c and we consider the typical case of $T_c = 1$. A snapshot of the time-varying Lyapunov vector at four consecutive time instants, each a decorrelation time interval apart, is shown in Fig. 12. The zonal wavenumber is $k = 2$, the Reynolds number is Re = 800, $\beta = 0$, the rms velocity fluctuation is 0.16, and the associated Lyapunov exponent is $\lambda = 0.2$.

The Lyapunov vector assumes with time a statistically steady structure. We first determine the statistics of its structure on the basis of the eigenvectors of the mean operator. The streamfunction eigenvectors of the mean operator are highly nonnormal, and expansion of the Lyapunov vector in this basis should not be confused with projection on an orthogonal basis. We pursue this expansion following the discussion of section of 3b, in order to clarify the mechanism of destabilization. We

transform the time-dependent barotropic equation to the normal coordinates of the mean operator proceeding as in (20). In the variable $y = \mathbf{U}^{-1}\psi$, where $\mathbf{U}$ is the eigenvector matrix of the mean operator (36), the time-dependent perturbation vorticity equation is governed by a diagonal mean operator whose eigenvectors can serve as an orthogonal basis for expansion of the Lyapunov vector; it is the coordinate basis of the projection on the eigenvectors of $\overline{\mathbf{A}}$. The eigenvectors are ordered in ascending order of their decay rate. The distribution of the eigenvalues of the mean operator is shown in Fig. 13 for zonal wavenumber $k = 2$, $\beta = 0$, and Re = 800. The mean coefficient of projection of the normalized Lyapunov vector on the eigenvectors of the mean operator and its standard deviation is shown in Fig. 14. The Lyapunov vector is found to be primarily a superposition of the eigenvectors associated with the numbered eigenvalues shown in Fig. 13. These eigenvectors form the eigenvector subspace of the operator, which is most nearly linearly dependent. As a result their excitation leads to robust growth, which through time dependence, can lead in the manner discussed in section 3b, to a positive Lyapunov exponent. Perhaps the most direct way to measure the contribution of an eigenvector, $\mathbf{e}$, to the nonnormality of $\overline{\mathbf{A}}$ is to consider

$$\nu(\mathbf{e}) = \frac{|\mathbf{b}|\,|\mathbf{e}|}{(\mathbf{b},\,\mathbf{e})}, \tag{38}$$

where $\mathbf{b}$ is the biorthogonal[5] of $\mathbf{e}$ and the inner product

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[5] If $\mathbf{U}$ denotes the matrix with columns, the eigenvectors, the biorthogonal $\mathbf{b}$ of the eigenvector $\mathbf{u}$, corresponding to a given column of $\mathbf{U}$, is the corresponding column of the matrix $(\mathbf{U}^{-1})^{\dagger}$ ($\dagger$ denotes the Hermitian transpose). The biorthogonal of $\mathbf{u}$ is orthogonal to all other eigenvectors.
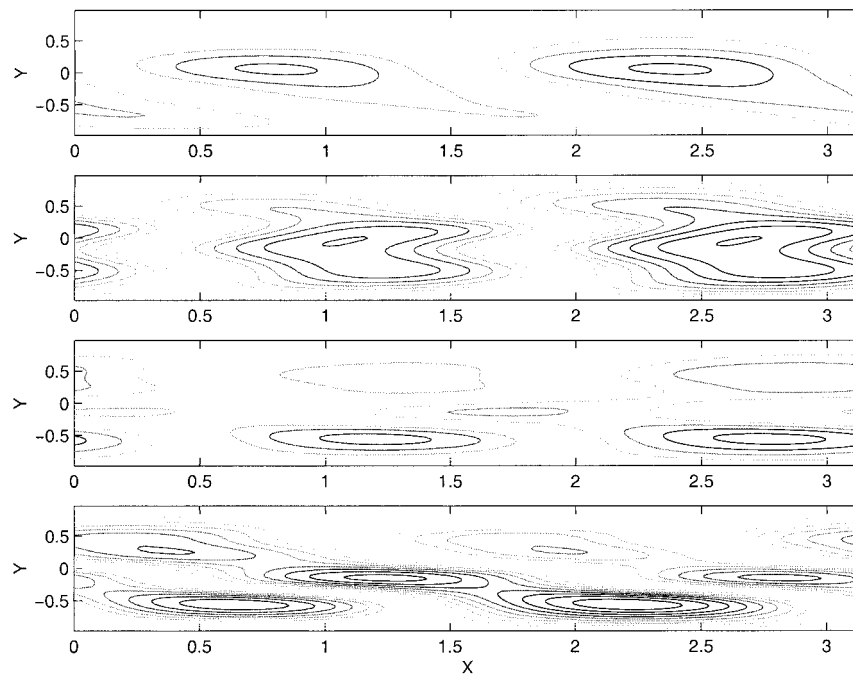
FIG. 12. The structure of the Lyapunov vector in the zonal (x), meridional (y) plane at four consecutive times separated by T_c. The rms velocity fluctuation is 0.16 and the noise decorrelation time is $T_c = 1$. The zonal wavenumber is $k = 2$, $\beta = 0$, and the Reynolds number is Re = 800. The Lyapunov exponent is $\lambda = 0.2$. At first (top panel) the Lyapunov vector is configured to grow producing an increase over T_c of 1.7, in the next period the Lyapunov vector has assumed a decay configuration (second panel from top) and suffers a decrease of 0.7, subsequently (third panel from top) it enjoys a slight growth of 1.1, and finally (bottom panel) a growth by 1.8.
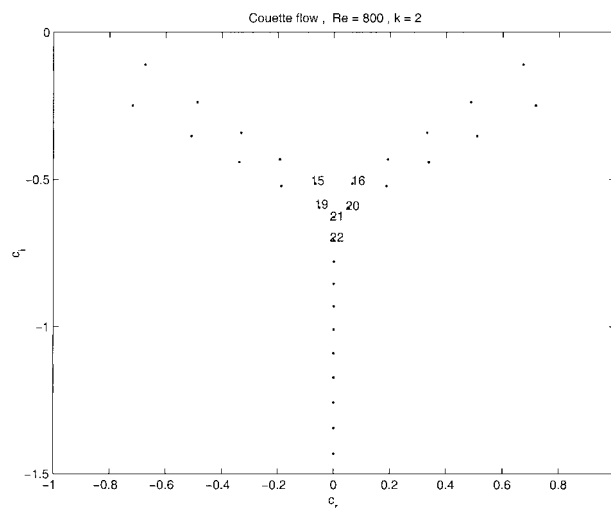

FIG. 13. Distribution of eigenvalues for the Couette flow at $k = 2$ and Re = 800. Only the least decaying eigenvalues are shown (dots). The remaining eigenvalues are highly damped diffusive modes with zero phase speed ($c_r = 0$). The Lyapunov vector for the time-dependent flow with velocity fluctuating about the constant shear profile as in (35) is predominantly a superposition of the numbered eigenvectors (cf. Fig. 14). These numbered eigenvalues denote the most nonorthogonal eigenvectors (cf. Fig. 15).
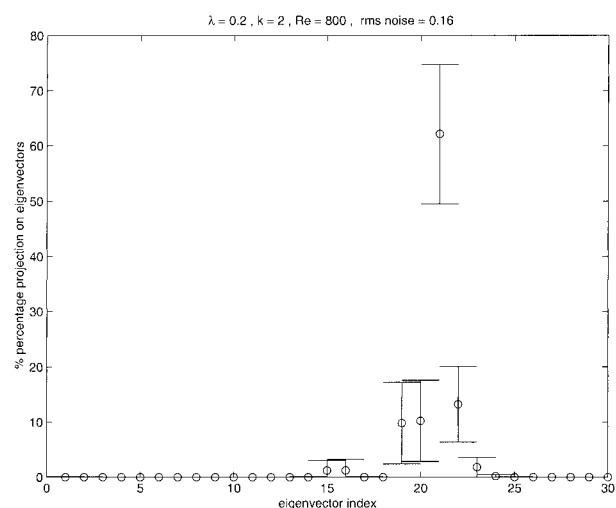

FIG. 14. Mean projection of the Lyapunov vector on the eigenvectors of the mean operator (36). The eigenvectors are indexed in ascending order of their decay rates. The error bars show the standard deviation of the projection coefficients. The Reynolds number is Re = 800, the zonal wavenumber is $k = 2$, $\beta = 0$, and the rms velocity fluctuation about the constant shear is 0.16. Note that the Lyapunov vector consists primarily of superposition of eigenvectors with eigenvalues denoted by their order number in Fig. 13.
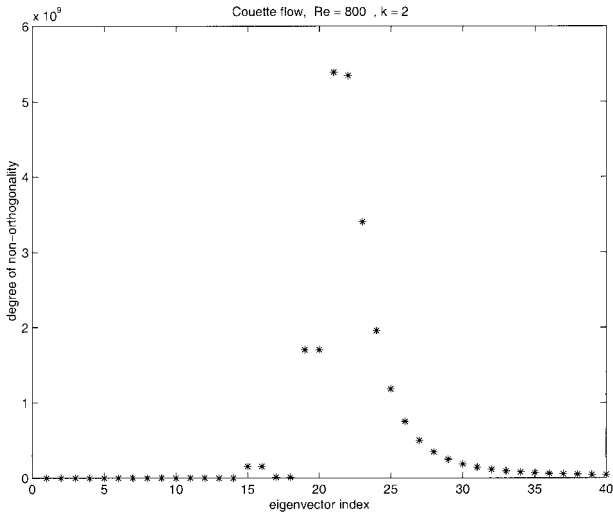
FIG. 15. Degree of nonorthogonality as defined in (38) for the first 40 eigenvectors (ordered in ascending order of their decay rate) of the mean Couette flow at $k = 2$ and Re $= 800$. The most nonnormal eigenvectors are those that correspond to eigenvalues denoted by their number in Fig. 13.

in the denominator of (38) is the dot product of the two vectors. For real eigenvectors we recognize that (38) is the secant of the angle between the eigenvector and its biorthogonal. We also recognize that $\nu(\mathbf{e})$ gives the gain in initial projection obtained when an eigenvector is

optimally excited by introducing at the initial time its biorthogonal rather than the eigenvector itself (cf. Farrell and Ioannou 1996a). Figure 15 shows the contribution of the eigenvectors of the mean operator to the nonnormality of $\overline{\mathbf{A}}$ as measured by ν. Comparison of Figs. 14 and Fig. 15 confirms that the Lyapunov vector results from destabilization of the dominant nonnormal subspace of the mean operator. The probability distribution of the three dominant eigenvector components of the Lyapunov vector is shown in Fig. 16. It should be noticed that the probability distribution is not normal, but similar to the distribution expected if the Lyapunov vector were the sum of a dominant vector (the common structure of the nonorthogonal vectors) and a vector in a random direction (see appendix C).

The nonorthogonal eigenvectors provide insight into the mechanism of time-dependent instabilities but because of their nonorthogonality the eigenvectors do not provide an economical basis for investigating the structure of the Lyapunov vector. We inquire now how Lyapunov vectors project on the orthogonal set of the optimal vectors and evolved optimal vectors (the right and left singular vectors). Projections of the Lyapunov vector on the optimal vectors and on the time-evolved optimal vectors of the mean flow propagator over a period T_c in the energy inner product are shown in Fig. 17. We observe that the Lyapunov vector does not project uniformly on the optimal vectors. In the energy inner product the Lyapunov vector projects primarily on the top
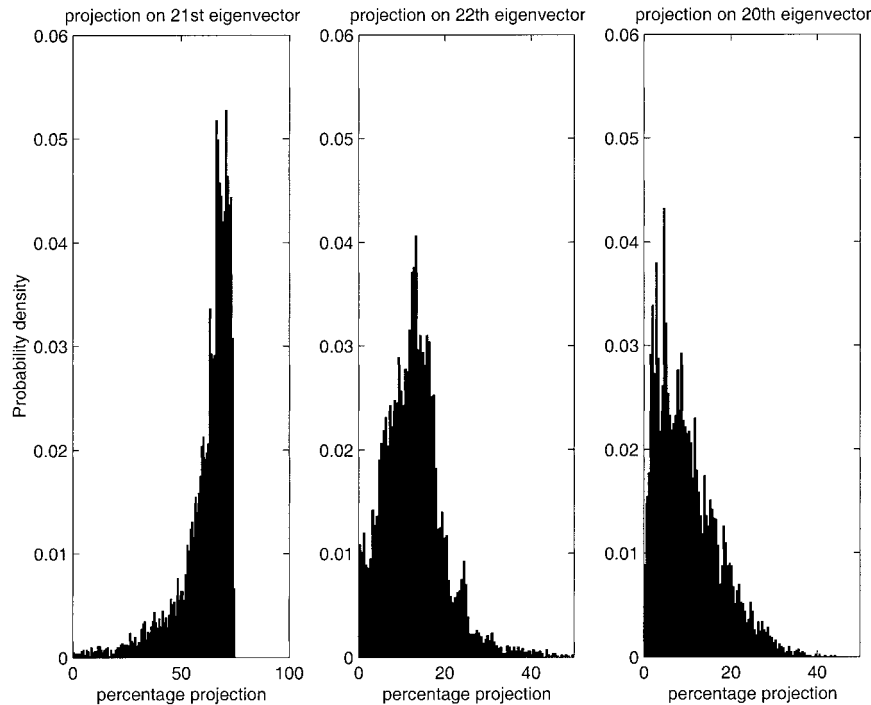

FIG. 16. Probability distribution of the percentage projection of Lyapunov vector on its three dominant eigenvector components: modes with index 21, 22, 20. The example is that of Figs. 13, 14, and 15.
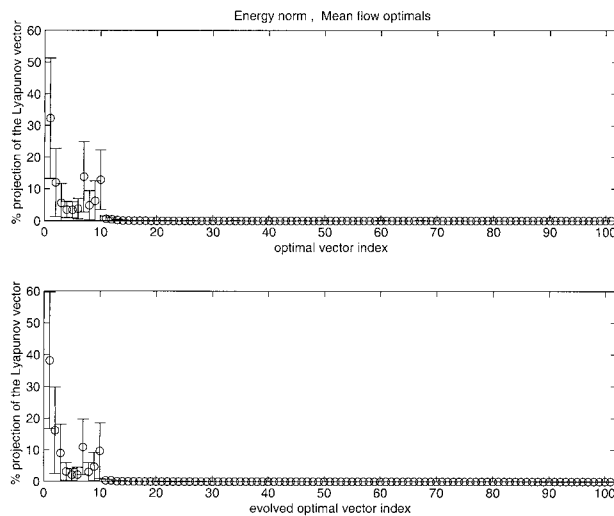
FIG. 17. (top) Mean projection and standard deviation of the Lyapunov vector on the optimal vectors of the mean flow calculated for a time interval equal to T_c in the energy norm. (bottom) The mean projection and standard deviation of the Lyapunov vector on the T_c evolved optimal vectors of the mean flow in the energy norm.

FIG. 18. Probability distribution of the magnitude of the projection on a coordinate axis of a vector in n-space distributed uniformly on the surface of the unit hypersphere. The case $n = 3$ yields a uniform distribution. Cases $n = 2$ and $n = 20$ are also shown.

10 optimal vectors. It follows that we can obtain an improved upper bound on the Lyapunov exponent if we take account in the calculation (33) of the probability distribution of the projection of the Lyapunov vector on these optimal vectors of the propagator over a decorrelation time. This suggests that it is possible to obtain a good upper-bound estimate for the Lyapunov exponent of the large-scale atmospheric flow using the routinely collected analyses of optimal growth distributions. We observe that the Lyapunov vector projects strongly on the top evolved optimal vectors, and consequently the Lyapunov vector can be usefully characterized statistically as having the spatial structure of the top optimal or time-evolved optimal vectors over the decorrelation time T_c.

7. Discussion and conclusions

It is useful to distinguish between two problems addressed by analysis of the linear stability of time-dependent operators: the growth of errors and the growth of perturbations. Calculation of error growth involves the tangent linear equations in which the linearization has been performed about a known time-dependent trajectory and the perturbation is regarded as a small error in specification of the initial conditions. The result of the calculation is the difference between the perturbed and the unperturbed trajectories and for an initial error of specified magnitude this trajectory difference is valid until nonlinear effects become important. If a positive Lyapunov exponent exists then an arbitrarily small perturbation to initial conditions assumes after a period of adjustment the form of the time-dependent first Lyapunov vector after which it proceeds to grow at the mean
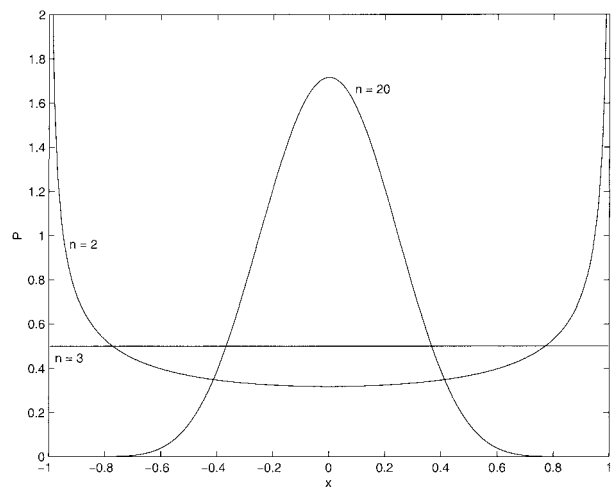
rate of the first Lyapunov exponent. In this case the asymptotic stability calculation is interpreted as constraining the predictability of the system and the Lyapunov exponent, the structure of the Lyapunov vector, and the time interval required for it to be established as an asymptotic are of practical importance for forecast.

The growth of perturbation problems by contrast envisions development to finite amplitude of a disturbance on a time-dependent flow such as the stratospheric flow forced from below by the temporally varying tropospheric flow and the result is interpreted in this context as a generalization of the transient growth to finite amplitude of perturbations on stationary flows. Another example of this type of problem would be an instability calculation studied as a model for cyclogenesis. Such instability problems have almost always been examined assuming that the underlying operator is autonomous. This assumption is seldom critically examined and it would be of interest, for example, to find that the midlatitude jet while stable if its time mean were analyzed were found to be unstable if realistic variation of the jet with time were included in its specification. It could be that stable mean jets support unstable perturbations in the form of the Lyapunov vector when their temporal variability is included. The nature of these perturbations and their relationship to cyclogenesis and the maintenance of waves of synoptic and planetary scale would be of great interest.

In a more general context, given that all physical problems are to a greater or lesser extent time dependent, the stability of realistic flows whether interpreted as an analysis of error or as the development of a perturbation must, if it is to be comprehensive, be assessed taking into account the time dependence of the flow.

For short enough time intervals, the time dependence of forecast model tangent linear trajectories may be ig-

nored without unacceptable effect on the growth of errors in synoptic forecast (Vukicevic 1991; Errico et al. 1993). In addition, the dominant energetics of the midlatitude atmosphere appear to be associated with growth on synoptic timescales so that the statistics of the atmosphere can be accurately modeled using autonomous time-mean operators (Farrell and Ioannou 1994, 1995; Whitaker and Sardeshmukh 1998). For longer time periods, such as are associated with medium and longer range forecast, explicit account of time dependence needs to be taken. In addition, time-dependent instabilities while not dominant in terms of variance and fluxes at synoptic and planetary scale may be important episodically in the growth of some perturbations to finite amplitude.

For the purpose of understanding these phenomena the methods of nonnormal dynamics can be extended from the study of perturbation growth over finite time in autonomous systems to address the perturbation growth over finite time in nonautonomous systems (F&I). In the limit of long time the analog of the most rapidly growing normal mode, which asymptotically dominates in the autonomous system, is the first Lyapunov vector that asymptotically dominates in the nonautonomous system. This asymptotic growth in the nonautonomous system can also be analyzed through the nonnormal dynamics of the underlying time-dependent dynamical operator. But this raises an issue concerning methods in the study of nonautonomous operators. While a stationary state can be easily specified by a deterministic function, a time-dependent system such as the atmosphere can be examined only in generality through its statistical properties. We have turned this to advantage to obtain results that transcend the particularities of a realization of the system but depend rather on the general statistical properties of the system's time dependence. We have studied this problem by using stochastic perturbation of the mean operator with the perturbations chosen to model statistically the observed deviations from the mean. We have found that the asymptotic stability properties of time-dependent systems with stable time-mean operators can be understood in general terms through considerations employed by Zel'dovich et al. (1984) in his analysis of vector growth in random unitary systems; that is instability in time-dependent systems results because transient growth given by the propagator over a time interval can dominate over decay even if the system is stable at each instant, a mechanism that can be traced to the general property of the convexity of the logarithm. We have seen that this general mechanism of destabilization in time-dependent systems can be modeled by extension of Floquet analysis to approximate aperiodic systems.

Using these methods and with knowledge of the statistical properties of the time dependence of the operator, the asymptotic stability properties and the nature of the associated Lyapunov vector can be obtained. This allows us to evaluate circumstances under which desta-

bilization of the system due to time dependence occurs and to understand the nature of the resulting growing structures. For the example of the barotropic jet we have found that increasing the amplitude of jet vacillation is destabilizing provided the vacillation does not have the same functional form as the mean flow. We found that increasing β is stabilizing because nonnormality of the mean operator decreases with increased β, although no generalization of the Rayleigh theorem exists: examples demonstrate that it is possible for a time-dependent system that satisfies necessary conditions for stability at each instant of time to be unstable. Sufficient spatial and temporal variation of effective β was found to be destabilizing and this is identified as the mechanism by which variation of jet structure with time produces instability. In the case of a stable mean operator, asymptotic stability is obtained when the temporal correlation of the time dependence of the jet vacillation is either too long or too short, while the optimal correlation time for growth is at an intermediate value.

Identification of the mechanism by which stable mean operators are destabilized by mode mixing induced by time dependence of stable mean operators shows that the Lyapunov vector is not in general the least stable mode of the mean operator; rather mode mixing in the nonorthogonal subspace of the operator produces the unstable Lyapunov vector, which includes structural characteristics from the entire excited subspace.

If the atmosphere is asymptotically unstable due to time dependence of the jet[6] those instabilities may explain the existence of structures that are not easily explained by appeal to the modes of the mean jet—the origin of short waves being one particular example. A more detailed analysis of the asymptotic stability of realistic flows is needed to answer this and other questions concerning the asymptotic stability of the atmosphere.

APPENDIX A

The Lyapunov Exponent is Independent of the Norm

The Lyapunov exponent in a norm p is given by

$$\lambda_p = \lim_{t \to \infty} \frac{\ln(\|x(t)\|_p)}{t}, \qquad (A1)$$

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[6] It is not necessary for a turbulent flow to have a positive Lyapunov exponent. Consider a forced but highly damped flow; the forcing can produce an arbitrarily complex flow field that is completely determined by the, in principle, known forcing. Meanwhile the damping can be sufficiently great that no positive Lyapunov exponent exists. While this would mean that an arbitrarily small perturbation would not result in a completely different state after sufficient time has passed, it would not mean that the state would be predictable given that the forcing is unknown. It may be of interest to remark in this connection that even unforced turbulence has not been established to exist on a chaotic attractor and therefore to have a positive Lyapunov exponent (Brosa 1989).

where $x(t)$ is the state at time t. We show that the Lyapunov exponent does not depend on the norm.

Consider any two norms (not necessarily associated with an inner product). In finite-dimensional spaces, the two norms are equivalent so that there are constants α and β for which

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1. \tag{A2}$$

Consequently

$$\frac{\ln(\alpha\|x(t)\|_1)}{t} \leq \frac{\ln(\|x(t)\|_2)}{t} \leq \frac{\ln(\beta\|x(t)\|_1)}{t}, \tag{A3}$$

and in the limit $t \to \infty$, we have $\lambda_1 \leq \lambda_2 \leq \lambda_1$ and the Lyapunov exponent is independent of norm.

APPENDIX B

Schematic Example of Violation of the Momentum Integral Constraint in Time-Dependent Flows

If $Q_y(y)$ does not depend on time, the time derivative can be moved out of the integrand in (25). We thus obtain that $\zeta(t, y)$ satisfies

$$\int_{-1}^{1} \frac{[\zeta^2(t, y)]}{Q_y(y)} \, dy = \int_{-1}^{1} \frac{[\zeta^2(0, y)]}{Q_y(y)} \, dy. \tag{B1}$$

If $Q_y(y)$ has no zero in the flow, then for all times

$$\int_{-1}^{1} [\zeta^2(t, y)] \, dy < \frac{Q_y^+}{Q_y^-} \int_{-1}^{1} [\zeta^2(0, y)] \, dy, \tag{B2}$$

where Q_y^- is the minimum value of $|Q_y(y)|$ and Q_y^+ the maximum value of $|Q_y(y)|$. This proves Rayleigh's theorem: if $Q_y(y) \neq 0$, perturbations with nonvanishing vorticity cannot grow without bound in the integral sense of (B2).

This proof does not carry over to time-dependent flows because the inequality needed,

$$\int_a^b f(y, t)/Q_y^- > \int_a^b f(y, t)/Q_y(y, t)$$

$$> \int_a^b f(y, t)/Q_y^+, \tag{B3}$$

in which $Q_y(y, t) > 0$ everywhere over the interval $[a, b]$ and $Q_y^- \leq Q_y(y, t) \leq Q_y^+$, is true only when $f(y, t)$ is everywhere positive. For example, consider $f(y) = -y \sin y$ and $Q_y(y) = y$ on the interval $[\pi/2, 3\pi/2]$, then $\int_{\pi/2}^{3\pi/2} f(y)/G(y) = 0$ but $\int_{\pi/2}^{3\pi/2} f(y) > 0$, contradicting the inequality. Consequently in time-dependent flows episodic decrease of $[\zeta^2]$ at locations with small values of Q_y can lead to exponential growth of perturbations while maintaining the constraint (25) obtained from momentum conservation.

Consider the following discrete two-state schematic that illustrates such a situation. Assume $A(t, y) = 1/Q_y(t, y)$ takes the following discrete values:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \tag{B4}$$

where the rows correspond to time intervals and the columns to spatial regions. An interpretation might be that during half the year A is 1 at low latitudes and 2 at high latitudes, while during the other half A is 2 at low latitudes and 1 at high latitudes. Observe that Q_y has been chosen to be always positive.

It is demanded from momentum conservation (25) that the enstrophy growth weighted by $1/Q_y$ for each of the time period vanishes. An enstrophy growth $B(t, y) = \partial[\zeta^2]/\partial t$ that satisfies the above condition is

$$B = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}. \tag{B5}$$

Indeed, during half the year the weighted enstrophy growth is $A(1, 1)B(1, 1) + A(1, 2)B(1, 2) = 1 - 1 = 0$, and during the other half we also have $A(2, 1)B(2, 1) + A(2, 2)B(2, 2) = -1 + 1 = 0$. In this example we have enstrophy growth in the time mean at each latitude, that is, $B(1, 1) + B(2, 1) = 1 - \frac{1}{2} > 0$ and $B(1, 2) + B(2, 2) = -\frac{1}{2} + 1 > 0$. Clearly, positivity of the mean vorticity gradient in time-dependent systems does not rule out the existence of mean growth at every latitude.

While this example demonstrates that it is impossible to exclude the possibility of growth when the potential vorticity is of one sign in time-dependent flows, it does not prove that exponential growth is possible in a dynamical system corresponding to a fluid stability problem. A constructive example of asymptotic exponential growth in a two-state periodic system is given in section 3c.

APPENDIX C

Mechanism of Universal Destabilization

Consider singular value decomposition of the propagator for a finite interval of time τ. The action of the propagator on a unit sphere is to distort the sphere in the direction of the optimal vectors producing an ellipse with semiaxes given by the corresponding optimal growths σ_i. We order the optimal growths in descending order: $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$.

Consider now the action of this propagator on a random unit vector. We show that the amplification of unit perturbations is a monotonic function of the optimal growths. The random vectors are considered to be equally distributed on the unit sphere. The mean growth rate over the time interval τ is given by

$$\overline{\lambda} = \frac{1}{2\tau|\Sigma|} \int_{\Sigma} \ln\left(\sum_{i=1}^{n} \alpha_i^2 \sigma_i^2\right) d\Sigma, \tag{C1}$$

where α_i are the projections of a state vector on the coordinate axes of the optimal vectors, σ_i are the optimal

growths, and Σ is the surface of the unit hypersphere of area $|\Sigma|$.

Assuming $\Pi_{i=1}^{n} \sigma_i = p$ only $n - 1$ of the optimal growths are independent. Without loss of generality, we choose to eliminate σ_2. Then

$$\frac{\partial \overline{\lambda}}{\partial \sigma_1} = \frac{1}{\sigma_1 \tau |\Sigma|} \int_{\Sigma} \frac{\alpha_1^2 \sigma_1^2 - \alpha_2^2 \sigma_2^2}{\displaystyle\sum_{i=1}^{n} \alpha_i^2 \sigma_i^2} \, d\Sigma \geq 0. \quad \text{(C2)}$$

The inequality follows from the observation:

$$\frac{1}{\sigma_2^2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 \sigma_3^2/\sigma_1^2 + \cdots + \alpha_n^2 \sigma_n^2/\sigma_1^2)}$$

$$\geq \frac{1}{\displaystyle\sum_{i=1}^{n} \alpha_i^2 \sigma_i^2}$$

$$\geq \frac{1}{\sigma_1^2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 \sigma_3^2/\sigma_1^2 + \cdots + \alpha_n^2 \sigma_n^2/\sigma_1^2)}. \quad \text{(C3)}$$

This demonstrates that $\overline{\lambda}$ is an increasing function of σ_1, the greatest optimal growth. (Note that a decrease in σ_3 also leads to an increase of the growth rate, as then σ_2 must increase to keep the product constant.)

It is immediate from (C2) that the minimum growth rate is attained when all the optimal growths are equal to $p^{1/n}$. The minimum is $\lambda_{\min} = 1np^{1/N}/\tau$.

For nondissipative dynamics ($p = 1$) these results demonstrate that when all the optimal growths are equal to unity the growth rate is zero, and that the slightest distribution of the optimal growths around unity guarantees growth.

The mean growth rate (C1) is the mean over all the growth rates of unit vectors uniformly distributed on the unit sphere. This growth rate is smaller than the growth rate of the vector that projects equally on each of the optimal vectors, that is,

$$\frac{1}{2\tau} \ln\left(\frac{\displaystyle\sum_{i=1}^{n} \sigma_i^2}{n}\right) \geq \overline{\lambda}. \quad \text{(C4)}$$

It is a remarkable fact that a vector projecting equally on each of the optimal vectors will by necessity amplify if the operator is nondissipative and the optimal growths are not all equal. It should be noted that equal projection on another orthogonal basis distinct from the basis of the optimal vectors does not ensure growth.

Inequality (C4) follows from the convexity of the logarithmic function and the observation that the average square projection of a uniformly distributed vector on a given axis in n dimensions is for all i:

$$\frac{1}{|\Sigma|} \int_{\Sigma} \alpha_i^2 \, d\Sigma = \frac{1}{n}. \quad \text{(C5)}$$

The above statement follows from the realization that the probability distribution, P, of the magnitude, x, of

the projection on a coordinate axis of a vector uniformly distributed on the sphere obeys the following distribution on $[-1, 1]$:

$$P(x) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\dfrac{n}{2}\right)}{\Gamma\left(\dfrac{n-1}{2}\right)} (1 - x^2)^{(n-3)/2}, \quad \text{(C6)}$$

where n is the dimension of the space ($n \geq 2$). The probability distribution is shown in Fig. 18. It is the ratio of the area of the projection of a coordinate axis increment dx onto the shell of a hypersphere to the total area of the hypersphere.

It is remarkable that while in two dimensions it is more probable for a random unit vector to project along the coordinate axes, as the dimension increases the probability distribution becomes highly concentrated near zero indicating that projection on a coordinate axis is increasingly unlikely (in three dimensions the distribution is uniform).

This probability distribution is useful in assessing the randomness of a given perturbation distribution. If the mean projections of the normalized perturbations to any orthogonal basis are distributed as in (C6), then the perturbations are uniformly distributed on the unit sphere. In practice uniform distributions over the unit hypersphere surface in n-dimensions can be obtained by choosing n uniformly distributed components on $[-1, 1]$ and normalizing them to unity.

APPENDIX D

An Approximate Expression for the Propagator

It can be easily verified that the product of two exponentials in the limit $\tau \to 0$ can be written as

$$\exp(\mathbf{A}_1 \tau) \exp(\mathbf{A}_2 \tau)$$

$$= \exp((\mathbf{A}_1 + \mathbf{A}_2)\tau) \exp\left(\frac{1}{2}[\mathbf{A}_1, \mathbf{A}_2]\tau^2\right)$$

$$+ O(\tau^3). \quad \text{(D1)}$$

Application of (D1) for a finite number of times n produces the following small τ approximation of the propagator that advances a system state over the interval $n\tau$:

$$\Phi(n\tau) = \prod_{i=1}^{n} e^{\mathbf{A}_i \tau}$$

$$= \exp\left(\sum_{i=1}^{n} \mathbf{A}_i \tau\right) \exp\left(\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{i} [\mathbf{A}_i, \mathbf{A}_j]\tau^2\right)$$

$$+ O(\tau^3). \quad \text{(D2)}$$

This expression shows that if the number of states is fixed at n the propagator of the time-dependent flow in the limit $\tau \to 0$ approaches the propagator of the mean

flow with operator $\Sigma_{i=1}^{n} \mathbf{A}_i/n$. Consequently if the flow is periodic with period $n\tau$, with n fixed, and $\tau \to 0$ the stability of the periodic flow will be determined from the stability of the mean flow, and consequently if the mean flow is stable the periodic flow will also be stable. Expression (D2) also shows that the next-order correction to the stability of the time-dependent flow involves the commutators among the operators of the flows.

The propagator of time-dependent flows on an arbitrary time interval with uncorrelated time-dependent deviations also approaches the propagator of the mean as the correlation time of the derivations $\tau \to 0$. This can be seen by considering a subdivision of a finite time interval T in n subintervals of duration τ and assuming that as $\tau \to 0$ the operators $\mathbf{A}_i$ remain uncorrelated. The reason is that as $n \to \infty$ the $O(n^2)$ terms in the sum of commutators in the second exponential of (D2) add to a sum that by the central limit theorem is $O(n)$ while the τ^2 term multiplying this sum decreases as $O(1/n^2)$ making the second exponential of (D2) inferior to the first. This shows that the stability of a time-dependent state consisting of a sum of a mean operator and uncorrelated time-dependent operators is governed by the stability of the mean operator.

Note that the above argument is not valid if the time-dependent part of the operators $\mathbf{A}_i$ are correlated. The reason is that in that case

$$\lim_{n\to\infty} \sum_{i=1}^{n} A_i \tau = \int_0^T dt \mathbf{A}(t), \tag{D3}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{i} [\mathbf{A}_i, \mathbf{A}_j]\tau^2 = \int_0^T dt \int_0^t ds[\mathbf{A}(t), \mathbf{A}(s)], \tag{D4}$$

where $T = n\tau$. For bounded $\mathbf{A}$ (D3) is order $O(T)$ while (D4) is $O(T^2)$, which is inferior to the first term only in the usually uninteresting case of $T \ll 1$.

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