

The mechanism by which nonlinearity sustains turbulence in plane Couette flow

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Abstract. Turbulence in wall-bounded shear flow results from a synergistic interaction between linear non-normality and nonlinearity in which non-normal growth of a subset of perturbations configured to transfer energy from the externally forced component of the turbulent state to the perturbation component maintains the perturbation energy, while the subset of energy-transferring perturbations is replenished by nonlinearity. Although it is accepted that both linear non-normality mediated energy transfer from the forced component of the mean flow and nonlinear interactions among perturbations are required to maintain the turbulent state, the detailed physical mechanism by which these processes interact in maintaining turbulence has not been determined. In this work a statistical state dynamics based analysis is performed on turbulent Couette flow at $R = 600$ and a comparison to DNS is used to demonstrate that the perturbation component in Couette flow turbulence is replenished by a non-normality mediated parametric growth process in which the fluctuating streamwise mean flow has been adjusted to marginal Lyapunov stability. It is further shown that the alternative mechanism in which the subspace of non-normally growing perturbations is maintained directly by perturbation-perturbation nonlinearity does not contribute to maintaining the turbulent state. This work identifies parametric interaction between the fluctuating streamwise mean flow and the streamwise varying perturbations to be the mechanism of the nonlinear interaction maintaining the perturbation component of the turbulent state, and identifies the associated Lyapunov vectors with positive energetics as the structures of the perturbation subspace supporting the turbulence.

1. Introduction

Turbulence is widely regarded as the primary exemplar of an essentially nonlinear phenomenon. However, the mechanism by which energy is transferred in shear flows from the externally forced component of the flow to the broad spectrum of spatially and temporally varying perturbations is through linear non-normal interaction between these flow components [1, 2]. Nevertheless, nonlinearity participates in an essential way in this cooperative interaction by which turbulence self-sustains. Our goal in this study is to provide a more comprehensive understanding of the role of nonlinearity and its interaction with linear non-normality in the maintenance of turbulence.

In order to study the mechanism by which nonlinearity between streamwise varying components participates in the maintenance of turbulence in shear flow we begin by partitioning the velocity field of plane parallel Couette flow into streamwise mean and perturbation components, or equivalently into the $k_x = 0$ and the $k_x \neq 0$ components of the Fourier decomposition of the flow field, where k_x is the wavenumber in the streamwise, x , direction. In this decomposition the flow



field is partitioned as:

$$\mathbf{u} = \mathbf{U}(y, z, t) + \mathbf{u}'(x, y, z, t), \quad (1)$$

with cross-stream direction y and spanwise direction z . Note that in this decomposition the mean flow retains temporal variation in its spanwise structure as would occur e.g. in the presence of time-dependent streaks.

The non-dimensional Navier-Stokes equations expressed using this mean and perturbation partition are:

$$\partial_t \mathbf{U} + \underbrace{\mathbf{U} \cdot \nabla \mathbf{U}}_{N_1} + \nabla P - \Delta \mathbf{U} / R = - \underbrace{\langle \mathbf{u}' \cdot \nabla \mathbf{u}' \rangle_x}_{N_2}, \quad (2a)$$

$$\partial_t \mathbf{u}' + \underbrace{\mathbf{U} \cdot \nabla \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{U}}_{N_3} + \nabla p' - \Delta \mathbf{u}' / R = - \underbrace{(\mathbf{u}' \cdot \nabla \mathbf{u}' - \langle \mathbf{u}' \cdot \nabla \mathbf{u}' \rangle_x)}_{N_4}, \quad (2b)$$

$$\nabla \cdot \mathbf{U} = 0, \quad \nabla \cdot \mathbf{u}' = 0, \quad (2c)$$

where $R = U_w h / \nu$ is the Reynolds number and $\pm U_w$ the wall velocity at $y = \pm h$. The flow satisfies no-slip boundary conditions in the cross-stream direction: $\mathbf{U}(x, \pm h, z, t) = (\pm U_w, 0, 0)$, $\mathbf{u}'(x, \pm h, z, t) = (0, 0, 0)$ and periodic boundary conditions in the z and x directions. From now on, lengths will be normalised with h , velocities with U_w , and time with h/U_w . Averaging is denoted with a bracket $\langle \cdot \rangle$ with the bracket subscript indicating the averaging variable, so that e.g. the streamwise mean velocity is $\mathbf{U} \equiv \langle \mathbf{u} \rangle_x = L_x^{-1} \int_0^{L_x} \mathbf{u} dx$, where L_x is the streamwise length of the channel. The Navier-Stokes equations with this decomposition will be referred to as the NL system and direct numerical simulations of this system as NL. We have indicated with an underbrace the terms in the NL system of primary relevance to our study. The advective term in the streamwise mean flow equation (2a) comprising nonlinear interactions among $k_x = 0$ flow components will be referred to as N_1 , the Reynolds stress divergence term in (2a), produced by nonlinear interaction between the k_x and $-k_x$ flow components with $k_x \neq 0$, will be referred to as N_2 . In the perturbation equation (2b) N_3 gives the linear interaction between the mean flow and the $k_x \neq 0$ flow components, which is the source of the non-normality giving rise to the transfer of energy from the mean flow to the perturbations, and N_4 is the nonlinear interactions between $k_{x_1} \neq 0$ and $k_{x_2} \neq 0$, with $k_{x_1} \neq -k_{x_2}$.

Transition to and maintenance of a self-sustained turbulent state results even when only nonlinearity N_2 is retained [3]. By retaining nonlinearities N_1 and N_2 we obtain the restricted non-linear system (RNL):

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} + \nabla P - \Delta \mathbf{U} / R = - \langle \mathbf{u}' \cdot \nabla \mathbf{u}' \rangle_x, \quad (3a)$$

$$\partial_t \mathbf{u}' + \mathbf{U} \cdot \nabla \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{U} + \nabla p' - \Delta \mathbf{u}' / R = 0, \quad (3b)$$

$$\nabla \cdot \mathbf{U} = 0, \quad \nabla \cdot \mathbf{u}' = 0. \quad (3c)$$

It has been confirmed that this RNL system supports a realistic self-sustaining process (SSP) which maintains a turbulent state in minimal channels [3, 4], in channels of moderate sizes at both low and high Reynolds numbers (at least for $R_\tau \leq 1000$) [5–7], and also in very long channels [8].

Consider in isolation the time varying mean flow \mathbf{U} obtained from a state of turbulence either of the NL or of the RNL system. Sufficiently small perturbations, \mathbf{u}' , on this mean flow evolve according to

$$\partial_t \mathbf{u}' + \mathbf{U} \cdot \nabla \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{U} + \nabla p' - \Delta \mathbf{u}' / R = 0, \quad \nabla \cdot \mathbf{u}' = 0, \quad (4)$$

which is the perturbation equation (3b) of the RNL system, while in the NL system the finite perturbations \mathbf{u}' obey the different equation (2b) with the N_4 term included. In the self-sustained RNL turbulent state the perturbations, \mathbf{u}' , that evolve under the linear dynamics

(3b) or equivalently under (4) remain finite and bounded. Therefore the mean-flow, \mathbf{U} , of the RNL turbulent state is stable in the sense that perturbations, i.e. the streamwise varying flow components, \mathbf{u}' , that evolve under (4), have zero asymptotic growth rate and the mean flow can be considered to be in the critical state of neutrality, poised between stability and instability. A question that will be addressed in this paper is whether the mean flow, \mathbf{U} , that is obtained from a simulation of NL turbulence shares this property of being adjusted similarly to neutrality in the sense that perturbations, \mathbf{u}' , that evolve under (4), remain bounded and therefore have vanishing asymptotic growth rate and the mean flow of the NL system can therefore be considered to be similarly in a critical state of neutrality when proper account is taken of dissipation in the NL system. If the turbulent mean flow \mathbf{U} of NL can be shown to be neutral, in this sense of parametric neutrality, then the mechanism of turbulence identified analytically in the RNL system, in which the perturbations arise from parametric instability of the mean flow with the mean flow being regulated to neutrality through quasi-linear interaction with the perturbation field, will have been extended to NL and the turbulent dynamics of NL identified to be essentially the same as that of the analytically characterized RNL dynamics. For this program to succeed it is required to show that the dynamically substantive difference between the RNL system and the NL system, which is the appearance in the NL system of the perturbation-perturbation nonlinearity N_4 , does not fundamentally change the dynamics of turbulence operating in the RNL system.

The mechanism determining the statistical state of turbulence in the RNL system and its extension to the NL system we are studying can be related to an influential conjecture of Malkus: “First, that the mean flow will be statistically stable if an Orr-Sommerfeld type equation is satisfied by fluctuations of the mean; second, that the smallest scale of motion that can be present in the spectrum of the momentum transport is the scale of the marginally stable fluctuations of the mean” [9]. In this quote, Malkus conjectures that the turbulent mean state is adjusted to a state of neutrality, as he defines neutrality, and the turbulent perturbations responsible for the Reynolds stresses are in the subspace of the neutral modes of the Orr-Sommerfeld operator about the mean flow, as he defines it. We have in common with Malkus the concept of adjustment by quasi-linear interaction between the mean flow and perturbations as the general mechanism determining the statistical state of turbulence in shear flow and we have succeeded to show that RNL turbulence operates with this program. Extension of the program to the NL system is the goal of the present work. We differ with Malkus in the particulars of this general program, including considering the parametric instability of the time dependent streamwise mean state as that being adjusted to neutrality rather than the time, spanwise and streamwise mean flow. While e.g. turbulent convection [10, 11] and the baroclinic turbulence in the midlatitude atmosphere [12] display a usefully close approximate adherence to this conjecture when both the spatial and temporal means are taken to define the mean flow, the turbulent mean state of wall-bounded turbulence, defined as the streamwise, spanwise and temporal mean, $\langle \mathbf{U} \rangle_{z,t}$, is hydrodynamically stable and far from neutrality in apparently strong violation of the Malkus conjecture [13]. However, RNL turbulence suggests that the program of Malkus [9] was essentially correct and can be extended to wall-turbulence requiring only the additional recognition that the instability to be equilibrated is the instability of the time-dependent operator associated with linearization about the temporally varying streamwise mean flow, \mathbf{U} .

The maximum growth rate of perturbations to the streamwise mean \mathbf{U} that are governed by the linear dynamics of (4) is given by the top Lyapunov exponent of \mathbf{u}' defined as:

$$\lambda_{Lyap} = \lim_{t \rightarrow \infty} \frac{\log |\mathbf{u}'|}{t} . \quad (5)$$

RNL turbulence with the definition of mean (1) satisfies the Malkus conjecture precisely under

our interpretation because for RNL

$$\lambda_{Lyap} = 0 . \quad (6)$$

An issue we wish to examine in this work is whether NL turbulence (with the N_4 term included) is similarly neutral in the Lyapunov sense with its perturbations being similarly supported by parametric growth on its fluctuating mean flow with perturbation structure being associated with the predicted Lyapunov vector structure. Specifically, we mean whether fluctuations, \mathbf{u}' , evolving under (4) on the time dependent mean flow, \mathbf{U} , that has been obtained from a turbulent simulation of the NL equations, have λ_{Lyap} , as defined in (5), zero when proper account is taken of dissipative processes and the predicted Lyapunov structure can be verified to be maintaining the perturbations. We caution the reader that the Lyapunov exponents we are calculating are not the familiar Lyapunov exponents of small perturbations from a turbulent trajectory, which is associated with the growth of perturbations $\delta\mathbf{U}$, $\delta\mathbf{u}'$ to the tangent linear dynamics of the full NL system linearized about a turbulent trajectory \mathbf{U} , \mathbf{u}' . The turbulent trajectory $\delta\mathbf{U}$, $\delta\mathbf{u}'$ is chaotic and will have typically many positive Lyapunov exponents. We instead calculate only the Lyapunov exponents and structures of perturbations, \mathbf{u}' , evolving under the linear dynamics (4) with time dependent mean flow \mathbf{U} . It is important to recognize that the parametric perturbation evolution equation (4) governing the perturbation Lyapunov vector dynamics of RNL is not limited to small perturbation amplitude and the nonlinearity required to regulate the perturbations to their finite statistical equilibrium state, although not present in (4), is contained in the Reynolds stress feedback term N_2 appearing in the mean equation.

Choice of the mean used in the cumulant expansion is fundamental to formulating an SSD to gain an understanding of the mechanism of turbulence in shear flow. SSD analysis reveals that transition to and maintenance of turbulence occurs in association with the breaking of statistical symmetries of the laminar state through a sequence of bifurcations. The laminar state in Couette flow has spanwise, streamwise and temporal statistical homogeneity. When the SSD of Couette flow is closed at second order it has been shown that at a first critical Reynolds number the spanwise symmetry is broken, and at a second higher Reynolds number the temporal statistical homogeneity is broken, coincident with transition to the turbulent state [3, 4]. The centrality of spanwise variation of the mean flow, which is associated with the streak component, to the maintenance of turbulence has been demonstrated by numerical experiments that show turbulence is not sustained when the streaks are sufficiently damped or removed [14]. Given that in the SSD turbulent state the spanwise and temporal homogeneity are broken, it is dynamically inconsistent to use either the spanwise or the temporal mean to separate the flow field into mean and perturbation components. It is necessary to allow both spanwise and temporal variations in the mean structure which requires representation (1).

It remains an open question whether in the turbulent state the streamwise statistical homogeneity is also broken. The evidence that is usually presented to support breaking of the streamwise statistical homogeneity is that no arbitrarily long streamwise structures are observed in simulations, but this evidence is based on individual realizations which may not reflect the symmetries of the statistical state. The evidence supporting the non-breaking of the streamwise symmetry is that RNL has been shown to support a realistic turbulent state under the assumption of a streamwise mean ($k_x = 0$) in long turbulent channels, demonstrating that there exists a statistical turbulent state with statistical streamwise homogeneity [8].

Comparing RNL and NL dynamics provides a new perspective on the role of the perturbation-perturbation nonlinearity N_4 in the maintenance and regulation of turbulence. The N_4 term in (2b) does not contribute directly to maintaining the perturbation energy because the perturbation-perturbation interactions redistribute energy internally among the streamwise $k_x \neq 0$ components of the flow and the term $\langle \mathbf{u}' \cdot N_4 \rangle_{x,y,z}$ is zero¹. Consequently, from (2b) we obtain that the

¹ In our simulations time discretization produces a $\langle \mathbf{u}' \cdot N_4 \rangle_{x,y,z,t}$ of the order of $-0.0005U_w^3/h$ which provides an

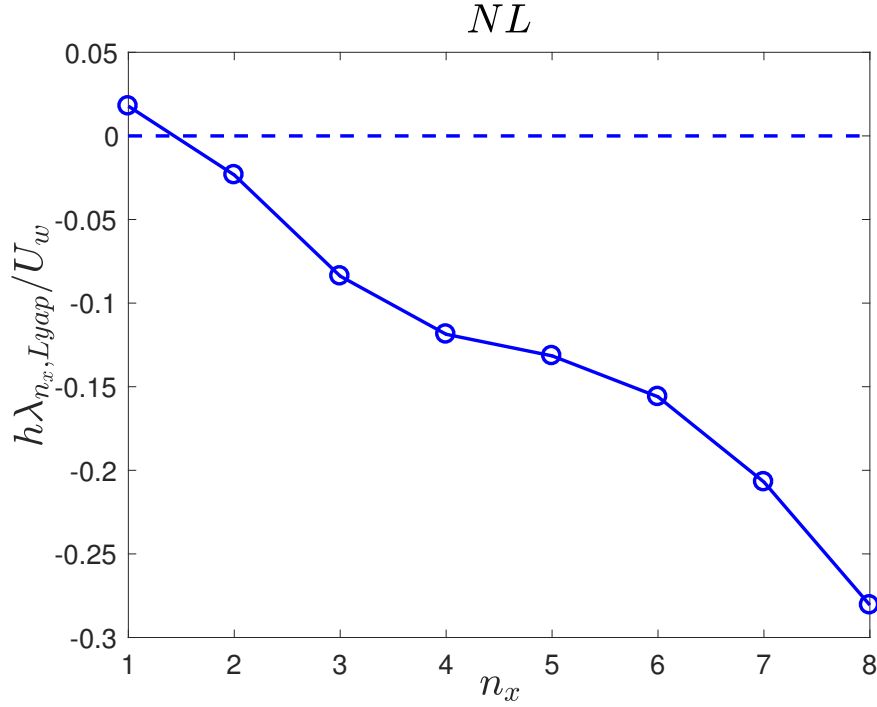


Figure 1. The top Lyapunov exponent of perturbations with channel wavenumbers $n_x = 1, \dots, 8$ evolving under the time-dependent turbulent mean flow in NL, \mathbf{U} . The Lyapunov exponent of all $n_x \geq 2$ components of \mathbf{u}' is negative. For comparison, the least stable mode of the streamwise-spanwise-temporal mean of \mathbf{U} has decay rate $\sigma = -0.12 U_w/h$ at $n_x = 1$ and $n_z = 3$. The Lyapunov exponent was nondimensionalized using advective time units, h/U_w . A plane Couette channel at $R = 600$ was used.

perturbation energy density, $E_p = \langle |\mathbf{u}'|^2/2 \rangle_{x,y,z}$, evolves according to:

$$\frac{dE_p}{dt} = \underbrace{\langle \mathbf{u}' \cdot (-\mathbf{U} \cdot \nabla \mathbf{u}' - \mathbf{u}' \cdot \nabla \mathbf{U} + \Delta \mathbf{u}'/R) \rangle_{x,y,z}}_{\dot{E}_{linear}}. \quad (7)$$

just as in RNL turbulence. The term \dot{E}_{linear} gives the energy transfer to the streamwise-varying perturbations by interaction with the mean \mathbf{U} . The Lyapunov exponent of the \mathbf{u}' associated with the mean flow, \mathbf{U} , taken from an NL simulation, and defined in (5) is also given by the time-average of the instantaneous energy growth rates:

$$\lambda_{Lyap} = \left\langle \frac{1}{2E_p} \frac{dE_p}{dt} \right\rangle_t. \quad (8)$$

Equation (8) gives the top Lyapunov exponent of the NL perturbation dynamics, but a full spectrum of exponents can also be obtained using orthogonalization techniques [15].

This top Lyapunov exponent should be contrasted with the exponent obtained by inserting into (7) and (8) the \mathbf{u}' taken from a simulation of the NL turbulent state. This \mathbf{u}' is bounded because it is the perturbation state vector and therefore this exponent is exactly $\lambda_{state} = 0$. While only the top Lyapunov vector is excited in RNL, in NL a spectrum of Lyapunov vectors

error estimate for the accuracy of our results.

comprise the \mathbf{u}' of the NL state and because these are orthogonal in Fourier representation it is useful to consider the energetics of each of the streamwise Fourier components of \mathbf{u}' separately. If we decompose the perturbation field into its streamwise components:

$$\mathbf{u}' = \sum_{n_x=1}^N \mathbf{u}'_{n_x}(y, z, t) e^{ik_x x}, \quad (9)$$

with $k_x = 2\pi n_x/L_x$, the effective growth rate of each of the streamwise components \mathbf{u}'_{n_x} will also be individually zero. This requires that

$$\lambda_{n_x, state} = \left\langle \frac{\text{Re} \langle \mathbf{u}'_{n_x} \cdot (-\mathbf{U} \cdot \nabla \mathbf{u}'_{n_x} - \mathbf{u}'_{n_x} \cdot \nabla \mathbf{U} + \Delta \mathbf{u}'_{n_x}/R + N_{4, n_x}) \rangle_{y,z}}{2E_{n_x}} \right\rangle_t = 0. \quad (10)$$

where $E_{n_x} = \langle |\mathbf{u}'_{n_x}|^2/2 \rangle_{y,z}$ is the kinetic energy of the n_x streamwise component and N_{4, n_x} is the n_x streamwise component of the N_4 term in (2b) (Re denotes the real part of a quantity). In the energetics of NL in addition to the rate of energy transfer to the perturbations from the mean flow:

$$\dot{E}_{def, n_x} = \text{Re} \langle \mathbf{u}'_{n_x} \cdot (-\mathbf{U} \cdot \nabla \mathbf{u}'_{n_x} - \mathbf{u}'_{n_x} \cdot \nabla \mathbf{U}) \rangle_{y,z}, \quad (11)$$

and perturbation energy dissipation rate:

$$\dot{E}_{dissip, n_x} = \frac{1}{R} \text{Re} \langle \mathbf{u}'_{n_x} \cdot \Delta \mathbf{u}'_{n_x} \rangle_{y,z}, \quad (12)$$

which are the only terms in (7) that are involved in the determination of the Lyapunov exponent, there is the additional term

$$\dot{E}_{nonlin, n_x} = \text{Re} \langle \mathbf{u}'_{n_x} \cdot N_{4, n_x} \rangle_{y,z}, \quad (13)$$

that is only present in the NL equations and specifies the net energy transfer rate to the other nonzero streamwise components.

The dynamical significance of N_4 in sustaining the turbulent state is revealed by comparing the perturbation energetics under the influence of the NL mean flow \mathbf{U} with and without the term N_4 . This can be achieved by calculating the Lyapunov exponents λ_{Lyap} of the \mathbf{U} obtained from an NL simulation and the associated Lyapunov vectors together with the contributions of each to the growth rate from \dot{E}_{def, n_x} , \dot{E}_{dissip, n_x} , \dot{E}_{nonlin, n_x} and comparing these rates to those obtained when the term N_4 is included. Although the N_4 term is energetically neutral it may have a profound impact on the energetics by modifying the perturbations to extract more or less energy from the mean flow. If the term N_4 is not fundamental to sustaining the turbulence and regulating the mean flow but instead the mechanism of RNL is fundamentally responsible for maintaining NL turbulence the following three conditions should be satisfied: (i) the top Lyapunov exponent, λ_{Lyap, n_x} , associated with streamwise components n_x of the turbulent field should be neutral after accounting for the transfer of energy to the other streamwise perturbation components, which would indicate that the turbulent state has been regulated to neutralize the top Lyapunov vector growth rate (maximum Lyapunov exponent) coincident with the (necessary) neutralization of the state vector, (ii) the transfer of energy from the mean flow by the top Lyapunov vector \dot{E}_{def, n_x} should exceed that by the state vector indicating that N_4 has disrupted the Lyapunov vector making it less effective at transferring energy from the mean flow, and (iii) the Lyapunov vectors span the energy and the energetics of the NL perturbation field in a convincingly efficient manner, most tellingly if they span it in the order of their growth rate. Satisfying these conditions would strongly support the conclusion that the turbulence

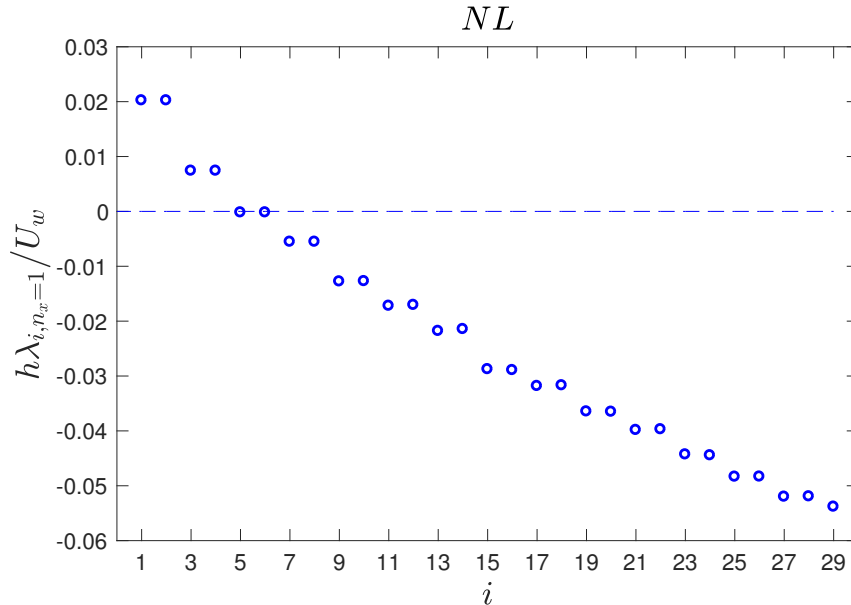


Figure 2. The first 30 Lyapunov exponents of $n_x = 1$ perturbations to the NL turbulent flow. Note that each exponent corresponds to two states that allow for streamwise translation of the Lyapunov vector.

Table 1. The channel is periodic in the streamwise, x , and spanwise, z , direction and at the channel walls $y = \pm h$ the velocity is $\mathbf{u} = (\pm U_w, 0, 0)$. The channel length is L_x and L_z in the streamwise and spanwise directions respectively. The number of streamwise and spanwise Fourier components is N_x and N_z after dealiasing in the streamwise and spanwise direction by the 2/3 rule, and we use N_y grid points in the wall-normal direction. $R = U_w h / \nu$ is the Reynolds number of the simulation, with ν the kinematic viscosity.

Parameter	$[L_x, L_z]/h$	$N_x \times N_z \times N_y$	R
NS600	$[1.75\pi, 1.2\pi]$	$17 \times 17 \times 35$	600

is being maintained primarily through the parametric perturbation growth process associated with the temporal variation of \mathbf{U} that supports RNL, without substantial contribution from the N_4 nonlinearity. The alternative is that the N_4 term has a first-order effect on the energetics which would imply centrality in the dynamics of turbulence of the alternative role for N_4 which is to replenish the subset of perturbations lying in the directions of growth. This distinction in mechanism can be clarified by observing that, if the mean flow is chosen to be the time-independent streamwise-spanwise-temporal mean, which in a boundary layer flow is the stable Reynolds-Tiederman profile [13], the N_4 nonlinearity must assume this role if turbulence is to be sustained as the parametric mechanism is not available. Turbulence could be in principle sustained by this mechanism if N_4 were sufficiently effective in scattering perturbations back into the directions of non-normal growth as is commonly hypothesized in toy models [16–19]. However, it is known that this is not the case [14].

2. The Lyapunov exponent of the mean flow in Couette turbulence at $R = 600$

Consider a Couette turbulence simulation at $R = 600$ in a periodic channel with parameters

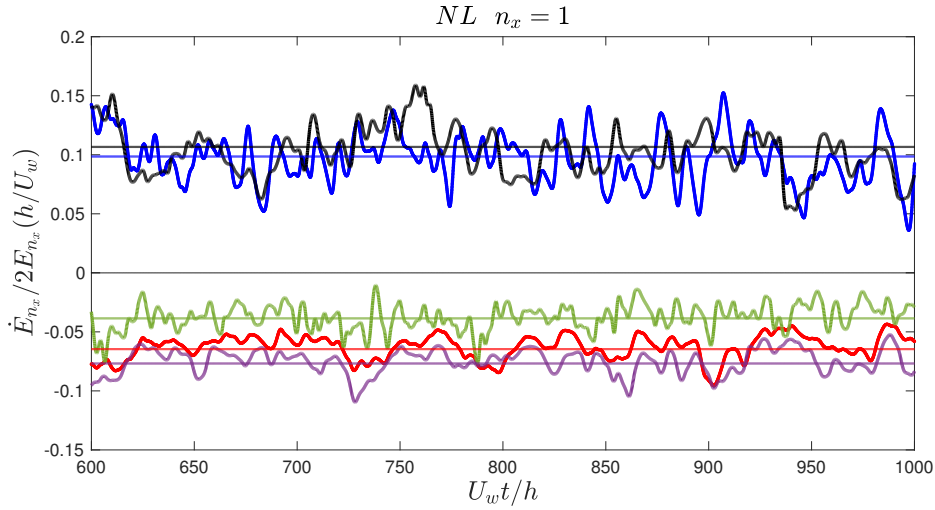


Figure 3. Contribution to the instantaneous energy growth rate of the $n_x = 1$ perturbation component of the NL simulation: extraction from the fluctuating $n_x = 0$ mean component $\dot{E}_{def,n_x}/(2E_{n_x})$ (blue, solid); loss to dissipation $\dot{E}_{dissip,n_x}/(2E_{n_x})$ (red, solid); transfer to the other $n_x > 1$ streamwise components $\dot{E}_{nonlin,n_x}/(2E_{n_x})$ (green). The mean values of these rates are indicated with the dashed lines with the corresponding color. These average rates sum to $\lambda_{state} = 0$. The corresponding rates for the first Lyapunov vector are shown in black and purple dash-dotted lines (there is no energy transfer to the other components as N_4 is absent in this calculation). These rates sum to the Lyapunov exponent $\lambda_{Lyap} = 0.02U_w/h$, which is comparable to $\dot{E}_{nonlin,n_x}/(2E_{n_x})$.

given in Table 1. This is a larger channel compared to the minimal Couette flow that was studied by Hamilton, Kim & Waleffe [20] at $R = 400$. RNL turbulence with these parameters at $R = 600$ was systematically examined recently [21].

We first calculate the Lyapunov exponent λ_{Lyap} of the NL mean flow by estimating (8) from a long integration of (4) with the mean flow \mathbf{U} obtained from a turbulent NL simulation. The initial state \mathbf{u}' is inconsequential because, with measure zero exception, any random initial condition converges in this system with exponential accuracy to the structure associated with the largest Lyapunov exponent. The full spectrum of Lyapunov exponents can be obtained by an orthogonalization procedure. For a discussion of the calculation and properties of Lyapunov exponents and the structures associated with them refer to Refs. [15, 21–24]. Because of the streamwise independence of \mathbf{U} , the different streamwise Fourier components of \mathbf{u}' in this Lyapunov exponent calculation, in which the N_4 term is absent, evolve independently and the structure (the Lyapunov vector) associated with a given Lyapunov exponent has streamwise structure confined to a single streamwise wavenumber $k_x = 2\pi n_x h/L_x$, corresponding to the n_x streamwise wavenumber.

The top Lyapunov exponent at each n_x is shown in Fig. 1. This plot reveals that the time dependent streamwise mean flow \mathbf{U} is asymptotically stable to all perturbations with $n_x > 1$ with only the $n_x = 1$ streamwise component supporting a positive Lyapunov exponent of $\lambda_{Lyap} \approx 0.02U_w/h$. Recall that in RNL the top Lyapunov exponent also has wavenumber $n_x = 1$ and is exactly zero consistent with mean \mathbf{U} being adjusted by the feedback through the Reynolds stress term N_2 to exact neutrality. The top Lyapunov exponent obtained using the \mathbf{U} of NL must be positive only to the degree required to account for the energy exported to other perturbations. The degree of positivity of λ_{Lyap} is necessarily adjusted by feedback through N_2 to account for this loss. With this consideration the NL mean flow \mathbf{U} can be verified to be neutral.

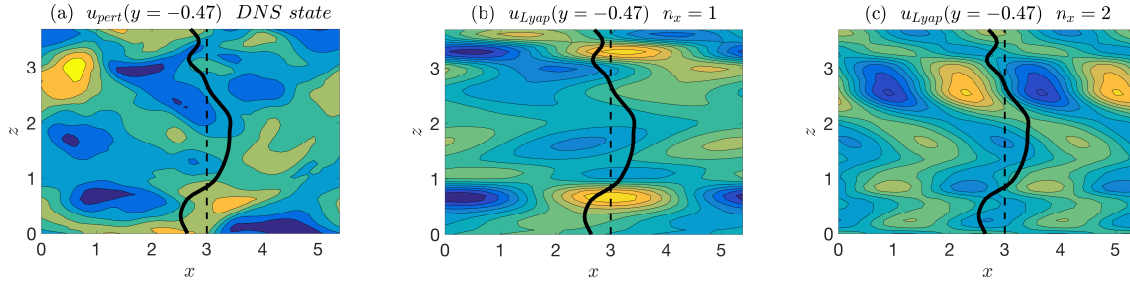


Figure 4. Snapshots at the same time showing the streamwise perturbation velocity u' of the NL state (panel (a)), of the top Lyapunov vector which has streamwise channel wavenumber $n_x = 1$ (panel (b)) and the decaying top Lyapunov vector associated with $n_x = 2$ (panel (c)). Also shown is the streak component of the streamwise mean flow $U - \langle U \rangle_z$ (black line).

Strictly speaking, the $n_x = 1$ component is associated with a pair of Lyapunov exponents corresponding to the real and imaginary part of a single structure. This degeneracy of the Lyapunov exponents results from the streamwise homogeneity of the linear operator which implies that the streamwise phase of the Lyapunov vectors is arbitrary. The growth rates of the first 30 Lyapunov exponents associated with $n_x = 1$ are shown in Fig. 2. Specifically, the perturbation subspace is spanned by a set of Lyapunov vectors comprised of pairs of structures sharing a single spanwise/cross-stream structure but having respectively $\sin(k_x x)$ and $\cos(k_x x)$ streamwise dependence. From Fig. 2 it is apparent that at this Reynolds number and for this minimal channel the NL turbulent mean flow supports two positive Lyapunov exponent pairs.

Contributions to the Lyapunov exponent from mean flow energy transfer and from dissipation are plotted as a function of time in Fig. 3. The growth rate associated with energy transfer from the fluctuating streamwise mean is on average $0.1066U_w/h$, while the dissipation rate is on average $0.0862U_w/h$ resulting in the positive Lyapunov exponent $\lambda_{Lyap} = 0.0204U_w/h$. These transfers occur when perturbations evolve under the dynamics of the fluctuating streamwise mean flow \mathbf{U} associated with NL turbulence but in the absence of two effects: (i) disturbance to the perturbation structure by the perturbation-perturbation nonlinearity N_4 and (ii) transfer of energy to other perturbations by N_4 . This result demonstrates that the parametric growth mechanism is able to maintain the perturbation turbulence component against dissipation with sufficient additional energy extraction to account for transfer to the other scales. The last can be roughly estimated by the energy transfer rate from $n_x = 1$ to $n_x \geq 2$ in the full NL simulation, which is on average $-0.0386 U_w/h$ or $0.8u_\tau/h$ in terms of friction velocity, as shown in Fig. 3.

We now contrast the energetics of the Lyapunov vectors on the turbulent NL mean flow just shown with the corresponding energetics of the $n_x = 1$ Fourier component of the state vector obtained from the turbulent NL simulation itself in order to determine whether the N_4 term has the effect of influencing the perturbations to be in a more or less favorable configuration for extracting energy from the mean flow. These results are also shown in Fig. 3 from which it can be seen that the transfer from the mean \mathbf{U} with the influence of the N_4 term included is slightly less than that achieved by the first Lyapunov vector in the absence of the influence of N_4 : the transfer to the NL state produces growth rate $0.0985U_w/h$ compared to $0.1066U_w/h$ for the case of the unperturbed Lyapunov vector on the NL mean flow. This demonstrates that the nonlinear term does not configure the flow states so as to extract more energy from the mean. However, despite the fact that the energy extracted from the mean flow by the NL perturbation state and the first Lyapunov vector are nearly equal when averaged over time, the correlation coefficient of their time series, shown in Fig. 3, is low (0.26) indicating that the N_4 term has disrupted the first Lyapunov vector and spread its energy to other Lyapunov vectors. The fact that this

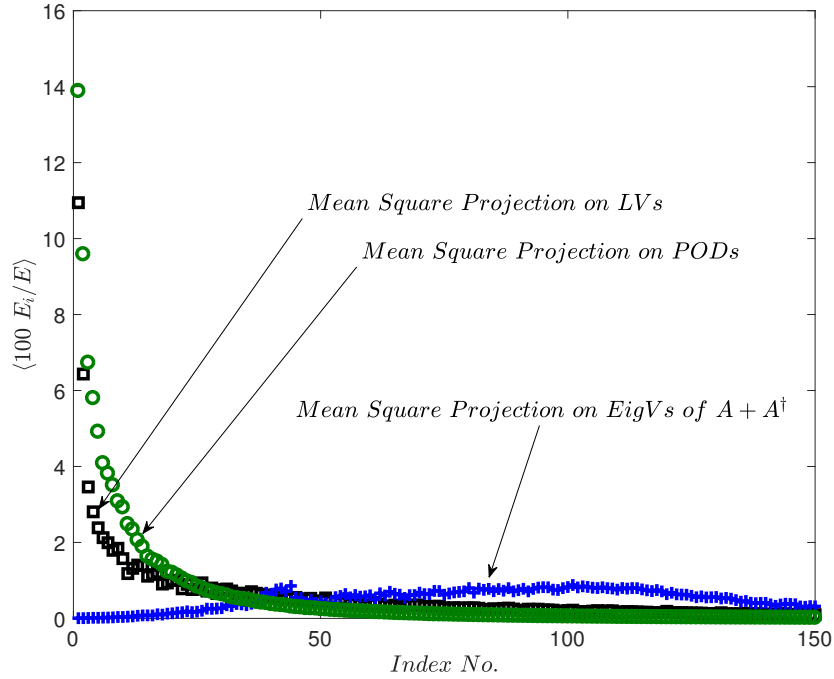


Figure 5. Average energy percentage of the $n_x = 1$ flow accounted for by each pair of Lyapunov vectors (black squares). Average energy percentage of the $n_x = 1$ flow accounted for by the eigenvectors of $A + A^\dagger$ ordered in descending order of their eigenvalue (blue crosses). A is the operator in (4) governing the linear evolution of the perturbations about \mathbf{U} . The eigenvectors of $A + A^\dagger$ are the orthogonal directions of stationary instantaneous perturbation energy growth rate, with this growth rate given by the corresponding eigenvalue. The perturbation component of the turbulent flow is adjusted so as to have a small projection on the first eigenvectors of $A + A^\dagger$ associated with large instantaneous energy growth rates. Also shown is the energy percentage accounted for by the PODs of the $n_x = 1$ component of the NL perturbation state (green circles).

disruption does not substantially alter the time-mean energy extraction from the streamwise mean flow suggests that the time mean energetics resulting from projection on the Lyapunov vectors of \mathbf{U} is not substantially altered by N_4 while the projection at an instant in time is (recall that in RNL N_4 is absent and this projection is onto the single top Lyapunov vector). To investigate this possibility we need to project the energetics on the Lyapunov vectors rather than on the growth directions associated with the eigenvectors of $A + A^\dagger$, with A the operator in (4) governing the linear evolution of the perturbations about \mathbf{U} . Note also that the decay rate of the NL flow state due to dissipation is $0.0646U_w/h$ which is less than the corresponding decay rate of the first Lyapunov vector. This difference in decay rate is explained by reference to Fig. 4, from which it is apparent that the $n_x = 1$ perturbation component of the NL state is of larger scale than the $n_x = 1$ Lyapunov vector. Also, consider that, in the energetics of the $n_x = 1$ perturbation component in NL there is a term not present in the corresponding Lyapunov vector: the energy interchanged with the remaining $n_x \neq 0$ components, which is also shown in Fig. 3. The $n_x = 1$ perturbation component of the NL state exports energy to the other streamwise components of the flow and this nonlinear transfer contributes $0.0386U_w/h$ at this wavenumber to the decay rate. This additional decay is just sufficient to reduce the mean growth rate of the NL state to the expected value $\lambda_{state} = 0$.

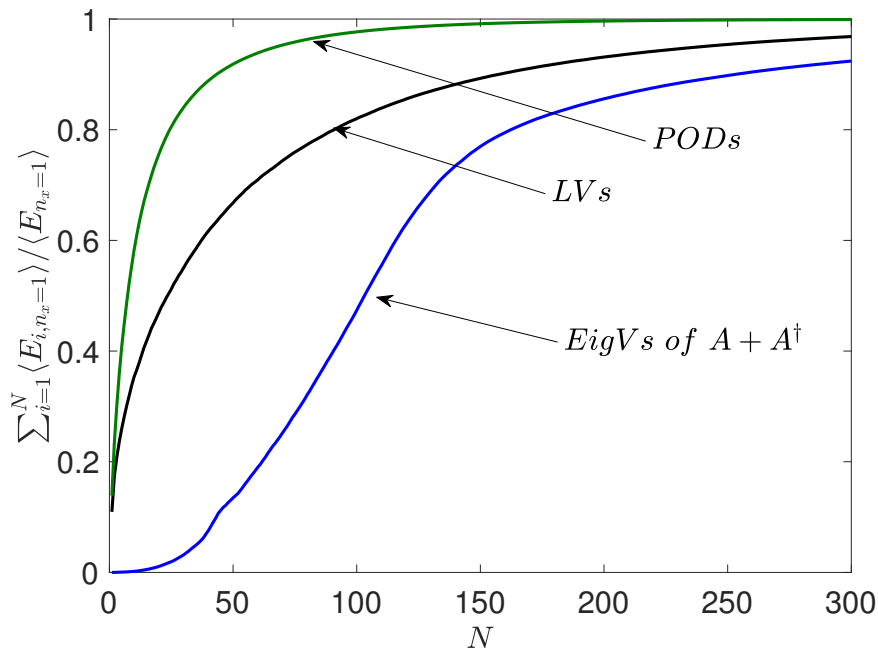


Figure 6. Average fraction of the energy of the $n_x = 1$ flow accounted for by the first N pairs of Lyapunov vectors (black), by the first N pairs of eigenvectors of of $A + A^\dagger$ (blue) ordered in descending order of their eigenvalue, and by the first N pairs of POD's of the $n_x = 1$ component of the NL state (green).

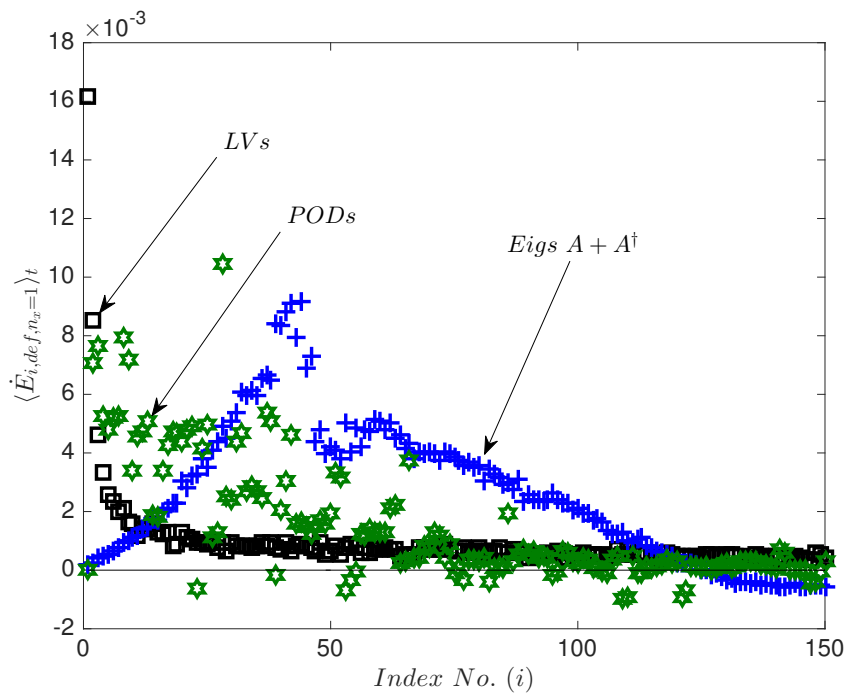


Figure 7. Contribution of the Lyapunov vectors (black squares), the PODs (green stars) and the eigenvectors of $A + A^\dagger$ (blue crosses) to the energy extraction rate, \dot{E}_{def} , from the mean flow.

We conclude that the Lyapunov exponent of the fluctuating streamwise mean flow \mathbf{U} in NL turbulence has been adjusted to near neutrality which is consistent with the parametric growth mechanism fully accounting for the maintenance of the perturbation component of the turbulent state. The perturbation-perturbation nonlinearity, N_4 , does not configure the perturbations to extract more energy from the mean flow than they would in the absence of this term implying that N_4 acts as a disruption to the parametric growth process supporting the Lyapunov vector rather than augmenting the perturbation maintenance by the frequently hypothesized mechanism in which perturbation-perturbation nonlinearity replenishes the subspace of perturbations configured to transfer energy from the mean flow to the perturbations. The fact that the mean NL flow has been adjusted to near neutrality indicates that the first Lyapunov vector should be a dominant component of the NL perturbation state. This will be examined in the next section.

3. Analysis of perturbation energetics by projection onto the Lyapunov vector basis

Despite the persuasive correspondence between the mean energetics of the NL perturbation state and the mean energetics of the top Lyapunov vector calculated using the associated fluctuating streamwise mean flow, \mathbf{U} , time series of perturbation growth rate for these shown in Fig. 3 and snapshots of the perturbation state and the Lyapunov vectors shown in Fig. 4 reveal considerable differences. This suggests further analysis to clarify the relation between the perturbation state and the Lyapunov vectors. The orthogonality property imposed on the Lyapunov vectors makes them an attractive basis for decomposing the $n_x = 1$ component of the NL perturbation state for the purpose of analyzing the relation between perturbation structure and energetics. Expanding the Fourier amplitude of the $n_x = 1$ NL perturbation state $\hat{\mathbf{u}}'$ in the basis of the orthonormal in energy $n_x = 1$ Lyapunov vectors:

$$\hat{\mathbf{u}}'(t) = \sum_i a_i(t) \mathbf{u}_{LV_i}(t), \quad (14)$$

with projection coefficient:

$$a_i(t) = \langle \mathbf{u}_{LV_i}(t) \cdot \hat{\mathbf{u}}'(t) \rangle_{x,y,z}, \quad (15)$$

we obtain that the contribution to the energy of the perturbations accounted for by projection of the perturbation state on Lyapunov vector u_{LV_i} is $E_i = a_i^2(t)/2$. The projection of the energy of the $n_x = 1$ component of the perturbation state on the first 150 Lyapunov vectors is shown in Fig. 5. The percentage of energy accounted for by projection on the most unstable Lyapunov vector is 11%, significantly larger than the energy in each of the remaining Lyapunov vectors. Adding the second unstable Lyapunov vector raises this value to 17.4% and the first 100 $n_x = 1$ Lyapunov vectors account for 82% of the energy of the $n_x = 1$ component of the perturbation state as seen in Fig. 6. In order to understand the significance of the Lyapunov vectors as a basis for representing the NL perturbation state we have determined the orthonormal structures of the proper orthogonal decomposition (PODs) of the $n_x = 1$ component of the NL flow with the methods discussed in [25]. Fig. 5 and Fig. 6 demonstrates that the Lyapunov vectors provide a good representation of the NL perturbation state. We conclude that the energy of the perturbation state is partitioned into the Lyapunov vectors in the order of their Lyapunov exponent and that the Lyapunov vectors are a good basis to represent the perturbation turbulent state. While the energy accounted by the POD basis is by construction expected to decrease monotonically (but not necessarily strictly monotonically) with the order of the POD this is not required of the Lyapunov vectors. Also note that the Lyapunov vectors are not constrained by the optimality of the POD basis to be an inferior basis for spanning the energy, because the Lyapunov vectors are time dependent and could theoretically span all the perturbation energy,

as indeed is the case in RNL for which the entire energetics is explained by the first Lyapunov vector.

In Fig. 5 we also show the average projection of the NL state on the eigenvectors of the time-mean operator $A + A^\dagger$ ordered in descending order of its eigenvalue. A is the linear operator in (4) governing the linear evolution of the perturbations about \mathbf{U} . The eigenvectors of $A + A^\dagger$ are the orthogonal directions associated with instantaneous energy growth rate of perturbations on the flow, \mathbf{U} . The perturbation component has small projection on the first eigenvectors of $A + A^\dagger$ which are the structures producing greatest instantaneous energy growth rates. The turbulent mean flow \mathbf{U} is such that perturbations that lead to large instantaneous growth rate have large spanwise wavenumber and are located at the very high shear regions next to the channel boundaries. Consequently, the state has a small projection on the structures producing the most rapid instantaneous growth and the turbulent perturbation component is concentrated on the directions with small positive instantaneous growth rate. Small projection of the perturbation state on the directions of maximum instantaneous growth rate was also seen in RNL simulations at $R = 600$ [21]. What is remarkable and indicative of the fundamental role of the Lyapunov vectors in the dynamics of NL is the monotonic ordering of the perturbation energy in the Lyapunov basis; comparable ordering does not occur in the case of e.g. the dynamically important basis of the $A + A^\dagger$.

In RNL simulations at $R = 600$ the perturbation turbulent state is entirely supported by the top Lyapunov vector and the energetics of the perturbation state consequently are the energetics of this single Lyapunov vector. The N_4 nonlinearity distributes the perturbation energy over a subspace spanned primarily by the first few Lyapunov vectors, as shown in Fig. 5. We can determine the distribution of the first N Lyapunov vectors ordered in contribution to the perturbation state energy growth rate, $\dot{E}_{i,def}$ by calculating

$$\sum_{i=1}^N \dot{E}_{i,def} \equiv \langle \mathbf{u}'_N \cdot (-\mathbf{U} \cdot \nabla \mathbf{u}'_N - \mathbf{u}'_N \cdot \nabla \mathbf{U}) \rangle_{x,y,z,t}, \quad (16)$$

where

$$\mathbf{u}'_N(t) = \text{Re} \left(\sum_{\alpha=1}^N a_\alpha(t) \mathbf{u}_{LV_\alpha}(t) e^{2\pi i x / L_x} \right), \quad (17)$$

is the truncation of the representation of the $n_x = 1$ perturbation state, given in (14), to the first N Lyapunov vectors. From that calculation, we can then obtain the incremental contribution, $E_{i,def}$, of each Lyapunov vector. We can similarly determine the contribution of each of the eigenvectors of $A + A^\dagger$ and of the PODs to the energetics of the perturbation state. The results, shown in Fig. 7, reveal that the Lyapunov vectors provide the primary support for the perturbation energetics and their energetic contribution follows the Lyapunov vector growth rate ordering. If the N_4 term were dominant in determining the structures supporting the perturbation state the energetics of the NL state would not be expected to so closely reflect the asymptotic structures of the Lyapunov vectors.

4. Conclusions

Statistical state dynamics (SSD) based analysis of the transition to turbulence in Couette flow has revealed the sequence of symmetry breaking bifurcations of the statistical state of the flow as the Reynolds number increases [3, 4]. First the spanwise statistical symmetry of the laminar state is broken and later the temporal statistical symmetry is broken coincident with transition to the turbulent state. Analyses made using SSD systems closed at second order have also demonstrated that a realistic self-sustained turbulent state is maintained by the parametric growth mechanism

arising from interaction between the temporally and spanwise spatially varying streamwise-mean flow and the associated perturbation component of the flow [5, 6, 8]. In these second order SSD systems the streamwise-mean flow is necessarily adjusted exactly to neutral stability, with the understanding that the time dependent streamwise mean flow is considered neutral when the first Lyapunov exponent is zero. This result allows reinterpretation of the Malkus conjecture that the statistical state of inhomogeneous turbulence has mean flow adjusted to neutral hydrodynamic stability.

In this work the identification of the parametric mechanism supporting the perturbation component of turbulence obtained using statistical state dynamics in the RNL system has been extended to NL. While support of both the energy and energetics is on a single feedback neutralized Lyapunov vector in the case of RNL, in the case of NL the energy and energetics are not confined to a single Lyapunov vector but rather are spread by nonlinearity over the Lyapunov vectors. Importantly, this support of the perturbation structure and energetics is ordered in the Lyapunov vectors descending in their associated exponents. The neutrality of the top Lyapunov vector in both RNL and NL, when account is taken of the transfer of energy to other scales, is interpreted as implying that the Malkus conjecture is valid if neutrality of the mean state is interpreted as neutrality of the top Lyapunov vector(s). Consistent with the parametric mechanism sustaining the turbulence, the perturbation structure is concentrated on the top Lyapunov vectors of the time varying streamwise-mean flow and ordered in their Lyapunov exponents. Identification of the dynamical support of RNL and NL turbulence by the marginally stable Lyapunov vectors with associated parametric growth mechanism vindicates the conjecture that the dynamically relevant mean flow in turbulent shear flow is not the streamwise-spanwise-temporal mean flow but rather the time and spanwise varying streamwise-mean flow which incorporates the statistical symmetries obtained from study of second order closures of the statistical state dynamics of wall-bounded turbulence. The perturbation-perturbation nonlinearity, does not configure the perturbations to extract more energy from the mean flow than they would in the absence of this term implying that the nonlinearity acts as a disruption to the parametric growth process supporting the perturbation field rather than augmenting the perturbation maintenance by the frequently hypothesized mechanism in which perturbation-perturbation nonlinearity replenishes the subspace of perturbations configured to transfer energy from the mean flow. The fact that the mean NL flow has been adjusted to Lyapunov neutrality and that the Lyapunov vectors support both the energy and the energetics of the perturbation component of the turbulent state indicates that the parametric growth mechanism on the fluctuating streamwise mean flow and its regulation by Reynolds stress feedback which has been identified to support RNL turbulence is also the mechanism underlying the support of NL turbulence.

Acknowledgments

This work was funded in part by the Coturb program of the European Research Council. We thank Javier Jiménez for his reviewing comments. Marios-Andreas Nikolaidis acknowledges the support of the Hellenic Foundation for Research and Innovation (HFRI) and the General Secretariat for Research and Technology (GSRT). Brian F. Farrell was partially supported by the U.S. National Science Foundation under Grant Nos. NSF AGS-1246929 and NSF AGS-1640989.

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