Transient and asymptotic growth of two-dimensional perturbations in viscous compressible shear flow

B. F. Farrell
Division of Engineering and Applied Sciences, Harvard University, Cambridge, Massachusetts 02138

P. J. Ioannou
Physics Department, National and Capodistrian University of Athens, 15784 Athens, Greece

(Received 7 October 1999; accepted 26 July 2000)

A comprehensive assessment is made of transient and asymptotic two-dimensional perturbation growth in compressible shear flow using unbounded constant shear and the Couette problem as examples. The unbounded shear flow example captures the essential dynamics of the rapid transient growth processes at high Mach numbers, while excitation by nonmodal mechanisms of nearly neutral modes supported by boundaries in the Couette problem is found to be important in sustaining high perturbation amplitude at long times. The optimal growth of two-dimensional perturbations in viscous high Mach number flows in both unbounded shear flow and the Couette problem is shown to greatly exceed the optimal growth obtained in incompressible flows at the same Reynolds number. © *2000 American Institute of Physics.* [S1070-6631(00)00211-7]

I. INTRODUCTION

A comprehensive understanding of the stability of shear flow in general and of high Mach number shear flow in particular requires taking into account perturbation growth due to both modal and nonmodal processes. Flows confined by boundaries support normal modes and eigenanalysis of the linearized equations of fluid motion for such flows provides information about the rate of perturbation energy growth in the limit of long time while singular value decomposition of the system propagator provides information about transient growth on shorter time scales. These two stability analysis methods can be combined to form a generalized stability theory that synthesizes both the modal and nonmodal aspects of perturbation growth.[1]

High Mach number shear flow occurs in technological contexts such as hypersonic aircraft and in astrophysical phenomena such as accretion disks[2] and at least in the case of accretion disks the dynamics is confined to two dimensions. Transient nonmodal growth generally dominates over modal growth in highly non-normal systems such as fluid shear flow except in the limit of long time in idealized models. Moreover, in realistic physical shear flow problems it is often possible to exclude long time asymptotic growth either because the linear operator associated with the mean flow is asymptotically stable or because disruption of the coherence of the mode by turbulent motions limits the time available for modal growth so that emergence of the modal asymptote is never attained and growth instead occurs by repeated rapid but transient nonmodal processes.[1]

Because of the universality of nonmodal growth processes in shear flow[3] transient growth can be understood to a first approximation through study of the relatively simple unbounded constant shear flow model. However, unbounded constant shear flow does not support modal solutions and so cannot accurately model the behavior of any bounded viscous flow in the limit $t\rightarrow\infty$ in which limit the least damped mode supported by the boundary conditions determines the asymptotic growth or decay rate which must be exponential in time. Nevertheless, for studying perturbation growth over the interval of time in which transient growth dominates the unbounded constant shear flow has the great advantage of having analytic solutions for incompressible shear flow,[4–8] for stably stratified flows,[9] and for compressible inviscid flows.[10,11]

Using convected coordinates Chagelishvili *et al.*[10,11] showed that inviscid divergent acoustic perturbations are always eventually excited, even if the initial conditions are of nondivergent vortical form, and that the energy of acoustic perturbations grows linearly with time as $t\rightarrow\infty$. However, as $t\rightarrow\infty$ the total wave number of these growing disturbances also increases linearly with time suggesting an accelerating viscous damping rate and the question arises as to whether the inviscid asymptotic growth obtained in the inviscid model is sustained at finite Reynolds number. We show by integration of the viscous extension of the compressible shear wave solutions in unbounded constant shear flow that the inviscid growth is not sustained and that all solutions in viscous unbounded constant shear flow are damped as $t\rightarrow\infty$. Nevertheless, we find that the maximum attainable growth is greater in compressible constant shear flow than that obtained in incompressible flow at the same Reynolds number. Having established that nonmodal growth remains robust at finite Reynolds number in the unbounded constant shear flow we turn to the issue of the influence of boundaries on the perturbation dynamics. Imposition of no slip rigid boundaries in the viscous problem induces an eigenvalue problem with a discrete basis of modes. The perturbation dynamical system plus the boundary conditions is non-normal and consequently these modes are not orthogonal.

Constant shear flow with rigid no slip boundaries constitutes the plane Couette flow the stability of which has long been an object of study in hydrodynamics. Couette flow does not support modal instability in the incompressible limit[12] while the unstable modes which exist for some sufficiently small Mach numbers in compressible Couette flow are weakly growing.[13]

We find weak modal instability in the viscous compressible Couette flow in agreement with previous results. However, we also find that at high Mach numbers two-dimensional perturbations produce rapid nonmodal growth and that this nonmodal growth can strongly excite the persistent modes, a process that cannot occur in unbounded flow due to the absence of modes in unbounded flow.

These two-dimensional growth mechanisms at high Mach numbers have implications for the maintenance of turbulence in accretion disks around massive bodies. The flow in accretion disks is nearly two dimensional and Keplerian (i.e., the rotational frequency $\Omega \propto R^{-3/2}$, where R is the distance from the massive body) and the Mach number is very large.[2] Nonmodal processes may be key to producing the outward angular momentum flux required to support the observationally inferred rates of mass inflow.[14]

We begin by introducing the viscous compressible equations; optimal perturbations in unbounded shear flow are then obtained using convected coordinate solutions, after which boundary conditions are introduced and the resulting Couette problem is formulated as a matrix dynamical system and the optimal perturbations obtained. Finally some physical implications of these solutions for perturbation growth processes in bounded and unbounded shear flow are discussed.

II. GROWTH OF TWO-DIMENSIONAL PERTURBATIONS IN COMPRESSIBLE CONSTANT SHEAR FLOW

The two-dimensional continuity and momentum equations for harmonic perturbations of reduced pressure, p, streamwise (x) velocity, u, and cross-stream (y) velocity, v, of the form $[p,u,v]=[\hat{p},\hat{u},\hat{v}]e^{ikx}$, in a polytropic fluid (i.e., pressure related to density by $P=K\rho^{\gamma}$) with constant shear $U=y$ and spatially uniform mean density are

$$\frac{\partial \hat{p}}{\partial t}+iky\hat{p}=-\frac{\hat{\Delta}}{M^2}, \tag{2.1}$$

$$\frac{\partial \hat{u}}{\partial t}+iky\hat{u}=-\hat{v}-ik\hat{p}+\frac{1}{\mathrm{Re}}D^2\hat{u}+\frac{ik}{3\,\mathrm{Re}}\hat{\Delta}, \tag{2.2}$$

$$\frac{\partial \hat{v}}{\partial t}+iky\hat{v}=-\frac{d\hat{p}}{dy}+\frac{1}{\mathrm{Re}}D^2\hat{v}+\frac{1}{3\,\mathrm{Re}}\frac{d\hat{\Delta}}{dy}, \tag{2.3}$$

where Δ denotes the velocity divergence which has the streamwise Fourier coefficient:

$$\hat{\Delta}=ik\hat{u}+\frac{d\hat{v}}{dy}, \tag{2.4}$$

and D^2 denotes the Laplacian operator $D^2=d^2/dy^2-k^2$. In (2.1), (2.2), (2.3) time has been nondimensionalized by the inverse shear, τ; and space by the channel half-width for bounded channel flow, and by $1/k$ for unbounded flow. The Reynolds number is defined as $\mathrm{Re}=L^2/\nu\tau$, where ν is the coefficient of shear viscosity, the coefficient of second viscosity is assumed to be zero, and L is the space scale for the unbounded or channel flow. The Mach number, measuring the ratio of the characteristic flow speed to the speed of sound c_s, is $M=L/\tau c_s$.

In the case of unbounded flow, for which due to the nondimensionalization $k=1$, it is useful to transform to convected coordinates $x-yt$ and seek solutions to (2.1), (2.2), and (2.3) of form $[\hat{p}(y,t),\hat{u}(y,t),\hat{v}(y,t)]=[\tilde{p}(t),\tilde{u}(t),\tilde{v}(t)]e^{im(t)y}$, with time varying cross-stream wave number $m(t)=m(0)-t$ (cf. Refs. 4–11). Substituting this solution form transforms (2.1), (2.2), and (2.3) to

$$\frac{d\tilde{p}}{dt}=-\frac{i}{M^2}\tilde{u}-\frac{im(t)}{M^2}\tilde{v}, \tag{2.5}$$

$$\frac{d\tilde{u}}{dt}=-i\tilde{p}-\left(\frac{K^2(t)}{\mathrm{Re}}+\frac{1}{3\,\mathrm{Re}}\right)\tilde{u}-\left(1+\frac{m(t)}{3\,\mathrm{Re}}\right)\tilde{v}, \tag{2.6}$$

$$\frac{d\tilde{v}}{dt}=-im(t)\tilde{p}-\frac{m(t)}{3\,\mathrm{Re}}\tilde{u}-\left(\frac{K^2(t)}{\mathrm{Re}}+\frac{m(t)}{3\,\mathrm{Re}}\right)\tilde{v}, \tag{2.7}$$

where $K^2(t)=1+m^2(t)$ is the total time varying wave number. The original partial differential equations have been reduced to a set of three ordinary differential equations which can be readily integrated to determine the propagator matrix that advances the initial state of the system to the state at a later time t.

We choose to scale pressure by Mach number so that the system state is $\mathbf{x}$, where $\mathbf{x}$ denotes the column vector $[M\tilde{p},\tilde{u},\tilde{v}]^T$. The evolution of the state $\mathbf{x}$ is then governed by the time-dependent matrix equation:

$$\frac{d\mathbf{x}}{dt}=\mathbf{A}(t)\mathbf{x}, \tag{2.8}$$

where $\mathbf{A}(t)$ is given by

$$\mathbf{A}(t)=\begin{pmatrix} 0 & -\frac{i}{M} & -\frac{im(t)}{M} \\ -\frac{i}{M} & -\left[\frac{K^2(t)}{\mathrm{Re}}+\frac{1}{3\,\mathrm{Re}}\right] & -\left[1+\frac{m(t)}{3\,\mathrm{Re}}\right] \\ -\frac{im(t)}{M} & -\frac{m(t)}{3\,\mathrm{Re}} & -\left[\frac{K^2(t)}{\mathrm{Re}}+\frac{m(t)}{3\,\mathrm{Re}}\right] \end{pmatrix}. \tag{2.9}$$

The state at t is related to the initial state by: $\mathbf{x}(t)=\mathbf{\Phi}(t)\mathbf{x}(0)$, where $\mathbf{\Phi}(t)$ is the finite time propagator

$$\mathbf{\Phi}(t)\equiv\lim_{\tau\to 0}\prod_{n=1}^{N}e^{\mathbf{A}(n\tau)\tau}, \tag{2.10}$$

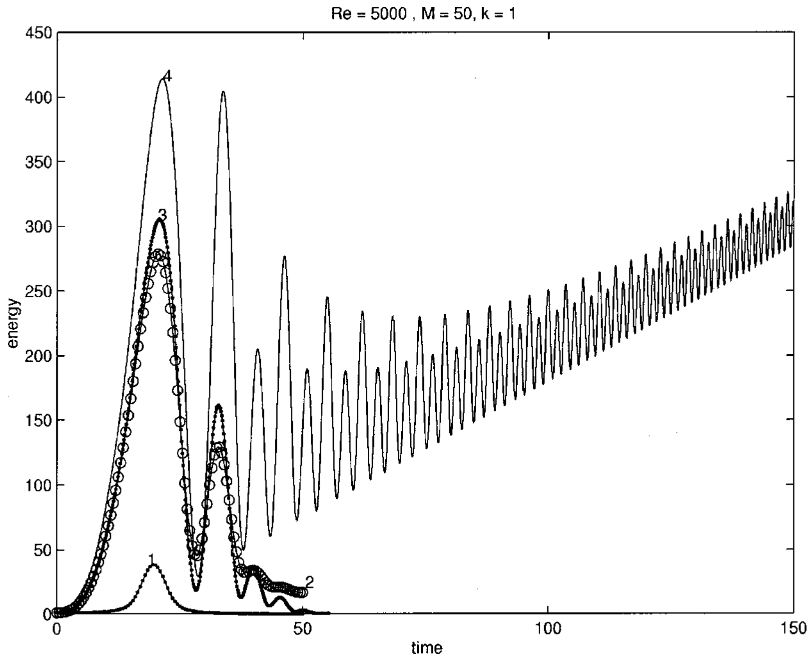

FIG. 1. Energy evolution of the initial perturbation of unit energy that leads to maximum energy growth at $t=20$ (with $k=1$). Curve 1 is for incompressible ($M=0$) viscous bounded Couette flow at Re=5000; curve 2 is for the same case except compressible ($M=50$); curve 3 is for unbounded constant shear compressible flow at $M=50$ and Re=5000; curve 4 is for the same case except the flow is inviscid.

obtained by N advances of the system by the infinitesimal propagators $e^{\mathbf{A}(n\tau)\tau}$, where N and τ satisfy the relation $t=N\tau$.

The perturbation energy density has both kinetic and potential energy components with the total energy density given by

$$E=\frac{|\hat{u}|^2+|\hat{v}|^2+M^2|\hat{p}|^2}{4}=\frac{1}{4}|\mathbf{x}|^2, \tag{2.11}$$

in which pressure has been scaled by Mach number in the definition of the state variable $\mathbf{x}$ so that the Euclidean inner product is proportional to perturbation energy. The maximum factor of increase in energy density that can be achieved at time t is then $||\mathbf{\Phi}(t)||^2$ where $||\cdot||$ denotes the L_2 matrix norm. This norm can be readily evaluated by singular value decomposition which for the propagator $\mathbf{\Phi}(t)$ is of the form: $\mathbf{\Phi}(t)=\mathbf{U}\mathbf{\Sigma}\mathbf{V}^\dagger$ where $\mathbf{\Sigma}$ is positive diagonal and $\mathbf{U}$, $\mathbf{V}$ are unitary. The square of the maximum element of $\mathbf{\Sigma}$ is the square of the L_2 norm of $\mathbf{\Phi}(t)$ which can be interpreted as the maximum factor of energy density increase that can be achieved at time t. This maximum energy growth factor is called the optimal growth, and the corresponding column of $\mathbf{V}$, referred to as the optimal perturbation, is the initial condition that results in this increase.

The energy density evolution of the initial perturbation that produces optimal energy growth at $t=20$ is shown in Fig. 1 for the inviscid and viscous (Re=5000) unbounded constant shear flows at Mach number $M=50$. In the inviscid case the energy grows asymptotically as $t^{10,11}$ (this is true for all initial perturbations, not only the optimal initial perturbation), but as can be seen from Fig. 1 this inviscid asymptotic growth is not sustained in the presence of viscous dissipation.

Optimal initial perturbations undergo two physically distinct growth phases: a rapid transient phase that peaks at a nondimensional time of the order of the initial cross-stream wave number and is robust in the presence of viscosity; and a slow secondary asymptotic growth phase that disappears when viscosity is included. The mechanism of growth in these two phases can be understood by considering the perturbation energy tendency equation:

$$\frac{dE}{dt}=-\int_0^{2\pi/m(t)}\overline{uv\,dy}-\frac{1}{\mathrm{Re}}\int_0^{2\pi/m(t)}\overline{(|\nabla u|^2+|\nabla v|^2)}dy - \frac{1}{3\,\mathrm{Re}}\int_0^{2\pi/m(t)}\overline{\Delta^2 dy}, \tag{2.12}$$

where the bar denotes average over a streamwise (x) wavelength and Δ is the velocity divergence. The only source of perturbation growth is the Reynolds stress which contributes the first term on the right-hand side of (2.12). By decomposing the velocity field into its irrotational and solenoidal parts $(u,v)=\nabla\phi+\nabla\times(\psi\mathbf{k})$ (where $\mathbf{k}$ is the unit vector perpendicular to the plane of the flow, ϕ is the potential of the irrotational velocity component, and ψ is the streamfunction of the solenoidal velocity component), the following proportional relation for the spatially averaged Reynolds stress in convected coordinates can be obtained:

$$-\int_0^{2\pi/m(t)}\overline{uv\,dy}\propto m(t)|\tilde{\psi}|^2-m(t)|\tilde{\phi}|^2 + \Re(\tilde{\phi}^*\tilde{\psi}-m^2(t)\tilde{\psi}^*\tilde{\phi}), \tag{2.13}$$

where $\Re$ denotes the real part of a complex number and the tildes denote the Fourier amplitude of the perturbation fields in the convected coordinate ansatz: $\phi=\tilde{\phi}(t)e^{ix+im(t)y}$, $\psi=\tilde{\psi}(t)e^{ix+im(t)y}$. Consider an initial perturbation with $m(0)>0$. Part of the contribution to energy growth due to the solenoidal component of the perturbation velocity field is the first rhs term in (2.13), $m(t)|\tilde{\psi}|^2$, which contributes substantially initially, while $m(t)>0$, but as the phase lines are

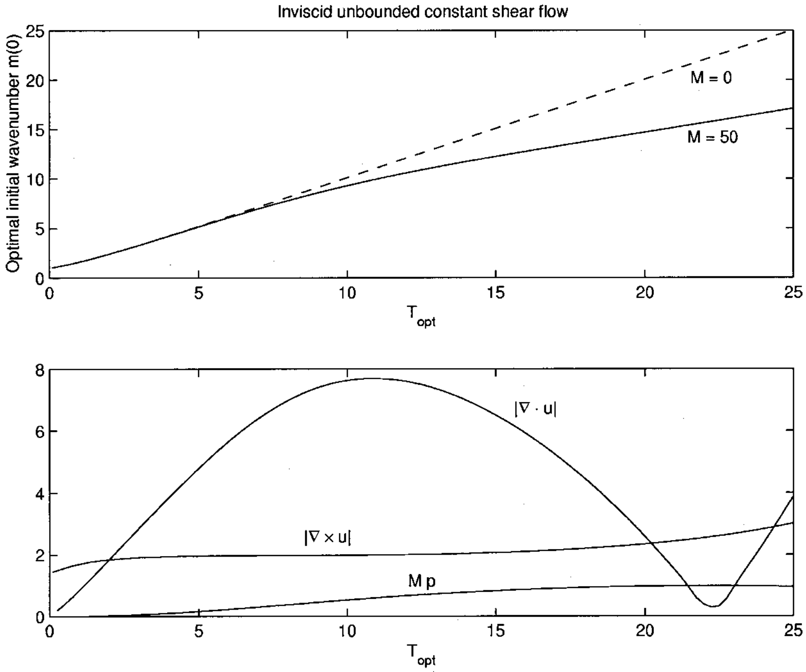

FIG. 2. Top panel: The optimal initial wave number $m(0)$ as a function of optimizing time for perturbations in inviscid unbounded constant shear flow. The dashed line shows $m(0)$ for incompressible flow ($M=0$); the full line shows the compressible flow $m(0)$ ($M=50$). As $\tan^{-1} m(0)$ is the angle between the phase lines and the vertical, the phase lines of optimal perturbations are seen to be less inclined to the vertical in the compressible case. Bottom panel: The magnitude of the vorticity, divergence, and pressure multiplied by M of the initial optimal perturbation as a function of the optimizing time. The phase of these quantities at $t=0$ is the same.

turned into the direction of the shear, so that eventually $m(t)<0$, it becomes an energy sink. This term, which is the sole energy source in incompressible shear flow,[15] vanishes in compressible flow as $m(t)|\tilde{\psi}|^2=O(t^{-2})$ at large times (see the Appendix), which is slower than the $O(t^{-3})$ rate of asymptotic decay of the Reynolds stress in unbounded inviscid incompressible constant shear flow.[15]

The second term on the rhs in (2.13), $-m(t)|\tilde{\phi}|^2$, arises from the irrotational component of the perturbation velocity field and exists only in compressible flow. It is initially an energy sink, while $m(t)>0$, but becomes an energy source when the phase lines have turned into the flow direction so that $m(t)<0$, leading to eventual energy growth linear in time in inviscid unbounded shear flows (in the Appendix it is shown that at large times $m(t)|\tilde{\phi}|^2\rightarrow C+D\cos(t^2/M+\theta)$ with C, D, and θ constants). This ability of perturbations in compressible shear flow to continue extracting energy from the mean flow after the perturbation phase lines have turned into the direction of shear is a new mechanism arising from compressibility. The fact that the only source of energy growth in incompressible shear flows is the solenoidal velocity component which extracts energy only when $m(t)>0$, and that in constant shear flow $m(t)=m(0)-t$ decreases with time, means that in order to maximize energy growth at time t the optimal perturbation must have large initial cross-stream wave number [further analysis shows that in inviscid incompressible shear flows the optimal wave number is $m(0)=t/2+\sqrt{t^2/4+1}$, where t is the optimizing time, so that at large times $m(0)\approx t$ (cf. Ref. 15)]. In compressible flow the perturbation can extract energy even when $m(t)<0$ and consequently optimal growth does not require that $m(0)$ be as large initially as in the case for incompressible flow. The initial $m(0)$ that leads to optimal growth is shown as a function of optimizing time T_{opt} in Fig. 2 for $M=0$ and $M=50$. Also shown in Fig. 2 are magnitudes of the vorticity, divergence, and pressure of the optimal initial condition as a function of T_{opt} for $M=50$.

Given that viscosity damps perturbations in proportion to the square of the total perturbation wave number, it follows that optimal perturbations in viscous compressible flow will be damped less than optimal perturbations in viscous incompressible flows because compressible flow optimals have smaller wave numbers during their growth period. Optimal energy growth for inviscid incompressible flows increases with T_{opt} as $E_{\text{opt}}(T_{\text{opt}})/E_{\text{opt}}(0)=1+T_{\text{opt}}^2$, where T_{opt} is the optimizing time,[8] and consequently optimal growth is greater at large times for inviscid incompressible than is attained in inviscid compressible flows, where the asymptotic growth is linear in time. However, due to their smaller wave numbers optimal perturbations in compressible viscous constant shear flow realize greater growth than is attained by optimals in incompressible viscous constant shear flow at the same Reynolds number (cf. an example in Fig. 1).

The third term in (2.13) is the energy tendency from interactions between the irrotational and solenoidal velocity components. It can be shown to oscillate with high amplitude at times exceeding the optimization time (see the Appendix). However, these oscillations, which are due to interference of oppositely traveling waves with both irrotational and solenoidal velocity components, do not contribute to the secular growth of energy at large times.

While the first stage of transient growth is only moderately affected by viscosity, the later stage, which is associated with secular growth in the inviscid problem, is eliminated by viscosity. For example, energy evolution of the optimal perturbation at $t=20$ for the unbounded constant shear flow at Re=5000 (curve 3 of Fig. 1) shows that the initial transient growth is largely retained in the presence of viscosity, along with a secondary maximum due mainly to the Reynolds stress from the irrotational velocity component,

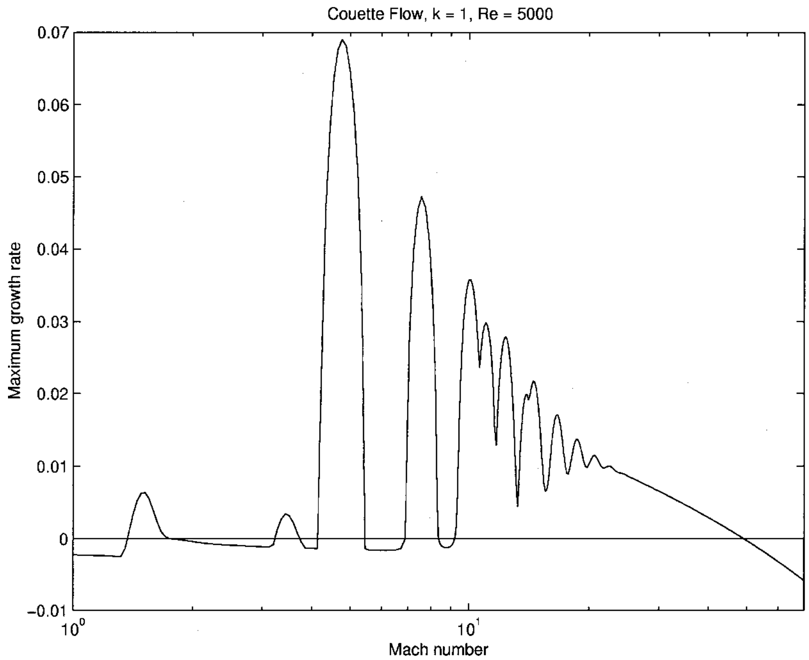

FIG. 3. Maximum modal growth rate as a function of Mach number for bounded Couette flow at Re$=$5000 and $k=1$.

while the secular energy growth at large t has been lost.

Consider now viscous Couette flow between two bounding surfaces with no slip boundary conditions $(u=v=0$ at $y=\pm 1)$. The perturbation dynamics are governed by Eqs. (2.1)–(2.3). In order to study the dynamics of perturbations in bounded flow we approximate the system (2.1)–(2.3) by discretizing the differential operators with central differences. It is known that in the presence of boundaries and at fixed Reynolds number the compressible flow may be weakly unstable for moderate Mach numbers,[13] while for sufficiently high Mach numbers the modal instability is lost. The growth rate of the most unstable mode for $k=1$ and Re$=$5000 as a function of Mach number is shown in Fig. 3.

Consider first a flow at Re$=$5000 and $M=50$ for which parameters the flow is almost neutral for $k=1$. The energy evolution of the optimal initial perturbation that leads to maximum growth at $t=20$ is shown in Fig. 1. Observe that the evolution of energy at first follows that in the viscous unbounded constant shear flow, as expected from the universality of the transient growth mechanism, but for large times the perturbation asymptotically decays at the exponential rate of the least damped mode as expected for a bounded viscous flow. For comparison curve 1 in Fig. 1 shows the energy evolution in an incompressible flow for the initial condition that maximizes energy growth at $t=20$. As discussed earlier the enhanced energy growth in the compressible flow is due to the fact that the irrotational component of the velocity field can continue to extract energy from the mean flow even after it has been convected into the direction of the shear, and consequently such perturbations can continue to grow for a longer period of time before being strongly affected by viscosity. The structure of the optimal perturbation at the initial time is indicated in Fig. 4 by its pressure, which accounts for 92% of the initial perturbation energy. The evolved optimal at the optimizing time $t=20$ is shown in Fig. 5. It is evident that the perturbation is leaning in the direction of the shear and is also extracting energy from the mean as indicated by its Reynolds stress.

Optimal energy growth as a function of time for various Mach numbers as well as for incompressible flow $(M=0)$ at Re$=$5000 and $k=1$ is shown in Fig. 6. While at small Mach number the optimal growth in compressible shear flow is less than that found in incompressible shear flow (this is also found to be the case for three-dimensional perturbations in boundary layer lows at low Mach numbers[16]), at large Mach numbers the optimal growth attained in compressible flow is an order of magnitude greater than that in incompressible flow.

The mechanism of capture of transient growth energy by the least damped mode is of particular significance because it produces robust excitation of a persistent structure which would otherwise be damped or at most weakly growing. For example, consider the optimal energy growth attained as a function of time at $M=20,50,100$, shown in Fig. 6; for large optimizing times the perturbation energy becomes large and nearly constant as the nonmodal processes have strongly excited the nearly neutral modes of the flow at these Mach numbers. Moreover, this mechanism is important even in cases for which the flow is unstable. Consider $M=4.75$ for which there is an unstable mode with nondimensional growth rate 0.0689. The optimal energy growth as a function of time and the energy evolution of the most unstable mode (dashed line) are shown in Fig. 6. Energy derived from transient growth is transferred to the weakly unstable mode as t increases resulting in greatly enhanced excitation of the unstable mode (by a factor of 20 in this case).

III. CONCLUSIONS

Perturbation growth in viscous compressible shear flow was examined using unbounded constant shear flow and the Couette flow as examples. These constant shear examples are

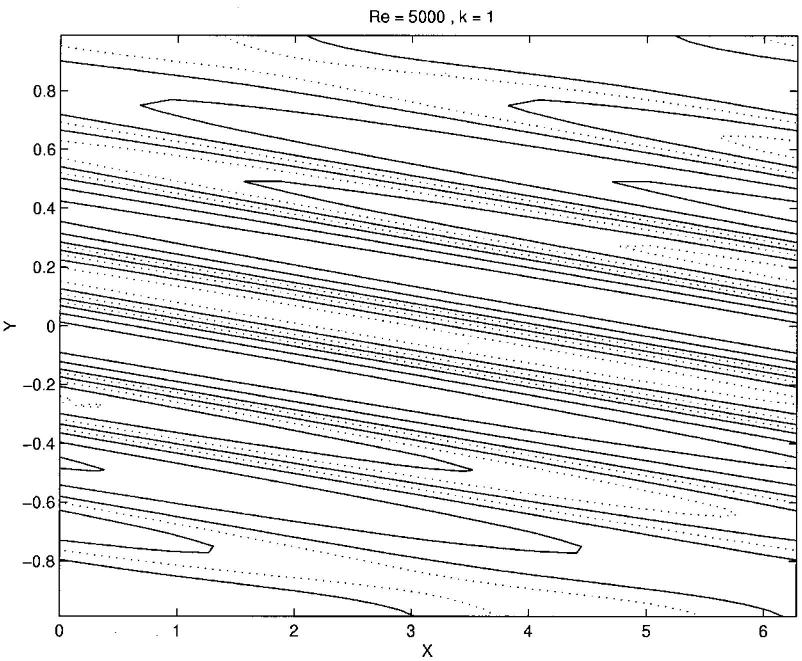

FIG. 4. Structure of the pressure field component of the optimal initial perturbation that leads to the maximum energy growth at $t=20$. The Mach number is $M=50$ and the Reynolds number is $\mathrm{Re}=5000$ and $k=1$. The maximum energy growth at $t=20$ is $E(20)/E(0)=277$ (the energy evolution of this perturbation is shown in Fig. 1). Approximately 92% of the optimal perturbation energy is in the pressure field.

particularly useful in revealing the fundamental properties of nonmodal growth[1] while retaining the simplicity of the analytic convected coordinate solutions in the case of the unbounded shear flow and the familiarity of the Couette problem in the case of the bounded flow. Nonmodal perturbation growth was found to be enhanced in compressible shear flow compared to that found in incompressible shear flow because irrotational velocity fields sustain downgradient Reynolds stresses after the Reynolds stress from solenoidal motions has reversed. However, growth by this mechanism is not sustained as $t\rightarrow\infty$ because viscous damping rapidly increases as the wave number of the solution increases.

In order to quantify in a canonical manner the potential for perturbation growth in compressible shear flow initial perturbations producing optimal energy growth over specified time intervals were identified using singular value decomposition of the system propagator. The optimal perturbations reveal that in contrast to what is found in inviscid flow nonmodal growth can be increased by compressibility in viscous flow. In addition, asymptotic excitation of a nearly neutral mode by its optimal was found to excite the mode at more than an order of magnitude greater energy compared to direct excitation of the mode itself, which demonstrates the importance of nonmodal growth in the asymptotic regime even in flows which support an unstable mode.

Lastly, we remark that the great amplification of a subset of perturbations in viscous compressible flow suggests that continual excitation by intrinsic or extrinsic sources would

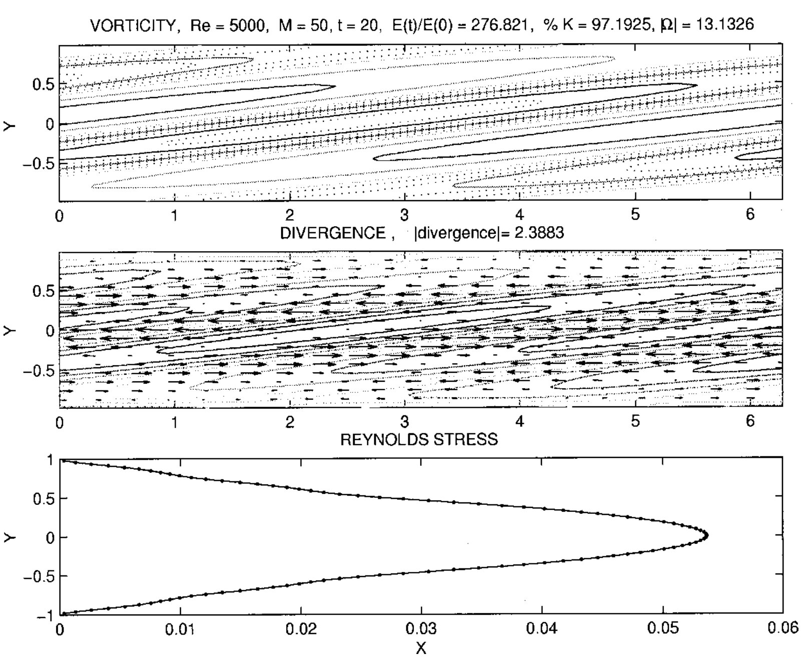

FIG. 5. Structure of the evolved optimal perturbation associated with optimal growth at $T_{\mathrm{opt}}=20$ viewed at $t=20$. The Mach number is $M=50$ and the Reynolds number is $\mathrm{Re}=5000$ and $k=1$. The top panel shows perturbation vorticity, the middle panel shows perturbation divergence with arrows indicating the total velocity. The bottom panel shows the vertical distribution of $-\overline{uv}$. Note that although at the initial time most of the energy is concentrated in potential form, at the optimal time 97% of the perturbation energy is in kinetic form. Note also that although the perturbation is oriented with lines of constant phase in the direction of the shear the Reynolds stress is positive indicative of energy transfer from the mean flow to the perturbation which would not be the case for incompressible flow.

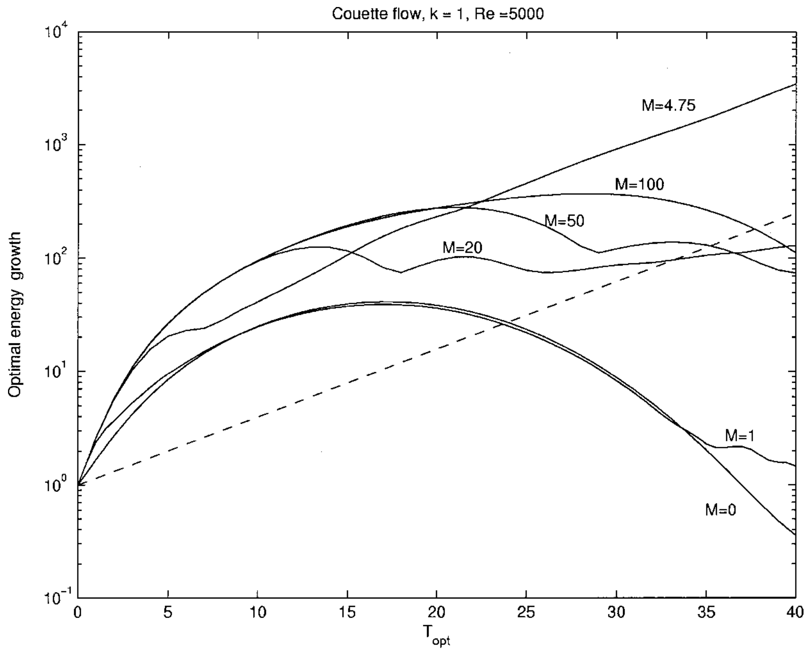

FIG. 6. Optimal energy growth attained by two-dimensional perturbations as a function of optimizing time in incompressible flow ($M=0$), and for Mach numbers $M=1,4.75,20,50,100$. The flow is bounded Couette at Re=5000, and $k=1$. The dashed curve shows the growth associated with the mode with the fastest exponential growth rate (growth rate 0.0689) which occurs when $M=4.75$.

support a statistically steady state with greatly enhanced perturbation variance compared to the variance that would be excited in a system with equivalent damping but without transient amplification, a result that has been demonstrated for similar systems in incompressible flows.[17]

ACKNOWLEDGMENT

This work was supported in part by NSF ATM-9623539.

APPENDIX: ASYMPTOTIC ENERGETICS OF TWO-DIMENSIONAL INVISCID PERTURBATIONS IN COMPRESSIBLE UNBOUNDED CONSTANT SHEAR FLOW

Evolution equations for the irrotational and solenoidal components of the velocity field $(u,v)=\nabla\phi+\nabla\times(\psi\mathbf{k})$ can be derived from (2.5) to (2.7). Using the same convected coordinate ansatz, $\phi=\tilde{\phi}(t)e^{ix+im(t)y}$, $\psi=\tilde{\psi}(t)e^{ix+im(t)y}$, the following governing equations are obtained in the absence of viscosity:

$$\frac{d\tilde{p}}{dt}=-\frac{K^2(t)}{M^2}\tilde{\phi}, \tag{A1}$$

$$\frac{d(K^2(t)\tilde{\psi})}{dt}=-K^2(t)\tilde{\phi}, \tag{A2}$$

$$\frac{d\tilde{\phi}}{dt}=-\tilde{p}-\frac{2}{K^2(t)}\tilde{\psi}, \tag{A3}$$

where $K^2(t)=1+m^2(t)$ is the time varying total wave number, and $\tilde{p}$ is the amplitude of the pressure perturbation. It is easy to show that

$$K^2(t)\tilde{\psi}+M^2\tilde{p}=C \tag{A4}$$

is a conserved quantity. Writing $f(t)=-K^2(t)\tilde{\psi}$ and $\tilde{p}=C/M^2+f(t)/M^2$ the evolution equations (A1)–(A3) reduce to the following two equations in the variables $f(t)$ and $\tilde{\phi}$:

$$\frac{df}{dt}=K^2(t)\tilde{\phi}, \tag{A5}$$

$$\frac{d\tilde{\phi}}{dt}=-\frac{C}{M^2}-\frac{f}{M^2}+\frac{2}{K^4}f. \tag{A6}$$

At large times (A6) can be approximated as

$$\frac{d\tilde{\phi}}{dt}=-\frac{C}{M^2}-\frac{f}{M^2}, \tag{A7}$$

because $K^2(t)=O(t^2)$, and then (A5) and (A7) can be combined to yield the following differential equation governing the large t behavior of the irrotational velocity potential $\tilde{\phi}$:

$$\frac{d^2\tilde{\phi}}{dt^2}=-\frac{K^2(t)}{M^2}\tilde{\phi}, \tag{A8}$$

which admits the asymptotic Wentzel–Kramers–Brillouin solution:

$$\tilde{\phi}=\frac{A^\pm}{\sqrt{K(t)}}\exp\left(\pm i\int^t\frac{K(s)}{M}ds\right), \tag{A9}$$

which in the $t\to\infty$ limit behaves as $|\tilde{\phi}|=O(t^{-1/2})$. The asymptotic amplitude of the streamfunction is

$$\tilde{\psi}=\pm i\frac{MA^\pm}{(K(t))^{3/2}}\exp\left(\pm i\int^t\frac{K(s)}{M}ds\right), \tag{A10}$$

which in the $t\to\infty$ limit behaves as $|\tilde{\psi}|=O(t^{-3/2})$. Note that in incompressible flows the streamfunction decays at the rate $|\tilde{\psi}|=O(t^{-3})$.

We have shown in (2.13) that the instantaneous perturbation energy density tendency is determined by the sum of three terms: the Reynolds stress due to the solenoidal part of the velocity field given by $A=m(t)|\tilde{\psi}|^2$, the Reynolds stress due to the irrotational part of the velocity field given by $B=-m(t)|\tilde{\phi}|^2$ and an interaction term between the solenoidal and irrotational parts. The above-given asymptotic expressions show that at large times the Reynolds stress, A, due to the solenoidal part vanishes. The Reynolds stress due to the interaction can be shown to be of the form $C_1\cos(t^2/M+\theta_1)$ where C_1 and θ_1 are constants. The contribution to the energetics from the Reynolds stress due to the irrotational part of the velocity field, B, can be shown to be $B=|A^+|^2+|A^-|^2+C_2\cos(t^2/M+\theta_2)$ where C_2 and θ_2 are constants. Consequently, the energy tendency is

$$\frac{d\overline{E}}{dt}\propto|A^+|^2+|A^-|^2+C\cos(t^2/M+\theta), \tag{A11}$$

where C and θ constants, which implies that as $t\to\infty$:

$$E(t)\propto(|A^+|^2+|A^-|^2)t+D, \tag{A12}$$

where D is a constant. The energy grows linearly with time and the slope of this linear energy growth is proportional to the amplitude of the solenoidal part of the velocity field at large times. It should be noted that even if the perturbation field starts off with no solenoidal perturbations, these will emerge as the plane wave tilts over, and the perturbation field will acquire solenoidal character, thus exciting the unbounded linear energy growth. The amplitude of the solenoidal field produced depends delicately on the initial conditions. For a discussion of these points and also an alternative derivation of the asymptotic solutions see Ref. 11.

[1]B. F. Farrell and P. J. Ioannou, ''Generalized stability theory. Part I: Autonomous operators,'' J. Atmos. Sci. **53**, 2025 (1996).
[2]J. E. Pringle, ''Accretion discs in astrophysics,'' Annu. Rev. Astron. Astrophys. **19**, 137 (1981); J. Frank, A. King, and D. Raine, *Accretion Power in Astrophysics* (Cambridge University Press, Cambridge, 1995).
[3]B. F. Farrell and P. J. Ioannou, ''Perturbation growth in shear flow exhibits universality,'' Phys. Fluids A **5**, 2298 (1993).
[4]Lord Kelvin, ''On the stability of steady and of periodic fluid motion,'' Philos. Mag. **23**, 459 (1887).
[5]K. Moffatt, ''The interaction of turbulence with strong wind shear,'' in *Atmospheric Turbulence and Radio Wave Propagation*, edited by A. M. Yaglom and V. I. Tatarskii (Nauka, Moscow, 1967).
[6]A. D. D. Craik and W. O. Criminale, ''Evolution of wavelike disturbances in shear flow: A class of exact solutions of the Navier–Stokes equations,'' Proc. R. Soc. London, Ser. A **406**, 13 (1986).
[7]K. K. Tung, ''Initial-value problems for Rossby waves in a shear flow with critical level,'' J. Fluid Mech. **133**, 443 (1983).
[8]B. F. Farrell and P. J. Ioannou, ''Optimal excitation of perturbations in viscous shear flow,'' Phys. Fluids A **5**, 1390 (1993).
[9]B. Farrell and P. Ioannou, ''Transient development of perturbations in stratified shear flow,'' J. Atmos. Sci. **50**, 2201 (1993).
[10]G. D. Chagelishvili, A. G. Tevzadze, G. Bodo, and S. S. Moiseev, ''Linear mechanism of wave emergence from vortices in smooth shear flows,'' Phys. Rev. Lett. **79**, 3178 (1997).
[11]G. D. Chagelishvili, G. R. Khujadze, J. G. Lominadze, and A. D. Rogava, ''Acoustic waves in unbounded shear flows,'' Phys. Fluids **9**, 1955 (1997).
[12]P. G. Drazin and W. H. Reid, *Hydrodynamic Stability* (Cambridge University Press, Cambridge, 1981); A. P. Gallagher and A. McD. Mercer, ''On the behavior of small disturbances in plane Couette flow,'' J. Fluid Mech. **13**, 91 (1962); L. A. Dikii, ''On the nonlinear theory of hydrodynamic stability,'' J. Appl. Math. Mech. **29**, 1009 (1965); A. Davey, ''On the stability of plane Couette flow,'' J. Fluid Mech. **57**, 369 (1973); V. A. Romanov, ''Stability of plane-parallel Couette flow,'' Funct. Anal. Appl. **7**, 137 (1973).
[13]W. Glatzel, ''Sonic instability in supersonic shear flows,'' Mon. Not. R. Astron. Soc. **231**, 795 (1988); ''The linear stability of viscous compressible plane Couette flow,'' J. Fluid Mech. **202**, 515 (1989); L. M. Mack, ''On the inviscid acoustic-mode instability of shear flows. I. Two dimensional waves,'' Theor. Comput. Fluid Dyn. **2**, 97 (1990); P. W. Duck, G. Erlebacher, and M. Y. Hussaini, ''On the linear stability of compressible plane Couette flow,'' J. Fluid Mech. **258**, 131 (1994); S. Hu and X. Zhong, ''Linear stability of viscous supersonic plane Couette flow,'' Phys. Fluids **10**, 709 (1998).
[14]P. J. Ioannou and A. Kakouris, '' Stochastic dynamics of Keplerian accretion disks,'' submitted to Astrophys. J.
[15]B. F. Farrell and P. J. Ioannou, ''Stochastic forcing of perturbation variance in unbounded shear and deformation flows,'' J. Atmos. Sci. **50**, 200 (1993).
[16]A. Hanifi, P. J. Schmid, and D. S. Henningson, ''Transient growth in compressible boundary layers,'' Phys. Fluids **8**, 826 (1996).
[17]B. F. Farrell and P. J. Ioannou, ''Variance maintained by stochastic forcing of nonnormal dynamical systems associated with linearly stable shear flow,'' Phys. Rev. Lett. **72**, 1188 (1994); P. J. Ioannou, ''Non-normality increases variance,'' J. Atmos. Sci. **51**, 998 (1995).