Structure and Stability of Low Amplitude Jet Equilibria in Barotropic Turbulence

N.C. Constantinou, P.J. Ioannou and N.A. Bakas

Abstract Planetary turbulent flows are observed to self-organize into large scale structures such as zonal jets and coherent vortices. Recently, it was shown that a comprehensive understanding of the properties of these large scale structures and of the dynamics underlying their emergence and maintenance is gained through the study of the dynamics of the statistical state of the flow. Previous studies addressed the emergence of the coherent structures in barotropic turbulence and showed the zonal jets emerge as an instability of the Statistical State Dynamics (SSD). In this work, the equilibration of the incipient instabilities and the stability of the equilibrated jets near onset is investigated. It is shown through a weakly nonlinear analysis of the SSD that the amplitude of the jet evolves according to a Ginzburg-Landau equation. The equilibrated jets were found to have a harmonic structure and an amplitude that is an increasing function of the planetary vorticity gradient. It is also shown that most of the equilibrated jets are unstable and will evolve through jet merging and branching to the stable jets that have a scale close to the most unstable emerging jet.

1 Introduction

Atmospheric turbulence is commonly observed to be organized into two large scale zonal jets per hemisphere with long-lasting coherent waves embedded in them. The jets and their associated storm tracks control the transport of heat, while the coherent waves produce significant spatiotemporal variability of weather patterns.

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It is therefore important to understand the mechanisms for the emergence, equilibration, and maintenance of these coherent structures.

The simplest approximation that retains the relevant self-organization dynamics is a turbulent barotropic flow on a β-plane. Numerical simulations of this model have shown that robust, large scale zonal jets emerge in the flow out of a homogeneous background of turbulence and oftentimes coexist with westward propagating coherent waves embedded in them (Galperin et al. 2010). In addition, as the energy input rate is increased the initially emerging jets merge into larger scale jets before they equilibrate at large amplitude. Recently, a theory that accurately predicts the formation and nonlinear equilibration of large scale coherent structures in barotropic β-plane turbulence was developed (Bakas and Ioannou 2013). The theory is based on the investigation of the dynamics of the statistical state of the flow as obtained in the Stochastic Structural Stability Theory (S3T; Farrell and Ioannou 2003). That is, instead of studying the evolution of the complex flow itself, the goal is to study the evolution of the flow statistics, the dynamics of the corresponding equations and the stability of the statistical equilibria that emerge. Bakas and Ioannou (2013) have shown that the coherent structures emerge as an instability of the homogeneous equilibrium of the S3T dynamical system. In this work, we focus on the unstable zonal jet structures and study the equilibration of the incipient instabilities and the stability of the equilibrated jets near onset, seeking a theory that predicts the amplitude for the equilibrated jets as well as when the jet merging process occurs.

2 Formulation of Stochastic Structural Stability Theory

Consider a non-divergent barotropic flow with a velocity field $\mathbf{u} = (u, v)$ on a β-plane with cartesian coordinates $\mathbf{x} = (x, y)$. The relative vorticity $\zeta = \partial_x v - \partial_y u$, evolves according to the non-linear equation:

$$(\partial_t + \mathbf{u} \cdot \nabla)\zeta + \beta v = -r\zeta + f \tag{1}$$

where β is the gradient of planetary vorticity, r is the coefficient of linear dissipation that typically parameterizes Ekman drag in the Earth's atmosphere and f is a random stirring that parameterizes processes such as small scale convection or baroclinic instability, that are missing from the barotropic dynamics. We assume that f is a temporally delta correlated and spatially homogeneous random stirring. We also assume that the forcing is isotropic, injecting energy at a rate ε in a narrow ring of wavenumbers with radius K_f.

S3T describes the statistical dynamics of the first two same time moments of (1): the mean of the vorticity field $Z = \overline{\zeta(\mathbf{x}, t)}$ and the two point correlation function $C = \overline{\zeta'_a \zeta'_b}$ of the vorticity deviation from the mean $\zeta'_i \equiv \zeta_i - Z_i$, where the subscript i refers to the value of the relative vorticity at the two points $\mathbf{x}_i$. The overbar in the definition of the moments denotes a proper averaging operator that identifies the coherent motions of the turbulent flow and additionally satisfies the ergodic property that the average of any flow quantity is equal to an ensemble average over the forcing realizations $\bar{\cdot} = \langle\cdot\rangle$. Since we are interested in this study in the emergence of zonal jets, we consider the overbar as a zonal average. With this definition of the mean, the equations governing the evolution of the two moments of the flow are:

$$\partial_t U + rU = \overline{v'\zeta'} = G(C) \quad (2)$$

$$\partial_t C + (A_a + A_b)C = \Xi \quad (3)$$

where U is the zonal mean flow of the jet $\left(Z = -\partial_y U\right)$, $\overline{v'\zeta'}$ is the eddy vorticity flux that can be expressed as a function of the vorticity covariance $\overline{v'\zeta'} = G(C)$,

$$\mathrm{A} = -U\partial_x - \left(\beta - \partial^2_{yy}U\right)\partial_x\Delta^{-1} - r$$

governs the linear dynamics of eddies about the instantaneous mean flow U and Ξ is the spatial correlation function of the forcing. In obtaining (3), we have ignored the third cumulant, that is we have ignored the non-linear term describing the eddy-eddy interactions, so that (2–3) form a closed deterministic system that governs the joint evolution of the zonal jet and of the eddy statistics. The S3T system has bounded solutions and the fixed points U^E and C^E, if they exist, define statistical equilibria of zonal jets with velocity U^E, in the presence of an eddy field with covariance C^E.

3 Weakly Non-linear Dynamics of the Jet Forming Instability

The S3T system (2), (3) admits the homogeneous equilibrium $U^E = 0$, $C^E = \Xi/2r$, with no jets and a homogeneous eddy field with the spatial covariance of the forcing. The homogeneous equilibrium becomes unstable when the energy input rate passes a certain threshold ε_t and bifurcates to inhomogeneous equilibria in the form of zonal jets (Farrell and Ioannou 2007). The linear stability analysis of the S3T dynamical system (2–3) around the homogeneous equilibrium reveals that due to the homogeneity of the equilibrium, the eigenfunctions consist of a sinusoidal mean flow perturbation $\delta U = \mathrm{e}^{iny}$, with n the jet wavenumber. It was also shown that the minimum threshold $\varepsilon_c = min_n \varepsilon_t$ required for instability, as well as the scale of the jet $1/n_c$ emerging at ε_c is a function of the non-dimensional planetary vorticity gradient $\tilde{\beta} = \beta/K_f r$.

In this work, we focus near the onset of this jet forming instability. Near the threshold ε_c, the S3T dynamics can be approximated by performing a multi-scale perturbation expansion with respect to the supercriticality parameter $\mu = \sqrt{\varepsilon/\varepsilon_c - 1}$. That is, we expand the mean flow velocity and the corresponding covariance

$$U = \mu U_1 + \mu^2 U_2 + \cdots, \quad C = C^E + \mu C_1 + \mu^2 C_2 + \cdots$$

in the limit of small supercriticality $\mu \ll 1$. At leading order, the mean flow and the covariance is the eigenmode

$$U_1 = A(Y,T)\mathrm{e}^{in_c y} + c.c, \quad C_1 = A(Y,T)C^h(\mathbf{x}_a - \mathbf{x}_b)\mathrm{e}^{\frac{in_c(y_a+y_b)}{2}} + c.c,$$

which is harmonic with n_c the critical wavenumber at the onset of instability and has an amplitude $A(Y,T)$ that is slowly varying in latitude $Y = \mu y$ and in time $T = \mu^2 t$. At second order, a jet with the double harmonic is generated

$$U_2 = a_1 A^2(Y,T)\mathrm{e}^{2in_c y} + c.c$$

At next order there are secular terms produced that vanish only when the amplitude A satisfies the following real Ginzburg-Landau (G-L) equation:

$$c_1 \partial_T A = c_2 A + c_3 \partial^2_{YY} A - c_4 |A|^2 A, \tag{4}$$

The coefficients $c_i(\beta,\ r,\ n_c,\ C^E)$ and a_1 are real and positive for the isotropic forcing and are given in Constantinou et al. (2016). The linear terms are obtained from the linear stability analysis: the first term on the rhs gives the exponential growth (with rate c_2/c_1) of the unstable jet with wavenumber n_c, while the second term corrects for the growth of zonal jets with a slightly different wavenumber than n_c which is lower than the most unstable jet. The non-linear term is important since it controls the amplitude of the equilibrated jet. It can be shown that a harmonic mean flow $A = R_0 \mathrm{e}^{i\nu Y}$ is an equilibrium solution with amplitude

$$R_0 = \sqrt{(c_2 - \nu^2 c_3)/c_4}.$$

Figure 1 shows the amplitude R_0 as a function of $\tilde{\beta}$ for the most unstable jet ($\nu = 0$). For low $\tilde{\beta}$, the small scale jets ($n_c \approx K_f$ in this limit) equilibrate at low amplitude $R_0 \sim \tilde{\beta}^{5/8}$, while at large $\tilde{\beta}$, the large scale jets ($n_c \ll K_f$ in this limit) equilibrate at large amplitude $R_0 \sim \tilde{\beta}^{3/10}$.

The stability of these non-homogeneous zonal jet equilibria can be assessed by first rewriting the G-L equation as follows. We assume that $A(Y,T) = R(Y,T)\mathrm{e}^{i\theta(Y,T)}$ where R is the amplitude and θ is the phase and substitute into (4). Separating real and imaginary parts we obtain:

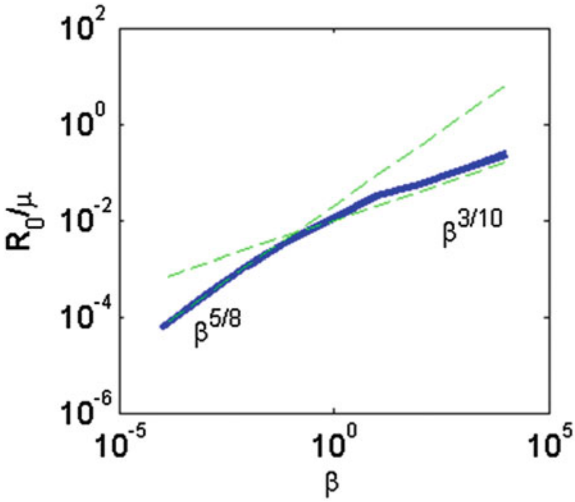

Fig. 1 Amplitude R_0 of equilibrated harmonic jets corresponding to the most unstable jet structure with wavenmber n_c as a function of the non-dimensional planetary vorticity gradient $\tilde{\beta} = \beta/K_f r$

$$c_1 \partial_T R = \left[c_2 - c_3 (\partial_Y \theta)^2 + c_3 \partial^2_{YY}\right] R - c_4 R^3, \tag{5}$$

$$c_1 R \partial_T \theta = 2c_3 \partial_Y R \partial_Y \theta + c_3 R \partial^2_{YY} \theta, \tag{6}$$

Assume now the equilibrium jet with constant amplitude R_0 and linearly varying phase $\theta = vY$ and small wavelike perturbations in the amplitude ρ and the phase φ of the form $[\rho, \varphi] = [\hat{\rho}, \hat{\varphi}] e^{iqY+\sigma t}$. Linearizing (5–6) around the harmonic equilibrium and solving for σ we obtain the dispersion relation:

$$\sigma = -\left(q^2 + R_0^2\right) \pm \sqrt{R_0^4 + 4c_3^2 v^2 q^2}$$

We have instability for $|q| \leq \sqrt{6v^2 - 2v_e^2}$, so only the zonal jets with $v \geq v_e/\sqrt{3}$ are unstable. The maximum instability occurs for $q = \left(\frac{v_e}{2}\right)\sqrt{3\left(\frac{v}{v_e}\right)^4 + 2\left(\frac{v}{v_e}\right)^2 - 1}$ with the growth rate $\sigma = c_3\left(3v^2 - v_e^2\right)^2/4v^2$. This is the Eckhaus instability first discovered in studies of pattern formation in convection and results in a transformation of the Eckhaus unstable jets into stable jets through a series of jet mergings or branchings depending on whether the jet has $n > n_c$ or $n < n_c$ respectively.

We will now test the accuracy of the G-L equation in describing the equilibration of the jet forming instability and the stability of the equilibrated jets. The quantitative accuracy of the G-L equation is unfortunately limited only to parameter values that are very close to the stability boundary, i.e., only for $0 < \mu < 0.1$ (not shown). However, the G-L dynamics show qualitatively the same behavior as the S3T dynamics for low supercriticalities (up to $\mu \approx 0.5$) and reveal the dynamics of the Eckhaus instability that attracts the finite amplitude states to the Ekchaus stable

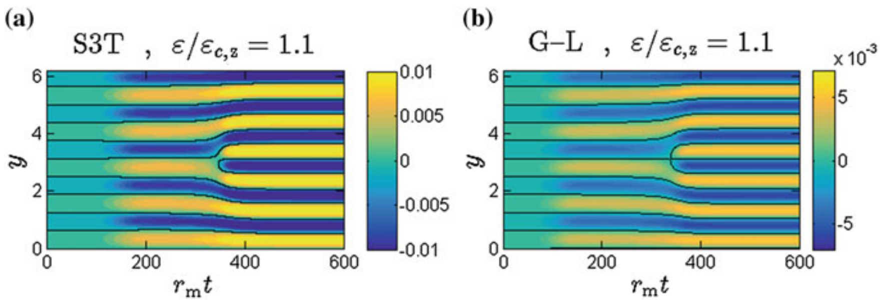

Fig. 2 Comparison of the evolution of the zonal mean flow U of the unstable jet structure with $n = 5$, as predicted by **a** S3T and **b** G-L dynamics. The *solid lines* mark the zero contour, while the parameters are $\beta = 9.49$, $K_f = 14$, $r = 0.1$

structures. To show this, we present a comparison of the predictions of the fully non-linear S3T dynamics integrated numerically in a $2\pi \times 2\pi$ doubly periodic box for a certain set of parameters ($\beta = 9.49$, $K_f = 14$, $r = 0.1$) and its approximation by G-L dynamics at supercriticality $\mu = 0.1$. For these parameters, the homogeneous equilibrium for the periodic box is unstable to only three jets with $n = 5$, 6 and 7 due to quantization of the wavenumbers. At this supercriticality the $n = 5$ jet equilibrium is Eckhaus unstable to $n = 6$. To illustrate the Eckhaus instability both simulations are initiated from the homogeneous state perturbed by an $n = 5$ jet mean flow perturbation that is contaminated with an $n = 6$ mean flow perturbation with an amplitude 10^{-5} times smaller. The jet evolution shown in Fig. 2 exhibits qualitatively the same behavior: the $n = 5$ jet mean flow perturbation initially grows exponentially and reaches finite amplitude. This equilibrium is Eckhaus unstable and both dynamics subsequently transition to the stable $n = 6$ jet state through jet branching. The only differences are quantitative and regard the amplitude of the finally equilibrated flow. Similar qualitative agreement between S3T and G-L dynamics is also found for the $n = 7$ jet equilibrium that transitions to the same $n = 6$ state through jet merging. Therefore we conclude the jet mergers/branchings at small supercriticalities are manifestations of the equilibration of the Eckhaus instabilities.

4 Conclusions

In summary, we studied jet formation based on a theory for the statistical state dynamics of the flow (S3T). We focused on the equilibration of the incipient instability of the S3T system governing the evolution of the flow statistics, that gives rise to zonal jets. We showed that the jet amplitude follows a Ginzburg-Landau equation and that the instabilities equilibrate as harmonic jets. The amplitude of the jets was found to be increasing with the non-dimensional planetary vorticity gradient $\tilde{\beta} = \beta/K_f r$. Only a band of wavenumbers close to the most unstable jet with wavenumber n_c was found to be stable. As a result of the secondary instability, the rest of the emerging jets are attracted to the stable equilibria through a series of jet mergings/branchings. The role of the Eckhaus instability in jet mergings observed at large energy input rates will be the subject of future research.

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