On the other hand, the capacitor (which may be thought of as two metal plates separated by some insulator; in the water model it is a tank) imposes the condition

$$C\,\frac{dv_C(t)}{dt} = i_C(t),$$

where $C$ is a positive constant called the capacitance.

We summarize our development so far: a state of our circuit is given by the six numbers $(i_R, i_L, i_C, v_R, v_L, v_C)$, that is, an element of $\mathbf{R}^3 \times \mathbf{R}^3$. These numbers are subject to three restrictions: Kirchhoff's current law, Kirchhoff's voltage law, and the resistor characteristic or "generalized Ohm's law." Therefore the space of physical states is a certain subset $\Sigma \subset \mathbf{R}^3 \times \mathbf{R}^3$. The way a state changes in time is determined by two differential equations.

Next, we simplify the state space $\Sigma$ by observing that $i_L$ and $v_C$ determine the other four coordinates, since $i_R = i_L$ and $i_C = -i_L$ by KCL, $v_R = f(i_R) = f(i_L)$ by the generalized Ohm's law, and $v_L = v_C - v_R = v_C - f(i_L)$ by KVL. Therefore we can use $\mathbf{R}^2$ as the state space, interpreting the coordinates as $(i_L, v_C)$. Formally, we define a map $\pi: \mathbf{R}^3 \times \mathbf{R}^3 \to R^2$, sending $(i, v) \in \mathbf{R}^3 \times \mathbf{R}^3$ to $(i_L, v_C)$. Then we set $\pi_0 = \pi \mid \Sigma$, the restriction of $\pi$ to $\Sigma$; this map $\pi_0: \Sigma \to \mathbf{R}^2$ is one-to-one and onto; its inverse is given by the map $\varphi: \mathbf{R}^2 \to \Sigma$,

$$\varphi(i_L, v_C) = (i_L, i_L, -i_L, f(i_L), v_C - f(i_L), v_C).$$

It is easy to check that $\varphi(i_L, v_C)$ satisfies KCL, KVL, and the generalized Ohm's law, so $\varphi$ does map $\mathbf{R}^2$ into $\Sigma$; it is also easy to see that $\pi_0$ and $\varphi$ are inverse to each other.

We therefore adopt $\mathbf{R}^2$ as our state space. The differential equations governing the change of state must be rewritten in terms of our new coordinates $(i_L, v_C)$:

$$L\,\frac{di_L}{dt} = v_L = v_C - f(i_L),$$

$$C\,\frac{dv_C}{dt} = i_C = -i_L.$$

For simplicity, since this is only an example, we make $L = 1$, $C = 1$.

If we write $x = i_L$, $y = v_C$, we have as differential equations on the $(x, y)$ Cartesian space:

$$\frac{dx}{dt} = y - f(x),$$

$$\frac{dy}{dt} = -x.$$

These equations are analyzed in the following section.



*PROBLEMS*

1. Find the differential equations for the network in Fig. D, where the resistor is voltage controlled, that is, the resistor characteristic is the graph of a $C^1$ function $g: \mathbf{R} \to \mathbf{R}$, $g(v_R) = i_R$.

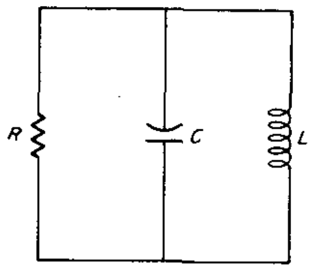


FIG. D

2. Show that the $LC$ circuit consisting of one inductor and one capacitor wired in a closed loop oscillates.

## §2. Analysis of the Circuit Equations

Here we begin a study of the phase portrait of the planar differential equation derived from the circuit of the previous section, namely:

$$\frac{dx}{dt} = y - f(x), \tag{1}$$

$$\frac{dy}{dt} = -x.$$

This is one form of *Lienard's equation*. If $f(x) = x^3 - x$, then (1) is a form of *Van der Pol's equation*.

First consider the most simple case of linear $f$ (or ordinary resistor of Section 1). Let $f(x) = Kx$, $K > 0$. Then (1) takes the form

$$z' = Az, \qquad A = \begin{bmatrix} -K & 1 \\ -1 & 0 \end{bmatrix}, \qquad z = (x, y).$$

The eigenvalues of $A$ are given by $\lambda = \frac{1}{2}[-K \pm (K^2 - 4)^{1/2}]$. Since $\lambda$ always has negative real part, the zero state $(0, 0)$ is an asymptotically stable equilibrium, in fact a sink. Every state tends to zero; physically this is the dissipative effect of the resistor. Furthermore, one can see that (0, 0) will be a spiral sink precisely when $K < 2$.

Next we consider the equilibria of (1) for a general $C^1$ function $f$.

There is in fact a unique equilibrium $\bar{z}$ of (1) obtained by setting

$$y - f(x) = 0,$$

$$-x = 0,$$

or

$$\bar{z} = (0, f(0)).$$

The matrix of first partial derivatives of (1) at $\bar{z}$ is

$$\begin{bmatrix} -f'(0) & 1 \\ -1 & 0 \end{bmatrix}$$

whose eigenvalues are given by

$$\lambda = \tfrac{1}{2}[-f'(0) \pm (f'(0)^2 - 4)^{1/2}].$$

We conclude that this equilibrium satisfies:

$$\bar{z} \quad \text{is a sink if} \quad f'(0) > 0,$$

and

$$\bar{z} \quad \text{is a source if} \quad f'(0) < 0$$

(see Chapter 9).

In particular for Van der Pol's equation ($f(x) = x^3 - x$) the unique equilibrium is a source.

To analyze (1) further we define a function $W: \mathbf{R}^2 \to \mathbf{R}^2$ by $W(x, y) = \frac{1}{2}(x^2 + y^2)$; thus $W$ is half of the norm squared. The following proposition is simple but important in the study of (1).

**Proposition** *Let $z(t) = (x(t), y(t))$ be a solution curve of Lienard's equation* (1). *Then*

$$\frac{d}{dt} W(z(t)) = -x(t)f(x(t)).$$

***Proof.*** Apply the chain rule to the composition

$$J \xrightarrow{z} \mathbf{R}^2 \xrightarrow{W} \mathbf{R}$$

to obtain

$$\frac{d}{dt} W(z(t)) = DW(z(t))(z'(t)) = x(t)x'(t) + y(t)y'(t);$$



suppressing $t$, this is equal to

$$x(y - f(x)) - yx = -xf(x)$$

by (1). Here $J$ could be any interval of real numbers in the domain of $z$.

The statement of the proposition has an interpretation for the electric circuit that gave rise to (1) and which we will pursue later: energy decreases along the solution curves according to the power dissipated in the resistor.

In circuit theory, a resistor whose characteristic is the graph of $f: \mathbf{R} \to \mathbf{R}$, is called *passive* if its characteristic is contained in the set consisting of (0, 0) and the interior of the first and third quadrant (Fig. A for example). Thus in the case of a passive resistor $-xf(x)$ is negative except when $x = 0$.

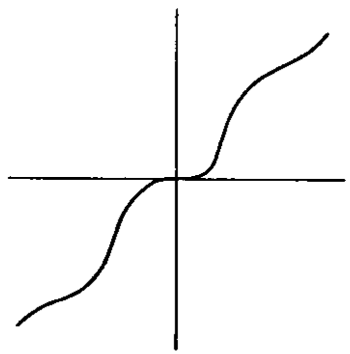

FIG. A

From Theorem 2 of Chapter 9, Section 3, it follows that the origin is asymptotically stable and its basin of attraction is the whole plane. Thus the word *passive* correctly describes the dynamics of such a circuit.

## §3. Van der Pol's Equation

The goal here is to continue the study of Lienard's equation for a certain function $f$.

$$\text{(1)} \qquad \frac{dx}{dt} = y - f(x), \qquad f(x) = x^3 - x,$$

$$\frac{dy}{dt} = -x.$$

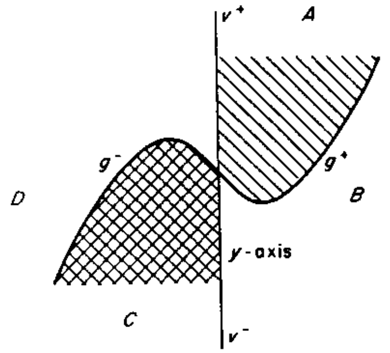


FIG. A

This is called *Van der Pol's equation*; equivalently

$$\frac{dx}{dt} = y - x^3 + x, \tag{2}$$

$$\frac{dy}{dt} = -x.$$

In this case we can give a fairly complete phase portrait analysis.

**Theorem** *There is one nontrivial periodic solution of* (1) *and every nonequilibrium solution tends to this periodic solution. "The system oscillates."*

We know from the previous section that (2) has a unique equilibrium at (0, 0), and it is a source. The next step is to show that every nonequilibrium solution "rotates" in a certain sense around the equilibrium in a clockwise direction. To this end we divide the $(x, y)$ plane into four disjoint regions (open sets) $A$, $B$, $C$, $D$ in Fig. A. These regions make up the complement of the curves

$$y - f(x) = 0, \tag{3}$$

$$-x = 0.$$

These curves (3) thus form the boundaries of the four regions. Let us make this more precise. Define four curves

$$v^+ = \{(x, y) \mid y > 0, x = 0\},$$

$$g^+ = \{(x, y) \mid x > 0, y = x^3 - x\},$$

$$v^- = \{(x, y) \mid y < 0, x = 0\},$$

$$g^- = \{(x, y) \mid x < 0, y = x^3 - x\}.$$



These curves are disjoint; together with the origin they form the boundaries of the four regions.

Next we see how the vector field $(x', y')$ of (1) behaves on the boundary curves. It is clear that $y' = 0$ at $(0, 0)$ and on $v^+ \cup v^-$, and nowhere else; and $x' = 0$ exactly on $g^+ \cup g^- \cup (0, 0)$. Furthermore the vector $(x', y')$ is horizontal on $v^+ \cup v^-$ and points right on $v^+$, and left on $v^-$ (Fig. B). And $(x', y')$ is vertical on $g^+ \cup g^-$, pointing downward on $g^+$ and upward on $g^-$. In each region $A$, $B$, $C$, $D$ the signs of $x'$ and $y'$ are constant. Thus in $A$, for example, we have $x' > 0$, $y' < 0$, and so the vector field always points into the fourth quadrant.

The next part of our analysis concerns the nature of the flow in the interior of the regions. Figure B suggests that trajectories spiral around the origin clockwise. The next two propositions make this precise.

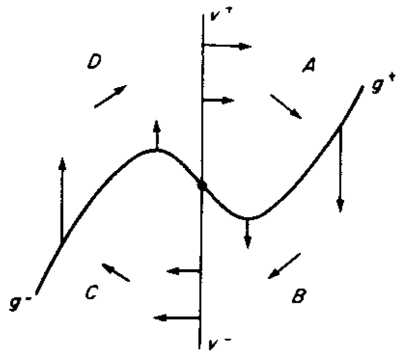


FIG. B

**Proposition 1** *Any trajectory starting on $v^+$ enters $A$. Any trajectory starting in $A$ meets $g^+$; furthermore it meets $g^+$ before it meets $v^-$, $g^-$ or $v^+$.*

***Proof.*** See Fig. B. Let $(x(t), y(t))$ be a solution curve to (1). If $(x(0), y(0)) \in v^+$, then $x(0) = 0$ and $y(0) > 0$. Since $x'(0) > 0$, $x(t)$ increases for small $t$ and so $x(t) > 0$ which implies that $y(t)$ decreases for small $t$. Hence the curve enters $A$. Before the curve leaves $A$ (if it does), $x'$ must become 0 again, so the curve must cross $g^+$ before it meets $v^-$, $g^-$ or $v^+$. Thus the first and last statements of the proposition are proved.

It remains to show that if $(x(0), y(0)) \in A$ then $(x(t), y(t)) \in g^+$ for some $t > 0$. Suppose not.

Let $P \subset \mathbf{R}^2$ be the compact set bounded by $(0, 0)$ and $v^+$, $g^+$ and the line $y = y(0)$ as in Fig. C. The solution curve $(x(t), y(t))$, $0 \leq t < \beta$ is in $P$. From Chapter 8, it follows since $(x(t), y(t))$ does not meet $g^+$, it is defined for all $t > 0$.

Since $x' > 0$ in $A$, $x(t) \geq a$ for $t > 0$. Hence from (1), $y'(t) \leq -a$ for $t > 0$.

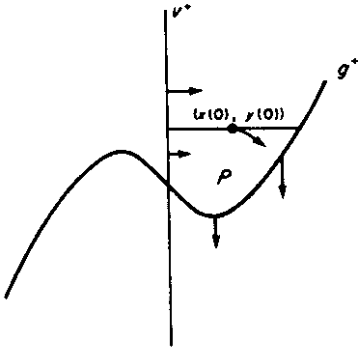


FIG. C

For these values of $t$, then

$$y(t) = \int_0^t y'(s)\, ds \leq y(0) - at.$$

This is impossible, unless our trajectory meets $g^+$, proving Proposition 1.

Similar arguments prove (see Fig. D):

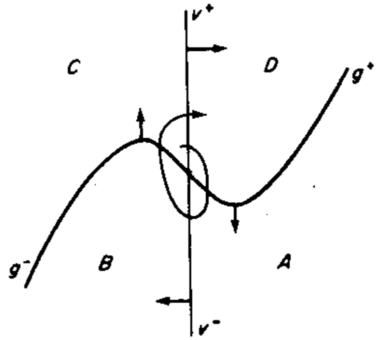


FIG. D. Trajectories spiral clockwise.

**Proposition 2** *Every trajectory is defined for (at least) all $t \geq 0$. Except for $(0, 0)$, each trajectory repeatedly crosses the curves $v^+$, $g^+$, $v^-$, $g^-$, in clockwise order, passing among the regions $A$, $B$, $C$, $D$ in clockwise order.*

To analyze further the flow of the Van der Pol oscillator we define a map

$$\sigma\colon v^+ \to v^+$$



as follows. Let $p \in v^+$; the solution curve $t \to \phi_t(p)$ through $p$ is defined for all $t \geq 0$. There will be a smallest $t_1(p) = t_1 > 0$ such that $\phi_{t_1}(p) \in v^+$. We put $\sigma(p) = \phi_{t_1}(p)$. Thus $\sigma(p)$ is the first point after $p$ on the trajectory of $p$ (for $t > 0$) which is again on $v^+$ (Fig. E). The map $p \to t_1(p)$ is continuous; while this should be intuitively clear, it follows rigorously from Chapter 11. Hence $\sigma$ is also continuous. Note that $\sigma$ is one to one by uniqueness of solutions.

The importance of this *section map* $\sigma\colon v^+ \to v^+$ comes from its intimate relationship to the phase portrait of the flow. For example:

**Proposition 3** *Let $p \in v^+$. Then $p$ is a fixed point of $\sigma$ (that is, $\sigma(p) = p$) if and only if $p$ is on a periodic solution of (1) (that is, $\phi_t(p) = p$ for some $t \neq 0$). Moreover every periodic solution curve meets $v^+$.*

***Proof.*** If $\sigma(p) = p$, then $\phi_{t_1}(p) = p$, where $t_1 = t_1(p)$ is as in the definition of $\sigma$. Suppose on the other hand that $\sigma(p) \neq p$. Let $v^* = v^+ \cup (0, 0)$. We observe first that $\sigma$ extends to a map $v^* \to v^*$ which is again continuous and one to one, sending $(0, 0)$ to itself. Next we identify $v^*$ with $\{y \in \mathbf{R} \mid y \geq 0\}$ by assigning to each point its $y$-coordinate. Hence there is a natural order on $v^*$: $(0, y) < (0, z)$ if $y < z$. It follows from the intermediate value theorem that $\sigma\colon v^* \to v^*$ is *order preserving*. If $\sigma(p) > p$, then $\sigma^2(p) > \sigma(p) > p$ and by induction $\sigma^n(p) > p$, $n = 1, 2, \ldots$. This means that the trajectory of $p$ never crosses $v^+$ again at $p$. Hence $\phi_t(p) \neq p$ for all $t \neq 0$. A similar argument applies if $\sigma(p) < p$. Therefore if $\sigma(p) \neq p$, $p$ is not on a periodic trajectory. The last statement of Proposition 3 follows from Proposition 2 which implies that every trajectory (except $(0, 0)$) meets $v^+$.

For every point $p \in v^+$ let $t_2(p) = t_2$ be the smallest $t > 0$ such that $\phi_t(p) \in v^-$. Define a continuous map

$$\alpha\colon v^+ \to v^-,$$

$$\alpha(p) = \phi_{t_2}(p).$$

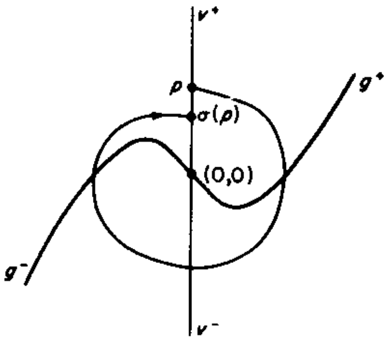


FIG. E. The map $\sigma\colon v^+ \to v^+$.

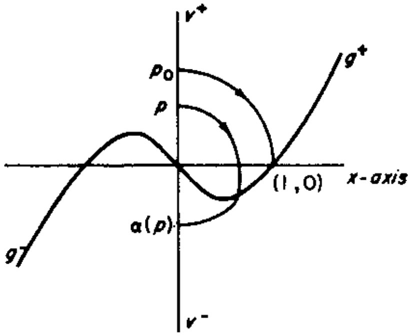


FIG. F. The map $\alpha: v^+ \to v^-$.

See Fig. F. The map $\alpha$ is also one to one by uniqueness of solutions and thus monotone.

Using the methods in the proof of Proposition 1 it can be shown that there is a unique point $p_0 \in v^+$ such that the solution curve

$$\{\phi_t(p_0) \mid 0 \leq t \leq t_1(p_0)\}$$

intersects the curve $g^+$ at the point $(1, 0)$ where $g^+$ meets the $x$-axis. Let $r = |p_0|$.

Define a continuous map

$$\delta: v^+ \to \mathbf{R},$$

$$\delta(p) = {}^2(|\alpha(p)|^2 - |p|^2)$$

where $|p|$ means the usual Euclidean norm of the vector $p$. Further analysis of the flow of (1) is based on the following rather delicate result:

**Proposition 4** (a) $\delta(p) > 0$ *if* $0 < |p| < r$;
(b) $\delta(p)$ *decreases monotonely to* $-\infty$ *as* $|p| \to \infty$, $|p| \geq r$.

Part of the graph of $\delta(p)$ as a function of $|p|$ is shown schematically in Fig. G. The intermediate value theorem and Proposition 4 imply that *there is a unique* $q_0 \in v^+$ *with* $\delta(q_0) = 0$.

We will prove Proposition 4 shortly; first we use it to complete the proof of the main theorem of this section. We exploit the *skew symmetry* of the vector field

$$g(x, y) = (y - x^3 + x, -x)$$

given by the right-hand side of (2), namely,

$$g(-x, -y) = -g(x, y).$$

This means that if $t \to (x(t), y(t))$ is a solution curve, so is $t \to (-x(t), -y(t))$. Consider the trajectory of the unique point $q_0 \in v^+$ such that $\delta(q_0) = 0$. This point has the property that $|\alpha(q_0)| = |q_0|$, hence that

$$\phi_{t_2}(q_0) = -q_0.$$

From skew symmetry we have also

$$\phi_{t_2}(-q_0) = -(-q_0) = q_0;$$

hence putting $\lambda = 2t_2 > 0$ we have

$$\phi_\lambda(q_0) = q_0.$$

Thus $q_0$ lies on a nontrivial periodic trajectory $\gamma$.

Since $\delta$ is monotone, similar reasoning shows that the trajectory through $q_0$ is the unique nontrivial periodic solution.

To investigate other trajectories we define a map $\beta: v^- \to v^+$, sending each point of $v^-$ to the first intersection of its trajectory (for $t > 0$) with $v^+$. By symmetry

$$\beta(p) = -\alpha(-p).$$

Note that $\sigma = \beta\alpha$.

We identify the $y$-axis with the real numbers in the $y$-coordinate. Thus if $p, q \in v^+ \cup v^-$ we write $p > q$ if $p$ is above $q$. Note that $\alpha$ and $\beta$ reverse this ordering while $\sigma$ preserves it.

Now let $p \in v_+$, $p > q_0$. Since $\alpha(q_0) = -q_0$ we have $\alpha(p) < -q_0$ and $\sigma(p) > q_0$. On the other hand, $\delta(p) < 0$ which means the same thing as $\alpha(p) > -p$. Therefore $\sigma(p) = \beta\alpha(p) < p$. We have shown that $p > q_0$ implies $p > \sigma(p) > q_0$. Similarly $\sigma(p) > \sigma^2(p) > q_0$ and by induction $\sigma^n(p) > \sigma^{n+1}(p) > q_0$, $n = 1, 2, \ldots$.

The sequence $\sigma^n(p)$ has a limit $q_1 \geq q_0$ in $v^+$. Note that $q_1$ is a fixed point of $\sigma$, for by continuity of $\sigma$ we have

$$\sigma(q_1) - q_1 = \lim_{n\to\infty} \sigma(\sigma^n(p)) - q_1$$
$$= q_1 - q_1 = 0.$$

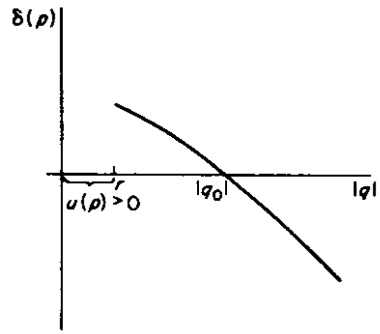


FIG. G

Since $\sigma$ has only one fixed point $q_1 = q_0$. This shows that *the trajectory of $p$ spirals toward $\gamma$ as $t \to \infty$*. The same thing is true if $p < q_0$; the details are left to the reader. Since every trajectory except $(0, 0)$ meets $v^+$, the proof of the main theorem is complete.

It remains to prove Proposition 4.

We adopt the following notation. Let $\gamma\colon [a, b] \to \mathbf{R}^2$ be a $C^1$ curve in the plane, written $\gamma(t) = (x(t), y(t))$. If $F\colon \mathbf{R}^2 \to \mathbf{R}$ is $C^1$, define

$$\int_\gamma F(x, y) = \int_a^b F(x(t), y(t))\, dt.$$

It may happen that $x'(t) \neq 0$ for $a \leq t \leq b$, so that along $\gamma$, $y$ is a function of $x$, $y = y(x)$. In this case we can change variables:

$$\int_a^b F(x(t), y(t))\, dt = \int_{x(a)}^{x(b)} F(x, y(x)) \frac{dt}{dx}\, dx;$$

hence

$$\int_\gamma F(x, y) = \int_{x(a)}^{x(b)} \frac{F(x, y(x))}{dx/dt}\, dx.$$

Similarly if $y'(t) \neq 0$.

Recall the function

$$W(x, y) = \tfrac{1}{2}(x^2 + y^2).$$

Let $\gamma(t) = (x(t), y(t))$, $0 \leq t \leq t_2 = t_2(p)$ be the solution curve joining $p \in v^+$ to $\alpha(p) \in v^-$. By definition $\delta(p) = W(x(t_2), y(t_2)) - W(x(0), y(0))$. Thus

$$\delta(p) = \int_0^{t_2} \frac{d}{dt} W(x(t), y(t))\, dt$$

By the proposition of Section 2 we have

$$\delta(p) = \int_0^{t_2} -x(t)(x(t)^3 - x(t))\, dt;$$

$$\delta(p) = \int_0^{t_2} x(t)^2(1 - x(t)^2)\, dt.$$

This immediately proves (a) of Proposition 4 because the integrand is positive for $0 < x(t) < 1$.

We may rewrite the last equality as

$$\delta(p) = \int_\gamma x^2(1 - x^2).$$

We restrict attention to points $p \in v^+$ with $|p| > r$. We divide the corresponding



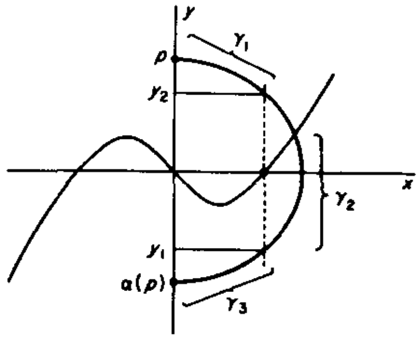


FIG. H

solution curve $\gamma$ into three curves $\gamma_1$, $\gamma_2$, $\gamma_3$ as in Fig. H. Then

$$\delta(p) = \delta_1(p) + \delta_2(p) + \delta_3(p),$$

where

$$\delta_i(p) = \int_{\gamma_i} x^2(1 - x^2), \qquad i = 1, 2, 3.$$

Notice that along $\gamma_1$, $y(t)$ is a function of $x(t)$. Hence

$$\delta_1(p) = \int_0^1 \frac{x^2(1 - x^2)}{dx/dt}\, dx$$

$$= \int_0^1 \frac{x^2(1 - x^2)}{y - f(x)}\, dx,$$

where $f(x) = x^3 - x$. As $p$ moves up the $y$-axis, $y - f(x)$ increases (for $(x, y)$ on $\gamma_1$). Hence $\delta_1(p)$ decreases as $|p| \to \infty$. Similarly $\delta_3(p)$ decreases as $|p| \to \infty$.

On $\gamma_2$, $x$ is a function of $y$, and $x \geq 1$. Therefore, since $dy/dt = -x$,

$$\delta_2(p) = \int_{y_2}^{y_1} -x(y)(1 - x(y)^2)\, dy$$

$$= \int_{y_1}^{y_2} x(y)(1 - x(y)^2)\, dy < 0.$$

As $|p|$ increases, the domain $[y_1, y_2]$ of integration becomes steadily larger. The function $y \to x(y)$ depends on $p$; we write it $x_p(y)$. As $|p|$ increases, the curves $\gamma_2$ move to the right; hence $x_p(y)$ increases and so $x_p(y)(1 - x_p(y)^2)$ decreases. It follows that $\delta_2(p)$ decreases as $|p|$ increases; and evidently $\lim_{|p|\to\infty} \delta_2(p) = -\infty$. This completes the proof of Proposition 4

*PROBLEMS*

1. Find the phase portrait for the differential equation

$$x' = y - f(x), \qquad f(x) = x^2,$$
$$y' = -x.$$

2. Give a proof of Proposition 2.

3. (Hartman [9, Chapter 7, Theorem 10.2]) Find the phase portrait of the following differential equation and *in particular show there is a unique nontrivial periodic solution*:

$$x' = y - f(x),$$
$$y' = -g(x),$$

where all of the following are assumed:
(i) $f, g$ are $C^1$;
(ii) $g(-x) = -g(x)$ and $xg(x) > 0$ for all $x \neq 0$;
(iii) $f(-x) = -f(x)$ and $f(x) < 0$ for $0 < x < a$;
(iv) for $x > a$, $f(x)$ is positive and increasing;
(v) $f(x) \to \infty$ as $x \to \infty$.
(*Hint*: Imitate the proof of the theorem in Section 3.)

4. (Hard!) Consider the equation

$$x' = y - f(x), \qquad f: \mathbf{R} \to \mathbf{R},\ C^1,$$
$$y' = -x.$$

Given $f$, how many periodic solutions does this system have? This would be interesting to know for many broad classes of functions $f$. Good results on this would probably make an interesting research article.

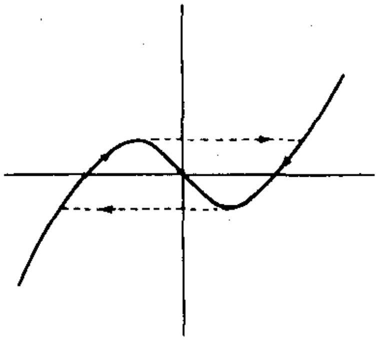

FIG. I



5. Consider the equation

$$x' = \mu(y - (x^3 - x)), \qquad \mu > 0,$$
$$y' = -x.$$

It has a unique nontrivial periodic solution $\gamma_\mu$ by Problem 3. Show that as $\mu \to \infty$, $\gamma_\mu$ tends to the closed curve consisting of two horizontal line segments and two arcs on $y = x^3 - x$ as in Fig. I.

## §4. Hopf Bifurcation

Often one encounters a differential equation *with parameter*. Precisely, one is given a $C^1$ map $g_\mu: W \to E$ where $W$ is an open set of the vector space $E$ and $\mu$ is allowed to vary over some parameter space, say $\mu \in J = [-1, 1]$. Furthermore it is convenient to suppose that $g_\mu$ is differentiable in $\mu$, or that the map

$$J \times W \to E, \qquad (\mu, x) \to g_\mu(x)$$

is $C^1$.

Then one considers the differential equation

$$x' = g_\mu(x) \quad \text{on} \quad W. \tag{1}$$

One is especially concerned how the phase portrait of (1) changes as $\mu$ varies. A value $\mu_0$ where there is a basic structural change in this phase portrait is called a *bifurcation* point. Rather than try to develop any sort of systematic bifurcation theory here, we will give one fundamental example, or a realization of what is called Hopf bifurcation.

Return to the circuit example of Section 1, where we now suppose that the resistor characteristic depends on a parameter $\mu$ and is denoted by $f_\mu: \mathbf{R} \to \mathbf{R}$, $-1 \leq \mu \leq 1$. (Maybe $\mu$ is the temperature of the resistor.) The physical behavior of the circuit is then described by the differential equation on $\mathbf{R}^2$:

$$\frac{dx}{dt} = y - f_\mu(x), \tag{2}$$

$$\frac{dy}{dt} = -x.$$

Consider as an example the special case where $f_\mu$ is described by

$$f_\mu(x) = x^3 - \mu x. \tag{2a}$$

Then we apply the results of Sections 2 and 3 to see what happens as $\mu$ is varied from $-1$ to 1.

For each $\mu$, $-1 \leq \mu \leq 0$, the resistor is passive and the proposition of Section 2 implies that *all solutions tend asymptotically to zero as* $t \to \infty$. Physically the circuit is dead, in that after a period of transition all the currents and voltages

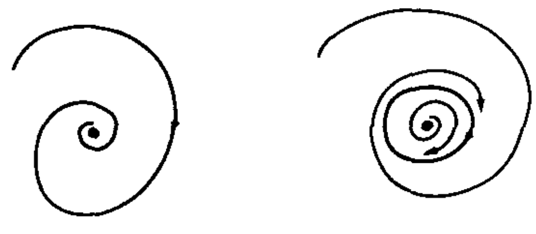


FIG. A. Bifurcation.

stay at 0 (or as close to 0 as we want). But note that as $\mu$ crosses 0, the circuit becomes alive. It will begin to oscillate. This follows from the fact that the analysis of Section 3 applies to (2) when $0 < \mu \leq 1$; in this case (2) will have a unique periodic solution $\gamma_\mu$ and the origin becomes a source. In fact every nontrivial solution tends to $\gamma_\mu$ as $t \to \infty$. Further elaboration of the ideas in Section 3 can be used to show that $\gamma_\mu \to 0$ as $\mu \to 0$, $\mu > 0$.

For (2), $\mu = 0$ is the bifurcation value of the parameter. The basic structure of the phase portrait changes as $\mu$ passes through the value 0. See Fig. A.

The mathematician E. Hopf proved that for fairly general one-parameter families of equations $x' = f_\mu(x)$, there must be a closed orbit for $\mu > \mu_0$ if the eigenvalue character of an equilibrium changes suddenly at $\mu_0$ from a sink to a source.

## *PROBLEMS*

1. Find all values of $\mu$ which are the bifurcation points for the linear differential equation:

$$\frac{dx}{dt} = \mu x + y,$$

$$\frac{dy}{dt} = x - 2y.$$

2. Prove the statement in the text that $\gamma_\mu \to 0$ as $\mu \to 0$, $\mu > 0$.

## §5. More General Circuit Equations

We give here a way of finding the ordinary differential equations for a class of electrical networks or circuits. We consider networks made up of resistors, capacitors, and inductors. Later we discuss briefly the nature of these objects, called the *branches* of the circuit; at present it suffices to consider them as devices with two



terminals. The circuit is formed by connecting together various terminals. The connection points are called *nodes*.

Toward giving a mathematical description of the network, we define in $\mathbf{R}^3$ a *linear graph* which corresponds to the network. This linear graph consists of the following data:

(a) A finite set $A$ of points (called nodes) in $\mathbf{R}^3$. The number of nodes is denoted by $a$, a typical node by $\alpha$.

(b) A finite set $B$ of line segments in $\mathbf{R}^3$ (called branches). The end points of a branch must be nodes. Distinct branches can meet only at a node. The number of branches is $b$; a typical branch is denoted by $\beta$.

We assume that each branch $\beta$ is *oriented* in the sense that one is given a direction from one terminal to the other, say from a $(-)$ terminal $\beta^-$ to a $(+)$ terminal $\beta^+$. The *boundary* of $\beta \in B$ is the set $\partial\beta = \beta^+ \cup \beta^-$

For the moment we ignore the exact nature of a branch, whether it is a resistor, capacitor, or inductor.

We suppose also that the set of nodes and the set of branches are ordered, so that it makes sense to speak of the $k$th branch, and so on.

A *current state* of the network will be some point $i = (i_1, \ldots, i_b) \in \mathbf{R}^b$ where $i_k$ represents the current flowing through the $k$th branch at a certain moment. In this case we will often write $\mathcal{I}$ for $\mathbf{R}^b$.

The *Kirchhoff current law* or KCL states that the amount of current flowing into a node at a given moment is equal to the amount flowing out. The water analogy of Section 1 makes this plausible. We want to express this condition in a mathematical way which will be especially convenient for our development. Toward this end we construct a linear map $d\colon \mathcal{I} \to \mathfrak{D}$ where $\mathfrak{D}$ is the Cartesian space $\mathbf{R}^a$ (recall $a$ is the number of nodes).

If $i \in \mathcal{I}$ is a current state and $\alpha$ is a node we define the $\alpha$th coordinate of $di \in \mathfrak{D}$ to be

$$(di)_\alpha = \sum_{\beta \in B} \epsilon_{\alpha\beta} i_\beta,$$

where

$$\epsilon_{\alpha\beta} = \begin{cases} 1 & \text{if } \beta^+ = \alpha, \\ -1 & \text{if } \beta^- = \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

One may interpret $(di)_\alpha$ as the net current flow into node $\alpha$ when the circuit is in the current state $i$.

**Theorem 1** *A current state $i \in \mathcal{I}$ satisfies KCL if and only if $di = 0$.*

***Proof.*** It is sufficient to check the condition for each node $\alpha \in A$. Thus $(di)_\alpha = 0$ if and only if

$$\sum_{\beta \in B} \epsilon_{\alpha\beta} i_\beta = 0,$$