CHAPTER

ELEVEN

MULTIPLE-SCALE ANALYSIS

And here—ah, now, this really is something a little recherché.

—Sherlock Holmes, *The Musgrave Ritual*
Sir Arthur Conan Doyle

(E) 11.1 RESONANCE AND SECULAR BEHAVIOR

Multiple-scale analysis is a very general collection of perturbation techniques that embodies the ideas of both boundary-layer theory and WKB theory. Multiple-scale analysis is particularly useful for constructing uniformly valid approximations to solutions of perturbation problems.

In this section we show how nonuniformity can appear in a regular perturbation expansion as a result of resonant interactions between consecutive orders of perturbation theory. To illustrate, we examine a simple perturbation problem, show how resonances occur and lead to a nonuniformly valid perturbation expansion, and finally show how to interpret and eliminate these nonuniformities. The formal development of multiple-scale analysis is postponed to Sec. 11.2.

Resonance

The phenomenon of resonance is nicely illustrated by the differential equation

$$\frac{d^2}{dt^2} y(t) + y(t) = \cos(\omega t). \tag{11.1.1}$$

This equation represents a harmonic oscillator of natural frequency 1 which is driven by a periodic external force of frequency ω. The general solution to this equation for $|\omega| \neq 1$ has the form

$$y(t) = A \cos t + B \sin t + \frac{\cos(\omega t)}{1 - \omega^2}, \qquad |\omega| \neq 1. \tag{11.1.2}$$

Observe that for all $|\omega| \neq 1$ the solution remains bounded for all t. If $|\omega|$ is close to 1, the amplitude of oscillation becomes large because the system absorbs large amounts of energy from the external force. Nevertheless, the amplitude of the system is still bounded when $|\omega| \neq 1$ because the system is oscillating out of phase with the driving force.

C. M. Bender et al., *Advanced Mathematical Methods for Scientists and Engineers I*

The solution in (11.1.2) is incorrect when $|\omega| = 1$. The correct solution has an amplitude which grows with t:

$$y(t) = A \cos t + B \sin t + \tfrac{1}{2}t \sin t, \qquad |\omega| = 1. \tag{11.1.3}$$

The amplitude of oscillation of this solution is unbounded as $t \to \infty$ because the oscillator continually absorbs energy from the periodic external force. This system is in *resonance* with the external force.

The term $\frac{1}{2}t \sin t$, whose amplitude grows with t, is said to be a *secular* term. The secular term $\frac{1}{2}t \sin t$ has appeared because the inhomogeneity $\cos t$ in (11.1.1) with $|\omega| = 1$ is itself a solution of the homogeneous equation associated with (11.1.1): $d^2y/dt^2 + y = 0$. In general, secular terms always appear whenever the inhomogeneous term is itself a solution of the associated homogeneous *constant-coefficient* differential equation. A secular term always grows more rapidly than the corresponding solution of the homogeneous equation by at least a factor of t.

Example 1 *Appearance of secular terms.*

(*a*) The solution to the differential equation $d^2y/dt^2 - y = e^{-t}$ has a secular term because e^{-t} satisfies the associated homogeneous equation. The general solution is $y(t) = Ae^{-t} + Be^t - \frac{1}{2}te^{-t}$. The particular solution $-\frac{1}{2}te^{-t}$ is secular relative to the homogeneous solution Ae^{-t}; we must regard the term $-\frac{1}{2}te^{-t}$ as secular even though it is negligible compared with the homogeneous solution Be^t as $t \to \infty$.

(*b*) The solution to the differential equation $d^2y/dt^2 - 2dy/dt + y = e^t$ has a secular term because e^t satisfies the associated homogeneous equation. The general solution is $y(t) = Ae^t + Bte^t + \frac{1}{2}t^2e^t$. In this case, the particular solution $\frac{1}{2}t^2e^t$ is secular with respect to all solutions of the associated homogeneous equation.

Nonuniformity of Regular Perturbation Expansions

The appearance of secular terms signals the nonuniform validity of perturbation expansions for large t. The nonlinear oscillator equation (Duffing's equation)

$$\frac{d^2y}{dt^2} + y + \varepsilon y^3 = 0, \qquad y(0) = 1,\ y'(0) = 0, \tag{11.1.4}$$

provides a good illustration of what we mean by nonuniformity. A perturbative solution of this equation is obtained by expanding $y(t)$ as a power series in ε:

$$y(t) = \sum_{n=0}^{\infty} \varepsilon^n y_n(t), \tag{11.1.5}$$

where $y_0(0) = 1$, $y_0'(0) = 0$, $y_n(0) = y_n'(0) = 0$ $(n \geq 1)$. Substituting (11.1.5) into the differential equation (11.1.4) and equating coefficients of like powers of ε gives a sequence of linear differential equations of which all but the first are inhomogeneous:

$$y_0'' + y_0 = 0, \tag{11.1.6a}$$

$$y_1'' + y_1 = -y_0^3, \tag{11.1.6b}$$

and so on.

The solution to (11.1.6*a*) which satisfies $y_0(0) = 1$, $y_0'(0) = 0$ is

$$y_0(t) = \cos t.$$

To solve (11.1.6b) we invoke the trigonometric identity $\cos^3 t = \frac{1}{4}\cos 3t + \frac{3}{4}\cos t$ to rewrite the inhomogeneous term. The formulas in (11.1.2)–(11.1.3) then provide the general solution to (11.1.6b):

$$y_1(t) = A\cos t + B\sin t + \tfrac{1}{32}\cos 3t - \tfrac{3}{8}t\sin t;$$

the particular solution satisfying $y_1(0) = y_1'(0) = 0$ is

$$y_1(t) = \tfrac{1}{32}\cos 3t - \tfrac{1}{32}\cos t - \tfrac{3}{8}t\sin t.$$

We observe that $y_1(t)$ contains a secular term. This secularity necessarily occurs because $\cos^3 t$ contains a component, $\frac{3}{4}\cos t$, whose frequency equals the natural frequency of the unperturbed oscillator.

In summary, the first-order perturbative solution to (11.1.4) is

$$y(t) = \cos t + \varepsilon[\tfrac{1}{32}\cos 3t - \tfrac{1}{32}\cos t - \tfrac{3}{8}t\sin t] + O(\varepsilon^2), \qquad \varepsilon \to 0+. \qquad (11.1.7)$$

We emphasize that the term $O(\varepsilon^2)$ in the above expression means that for *fixed* t the error between $y(t)$ and $y_0(t) + \varepsilon y_1(t)$ is at most of order ε^2 as $\varepsilon \to 0+$. The nonuniformity of this result surfaces if we consider large values of t—specifically, values of t of order $1/\varepsilon$ or larger as $\varepsilon \to 0+$. For such large values of t, the secular term in $y_1(t)$ suggests that the amplitude of oscillation grows with t. However, as we will now show, the exact solution $y(t)$ remains bounded for all t.

Boundedness of the Solution to (11.1.4)

To show that the solution to (11.1.4) is bounded for all t, we construct an integral of the differential equation. Multiplying (11.1.4) by the integrating factor dy/dt converts each term in the differential equation to an exact derivative:

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}y^2 + \frac{1}{4}\varepsilon y^4\right] = 0.$$

Thus,

$$\frac{1}{2}\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}y^2 + \frac{1}{4}\varepsilon y^4 = C, \qquad (11.1.8)$$

where C is a constant. Since $y(0) = 1$ and $y'(0) = 0$, $C = \frac{1}{2} + \frac{1}{4}\varepsilon$. When $\varepsilon > 0$, the integral in (11.1.8) shows that $\frac{1}{2}y^2 \le C$ for all t. Therefore, $|y(t)|$ is bounded for all t by $\sqrt{1 + \varepsilon/2}$.

The argument just given is frequently used in applied mathematics to prove boundedness of solutions to both ordinary and partial differential equations. The integral in (11.1.8) is called an *energy* integral. Equation (11.1.8) may be interpreted graphically as a closed bounded orbit in the phase plane whose axes are labeled by y and dy/dt (see Fig. 11.1).

Perturbative Construction of a Bounded Solution to (11.1.4)

We have arrived at an apparent paradox; we have shown that the exact solution $y(t)$ to (11.1.4) is bounded for all t but that the first-order perturbative solution in (11.1.7) is secular (grows with t for large t). The resolution of this paradox lies in

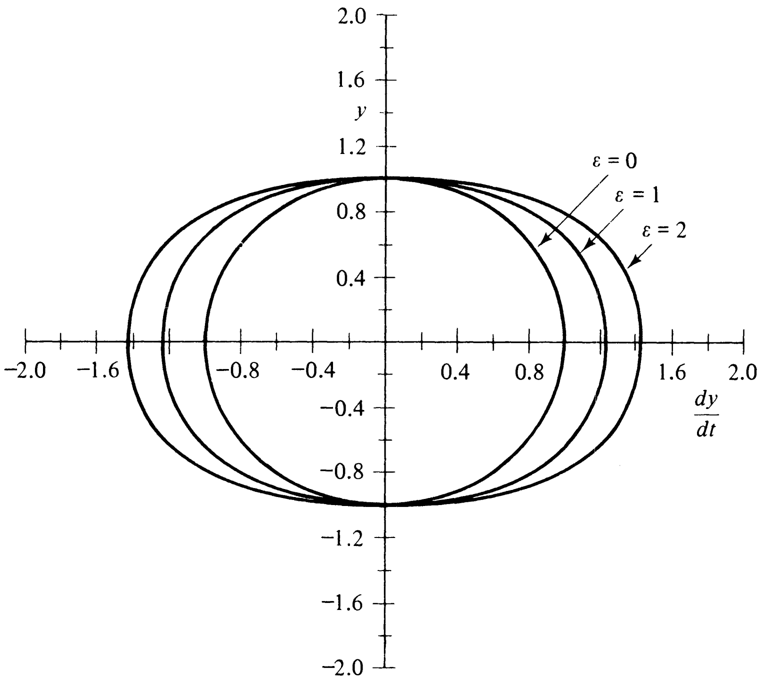

Figure 11.1 A phase-plane plot (y versus dy/dt) of solutions to Duffing's equation $d^2y/dt^2 + y + \varepsilon y^3 = 0$ [$y(0) = 1$, $y'(0) = 0$] for $\varepsilon = 0$, 1, and 2. The orbits shown are constant-energy curves [see (11.1.8)] which satisfy $(dy/dt)^2 + y^2 + \varepsilon y^4/2 = 1 + \varepsilon/2$.

the summation of the perturbation series (11.1.5). We know that the problem (11.1.4) is a regular perturbation problem as $\varepsilon \to 0+$ for fixed t (see Sec. 7.2). Therefore, the series (11.1.5) converges to the solution $y(t)$ for each t. We conclude that although order by order each term in the perturbation expansion may be secular, the secularity must disappear when the series is summed.

To illustrate how summing a perturbation series can eliminate secularity, consider the perturbation series

$$1 - \varepsilon t + \tfrac{1}{2}\varepsilon^2 t^2 - \tfrac{1}{6}\varepsilon^3 t^3 + \cdots + \varepsilon^n t^n[(-1)^n/n!] + \cdots, \qquad \varepsilon \to 0+.$$

Each term in this series is secular when t is of order $1/\varepsilon$ or larger. Nevertheless, the sum of the series $e^{-\varepsilon t}$ is bounded for all positive t!

We will now examine the more complicated perturbation series (11.1.5) and show that the sum of the most secular terms in each order in perturbation theory is actually not secular. We will show, using an inductive argument, that the most secular term in $y_n(t)$ has the form

$$A_n t^n e^{it} + A_n^* t^n e^{-it}, \tag{11.1.9}$$

where * denotes complex conjugation. There are less secular terms in $y_n(t)$ which grow like t^k ($k < n$), but we ignore such terms for now.

The final result of our calculations will be

$$A_n = \frac{1}{2}\frac{1}{n!}\left(\frac{3i}{8}\right)^n. \tag{11.1.10}$$

Using this formula for A_n we see that the sum of the most secular terms in the perturbation series (11.1.5) is a cosine function:

$$\sum_{n=0}^{\infty} \frac{1}{2} \varepsilon^n t^n \left[\frac{1}{n!}\left(\frac{3i}{8}\right)^n e^{it} + \frac{1}{n!}\left(-\frac{3i}{8}\right)^n e^{-it}\right] = \cos\left[t\left(1 + \frac{3}{8}\varepsilon\right)\right]. \quad (11.1.11)$$

Observe that this expression is not secular; it remains bounded for all t.

The expression (11.1.11) is a much better approximation to the exact solution $y(t)$ than $y_0(t) = \cos t$ because it is a good approximation to $y(t)$ for $0 \le t = O(1/\varepsilon)$. The difference between $y(t)$ and $\cos t$ is small so long as $0 \le t \ll 1/\varepsilon$ ($\varepsilon \to 0+$), while $\cos [t(1 + \frac{3}{8}\varepsilon)]$ is an accurate approximation to $y(t)$ over a much larger range of t. These assertions are explained as follows. In order that $y_0(t)$ be a good approximation to $y(t)$, it is necessary that $\varepsilon^n y_n(t) \ll y_0(t)$ ($\varepsilon \to 0+$) for all $n \ge 1$; this is true if $0 \le t \ll 1/\varepsilon$. On the other hand, the terms that we ignored in deriving (11.1.11) all have the form

$$\varepsilon[A\varepsilon^k(\varepsilon t)^l e^{imt} + A^*\varepsilon^k(\varepsilon t)^l e^{-imt}],$$

where k, l, m are nonnegative integers. Therefore, when $t = O(1/\varepsilon)$, each of these ignored terms is in fact negligible compared to at least one of the secular terms included in (11.1.11). We accept without proof the nontrivial result that the sum of all these small terms is still small. The higher-order terms are analyzed in Probs. 11.5 to 11.7.

We interpret the formula in (11.1.11) to mean that the cubic anharmonic term in (11.1.4) causes a shift in the frequency of the harmonic oscillator $y'' + y = 0$ from 1 to $1 + \frac{3}{8}\varepsilon$. This small frequency shift causes a phase shift which becomes noticeable when t is of order $1/\varepsilon$ (see Figs. 11.2 to 11.4 in Sec. 11.2).

Inductive Derivation of (11.1.10)

Comparing the first-order perturbation theory result in (11.1.7) with (11.1.9) verifies that the coefficient of the most secular terms in zeroth and first order are given correctly by (11.1.10). To establish (11.1.10) for all n, we proceed inductively. The $(n + 2)$th equation in the sequence of equations (11.1.6) determines $y_{n+1}(t)$:

$$y''_{n+1} + y_{n+1} = -I_{n+1}, \quad (11.1.12)$$

where the inhomogeneity I_{n+1} is the coefficient of ε^n in the expansion of $[\sum_{j=0}^{\infty} \varepsilon^j y_j(t)]^3$. Thus,

$$I_{n+1} = \sum_{j+k+l=n} y_j y_k y_l. \quad (11.1.13)$$

The most secular term in $y_{n+1}(t)$ is generated by the most secular terms in $y_j(t)$ for $0 \le j \le n$ (see Prob. 11.2). If we assume that (11.1.10) is valid for $A_0, A_1, A_2, \ldots, A_n$, then the coefficient of $t^n e^{it}$ in I_{n+1} is given by

$$\frac{1}{8}\left(\frac{3}{8}\right)^n \sum_{j+k+l=n} \frac{i^{j+k-l} + i^{j+l-k} + i^{k+l-j}}{j!\,k!\,l!} = \frac{1}{8}\left(\frac{3i}{8}\right)^n \sum_{j+k+l=n} \frac{(-1)^l + (-1)^k + (-1)^j}{j!\,k!\,l!}.$$

The sum in the above expression is just three times the coefficient of x^n in the Taylor expansion of $e^x e^x e^{-x}$ (see Prob. 11.3); therefore, it has the value $3/n!$. Thus, the terms in I_{n+1} which generate the most secular terms in $y_{n+1}(t)$ are

$$\tfrac{3}{8}(\tfrac{3}{8}t)^n[i^n e^{it} + (-i)^n e^{-it}]/n!.$$

Substituting these terms into the right side of (11.1.12) and solving for $y_{n+1}(t)$ gives

$$y_{n+1}(t) = (\tfrac{3}{8}t)^{n+1}[i^{n+1}e^{it} + (-i)^{n+1}e^{-it}]/(n+1)! + \text{less secular terms.}$$

By induction, we conclude that since (11.1.10) is true for $n = 0$, it remains true for all n.

(E) 11.2 MULTIPLE-SCALE ANALYSIS

In Sec. 11.1 we showed how to eliminate the most secular contributions to perturbation theory by simply summing them to all orders in powers of ε. The method we used works well but requires a lengthy calculation which can be avoided by using the methods of multiple-scale analysis that are introduced in this section.

Once again, we consider the nonlinear oscillator problem in (11.1.4):

$$\frac{d^2y}{dt^2} + y + \varepsilon y^3 = 0, \qquad y(0) = 1,\ y'(0) = 0. \tag{11.2.1}$$

The principal result of the last section is that when t is of order $1/\varepsilon$, perturbation theory in powers of ε is invalid. Secular terms appear in all orders (except zeroth order) and violate the boundedness of the solution $y(t)$.

A shortcut for eliminating the most secular terms to all orders begins by introducing a new variable $\tau = \varepsilon t$. τ defines a long time scale because τ is not negligible when t is of order $1/\varepsilon$ or larger. Even though the exact solution $y(t)$ is a function of t alone, multiple-scale analysis seeks solutions which are functions of both variables t and τ treated as *independent* variables. We emphasize that expressing y as a function of two variables is an artifice to remove secular effects; the actual solution has t and τ related by $\tau = \varepsilon t$ so that t and τ are ultimately not independent.

The formal procedure consists of assuming a perturbation expansion of the form

$$y(t) = Y_0(t, \tau) + \varepsilon Y_1(t, \tau) + \cdots. \tag{11.2.2}$$

We use the chain rule for partial differentiation to compute derivatives of $y(t)$:

$$\frac{dy}{dt} = \left(\frac{\partial Y_0}{\partial t} + \frac{\partial Y_0}{\partial \tau}\frac{d\tau}{dt}\right) + \varepsilon\left(\frac{\partial Y_1}{\partial t} + \frac{\partial Y_1}{\partial \tau}\frac{d\tau}{dt}\right) + \cdots.$$

However, since $\tau = \varepsilon t$, $d\tau/dt = \varepsilon$. Thus,

$$\frac{dy}{dt} = \frac{\partial Y_0}{\partial t} + \varepsilon\left(\frac{\partial Y_0}{\partial \tau} + \frac{\partial Y_1}{\partial t}\right) + O(\varepsilon^2). \tag{11.2.3}$$

Also, differentiating with respect to t again gives

$$\frac{d^2y}{dt^2} = \frac{\partial^2 Y_0}{\partial t^2} + \varepsilon\left(2\frac{\partial^2 Y_0}{\partial \tau\, \partial t} + \frac{\partial^2 Y_1}{\partial t^2}\right) + O(\varepsilon^2). \tag{11.2.4}$$

Substituting (11.2.4) into (11.2.1) and collecting powers of ε gives

$$\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \tag{11.2.5}$$

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -Y_0^3 - 2\frac{\partial^2 Y_0}{\partial \tau\, \partial t}. \tag{11.2.6}$$

The most general real solution to (11.2.5) is

$$Y_0(t, \tau) = A(\tau)e^{it} + A^*(\tau)e^{-it}, \tag{11.2.7}$$

where $A(\tau)$ is an arbitrary complex function of τ.

$A(\tau)$ will be determined by the condition that secular terms do *not* appear in the solution to (11.2.6). From (11.2.7), the right side of (11.2.6) is

$$e^{it}\left[-3A^2A^* - 2i\frac{dA}{d\tau}\right] + e^{-it}\left[-3A(A^*)^2 + 2i\frac{dA^*}{d\tau}\right] - e^{3it}A^3 - e^{-3it}(A^*)^3.$$

Note that e^{it} and e^{-it} are solutions of the homogeneous equation $\partial^2 Y_1/\partial t^2 + Y_1 = 0$. Therefore, if the coefficients of e^{it} and e^{-it} on the right side of (11.2.6) are nonzero, then the solution $Y_1(t, \tau)$ will be secular in t. To preclude the appearance of secularity, we require that the as yet arbitrary function $A(\tau)$ satisfy

$$-3A^2A^* - 2i\frac{dA}{d\tau} = 0, \tag{11.2.8}$$

$$-3A(A^*)^2 + 2i\frac{dA^*}{d\tau} = 0. \tag{11.2.9}$$

These two complex equations do not overdetermine $A(\tau)$ because they are redundant; one is the complex conjugate of the other. If (11.2.8) and (11.2.9) are satisfied, no secularity appears in (11.2.2), at least through terms of order ε.

To solve (11.2.8) for $A(\tau)$, we represent $A(\tau)$ in polar coordinate form:

$$A(\tau) = R(\tau)e^{i\theta(\tau)}, \tag{11.2.10}$$

where R and θ are real. Substituting into (11.2.8) and equating real and imaginary parts gives

$$\frac{dR}{d\tau} = 0, \tag{11.2.11a}$$

$$\frac{d\theta}{d\tau} = \frac{3}{2}R^2. \tag{11.2.11b}$$

Therefore,

$$A(\tau) = R(0)e^{i\theta(0) + 3iR^2(0)\tau/2} \tag{11.2.12}$$

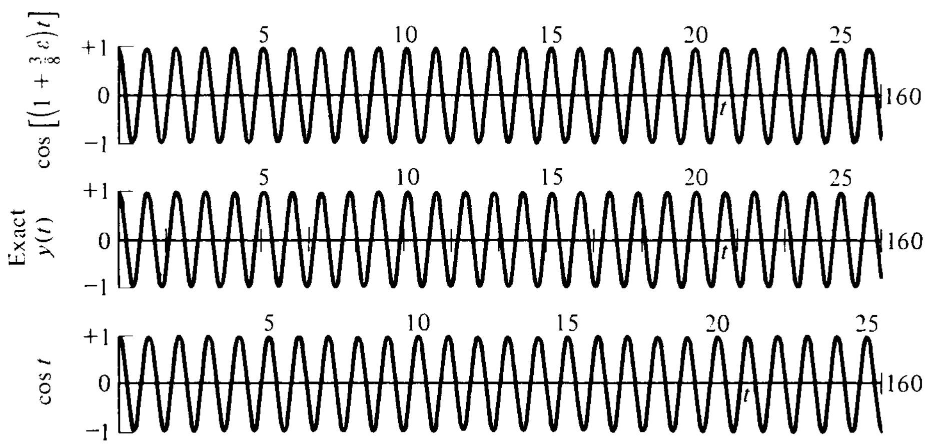

Figure 11.2 The exact solution $y(t)$ to Duffing's equation $d^2y/dt^2 + y + \varepsilon y^3 = 0$ $[y(0) = 1,\ y'(0) = 0]$ for $\varepsilon = 0.1$ (middle graph) compared with perturbative approximations to $y(t)$ (upper and lower graphs). The lower graph is a plot of $\cos t$, the first term in the regular perturbation series for $y(t)$, and the upper graph is a plot of $\cos[(1 + 3\varepsilon/8)t]$, the leading-order approximation to $y(t)$ obtained from multiple-scale methods. Both approximations, $\cos t$ and $\cos[(1 + 3\varepsilon/8)t]$, are correct up to additive terms of order ε, but $\cos t$ is not valid for large values of t; when $t = 160$, $\cos t$ is a full cycle out of phase with $y(t)$. The multiple-scale approximation closely approximates $y(t)$, even for large values of t.

and the zeroth-order solution (11.2.7) is

$$Y_0(t, \tau) = 2R(0) \cos [\theta(0) + \tfrac{3}{2}R^2(0)\tau + t]. \tag{11.2.13}$$

The initial conditions $y(0) = 1$, $y'(0) = 0$ determine $R(0)$ and $\theta(0)$. The condition $y(0) = 1$ becomes $Y_0(0, 0) = 1$, $Y_1(0, 0) = 0, \ldots$. From (11.2.3), $y'(0) = 0$ becomes $(\partial Y_0/\partial t)(0, 0) = 0$, $(\partial Y_1/\partial t)(0, 0) = -(\partial Y_0/\partial \tau)(0, 0), \ldots$. In order to satisfy these conditions, we must choose $R(0) = \frac{1}{2}$ and $\theta(0) = 0$. Therefore, the zeroth-order solution is $Y_0(t, \tau) = \cos[t + \frac{3}{8}\tau]$. Finally, since $\tau = \varepsilon t$,

$$y(t) = \cos[t(1 + \tfrac{3}{8}\varepsilon)] + O(\varepsilon), \qquad \varepsilon \to 0+,\ \varepsilon t = O(1), \tag{11.2.14}$$

and we have reproduced (11.1.11). In Figs. 11.2 to 11.4 we compare the exact solution to (11.2.1) with the approximation in (11.2.14).

A higher-order treatment of (11.2.1) is not completely straightforward. When more than two time scales are employed, there is so much freedom in the perturbation series representation that ambiguities can result (see Probs. 11.5 to 11.7).

(I) 11.3 EXAMPLES OF MULTIPLE-SCALE ANALYSIS

In this section we illustrate the formal multiple-scale technique that was developed in Sec. 11.2 by showing how to solve four elementary examples. The third and fourth of these examples are especially interesting because they show

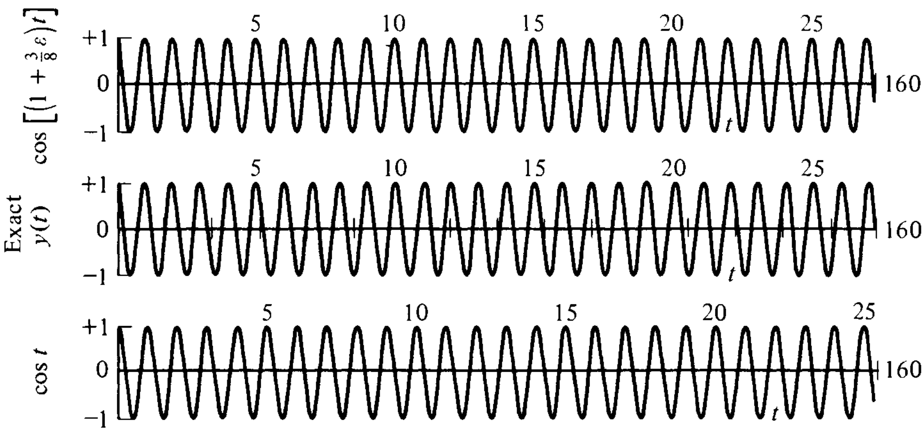

Figure 11.3 Same as in Fig. 11.2 but with $\varepsilon = 0.2$. Note that cos t is two cycles out of phase with $y(t)$ when $t = 160$.

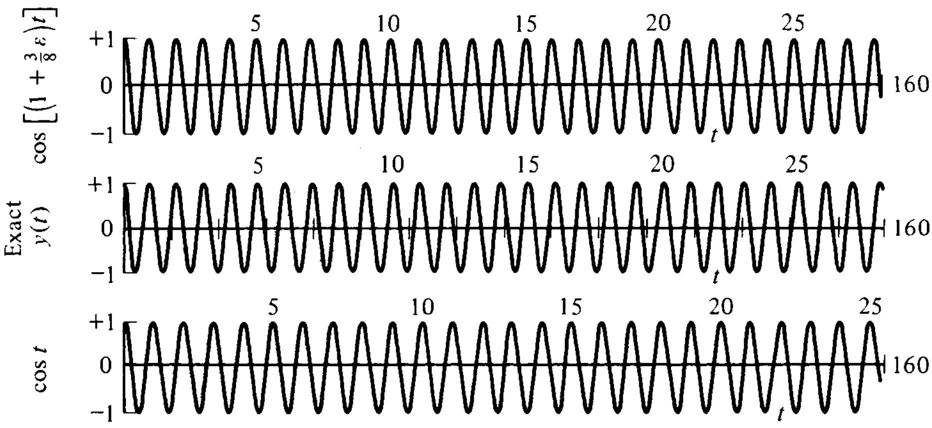

Figure 11.4 Same as in Fig. 11.2 but with $\varepsilon = 0.3$. Note that cos t is three cycles out of phase with $y(t)$ when $t = 160$.

how multiple-scale analysis can reproduce the results of boundary-layer and WKB analysis.

Example 1 *Multiple-scale analysis of a damped oscillator.* Let us consider an harmonic oscillator with a cubic damping term:

$$y'' + y + \varepsilon(y')^3 = 0, \qquad y(0) = 1,\ y'(0) = 0. \tag{11.3.1}$$

If $\varepsilon > 0$, the solution $y(t)$ must decay to 0 as $t \to \infty$. To prove this assertion, we multiply (11.3.1) by y' and construct an energy integral similar to that in (11.1.8):

$$\frac{d}{dt}\left[\frac{1}{2}(y')^2 + \frac{1}{2}y^2\right] = -\varepsilon(y')^4 \le 0. \tag{11.3.2}$$

This result shows that the energy $\frac{1}{2}(y')^2 + \frac{1}{2}y^2$ is a decreasing function of t unless $y'(t) = 0$ for all t. In Prob. 11.8 it is shown that the energy must decay to 0 as $t \to \infty$ and therefore that $y(t) \to 0$ as $t \to \infty$. [By contrast, when $\varepsilon < 0$, the energy argument just given shows that (11.3.1) represents a negatively damped system (like a self-propelled lawnmower that uses grass for fuel or a rocket with vacuum-cleaner drive that uses space dust for fuel) whose solutions grow explosively with t.]

Multiple-scale analysis may be used to study the behavior of $y(t)$ for large t. We begin by assuming a perturbation expansion for $y(t)$ in (11.3.1) of the form

$$y(t) \sim Y_0(t, \tau) + \varepsilon Y_1(t, \tau) + \cdots, \qquad \varepsilon \to 0+,$$

where $\tau = \varepsilon t$. Using (11.2.3) and (11.2.4) and equating coefficients of ε^0 and ε^1 gives two equations which correspond with (11.2.5) and (11.2.6):

$$\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \tag{11.3.3}$$

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -2\frac{\partial^2 Y_0}{\partial t\, \partial \tau} - \left(\frac{\partial Y_0}{\partial t}\right)^3. \tag{11.3.4}$$

The most general real solution to (11.3.3) is

$$Y_0(t, \tau) = A(\tau)e^{it} + A^*(\tau)e^{-it}. \tag{11.3.5}$$

Substituting this solution into the right side of (11.3.4) gives

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -e^{it}\left[2i\frac{dA}{d\tau} + 3iA^2A^*\right] - e^{-it}\left[-2i\frac{dA^*}{d\tau} - 3i(A^*)^2A\right]$$
$$+ ie^{3it}A^3 - ie^{-3it}(A^*)^3. \tag{11.3.6}$$

Since the solutions to the homogeneous equation (11.3.3) are $e^{\pm it}$, the solution to (11.3.6) is secular unless the expressions in the square brackets vanish; in order that Y_1 not be secular, we require that $A(\tau)$ satisfy the equations

$$2i\frac{dA}{d\tau} + 3iA^2A^* = 0, \tag{11.3.7a}$$

$$-2i\frac{dA^*}{d\tau} - 3i(A^*)^2A = 0. \tag{11.3.7b}$$

To solve (11.3.7) we set $A(\tau) = R(\tau)e^{i\theta(\tau)}$, where $R(\tau)$ and $\theta(\tau)$ are real. Substituting this expression into (11.3.7) gives equations for $R(\tau)$ and $\theta(\tau)$:

$$\frac{dR}{d\tau} = -\tfrac{3}{2}R^3, \qquad \frac{d\theta}{d\tau} = 0.$$

Therefore,

$$R(\tau) = \frac{R(0)}{\sqrt{3\tau R^2(0) + 1}}, \tag{11.3.8a}$$

$$\theta(\tau) = \theta(0). \tag{11.3.8b}$$

$R(0)$ and $\theta(0)$ are determined by the initial conditions $y(0) = 1$, $y'(0) = 0$. These conditions imply that $Y_0(0, 0) = 1$, $(\partial Y_0/\partial t)(0, 0) = 0$, whence $R(0) = \frac{1}{2}$, $\theta(0) = 0$. Thus, to leading order in ε,

$$y(t) \sim \frac{\cos t}{\sqrt{1 + 3\varepsilon t/4}}, \qquad \varepsilon \to 0+,\ \varepsilon t = O(1). \tag{11.3.9}$$

This result implies that when $\varepsilon > 0$ the solution decays like $t^{-1/2}$ for large t, and that when $\varepsilon < 0$ the solution becomes infinite at a finite value of t approximately equal to $-4/3\varepsilon$. Moreover, this solution does not exhibit any phase shift (or frequency shift) to leading order in ε. These qualitative conclusions are verified numerically in Figs. 11.5 to 11.7.

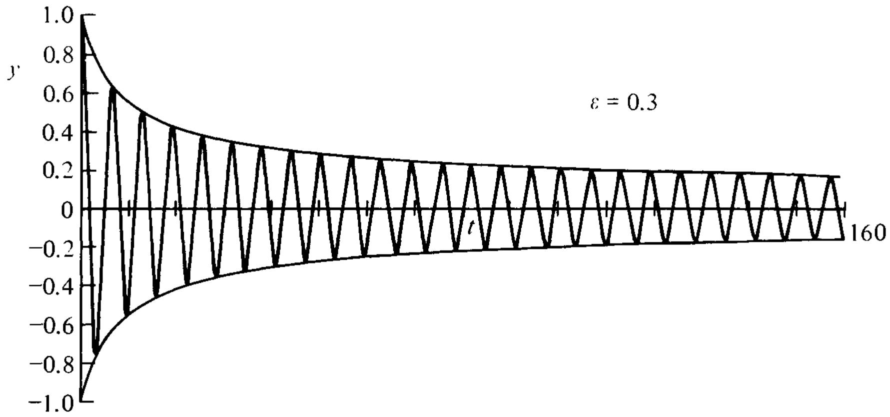

Figure 11.5 A plot of the exact solution to $y'' + y + \varepsilon(y')^3 = 0$ [$y(0) = 1$, $y'(0) = 0$] for $\varepsilon = 0.3$ [see (11.3.1)] together with a plot of the envelope $(1 + 3\varepsilon t/4)^{-1/2}$ of the leading-order multiple-scale approximation to $y(t)$ in (11.3.9). We have not plotted the full multiple-scale approximation to $y(t)$ because it is indistinguishable from the exact solution to within the thickness of the curve.

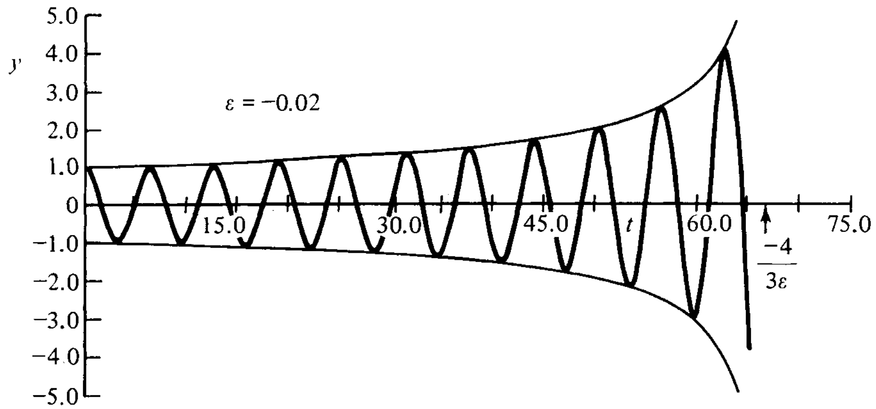

Figure 11.6 Same as in Fig. 11.5 except that $\varepsilon = -0.02$. Observe that the exact solution $y(t)$ and the multiple-scale approximation to it differ noticeably only when t is near the explosive singularity at $t = -4/3\varepsilon = 66\frac{2}{3}$.

Example 2 *Approach to a limit cycle.* The equation

$$y'' + y = \varepsilon[y' - \tfrac{1}{3}(y')^3], \qquad y(0) = 0,\ y'(0) = 2a, \tag{11.3.10}$$

known as the Rayleigh oscillator, is interesting because the solution approaches a limit cycle in the phase plane (see Sec. 4.4 and Example 3 of Sec. 9.7). Multiple-scale analysis determines the shape of this limit cycle and the rate of approach of $y(t)$ to the limit cycle.

As in Example 1, we assume a perturbation expansion for $y(t)$ in (11.3.10) of the form $y(t) \sim Y_0(t, \tau) + \varepsilon Y_1(t, \tau) + \cdots$ $(\varepsilon \to 0+)$, where $\tau = \varepsilon t$. Next we substitute (11.2.3) and (11.2.4)

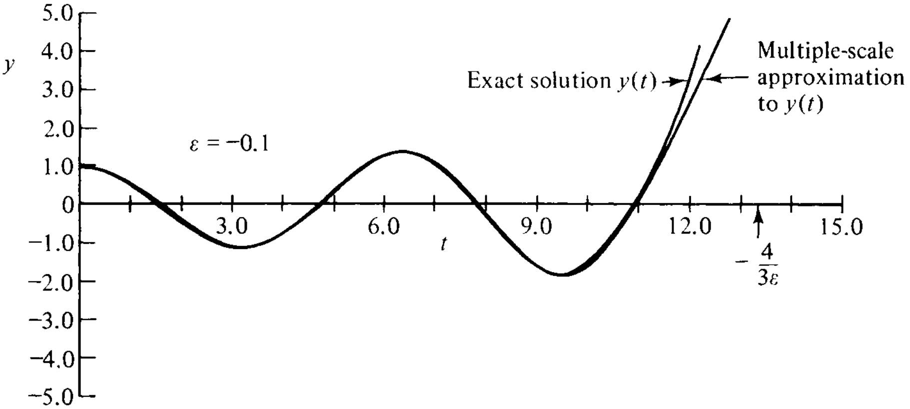

Figure 11.7 A comparison of the multiple-scale approximation and the exact solution to $y'' + y + \varepsilon(y')^3 = 0$ $[y(0) = 1, y'(0) = 0]$ for $\varepsilon = -0.1$. The approximation to $y(t)$ is extremely accurate except near the singularity at $t = -4/3\varepsilon = 13\frac{1}{3}$.

into (11.3.10) and equate coefficients of ε^0 and ε^1:

$$\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \tag{11.3.11}$$

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -2\frac{\partial^2 Y_0}{\partial t\, \partial \tau} + \frac{\partial Y_0}{\partial t} - \frac{1}{3}\left(\frac{\partial Y_0}{\partial t}\right)^3. \tag{11.3.12}$$

The solution to (11.3.11) is again

$$Y_0(t, \tau) = A(\tau)e^{it} + A^*(\tau)e^{-it}.$$

We substitute this expression into (11.3.12) and observe that secular terms in $Y_1(t, \tau)$ will arise unless the coefficients of $e^{\pm it}$ on the right side of (11.3.12) vanish. Thus, the conditions for the absence of secular behavior are

$$-2i\frac{dA}{d\tau} + iA - iA^2A^* = 0, \tag{11.3.13a}$$

$$2i\frac{dA^*}{d\tau} - iA^* + i(A^*)^2A = 0. \tag{11.3.13b}$$

To solve (11.3.13) we again set $A(\tau) = R(\tau)e^{i\theta(\tau)}$, where R and θ are real. The equations for R and θ are

$$2\frac{dR}{d\tau} = R - R^3, \tag{11.3.14a}$$

$$\frac{d\theta}{d\tau} = 0. \tag{11.3.14b}$$

The solutions are

$$R(\tau) = R(0)[e^{-\tau} + R^2(0)(1 - e^{-\tau})]^{-1/2}, \tag{11.3.15a}$$

$$\theta(\tau) = \theta(0). \tag{11.3.15b}$$

The initial conditions $y(0) = 0$, $y'(0) = 2a$ require that $R(0) = a$, $\theta(0) = -\frac{1}{2}\pi$. Thus, to leading order in ε, the solution to (11.3.10) is

$$y(t) \sim \frac{2a \sin t}{\sqrt{e^{-\tau} + a^2(1 - e^{-\tau})}}, \qquad \varepsilon \to 0+, \tau = \varepsilon t = O(1). \tag{11.3.16}$$

Observe that for all values of a, this approximate solution smoothly approaches the limit cycle $y(t) = 2 \sin t$ as $t \to \infty$. This limit cycle is represented as a circle of radius 2 in the phase plane of y and y'. If $a < 1$, the solution spirals outward to the limit cycle, and if $a > 1$, the solution spirals inward. A comparison of these asymptotic results and the numerical solution to (11.3.10) is given in Figs. 11.8 to 11.10.

Example 3 *Recovery of the WKB physical-optics approximation.* Let us consider the oscillator

$$y''(t) + \omega^2(\varepsilon t)y(t) = 0. \tag{11.3.17}$$

Note that the frequency $\omega(\varepsilon t)$ is a slowly varying function of time t.

It is easy to solve (11.3.17) using the WKB approximation. We simply introduce the new variable $\tau = \varepsilon t$ to convert (11.3.17) to standard WKB form:

$$\varepsilon^2 \frac{d^2y}{d\tau^2} + \omega^2(\tau)y = 0. \tag{11.3.18}$$

The physical-optics approximation to (11.3.18) [see (10.1.13)] is then

$$y(t) = [\omega(\tau)]^{-1/2} \exp\left[\pm i\varepsilon^{-1} \int^{\tau} \omega(s)\, ds\right]. \tag{11.3.19}$$

Now, let us rederive (11.3.19) using multiple-scale theory. The procedure requires a bit of subtlety. Suppose we naively assume that there is a linear relation $\tau = \varepsilon t$ between the appropriate long and short time scales. Then, letting $y(t) = Y_0(t, \tau) + \varepsilon Y_1(t, \tau) + \cdots$, we obtain

$$\frac{\partial^2 Y_0}{\partial t^2} + \omega^2(\tau)Y_0 = 0, \tag{11.3.20}$$

$$\frac{\partial^2 Y_1}{\partial t^2} + \omega^2(\tau)Y_1 = -2\frac{\partial^2 Y_0}{\partial t\, \partial \tau}. \tag{11.3.21}$$

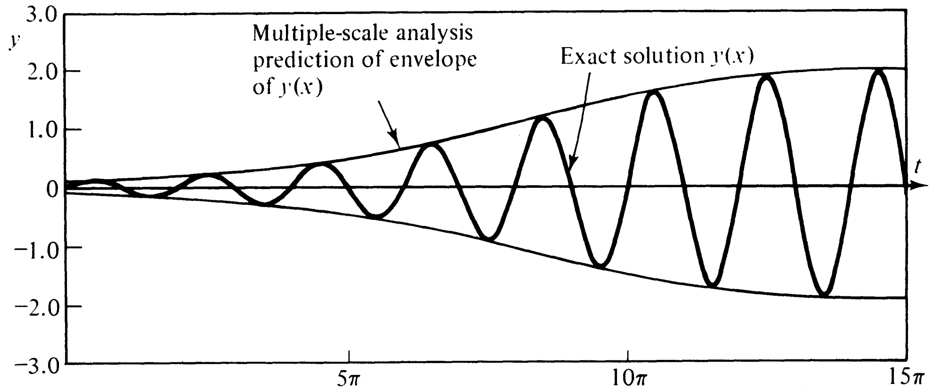

Figure 11.8 Approach to the limit cycle of the Rayleigh oscillator $y'' + y = \varepsilon[y' - \frac{1}{3}(y')^3]$ [$y(0) = 0$, $y'(0) = 2a$] [see (11.3.10)], where we have taken $\varepsilon = 0.2$ and $a = 0.05$. The oscillatory curve is the numerical solution to the differential equation; the envelope is the prediction of multiple-scale analysis [see (11.3.16)]. The two curves agree to better than their thicknesses.

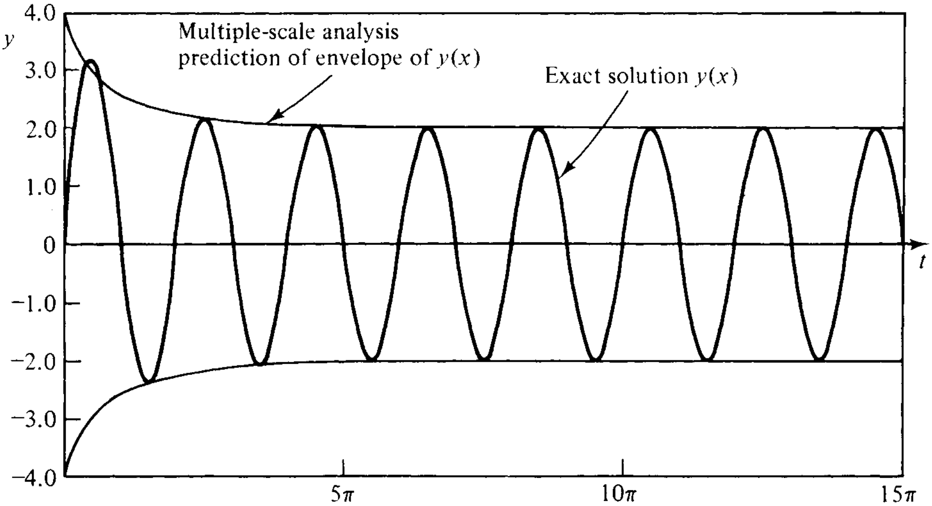

Figure 11.9 Approach to the limit cycle of the Rayleigh oscillator (11.3.10) (see Fig. 11.8). Here, $\varepsilon = 0.2$ and $a = 2.0$. Except for a small discrepancy at $t = \pi/2$ the exact and approximate solutions have nearly perfect agreement.

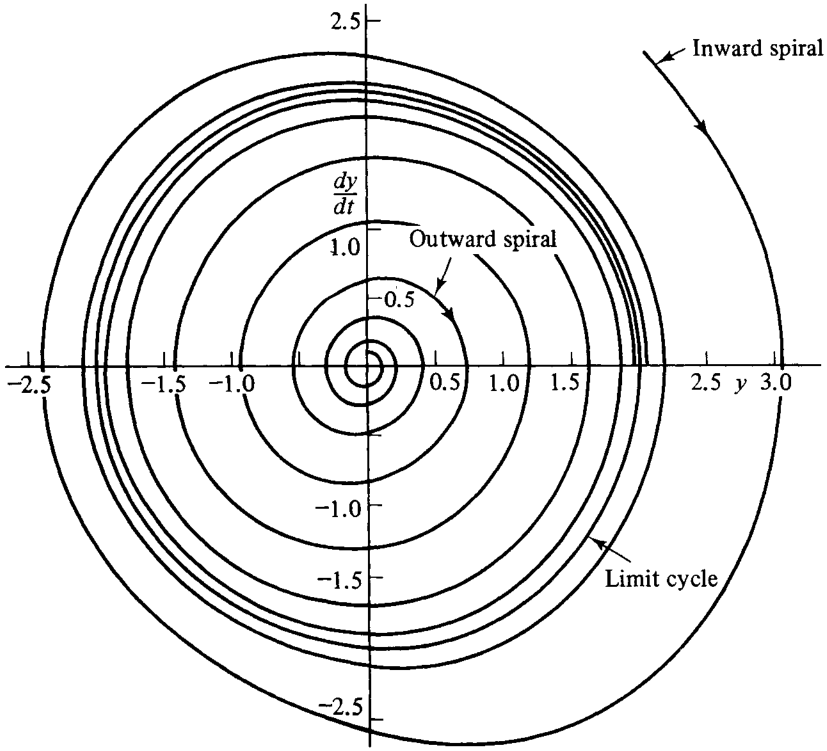

Figure 11.10 A phase-plane plot (y versus dy/dt) of three solutions to the Rayleigh oscillator (11.3.10) with $\varepsilon = 0.2$. Shown are the limit cycle solution which is approximately a circle of radius 2, the solution on Fig. 11.8 (spiraling outward toward the limit cycle), and the solution on Fig. 11.9 (spiraling inward toward the limit cycle).

The solution to (11.3.20) is $y_0 = A(\tau)e^{i\omega(\tau)t} + A^*(\tau)e^{-i\omega(\tau)t}$. Substituting this expression in the right side of (11.3.21) gives

$$\frac{\partial^2 Y_1}{\partial t^2} + \omega^2(\tau)Y_1 = -2ie^{i\omega(\tau)t}\left[\frac{d}{d\tau}(A\omega) + itA\omega\frac{d\omega}{d\tau}\right]$$

$$+ 2ie^{-i\omega(\tau)t}\left[\frac{d}{d\tau}(A^*\omega) - itA^*\omega\frac{d\omega}{d\tau}\right]. \tag{11.3.22}$$

The presence of the variable t in the square brackets implies that we cannot eliminate secularity without setting $A(\tau) \equiv 0$ (see Prob. 11.9).

This failure illustrates a crucial feature of multiple-scale perturbation methods. If the long-scale variable τ is linearly proportional to the short scale t ($\tau = \varepsilon t$), then multiple-scale methods will fail unless the frequency of the unperturbed oscillator is a constant; it must not vary even on the τ scale. Therefore, before we can apply multiple-scale methods to the oscillator (11.3.17), we must find a transformation which converts (11.3.17) to a fixed-frequency oscillator with a small perturbation term:

$$y'' + y + \varepsilon(\text{some function of } y) = 0. \tag{11.3.23}$$

With this in mind, we introduce a new time variable T:

$$T = f(t). \tag{11.3.24}$$

We will try to choose $f(t)$ to convert (11.3.17) to the form in (11.3.23). From (11.3.24) we have $d/dt = f'(t)\, d/dT$, $d^2/dt^2 = f''(t)\, d/dT + [f'(t)]^2\, d^2/dT^2$. Thus, (11.3.17) becomes

$$\frac{d^2}{dT^2}y + \frac{f''(t)}{[f'(t)]^2}\frac{d}{dT}y + \frac{\omega^2(\varepsilon t)}{[f'(t)]^2}y = 0.$$

We achieve the form in (11.3.23) if we choose $f'(t) = \omega(\varepsilon t)$. Thus,

$$T = f(t) = \int^t \omega(\varepsilon x)\, dx = \frac{1}{\varepsilon}\int^\tau \omega(s)\, ds. \tag{11.3.25}$$

In terms of T the differential equation now reads

$$\frac{d^2y}{dT^2} + y + \varepsilon\frac{\omega'(\tau)}{\omega^2(\tau)}\frac{d}{dT}y = 0. \tag{11.3.26}$$

This equation may be solved using multiple-scale methods. We expand

$$y = Y_0(T, \tau) + \varepsilon Y_1(T, \tau) + \cdots. \tag{11.3.27}$$

Using the relation $d\tau/dT = \varepsilon\, dt/dT = \varepsilon/f'(t) = \varepsilon/\omega(\tau)$, we substitute (11.3.27) into (11.3.26) and obtain, as usual, a sequence of partial differential equations:

$$\frac{\partial^2 Y_0}{\partial T^2} + Y_0 = 0, \tag{11.3.28}$$

$$\frac{\partial^2 Y_1}{\partial T^2} + Y_1 = -\frac{\omega'(\tau)}{\omega^2(\tau)}\frac{\partial Y_0}{\partial T} - \frac{2}{\omega}\frac{\partial^2 Y_0}{\partial T\,\partial\tau}. \tag{11.3.29}$$

Substituting the solution

$$Y_0 = A(\tau)e^{iT} + A^*(\tau)e^{-iT} \tag{11.3.30}$$

of (11.3.28) into the right side of (11.3.29) gives

$$\frac{\partial^2 Y_1}{\partial T^2} + Y_1 = -ie^{iT}\left[\frac{2}{\omega}\frac{dA}{d\tau} + \frac{\omega'(\tau)}{\omega^2(\tau)}A\right] + ie^{-iT}\left[\frac{2}{\omega}\frac{dA^*}{d\tau} + \frac{\omega'(\tau)}{\omega^2(\tau)}A^*\right].$$

To eliminate secularity we must require that the expressions in the square brackets vanish for all τ:

$$2\frac{dA}{d\tau} = -\frac{\omega'(\tau)}{\omega(\tau)}A,$$

$$2\frac{dA^*}{d\tau} = -\frac{\omega'(\tau)}{\omega(\tau)}A^*.$$

The solution for $A(\tau)$, apart from a multiplicative constant, is $1/\sqrt{\omega(\tau)}$. Inserting this solution into (11.3.30) gives

$$Y_0 = \frac{1}{\sqrt{\omega(\tau)}}e^{\pm iT},$$

and using the expression for T in (11.3.25) gives

$$Y_0 = \frac{1}{\sqrt{\omega(\tau)}}\exp\left[\pm\frac{i}{\varepsilon}\int^{\tau}\omega(s)\,ds\right].$$

We have reproduced the WKB result in (11.3.19).

Example 4 *Solution of a boundary-layer problem by multiple-scale perturbation theory.* Consider the elementary boundary-layer problem

$$\varepsilon y'' + ay' + by = 0, \qquad y(0) = A,\ y(1) = B,\ a > 0, \tag{11.3.31}$$

where a and b are constants. We know (see Fig. 9.4) that the solution to this problem has a boundary layer of thickness ε at $x = 0$ and is slowly varying in the range $\varepsilon \ll x \le 1$ $(\varepsilon \to 0+)$. Thus, there are two natural scales for this problem, a short scale t which describes the inner solution in the boundary layer and a long scale $x = \varepsilon t$ which describes the outer solution. Note that (11.3.31) is written in terms of the long scale. If we wish to use multiple-scale theory we must rewrite (11.3.31) in terms of the short scale t in order to eliminate secularity on the long scale:

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + \varepsilon by = 0. \tag{11.3.32}$$

Assuming that $y(t)$ in (11.3.32) has a perturbation expansion of the form $y(t) = Y_0(t, x) + \varepsilon Y_1(t, x) + \cdots$, we obtain the following sequence of equations:

$$\varepsilon^0\colon\ \frac{\partial^2 Y_0}{\partial t^2} + a\frac{\partial Y_0}{\partial t} = 0; \tag{11.3.33}$$

$$\varepsilon^1\colon\ \frac{\partial^2 Y_1}{\partial t^2} + a\frac{\partial Y_1}{\partial t} = -2\frac{\partial^2 Y_0}{\partial t\,\partial x} - a\frac{\partial Y_0}{\partial x} - bY_0. \tag{11.3.34}$$

The solution to (11.3.33) has the form

$$Y_0(t, x) = A_1(x) + A_2(x)e^{-at}. \tag{11.3.35}$$

Substituting (11.3.35) into (11.3.34) gives

$$\frac{\partial^2 Y_1}{\partial t^2} + a\frac{\partial Y_1}{\partial t} = -[aA_1'(x) + bA_1(x)] + [aA_2'(x) - bA_2(x)]e^{-at}.$$

The right side of this equation is a solution to the homogeneous equation in (11.3.33) and therefore gives rise to secular terms. To eliminate the secular term that grows like t (we know from our study of boundary-layer theory that no such term is present in leading order), we set

$$aA_1'(x) + bA_1(x) = 0.$$