(TD) 3.8 ASYMPTOTIC SERIES

In previous sections we have developed a formal procedure for finding asymptotic series representations of solutions to differential equations and have verified the validity of our results numerically. However, the approach has been intuitive. In this section, we outline the mathematical analysis necessary to justify the asymptotic methods we have used.

We begin by emphasizing the difference between convergent and asymptotic series. Then, we follow with several examples which illustrate what is involved in proving that a power series is asymptotic to a function and for these examples we show why the asymptotic series give good numerical approximations. Next, we review some of the mathematical properties of asymptotic series. We also show how to prove a formal power series is asymptotic to a solution of a differential equation. Finally, we consider asymptotic series in the complex plane and the Stokes phenomenon.

Convergent and Divergent Power Series

In Sec. 3.5 we defined $f(x) \sim \sum_{n=0}^{\infty} a_n(x-x_0)^n$ $(x \to x_0)$ to mean that for every N the remainder $\varepsilon_N(x)$ after $(N+1)$ terms of the series is much smaller than the last retained term as $x \to x_0$: $\varepsilon_N(x) \equiv f(x) - \sum_{n=0}^{N} a_n(x-x_0)^n \ll (x-x_0)^N$ $(x \to x_0)$.

Example 1 *Taylor series as asymptotic series.* If the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges for $|x-x_0| < R$ to the function $f(x)$, then the series is also asymptotic to $f(x)$ as $x \to x_0$: $f(x) \sim \sum_{n=0}^{\infty} a_n(x-x_0)^n$ $(x \to x_0)$. Since $a_n = f^{(n)}(x_0)/n!$, repeated application of l'Hôpital's rule gives

$$\lim_{x \to x_0} \frac{f(x) - \sum_{n=0}^{N} a_n(x-x_0)^n}{(x-x_0)^{N+1}} = a_{N+1}.$$

Thus, $\varepsilon_N(x) = f(x) - \sum_{n=0}^{N} a_n(x-x_0)^n \ll (x-x_0)^N$ $(x \to x_0)$. We conclude that asymptotic series are generalizations of Taylor series because they include Taylor series as special cases.

A series need not be convergent to be asymptotic. Indeed, most asymptotic series are not convergent. Let us contrast convergent and asymptotic series. If $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ is a convergent series for $|x-x_0| < R$, then the remainder $\varepsilon_N(x)$ goes to zero as $N \to \infty$ for any fixed x, $|x-x_0| < R$:

$$\boxed{\text{Convergent:}\quad \varepsilon_N(x) = \sum_{n=N+1}^{\infty} a_n(x-x_0)^n \to 0, \qquad N \to \infty;\ x \text{ fixed.}}$$

On the other hand, if the series is asymptotic to $f(x)$, $f(x) \sim \sum_{n=0}^{\infty} a_n(x-x_0)^n$ $(x \to x_0)$, then the remainder $\varepsilon_N(x)$ goes to zero faster than $(x-x_0)^N$ as $x \to x_0$, but need not go to zero as $N \to \infty$ for fixed x:

$$\boxed{\text{Asymptotic:}\quad \varepsilon_N(x) \ll (x - x_0)^N, \qquad x \to x_0;\ N \text{ fixed.}}$$

Convergence is an *absolute* concept; it is an intrinsic property of the expansion coefficients a_n. One can prove that a series converges without knowing the function to which it converges. However, asymptoticity is a *relative* property of the expansion coefficients *and* the function $f(x)$ to which the series is asymptotic. To prove that a power series is asymptotic to $f(x)$, one must consider *both* $f(x)$ and the expansion coefficients.

Let us clarify this distinction. Suppose you are given a power series and are asked to determine whether it is an asymptotic series as $x \to x_0$. The correct response is that you have been asked a stupid question! Why? Because every power series is asymptotic to some continuous function $f(x)$ as $x \to x_0$!

We present the construction of such a function as an example (see also Probs. 3.79 and 3.80).

Example 2 *Construction of a continuous function asymptotic to a given power series.* Given a formal power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$, we define the continuous function $\phi(x; \alpha)$, plotted in Fig. 3.20, as follows:

$$\phi(x;\alpha) = \begin{cases} 1, & |x| \le \frac{1}{2}\alpha, \\ 2\left(1 - \dfrac{|x|}{\alpha}\right), & \frac{1}{2}\alpha < |x| < \alpha, \\ 0, & \alpha \le |x|. \end{cases}$$

We also define a sequence of numbers $\alpha_n = \min\,(1/|a_n|,\ 2^{-n})$, where a_n are the arbitrary coefficients of the series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$. The function $f(x)$ defined by

$$f(x) = \sum_{n=0}^{\infty} a_n \phi(x - x_0; \alpha_n)(x - x_0)^n$$

is finite, continuous, and satisfies

$$f(x) \sim \sum_{n=0}^{\infty} a_n(x - x_0)^n, \qquad x \to x_0.$$

It is finite and continuous for any $x \ne x_0$ because the series defining $f(x)$ truncates after at most N terms, where N is the smallest integer satisfying $2^{-N} \le |x - x_0|$. Also, if $|x - x_0| \le \frac{1}{2}\min\,(2^{-N}, 1/|a_0|, \ldots, 1/|a_N|) = R_N$, then $\phi(x - x_0; \alpha_n) = 1$ for $n = 0, \ldots, N$. Thus, if $|x - x_0| \le R_N$,

$$f(x) = a_0 + a_1(x - x_0) + \cdots + a_N(x - x_0)^N + \sum_{n=N+1}^{\infty} a_n \phi(x - x_0; \alpha_n)(x - x_0)^n.$$

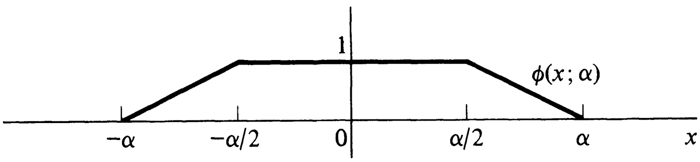

Figure 3.20 A plot of $\phi(x; \alpha)$ in Example 2.

Also, the definitions of α_n and $\phi(x; \alpha)$ imply that

$$|a_n\phi(x - x_0; \alpha_n)(x - x_0)| \le 1 \quad \text{and} \quad |\phi(x - x_0; \alpha_n)| \le 1$$

for all x and all n. Therefore, if $|x - x_0| \le R_N$, then

$$\begin{aligned}
|\varepsilon_N(x)| &= \left|f(x) - \sum_{n=0}^{N} a_n(x - x_0)^n\right| \\
&= \left|a_{N+1}\phi(x - x_0; \alpha_{N+1})(x - x_0)^{N+1} + \sum_{n=N+2}^{\infty} a_n\phi(x - x_0; \alpha_n)(x - x_0)^n\right| \\
&\le |a_{N+1}|\,|x - x_0|^{N+1} + \sum_{n=N+2}^{\infty} |x - x_0|^{n-1} \\
&= |x - x_0|^{N+1}[|a_{N+1}| + (1 - |x - x_0|)^{-1}] \ll (x - x_0)^N, \qquad x \to x_0.
\end{aligned}$$

Thus, $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is asymptotic to $f(x)$ as $x \to x_0$.

Since *every* power series is an asymptotic power series, it is vacuous to ask whether a given series is asymptotic. However, it is meaningful to ask whether a power series is asymptotic to a given function $f(x)$ as $x \to x_0$. This is the reason why the definition given at the beginning of this section includes *both* a series and a function.

Examples of Asymptotic Series

Example 3 *Stieltjes series.* The prototype of an asymptotic series is the so-called Stieltjes series

$$\sum_{n=0}^{\infty} (-1)^n n!\, x^n. \tag{3.8.1}$$

We saw earlier that this series is a formal power series solution to the differential equation (3.4.2). We will now *prove* that this series is really asymptotic to a solution of (3.4.2) by "summing" the series and thereby reconstructing the exact solution to (3.4.2).

Of course, one cannot actually add up all the terms of a divergent series because the sum does not exist. By "summing" we mean finding a function to which the series is asymptotic. "Summation" is the inverse of expanding a function into an asymptotic series.

To sum the series (3.8.1) we invoke the integral identity $n! = \int_0^\infty e^{-t}t^n\,dt$:

$$\sum_{n=0}^{\infty} (-x)^n n! \to \sum_{n=0}^{\infty} (-x)^n \int_0^\infty e^{-t}t^n\,dt.$$

Next, we execute several sleazy maneuvers. We interchange the order of summation and integration,

$$\sum_{n=0}^{\infty} (-x)^n \int_0^\infty e^{-t}t^n\,dt \to \int_0^\infty dt\, e^{-t} \sum_{n=0}^{\infty} (-xt)^n,$$

and we sum the geometric series $\sum_{n=0}^{\infty} (-xt)^n \to 1/(1 + xt)$, even though the sum diverges for those values of t such that $|xt| \ge 1$.

Despite these dubious manipulations, the resulting integral

$$y(x) = \int_0^\infty \frac{e^{-t}}{1 + xt}\,dt, \tag{3.8.2}$$

which is called a *Stieltjes integral*, exists and defines an analytic function of x for all $x > 0$. Moreover, $y(x)$ exactly satisfies the differential equation (3.4.2):

$$x^2y'' + (1 + 3x)y' + y = \int_0^\infty \left[\frac{2x^2t^2}{(1 + xt)^3} - \frac{(1 + 3x)t}{(1 + xt)^2} + \frac{1}{1 + xt}\right] e^{-t}\, dt$$

$$= \int_0^\infty \frac{d}{dt}\left[\frac{t}{(1 + xt)^2} e^{-t}\right] dt = 0, \qquad x > 0.$$

Finally, we show that $y(x)$ in (3.8.2) has an asymptotic power series expansion valid as $x \to 0+$ which is precisely the Stieltjes series in (3.8.1).

Integrating by parts, we obtain the identity

$$\int_0^\infty (1 + xt)^{-n}e^{-t}\, dt = 1 - nx \int_0^\infty (1 + xt)^{-n-1}e^{-t}\, dt.$$

Repeated application of this formula gives

$$\begin{aligned} y(x) &= \int_0^\infty (1 + xt)^{-1}e^{-t}\, dt \\ &= 1 - x \int_0^\infty (1 + xt)^{-2}e^{-t}\, dt \\ &= 1 - x + 2x^2 \int_0^\infty (1 + xt)^{-3}e^{-t}\, dt \\ &\ \vdots \\ &= 1 - x + 2!\, x^2 - 3!\, x^3 + \cdots + (-1)^N N!\, x^N + \varepsilon_N(x), \end{aligned}$$

where

$$\varepsilon_N(x) = (-1)^{N+1}(N + 1)!\, x^{N+1} \int_0^\infty (1 + xt)^{-N-2}e^{-t}\, dt.$$

Finally, we use the inequality

$$\int_0^\infty (1 + xt)^{-N-2}e^{-t}\, dt \le \int_0^\infty e^{-t}\, dt = 1, \qquad x \to 0+,$$

which holds because $1 + xt > 1$ if $x > 0$ and $t > 0$, to show that

$$|\varepsilon_N(x)| \le (N + 1)!\, x^{N+1} \ll x^N, \qquad x \to 0+.$$

This completes the demonstration that the Stieltjes series (3.8.1) is asymptotic to the Stieltjes integral solution to the differential equation (3.4.2).

For the behavior of $y(x)$ as $x \to +\infty$, see Prob. 3.39(i).

Example 4 *General Stieltjes series and integrals.* A generalization of the Stieltjes integral (3.8.2) is given by

$$f(x) = \int_0^\infty \frac{\rho(t)}{1 + xt}\, dt, \tag{3.8.3}$$

where the *weight function* $\rho(t)$ is nonnegative for $t > 0$ and approaches zero so rapidly as $t \to \infty$ that the *moment integrals*

$$a_n = \int_0^\infty t^n\rho(t)\, dt \tag{3.8.4}$$

exist for all positive integers n.

Every Stieltjes integral has an asymptotic power series expansion whose coefficients are $(-1)^n a_n$:

$$f(x) \sim \sum_{n=0}^{\infty} (-1)^n a_n x^n, \qquad x \to 0+. \tag{3.8.5}$$

To prove this assertion, we note that

$$\begin{aligned}\varepsilon_N(x) &\equiv f(x) - \sum_{n=0}^{N} (-1)^n a_n x^n \\ &= \int_0^\infty \rho(t) \left[\frac{1}{1+xt} - \sum_{n=0}^{N} (-xt)^n\right] dt \\ &= \int_0^\infty \frac{\rho(t)}{1+xt} (-xt)^{N+1}\, dt\end{aligned} \tag{3.8.6}$$

for all N. Thus,

$$|\varepsilon_N(x)| \le x^{N+1} \int_0^\infty \rho(t) t^{N+1}\, dt = a_{N+1} x^{N+1} \ll x^N, \qquad x \to 0+, \tag{3.8.7}$$

where we use $1 + xt > 0$ and $\rho(t) > 0$ for $x > 0$ and $t > 0$. This completes the verification of (3.8.5).

Example 5 *Stieltjes series with weight function $K_0(t)$.* If $\rho(t) = K_0(t)$, the modified Bessel function of order 0, then (3.8.6) becomes

$$\int_0^\infty \frac{K_0(t)}{1+xt}\, dt \sim \frac{1}{2} \sum_{n=0}^{\infty} (-2x)^n \left[\Gamma\left(\frac{1}{2} n + \frac{1}{2}\right)\right]^2 \tag{3.8.8}$$

(see Prob. 3.76).

Numerical Approximations Using Asymptotic Series: The Optimal Truncation Rule

It is possible to improve our error estimates for Stieltjes series. Equation (3.8.7) shows that the error between the Stieltjes integral (3.8.2) and the first N terms of the Stieltjes series in (3.8.1) is smaller than the absolute value of the next term in the series $N!\, x^N$. Also, the sign of the error is the same as the sign of the next term. The same is true for the general Stieltjes series (3.8.5); the error between the Stieltjes integral (3.8.3) and N terms of the Stieltjes series (3.8.5) has the same sign and is less than the $(N + 1)$th term of the series.

These error bounds imply that for any fixed x, truncating the Stieltjes series (3.8.5) just before the smallest term will give a good numerical estimate of the Stieltjes integral (3.8.4). It is more difficult to justify this optimal truncation rule for asymptotic series that are not Stieltjes series. However, we have had remarkable success with this rule for truncating asymptotic series (see Sec. 3.5). In fact, even though the asymptotic series (3.5.8*b*) and (3.5.9*b*) for $K_\nu(x)$ and (3.7.14) and (3.7.15) for $J_\nu(x)$ are not Stieltjes series, it is still true that the error after N terms is less than the $(N + 1)$th term provided that N is larger than some number depending on ν but not on x (see Prob. 3.77).

Of course, not every asymptotic series has the property that the error after N terms is less than the $(N + 1)$th term. For example, the error after N terms in the asymptotic series (3.5.8a) and (3.5.9a) for $I_\nu(x)$ is not similar in sign and smaller than the $(N + 1)$th term. Nevertheless, our truncation procedure gave very good results (see Figs. 3.5 and 3.6 and Table 3.1).

How well do these optimal asymptotic approximations really work? For Stieltjes series, we can provide an accurate asymptotic estimate of the difference between the exact value of the Stieltjes integral and the optimal truncation of the Stieltjes series as $x \to 0+$.

Example 6 *Error estimate for an optimally truncated Stieltjes series.* According to (3.8.6), the error after N terms of the Stieltjes series (3.8.1) for which the weight function $\rho(t) = e^{-t}$ is

$$(-x)^N \int_0^\infty t^N \frac{e^{-t}}{1 + xt}\, dt$$

for any N. The optimal truncation of (3.8.1) is obtained by choosing N equal to the largest integer less than or equal to $1/x$; this is true because the ratio of the $(n + 1)$th term to the nth term of (3.8.1) is $-nx$. If we approximate this integral representation for the error (see Probs. 5.25 and 6.37), we find that the optimal error $\varepsilon_{\text{optimal}}(x)$ satisfies

$$|\varepsilon_{\text{optimal}}(x)| \sim \left(\frac{\pi}{2x}\right)^{1/2} e^{-1/x}, \qquad x \to 0+. \tag{3.8.9}$$

We have checked the validity of (3.8.9) numerically. In Fig. 3.21 we plot the ratio of $|\varepsilon_{\text{optimal}}(x)|$ determined numerically by optimally truncating the series (3.8.1) to its leading behavior given in (3.8.9). Observe that this ratio approaches 1 as $x \to 0+$.

Properties of Asymptotic Series

(a) Nonuniqueness We have given a successful prescription for obtaining good numerical results from divergent asymptotic series. Strangely enough, one must use this technique with caution because it produces a *unique* numerical answer! Actually, the "sum" of a divergent power series is not uniquely determined. For example, if $f(x) \sim \sum_{n=0}^\infty a_n(x - x_0)^n$ $(x \to x_0)$, then it is also true that $f(x) + e^{-(x-x_0)^{-2}} \sim \sum_{n=0}^\infty a_n(x - x_0)^n$ $(x \to x_0)$ because $e^{-(x-x_0)^{-2}} \ll (x - x_0)^n$ as $x \to x_0$ for all n. In fact, the series $\sum_{n=0}^\infty a_n(x - x_0)^n$ is asymptotic as $x \to x_0$ to any function which differs from $f(x)$ by a function $g(x)$ so long as $g(x) \to 0$ as $x \to x_0$ more rapidly than all powers of $x - x_0$. Such a function $g(x)$ is said to be *subdominant to the asymptotic power series*; the asymptotic expansion of $g(x)$ is

$$g(x) \sim \sum_{n=0}^\infty 0(x - x_0)^n, \qquad x \to x_0.$$

In short, an asymptotic series is asymptotic to a whole class of functions that differ from each other by subdominant functions. We do not change the asymptotic series by adding a subdominant function, even if the subdominant function is multiplied by a huge numerical coefficient.

For example, e^{-x^4} is subdominant with respect to the asymptotic expansion

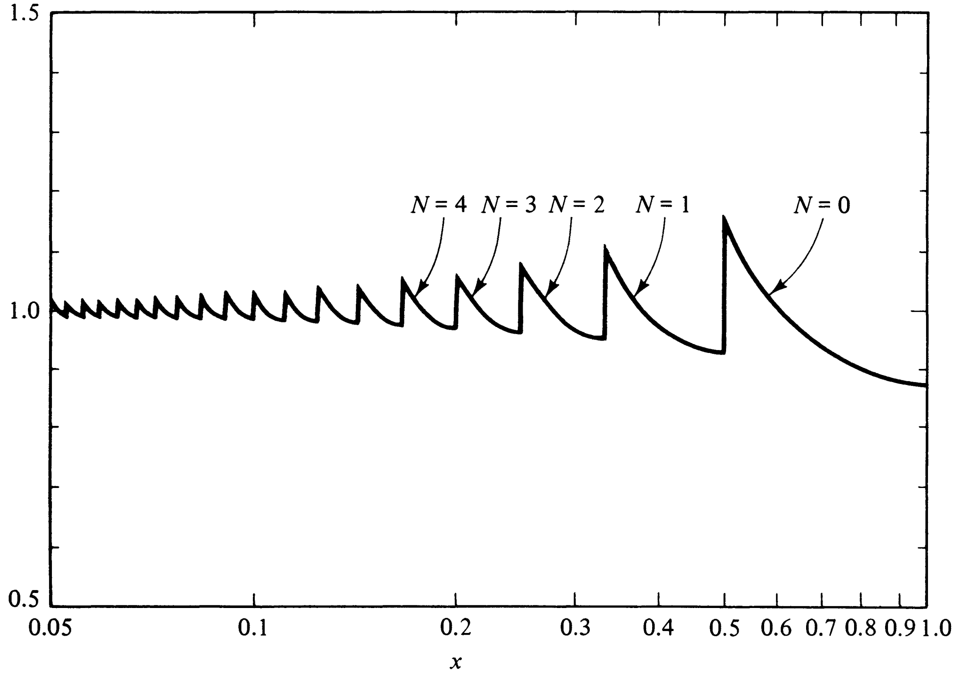

Figure 3.21 A computer plot of the ratio of $|\varepsilon_{\text{optimal}}(x)|/[(\pi/2x)^{1/2} \exp(-1/x)]$ for $0.05 \le x \le 1$. Here $\varepsilon_{\text{optimal}}(x)$ is the error in the optimal asymptotic approximation to the Stieltjes integral $\int_0^\infty e^{-t}(1+xt)^{-1}\,dt$ in (3.8.2); that is, it is the difference between the Stieltjes integral and the optimally truncated asymptotic series $\sum_{n=0}^{N} (-1)^n n!\, x^n$. Theoretically, the leading behavior of $|\varepsilon_{\text{optimal}}(x)|$ is $(\pi/2x)^{1/2} \exp(-1/x)$ as $x \to 0+$ [see (3.8.9)]. The graph clearly verifies this prediction.

(3.5.13) and (3.5.14) of $D_{3.5}(x)$ as $x \to +\infty$. Therefore, $f(x) = D_{3.5}(x) + 10^{10}e^{-x^4}$ has the same asymptotic expansion as $D_{3.5}(x)$ as $x \to +\infty$. What happens now if we compute the optimal asymptotic approximation to $f(x)$? We already know from Fig. 3.7 and Table 3.2 that the optimal asymptotic approximation is very close to $D_{3.5}(x)$ for $x > 1$. Therefore, since $10^{10}e^{-x^4} > |D_{3.5}(x)|$ for $0 \le x \le 2.1$, the optimal asymptotic approximation to $f(x)$ is not accurate for $1 \le x \le 2.1$. Nevertheless, when $x \ge 2.3$ the optimal asymptotic approximation is very close to $f(x)$.

The above discussion shows that the value of x for which the optimal asymptotic approximation becomes useful cannot be predicted from the asymptotic series itself. Rather, it depends on the admixture of subdominant functions. Thus, for any given problem we can never really know *a priori* whether or not asymptotic analysis will give good numerical results at a fixed value of x. However, experience has shown that asymptotic methods nearly always give spectacularly good results.

(b) Uniqueness Although there are many different functions asymptotic to a given power series, there is only *one* asymptotic power series for each function. Specifically, if a function $f(x)$ can be expanded as $f(x) \sim \sum_{n=0}^{\infty} a_n(x-x_0)^n$

$(x \to x_0)$, then the expansion coefficients are unique. The proof of uniqueness is given in Sec. 3.5.

(c) Equating coefficients in asymptotic series It is not strictly correct to write $\sum_{n=0}^{\infty} a_n(x-x_0)^n \sim \sum_{n=0}^{\infty} b_n(x-x_0)^n$ $(x \to x_0)$ because power series can only be asymptotic to functions, and not to other power series. However, we will occasionally use this notation; we define it to mean that the class of functions to which $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ and $\sum_{n=0}^{\infty} b_n(x-x_0)^n$ are asymptotic as $x \to x_0$ are the same. It follows from the uniqueness of asymptotic expansions that two power series are asymptotic if and only if $a_n = b_n$ for all n. Thus, we may equate coefficients of like powers of $x - x_0$ in power series that are "asymptotic to each other."

(d) Arithmetical operations on asymptotic series Arithmetical operations may be performed term by term on asymptotic series. Specifically, suppose

$$f(x) \sim \sum_{n=0}^{\infty} a_n(x-x_0)^n, \qquad x \to x_0,$$

$$g(x) \sim \sum_{n=0}^{\infty} b_n(x-x_0)^n, \qquad x \to x_0.$$

Then

$$\alpha f(x) + \beta g(x) \sim \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)(x-x_0)^n, \qquad x \to x_0,$$

$$f(x)g(x) \sim \sum_{n=0}^{\infty} c_n(x-x_0)^n, \qquad x \to x_0,$$

$$\frac{f(x)}{g(x)} \sim \sum_{n=0}^{\infty} d_n(x-x_0)^n, \qquad x \to x_0,$$

where $c_n = \sum_{m=0}^{n} a_m b_{n-m}$, and if $b_0 \neq 0$, $d_0 = a_0/b_0$ and

$$d_n = \frac{a_n - \sum_{m=0}^{n-1} d_m b_{n-m}}{b_0}, \qquad n \geq 1.$$

The proofs of these results are elementary. For example, let us prove that asymptotic series can be multiplied term by term. Using the above expression for c_n, we obtain

$$\begin{aligned} f(x)g(x) &- \sum_{n=0}^{N} c_n(x-x_0)^n \\ &= f(x)g(x) - \sum_{m=0}^{N} a_m(x-x_0)^m \sum_{n=m}^{N} b_{n-m}(x-x_0)^{n-m} \\ &= g(x)\left[f(x) - \sum_{m=0}^{N} a_m(x-x_0)^m\right] \\ &\quad + \sum_{m=0}^{N} a_m(x-x_0)^m \left[g(x) - \sum_{p=0}^{N-m} b_p(x-x_0)^p\right] \end{aligned}$$

for all N. Since $\lim_{x\to x_0} g(x) = b_0$, $|g(x)| \le 2|b_0|$ for x sufficiently close to x_0, say $|x - x_0| \le R$. Hence, by the definition of asymptotic series,

$$|f(x)g(x) - \sum_{n=0}^{N} c_n(x - x_0)^n| \ll (2|b_0| + |a_0| + |a_1| + \cdots + |a_N|)|x - x_0|^N, \qquad x \to x_0,$$

for all N. Thus, asymptotic series can be multiplied term by term.

(e) Integration of asymptotic series Any asymptotic series $f(x) \sim \sum_{n=0}^{\infty} a_n(x - x_0)^n$ $(x \to x_0)$ can be integrated term by term if $f(x)$ is integrable near x_0:

$$\int_{x_0}^{x} f(t)\,dt \sim \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1}, \qquad x \to x_0.$$

To prove this result we begin with the definition of an asymptotic power series: $|f(x) - \sum_{n=0}^{N} a_n(x - x_0)^n| \ll (x - x_0)^N$ $(x \to x_0)$. From this it follows that for any $\varepsilon > 0$ there exists an interval about x_0, say $|x - x_0| \le R$ (where R of course depends on ε), in which

$$|f(x) - \sum_{n=0}^{N} a_n(x - x_0)^n| \le \varepsilon|x - x_0|^N, \qquad |x - x_0| \le R.$$

Therefore,

$$\begin{aligned} \left|\int_{x_0}^{x}\left[f(t) - \sum_{n=0}^{N} a_n(t - x_0)^n\right]dt\right| &\le \int_{x_0}^{x}\left|f(t) - \sum_{n=0}^{N} a_n(t - x_0)^n\right|dt \\ &\le \varepsilon\int_{x_0}^{x}|t - x_0|^N\,dt \\ &= \frac{\varepsilon}{N+1}|x - x_0|^{N+1}, \qquad |x - x_0| \le R. \end{aligned}$$

Hence,

$$\left|\frac{\int_{x_0}^{x} f(t)\,dt - \sum_{n=0}^{N} [a_n/(n+1)](x - x_0)^{n+1}}{(x - x_0)^{N+1}}\right| \le \frac{\varepsilon}{N+1}, \qquad |x - x_0| \le R.$$

But $\varepsilon > 0$ is arbitrary, so

$$\int_{x_0}^{x} f(t)\,dt - \sum_{n=0}^{N} \frac{a_n}{n+1}(x - x_0)^{n+1} \ll (x - x_0)^{N+1}, \qquad x \to x_0,$$

for all N. Thus, asymptotic series can be integrated term by term.

If we wish to integrate an asymptotic series at infinity, there is a slight complication. The above argument can be extended to show that if $f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}$ $(x \to \infty)$, then

$$\int_{x}^{\infty} [f(t) - a_0 - a_1 t^{-1}]\,dt \sim \sum_{n=2}^{\infty} \frac{a_n}{n-1} x^{1-n}, \qquad x \to \infty.$$

(f) Differentiation of asymptotic series Asymptotic series cannot in general be differentiated term by term. For example, even if $f(x) \sim \sum_{n=0}^{\infty} a_n(x - x_0)^n$ $(x \to x_0)$, it does not necessarily follow that $f'(x) \sim \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1}$ $(x \to x_0)$. The problem with differentiation is connected with subdominance: the functions $f(x)$ and

$$g(x) = f(x) + e^{-1/(x-x_0)^2} \sin\left(e^{1/(x-x_0)^2}\right)$$

differ by a subdominant function and thus have the same asymptotic series expansion as $x \to x_0$. However, it is not necessarily true that $f'(x)$ and

$$g'(x) = f'(x) - 2(x - x_0)^{-3} \cos\left(e^{1/(x-x_0)^2}\right) + 2(x - x_0)^{-3} e^{-1/(x-x_0)^2} \sin\left(e^{1/(x-x_0)^2}\right)$$

have the same asymptotic power series expansion as $x \to x_0$. Therefore, term-by-term differentiation of an asymptotic series may not be valid for both $f(x)$ and $g(x)$; asymptotic series cannot be differentiated termwise without additional restrictions.

Termwise *integration* of an asymptotic series, which we justified above, is an example of an *Abelian* theorem. In an Abelian theorem, asymptotic information about an average of a function (its integral) is deduced from asymptotic information about the function itself. Differentiation of asymptotic series relates to the converse process; namely, deducing asymptotic information about a derivative from asymptotic information about a function. Converses to Abelian theorems are called *Tauberian* theorems. Tauberian theorems require conditions supplementary to those of corresponding Abelian theorems to be valid. In the case of termwise differentiation of asymptotic series, there are several situations in which Tauberian-like theorems provide justification for termwise differentiation.

One such result is as follows. Suppose $f'(x)$ exists, is integrable, and $f(x) \sim \sum_{n=0}^{\infty} a_n(x - x_0)^n$ $(x \to x_0)$. Then it follows that

$$f'(x) \sim \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1}, \qquad x \to x_0.$$

This result is an immediate consequence of the Abelian theorem for termwise integration of an asymptotic series proved above. To see this, suppose that $f'(x) \sim \sum_{n=0}^{\infty} b_n(x - x_0)^n$ $(x \to x_0)$. Then, integrating term by term gives

$$\begin{aligned} f(x) &= f(x_0) + \int_{x_0}^{x} f'(t)\, dt \\ &\sim f(x_0) + \sum_{n=0}^{\infty} \frac{b_n}{n+1}(x - x_0)^{n+1}, \qquad x \to x_0. \end{aligned}$$

But since $f(x) \sim \sum_{n=0}^{\infty} a_n(x - x_0)^n$ $(x \to x_0)$ and since asymptotic series are unique, we find that $a_0 = f(x_0)$, $a_{n+1} = b_n/(n + 1)$ $(n = 0, 1, \ldots)$. This proves the theorem.

There are other more technical Tauberian-like results for differentiation of asymptotic relations. A result of this kind is as follows. Suppose $f(x) \sim x^p$ $(x \to +\infty)$, where $p \geq 1$ and $f''(x)$ is positive. Then (see Prob. 3.84) $f'(x) \sim px^{p-1}$

$(x \to +\infty)$. This result concerns only the *leading* behavior of $f(x)$ and does *not* justify termwise differentiation of an asymptotic series (see Prob. 3.85).

Termwise differentiation of asymptotic series is much clearer in the complex domain. For example, suppose that $f(z)$ is analytic in the sector $\theta_1 \le \arg(z - z_0) \le \theta_2$, $0 < |z - z_0| < R$ and $f(z) \sim \sum_{n=0}^{\infty} a_n(z - z_0)^n$ $[z \to z_0;$ $\theta_1 \le \arg(z - z_0) \le \theta_2]$. Then (see Prob. 3.72)

$$f'(z) \sim \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}, \qquad z \to z_0;\, \theta_1 < \arg(z - z_0) < \theta_2.$$

It should be clear enough from these three special cases that the variety and complexity of Tauberian theorems for differentiation of asymptotic series is bewildering. Fortunately, the whole situation is greatly simplified if a function is known to satisfy a linear differential equation whose *coefficients* can be expanded in asymptotic series.

Asymptotic Expansions of Solutions to Differential Equations

The formal procedures given in Secs. 3.4 to 3.7 for calculating asymptotic expansions of solutions to differential equations require justification. There are several possible difficulties. First, we have always assumed that after the leading-order behavior is factored off, an asymptotic series expansion of the solution remains. However, not all functions can be expanded in asymptotic series. For example, consider the function $y(t) = t^2 + e^{-t^2(1 - \sin t)}$ which has leading behavior t^2 as $t \to \infty$. As $t \to \infty$, there are narrow regions that occur periodically in which $\sin t$ is near 1 and the term $e^{-t^2(1 - \sin t)}$ is not negligible with respect to 1. The existence of these regions implies that there does not exist any asymptotic power series representation for $y(t)$ as $t \to +\infty$. We shall see that this difficulty does not afflict solutions of differential equations whose *coefficients* themselves have asymptotic power series expansions.

Second, asymptotic series cannot, in general, be differentiated termwise. Thus, the formal differentiation of asymptotic series, which allowed us to determine the coefficients in the expansions of solutions to differential equations, needs to be justified.

The proof that our formal methods are correct has two parts. First, we argue that if $y(x)$ is the solution of $y'' + py' + qy = 0$ where $p(x)$, $p'(x)$, $q(x)$ are expandable in asymptotic power series as $x \to x_0$ and if we *assume* that $y(x)$ is also expandable in an asymptotic power series as $x \to x_0$, then the derivatives of $y(x)$ are also expandable and their asymptotic power series are obtained by termwise differentiation of the asymptotic power series representing $y(x)$. The proof is elementary. Consider the special case of the differential equation $y''(x) + q(x)y(x) = 0$. If $q(x)$ and $y(x)$ possess asymptotic power series representations as $x \to x_0$, then the differential equation itself ensures that $y''(x)$ does also (because multiplication of asymptotic power series is permissible). Integrating the asymptotic power series representing $y''(x)$ shows that $y'(x)$ also has an asymptotic series, so termwise differentiation is justified. The argument for a general nth-order differential equation is left for an exercise (Prob. 3.81).