NLS solitons

$$i u_t + \frac{1}{2} u_{xx} + \sigma |u|^2 u = 0 \quad (1)$$

Madelung: $u = \sqrt{\rho}\, e^{i\varphi}$

Hydrodynamic form:
$$\begin{cases} \rho_t + (\rho \varphi_x)_x = 0 & (2) \\ \varphi_t - \sigma\rho + \frac{1}{2}\varphi_x^2 - \frac{1}{2}\rho^{-1/2}(\rho^{1/2})_{xx} = 0 & (3) \end{cases}$$

We will study both focusing + defocusing cases

We seek solutions of stationary density: $\frac{\partial}{\partial t}|u|^2 = \frac{\partial \rho}{\partial t} = 0$

i.e. $\rho = \rho(x)$ and $\varphi = \varphi(x,t)$

The reason for this choice is the following:

> If $u(x,t)$ satisfies (1) then
> $$\phi(x,t) = u(x - vt, t)\exp[i(kx - \omega t)] \quad \text{with} \begin{cases} k = v \\ \omega = \frac{k^2}{2} \end{cases}$$
> also satisfies (1).

Proof: Use $x' = x - vt$, $t' = t$ $\Rightarrow$ $\frac{\partial}{\partial t} = \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = -v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}$

$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \underbrace{\cancel{\frac{\partial t'}{\partial x}}}_{0}\frac{\partial}{\partial t'} = \frac{\partial}{\partial x'}$

Then (1): $i(u_t - v u_x) + \frac{1}{2}u_{xx} + \sigma|u|^2 u = 0.$

Seek solution: $\phi = u(x,t)\exp[i(kx - \omega t)]$

$i\phi_t \sim i u_t + \omega u$; $\phi_{xx} \sim u_{xx} + 2iku_x - k^2 u$; $|\phi|^2\phi \sim |u|^2 u$

Thus: $i\phi_t + \frac{1}{2}\phi_{xx} + \sigma|\phi|^2\phi = iu_t + \omega u - ivu_x + \frac{1}{2}u_{xx} + iku_x - \frac{k^2}{2}u + \sigma|u|^2 u$

Choose $\boxed{k = v}$ and $\boxed{\omega = \frac{k^2}{2} = \frac{v^2}{2}}$ $\Rightarrow$ ϕ satisfies (1)

$$\rho_t + (\rho \varphi_x)_x = 0 \qquad (2)$$

$$\varphi_t - \sigma\rho + \frac{1}{2}\varphi_x^2 - \frac{1}{2}\rho^{-1/2}(\rho^{1/2})_{xx} = 0 \qquad (3)$$

For $\rho = \rho(x)$ (2) becomes:

$$(\rho\varphi_x)_x = 0 \Rightarrow \rho\varphi_x = C(t) \quad (4) \quad \Rightarrow \boxed{\varphi_x = \frac{C(t)}{\rho}} \quad (4')$$

We will show that $\boxed{C(t) = C = \text{const.}}$

Indeed (3): $\underbrace{\varphi_t + \frac{1}{2}\varphi_x^2 = \sigma\rho + \frac{1}{2}\rho^{-1/2}(\rho^{1/2})_{xx} \equiv f(x)}$ (5)

$$(4) \xrightarrow{\partial_t^2} \quad \rho\varphi_{xtt} = C_{tt} \Rightarrow \underbrace{\varphi_{xtt} = \frac{1}{\rho}C_{tt}} \quad (6)$$

$$(5) \xrightarrow{\partial_{xt}^2} \quad \varphi_{ttx} + \frac{1}{2}\left(\frac{C^2}{\rho^2}\right)_{tx} = 0 \Rightarrow$$

$$\Rightarrow \varphi_{ttx} + \frac{1}{\cancel{2}}(C^2)_t\left(-\frac{\cancel{2}\rho_x}{\rho^3}\right) = 0 \Rightarrow$$

$$\Rightarrow \underbrace{\varphi_{ttx} = +\frac{(C^2)_t\rho_x}{\rho^3}} \quad (7)$$

(6), (7): $\frac{1}{\rho}C_{tt} = (C^2)_t\frac{\rho_x}{\rho^3} \Rightarrow \frac{C_{tt}}{(C^2)_t} = \frac{\rho_x}{\rho^2} = \text{const.}$

Since $\rho^{-2}\rho_x \neq \text{const}$ we must have $\underbrace{C(t) = \text{const.}} = C_1$

Thus (4'): $\rho\varphi_x = C_1 \Rightarrow \varphi_x = \frac{C_1}{\rho} \Rightarrow$ [φ_x a function of x]

$$\Rightarrow \boxed{\varphi = \int \frac{C_1}{\rho}dx + A(t)} \quad (8)$$

So far: $\varphi_x = \text{function of } x$

Also $\varphi_t + \frac{1}{2}\varphi_x^2 = f(x)$

$$\left.\begin{array}{l} \varphi_x = \text{function of } x \\ \varphi_t + \frac{1}{2}\varphi_x^2 = f(x) \end{array}\right\} \Rightarrow \varphi_t = \text{function of } x$$

However $\varphi = \int \frac{c_1}{\rho}\,dx + A(t) \Rightarrow$

$$\left.\begin{array}{l} \varphi_t = \int \frac{c_1}{\rho}\,dx + A_t \\ \varphi_t = \text{function of } x \end{array}\right\} \Rightarrow A_t = \text{const} \equiv \Omega \Rightarrow$$

$$\Rightarrow \boxed{A(t) = \Omega t + \theta_0}$$

θ_0 → integration constant

Thus: $$\boxed{\varphi(x,t) = \int \frac{c_1}{\rho}\,dx + \Omega t + \theta_0}$$

Then (5): $$\varphi_t + \frac{1}{2}\varphi_x^2 = \sigma\rho + \frac{1}{2}\rho^{-1/2}\left(\rho^{1/2}\right)_{xx} \Rightarrow$$

$$\Rightarrow \Omega + \frac{1}{2}\frac{c_1^2}{\rho^2} = \sigma\rho + \frac{1}{2\sqrt{\rho}}\left(\sqrt{\rho}\right)_{xx} \Rightarrow$$

$$\Rightarrow \left(\rho^{1/2}\right)_{xx} = 2\Omega\rho^{1/2} - 2\sigma\rho^{3/2} + \frac{c_1^2}{\rho^{3/2}} \qquad (10)$$

Set: $\rho^{1/2} = q$; then

$$\boxed{q_{xx} = 2\Omega q - 2\sigma q^3 + \frac{c_1^2}{q^3} = -\frac{\partial V}{\partial q}} \qquad (11)$$

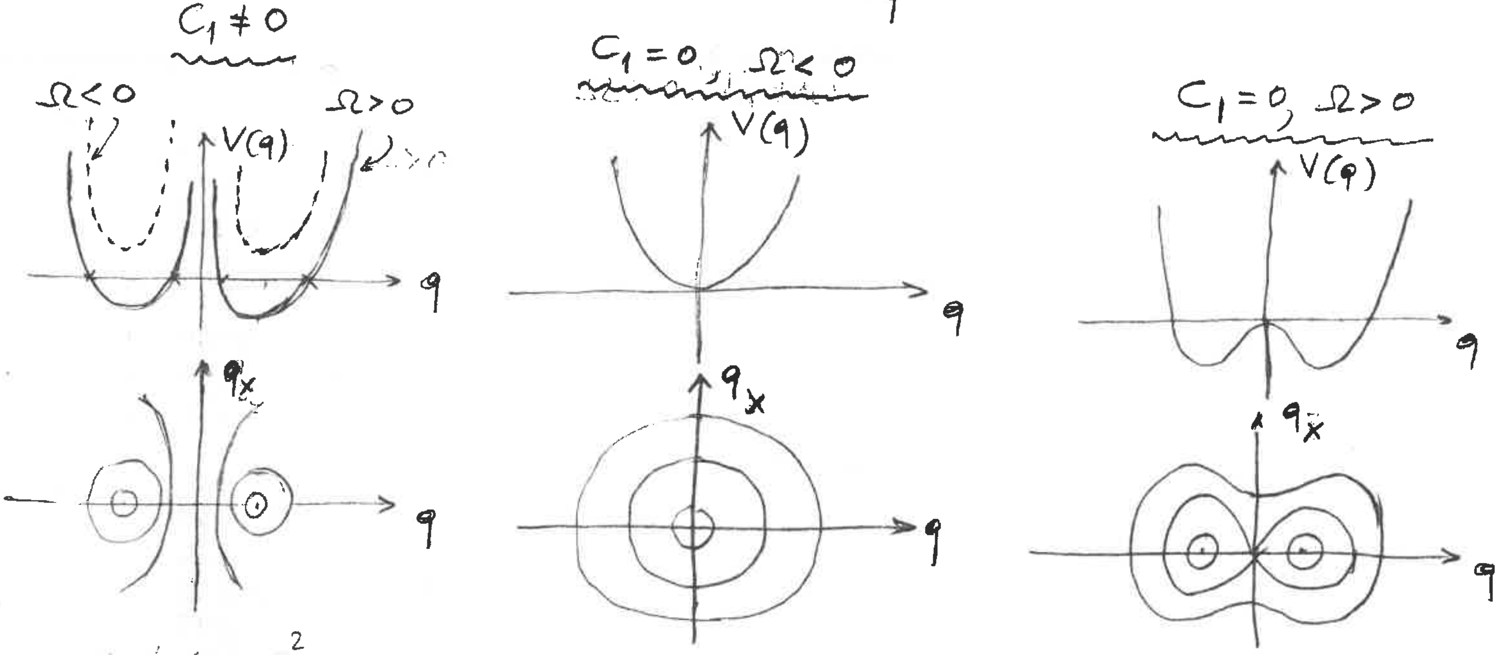
with $$\boxed{V(q) = -\Omega q^2 + \frac{\sigma}{2}q^4 + \frac{c_1^2}{2q^2}}$$

Then (11) becomes: $\boxed{\frac{1}{2}q_x^2 + V(q) = E}$ → energy level

$$\boxed{V(q) = -\Omega q^2 + \frac{\sigma}{2} q^4 + \frac{c_1^2}{2q^2}}$$

Case 1: $\sigma = +1$ (focusing)

$$V(q) = -\Omega q^2 + \frac{1}{2} q^4 + \frac{c_1^2}{2q^2}$$



For $c_1 = 0,\ \Omega > 0$: separatrix occurs for $E = 0$

$$\frac{1}{2}\left(\frac{dq}{dx}\right)^2 - \Omega q^2 + \frac{1}{2} q^4 = 0 \Rightarrow \frac{dq}{q\sqrt{2\Omega - q^2}} = dx$$

We know: $\boxed{\int \frac{\eta}{x(\eta^2 - x^2)^{1/2}}\, dx = \operatorname{sech}^{-1}\left(\frac{x}{\eta}\right)}$

Thus: $\int \frac{\eta\, dq}{q\sqrt{\eta^2 - q^2}} = \eta\, dx \Rightarrow \operatorname{sech}^{-1}\left(\frac{q}{\eta}\right) = \eta(x - x_0)$

we set: $\boxed{2\Omega = \eta^2}$

x_0: integration const.

$$\Rightarrow \quad q(x,t) = \sqrt{\rho(x,t)} = \eta \operatorname{sech}\left[\eta(x - x_0)\right]$$

and phase (for $c_1 = 0$) becomes $\varphi(x,t) = \Omega t + \theta_0 = \frac{\eta^2}{2} t + \theta_0$

Thus: bright soliton solution of the focusing NLS:

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = 0$$

▷▷ $$\boxed{u(x,t) = \eta \operatorname{sech}[\eta(x-x_0)] \exp\left[i\left(\frac{\eta^2}{2}t + \theta_0\right)\right]} \to \text{stationary soliton}$$

Traveling soliton: Galilean boost:

$$u(x,t) \longrightarrow u(x-vt,t)\exp[i(kx-\omega t)]\,; \quad k=v,\ \omega=\frac{k^2}{2}$$

Hence:

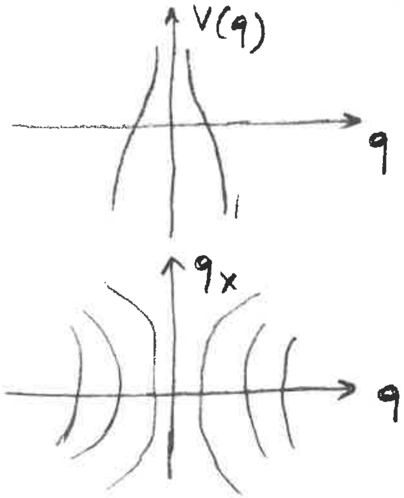
▷▷ $$\boxed{u(x,t) = \eta \operatorname{sech}[\eta(x-vt-x_0)] \exp\left\{i\left[kx + \frac{1}{2}(\eta^2-k^2)t + \theta_0\right]\right\}}$$

$$\boxed{V(q) = -\Omega q^2 + \frac{\sigma}{2} q^4 + \frac{C_1^2}{2q^2}}$$

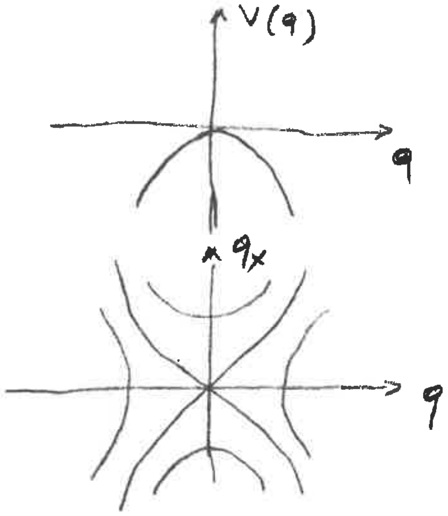
Case 2: $\sigma = -1$ (defocusing).

$$V(q) = -\Omega q^2 - \frac{1}{2} q^4 + \frac{C_1^2}{2q^2}$$

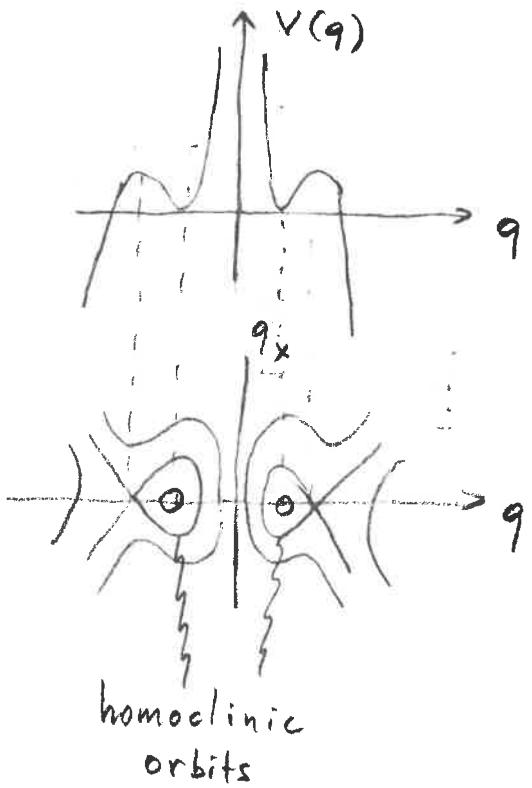
$\Omega > 0,\ C_1 \neq 0$



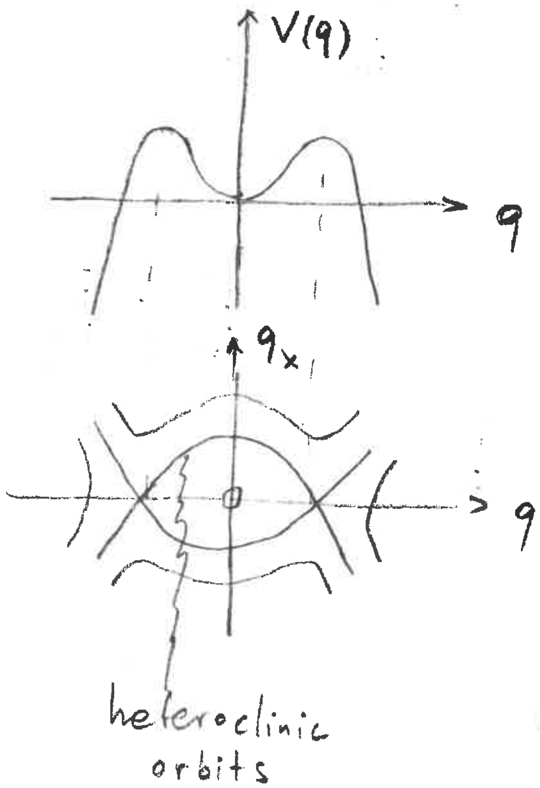
$\Omega > 0,\ C_1 = 0$



$\Omega < 0,\ C_1 \neq 0$



$\Omega < 0,\ C_1 = 0$



Asymptotics: $q \equiv \sqrt{\rho} \to \pm\sqrt{\rho_0}$ as $x \to \pm\infty$ $\longrightarrow$ steady-state density

Consider the simplest case $\Omega < 0$, $c_1 = 0$ which gives rise to heteroclinic orbits.

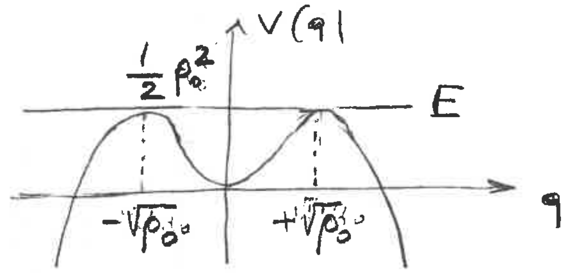
Since $\Omega < 0$ we set $\boxed{\Omega = -p_0}$ and $V(q)$ reads:

$$V(q) = -\Omega q^2 - \frac{1}{4} q^4 \Rightarrow \boxed{V(q) = p_0 q^2 - \frac{1}{2} q^4} \text{ and thus}$$

eq. of motion: $$\frac{1}{2}\left(\frac{dq}{dx}\right)^2 + p_0 q^2 - \frac{1}{2} q^4 = E \qquad (1)$$

and E must be chosen at the hyperbolic points

We find the potential max:



$$\frac{dV}{dq} = 0 \Rightarrow 2p_0 q - 2q^3 = 0 \Rightarrow$$

$\Rightarrow 2q(p_0 - q^2) = 0$ and max occurs at $q = \pm\sqrt{p_0}$

The max is: $$V(\pm\sqrt{p_0}) = p_0(\pm\sqrt{p_0})^2 - \frac{1}{2}(\pm\sqrt{p_0})^4 =$$
$$= p_0^2 - \frac{1}{2}p_0^2 = \frac{1}{2}p_0^2 \Rightarrow E = \frac{1}{2}p_0^2$$

Thus (1) becomes:

$$\frac{1}{2}\left(\frac{dq}{dx}\right)^2 + p_0 q^2 - \frac{1}{2} q^4 = \frac{1}{2}p_0^2 \Rightarrow$$

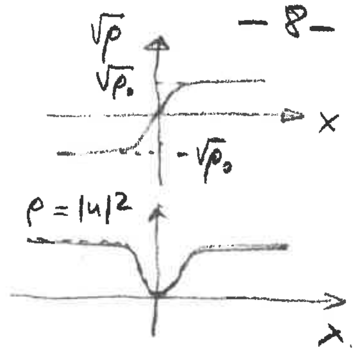
$$\Rightarrow \left(\frac{dq}{dx}\right)^2 = (p_0 - q^2)^2 \Rightarrow \frac{dq}{p_0 - q^2} = dx$$

We know: $$\boxed{\int \frac{\alpha}{\alpha^2 - x^2} dx = \tanh^{-1}\left(\frac{x}{\alpha}\right)}$$

$\Rightarrow$

$$\int \frac{\sqrt{p_0}\, dq}{(\sqrt{p_0})^2 - q^2} = \int \sqrt{p_0}\, dx \Rightarrow \tanh^{-1}\left(\frac{q}{\sqrt{p_0}}\right) = \sqrt{p_0}(x - x_0)$$

Thus: $q(x,t) = \sqrt{\rho(x,t)} = \sqrt{\rho_0}\tanh\left[\sqrt{\rho_0}(x-x_0)\right]$



Phase: $\varphi = +\Omega t + \theta_0 = -\rho_0 t + \theta_0$

"Black" soliton:

$$\boxed{u(x,t) = \sqrt{\rho_0}\tanh\left[\sqrt{\rho_0}(x-x_0)\right]\exp\left[i\left(-\rho_0 t + \theta_0\right)\right]}$$

We now consider the case $\Omega = -|\Omega| < 0$ and $C_1 \neq 0$

which gives rise to homoclinic orbits

$V(q) = |\Omega| q^2 - \frac{1}{2} q^4 + \frac{C_1^2}{2q^2}$ and eq. of motion reads:

$$\frac{1}{2}\left(\frac{dq}{dx}\right)^2 + |\Omega| q^2 - \frac{1}{2} q^4 + \frac{C_1^2}{2q^2} = E$$

We now recall that $\boxed{q^2 = \rho}$ $\Rightarrow$

$$\Rightarrow \frac{1}{2}\left(\frac{d}{dx}\rho^{1/2}\right)^2 + |\Omega|\rho - \frac{1}{2}\rho^2 + \frac{C_1^2}{2\rho} = E \Rightarrow$$

$$\Rightarrow \rho\left(\frac{d}{dx}\rho^{1/2}\right)^2 = -2|\Omega|\rho^2 + \rho^3 - C_1^2 + 2E\rho$$

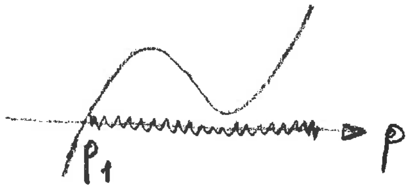
$$\rho\left(\frac{1}{2}\rho^{-1/2}\rho_x\right)^2 = \rho\left(\frac{1}{4}\rho^{-1}\rho_x^2\right) = \frac{1}{4}\rho_x^2$$

$\Rightarrow$

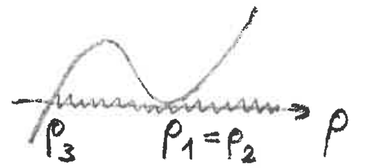
$$\Rightarrow \left(\frac{d\rho}{dx}\right)^2 = 4\rho^3 - 8|\Omega|\rho^2 + 8E\rho - 4C_1^2$$

$$\left(\frac{d\rho}{dx}\right)^2 = 4\left(\rho^3 - 2|\Omega|\rho^2 + 2E\rho - C_1^2\right) \equiv P(\rho) \quad (1)$$

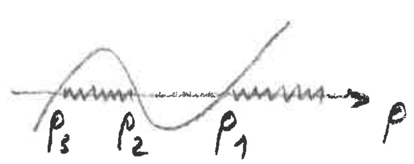
We must have $\left(\frac{d\rho}{dx}\right)^2 \geq 0$; Let ρ_1, ρ_2, ρ_3 be the roots of $P(\rho)$ with $\rho_3 < \rho_2 < \rho_1$



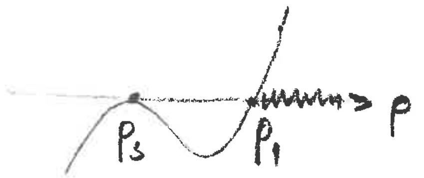
1 real root



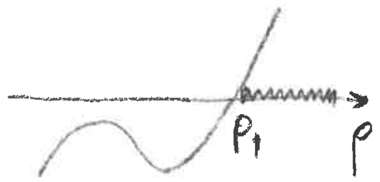
3 real roots, 1 double



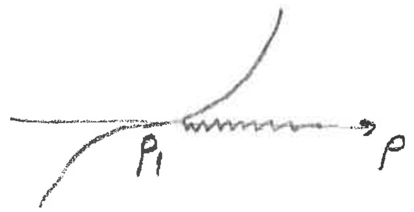
3 real roots



3 real roots, 1 double



1 real root



1 triple root

The saddle points correspond to the double roots of $P(\rho)$:

i.e., $\boxed{\begin{array}{l}\rho_1 = \rho_2 = \rho_0^2 \\ \rho_3 = \eta\end{array}}$

$$\left(\frac{d\rho}{dx}\right)^2 = 4(\rho - \rho_0^2)^2(\rho - \rho_3) =$$

$$= 4\left[\rho^3 - (2\rho_0^2 + \rho_3)\rho^2 + (\rho_0^4 + 2\rho_0^2\rho_3)\rho - \rho_3\rho_0^4\right] \equiv P(\rho)$$

Comparing with (1):

$$\therefore \quad 2|\Omega| = 2\rho_0^2 + \rho_3 \Rightarrow |\Omega| = \rho_0^2 + \frac{\rho_3}{2}$$

$$\therefore \quad C_1^2 = \rho_3\rho_0^4$$

$$\therefore \quad 2E = \rho_0^4 + 2\rho_0^2\rho_3 \Rightarrow E = \rho_0^2\left(\rho_3 + \frac{\rho_0^2}{2}\right)$$

$$\Rightarrow \boxed{\begin{array}{l}|\Omega| = \rho_0^2 + \dfrac{\eta}{2} \\ C_1^2 = \eta\rho_0^4 \\ E = \rho_0^2\left(\eta + \dfrac{\rho_0^2}{2}\right)\end{array}}$$

Integrate $\left(\frac{d\rho}{dx}\right)^2 = 4(\rho-\rho_0^2)(\rho-\eta) \Rightarrow$

$$\boxed{\rho(x) = \rho_0\left\{1-\alpha^2\operatorname{sech}^2[\rho_0\alpha(x-x_0)]\right\}^{1/2} \; ; \quad \alpha^2 = 1-\frac{\eta}{\rho_0^2}}$$

$$\varphi(x,t) = \int \frac{c_1}{\rho}\,dx - |\Omega|t + \theta_0 \Rightarrow$$

$\boxed{\text{Grey soliton}}$

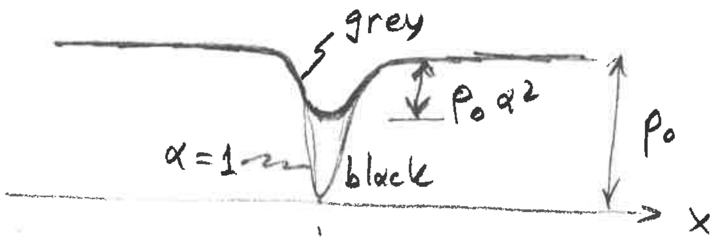
$$\Rightarrow \boxed{\varphi(x,t) = \rho_0\sqrt{1-\alpha^2}\,x + \tan^{-1}\left[\frac{\alpha}{\sqrt{1-\alpha^2}}\tanh(\rho_0\alpha x)\right] - \frac{\rho_0^2(3-\alpha^2)}{2}t + \theta_0}$$

When $\boxed{\alpha = 1}$:

$$\boxed{\begin{aligned}\rho(x) &= \rho_0\tanh[\rho_0(x-x_0)]\\ \varphi(x,t) &= -\rho_0^2 t + \frac{\pi}{2}\operatorname{sign}(x) + \theta_0\end{aligned}}$$

Black soliton
(heteroclinic orbit)

Density: $|u|^2 = \rho$



Phase: $\varphi(x)$