# Τρίτη 17/1/06

* Stokesian όριο (λύση προηγ. προβλήματος)

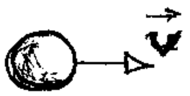
$$-\nabla p + \mu \nabla^2 u = 0 \quad \text{(ισορροπία)}$$

κ ασυμπ. ρευστό ($\nabla \cdot \vec{u} = 0$)

$$\Updownarrow$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \Leftrightarrow \frac{\nabla p}{\mu} = \nabla^2 u$$

Συνοριακή συνθήκη (σφαίρα)




στο σύνορο $\vec{u} = \vec{V}$

όλα τα μεγέθη εξαρτώνται γραμμικά από $\vec{V}$

$$\nabla^2 p = 0 \quad \text{κ} \quad \nabla^2 \vec{\omega} = \vec{0}$$

$$p = C \frac{\vec{V} \cdot \vec{x}}{r^3}$$

$$\vec{\omega} = D \frac{\vec{V} \times \vec{x}}{r^3} \quad \text{(πολικό διάνυσμα)}$$

Ερώτημα-Σχόλιο

Αν η σφαίρα αποτελείται από άλλο ρευστό με $\bar{\mu}$.

Πρέπει να λύσει κανείς τις ίδιες εξισώσεις και μέσα.

αλλά ομαλός στο κέντρο $p = C' \vec{V} \cdot \vec{x}$

$$\vec{\omega} = D' \vec{V} \times \vec{x}$$

$$\frac{\nabla p}{\mu} = C_1 \frac{\vec{V}}{r^3} + C(\vec{V} \cdot \vec{x}) \nabla \frac{1}{r^3} = \frac{C \vec{V}}{r^3} + -3C(\vec{V} \cdot \vec{x}) \frac{\vec{x}}{r^5}$$

$$\nabla \times (\nabla \times \vec{u}) = -\nabla^2 \vec{u} + \nabla(\cancelto{0}{\nabla \cdot \vec{u}}) = \nabla \times \vec{\omega} = \dots$$

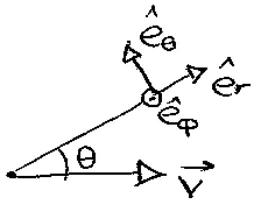
• Υπολογισμός αντίστασης

$$\vec{\omega} = \nabla\times\vec{u}$$

$$\nabla\cdot\vec{u} = 0 \Rightarrow \vec{u} = \nabla\times\vec{A} = \nabla\times\left(\frac{\psi\,\hat{e}_\varphi}{h_3}\right)$$

↑ ρευματοσυνάρτηση Stokes.

Εξήγηση:

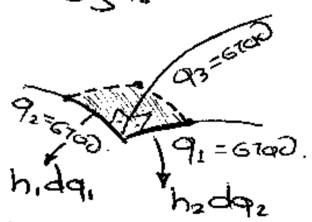



$$ds^2 = dq_1^2\, h_1^2 + dq_2^2\, h_2^2 + dq_3^2\, h_3^2$$

έτσι $h_3 = r\sin\theta$

$$\nabla\times\left(\frac{\psi\,\hat{e}_\varphi}{h_3}\right) = \begin{vmatrix} h_1\hat{e}_r & h_2\hat{e}_\theta & h_3\hat{e}_\varphi \\ \partial_r & \partial_\theta & \partial_\varphi \\ 0 & 0 & \psi \end{vmatrix} \frac{1}{h_1h_2h_3}$$

Απόδειξη:




$$\int_S (\vec{\nabla}\times\vec{A})\cdot\vec{n}_3\, dS = \int_\gamma \vec{A}\cdot d\vec{\ell}$$

$$(\vec{\nabla}\times\vec{A})_3\, h_1h_2\, dq_1 dq_2 = A_1 h_1 dq_1$$

$$- A_1(q_2+dq_2)\, h_1(q_2+dq_2)\, dq_1$$

$$- A_2 h_2 dq_2 +$$

$$+ A_2(q_1+dq_1)\, h_2(q_1+dq_1)\, dq_2$$

$$= \left[\frac{\partial}{\partial q_1}(A_2h_2) - \frac{\partial}{\partial q_2}(A_1h_1)\right] dq_1 dq_2$$

άσκηση: υπολογίστε την $\nabla^2$ σε καμπυλόγρ. συντεταγμένες.

$$\nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial q_1} + \frac{1}{h_2}\frac{\partial\phi}{\partial q_2} + \frac{1}{h_3}\frac{\partial\phi}{\partial q_3}$$

$$\vec{u} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\varphi \\ \partial_r & \partial_\theta & \partial_\varphi \\ 0 & 0 & \psi \end{vmatrix}$$

$$= \left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta},\ -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r},\ 0\right)$$

$$\vec{\omega} = \nabla\times\vec{u} = \dots = \frac{-\hat{e}_\varphi}{r\sin\theta}\left(\frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right)$$

όμως $\vec{\omega} = C\dfrac{\vec{V}\times\vec{x}}{r^3} = C\dfrac{V\sin\theta}{r^2}\hat{e}_\varphi$

$$\frac{\partial^2\psi}{\partial r^2} + \sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = -\frac{CV\sin^2\theta}{r}$$

Στο $r=a$ (στη σφαίρα)

$u_r = V\cos\theta$

$u_\theta = -V\sin\theta$

παρατήρηση: αν θέσω $\psi = f(r)\sin^2\theta$ λύνει το γωνιακό κομμάτι.

$$f'' - \frac{2f}{r^2} = -\frac{CV}{r} \qquad f = \frac{CV}{2}r + \frac{K_1}{r} + K_2 r^2$$

$K_2 = 0$ ώστε $\vec{u}(\infty) = 0$

$$\psi = \left(\frac{CVr}{2} + \frac{K_1}{r}\right)\sin^2\theta$$

$$u_r = \frac{2f(a)}{a^2}\cos\theta \qquad u_\theta = -\frac{\sin\theta}{a}f'(a) \quad \left.\begin{matrix} f(a) = V\frac{a^2}{2} \\ f'(a) = Va \end{matrix}\right\} \quad \frac{CVa}{2} + \frac{K_1}{a} = \frac{Va^2}{2}, \qquad \frac{CV}{2} - \frac{K_1}{a^2} = Va$$

$$CVa^{\cancel{2}} = \frac{3}{2}Va^{\cancel{3}} \Rightarrow$$

$$C = \frac{3}{2}a$$

$$2K_1 = -\frac{Va^3}{2} \Rightarrow$$

$$K_1 = -\frac{Va^3}{4}$$

$$u_r = V\left(3ar - \frac{a^3}{r}\right)\frac{\cos\theta}{2r^2}$$

$$u_\theta = \frac{-V}{2r}\sin\theta\left(\frac{3a}{2} + \frac{a^3}{2r^2}\right)$$

$u = O(1/r)$ λόγω στροβιλισμού

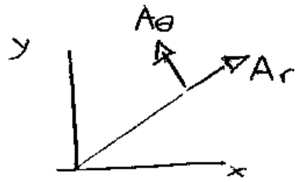
$$\vec{F} = \int_{S'}\left(\sigma_{r\theta}\,\hat{e}_\theta\, dS' + \sigma_{rr}\,\hat{e}_r\, dS'\right) = \int_{S_\infty}$$

αφού $\frac{\partial \sigma_{ij}}{\partial x_j} = 0$

ας το υπολογίσουμε στην επιφάνεια.

$$\left.\begin{matrix}\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij} \\ e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\end{matrix}\right\}$$

σε καρτ. συντεταγμένες.




$$A_x = A_r \cos\theta - A_\theta \sin\theta \qquad \frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$A_y = A_r \sin\theta + A_\theta \cos\theta \qquad \frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial u_x}{\partial x} = \cos\theta \frac{\partial}{\partial r}\left(u_r \cos\theta - u_\theta \sin\theta\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(u_r \sin\theta + u_\theta \cos\theta\right)$$

$$\sigma_{rr} = -p + 2\mu \left.\frac{\partial u_x}{\partial x}\right|_{\substack{x = \text{άξονας} \\ \text{με } \theta = 0}} = -p + 2\mu \frac{\partial u_r}{\partial r}$$

$$\sigma_{r\theta} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\Bigg|_{\substack{x = \text{άξονας} \\ \theta = 0}} \qquad \frac{\partial u_x}{\partial y} = \left[\sin\theta \frac{\partial}{\partial r}\left(u_r \cos\theta - u_\theta \sin\theta\right) + \cos\theta \frac{\partial}{r\partial\theta}\left(u_r \cos\theta - u_\theta \sin\theta\right)\right]$$

$$= \left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right]$$

$$\frac{\partial u_y}{\partial x} = \ldots \frac{\partial u_\theta}{\partial r}.$$