Note on a paper of John W. Miles

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The theorem X established by Miles in the preceding paper is here given a simpler and more general proof. Some further theoretical results concerning the stability of heterogeneous shear flows are also presented, in particular a demonstration that the complex wave velocity of any unstable mode must lie in a certain semi-circle.

1. Introduction

In his paper (1961; henceforth referred to as I), Miles has established the conjecture of G. I. Taylor that a sufficient condition for infinitesimal stability in a parallel, stratified, inviscid flow is that the local Richardson number should everywhere exceed $\frac{1}{4}$. Miles assumed in his proof that the velocity profile was monotonic, and that it and the density profile were analytic in a complex neighbourhood of the real flow domain. This note presents a simpler proof of Miles's theorem, which does not require these hypotheses. Some other related results are also given. For a derivation of the basic equations and references to previous work, one may consult Miles's paper.

2. Miles's theorem

To facilitate reference to I, the same notation will be used here. The basic velocity field is $U(y)$, and the density field $\rho(y)$ [$\rho_0(y)$ was used in I, but the subscript will not be needed here]. The stream-function perturbation is

$$(U-c)\,F(y)\,e^{ik(x-ct)},$$

and the linearized equations of motion lead to

$$[\rho(U-c)^2F']'+[\beta g-k^2(U-c)^2]\,F=0. \tag{2.1}$$

This is equation (3.3) of I; β is $-\rho'/\rho$, and g is the acceleration of gravity. It is always assumed here that the density stratification is statically stable, i.e. $\beta \geqslant 0$. The boundary conditions are that F vanish on $y=y_1$ and y_2 (rigid walls), which may recede to $\pm\infty$ in limiting cases. The flow is unstable if (2.1) and the boundary conditions have non-trivial solutions with $\operatorname{Im} c>0$. Set $c=c_r+ic_i$, and for brevity let $W=U-c$. Suppose now that F is such an unstable solution. Since $c_i>0$, W is never zero and one can select one branch of $W^{\frac{1}{2}}$ to be used consistently throughout (y_1,y_2); it will be as differentiable as U is, say at least piecewise twice continuously differentiable. Now set $G=W^{\frac{1}{2}}F$, and replace the variable F in (2.1) by G. This gives

$$(\rho WG')'-[\tfrac{1}{2}(\rho U')'+k^2\rho W+\rho W^{-1}(\tfrac{1}{4}U'^2-g\beta)]\,G=0, \tag{2.2}$$

and, of course, $G(y_1) = G(y_2) = 0$. Multiplication of (2.2) by the complex conjugate $\bar{G}$ of G and integration over (y_1, y_2) then leads to

$$\int \rho W[|G'|^2 + k^2|G|^2] + \int \tfrac{1}{2}(\rho U')'|G|^2 + \int \rho[\tfrac{1}{4}U'^2 - g\beta]\,\overline{W}\,|G/W|^2 = 0. \tag{2.3}$$

(Here and subsequently $\int$ is used to denote the definite integral over (y_1, y_2).) Recalling that $c_i > 0$, we see that the imaginary part of (2.3) implies

$$\int \rho[|G'|^2 + k^2|G|^2] + \int \rho[g\beta - \tfrac{1}{4}U'^2]\,|G/W|^2 = 0. \tag{2.4}$$

This is clearly impossible if $g\beta - \frac{1}{4}U'^2$ is non-negative throughout, so that a necessary condition for instability is that $g\beta - \frac{1}{4}U'^2$ be somewhere negative, or, as Miles expressed it, a sufficient condition for stability is that $g\beta - \frac{1}{4}U'^2$ should be everywhere non-negative. If the local Richardson number $J(y) = g\beta/U'^2$ is to be well defined everywhere, it is necessary to be cautious about points at which $U' = 0$, so that there is a minor difficulty in converting the dimensional stability condition $g\beta - \frac{1}{4}U'^2 \geqslant 0$ to the dimensionless one $J(y) \geqslant \frac{1}{4}$ if U is not strictly monotonic. This is obviously a trivial difficulty, however; if one prefers the statement $J \geqslant \frac{1}{4}$, it is probably simplest to define J at points at which $U' = 0$ to be its limiting value, $+\infty$ being allowed.

3. The semicircle theorem

Returning for a moment to the original equation (2.1), let us multiply it by $\bar{F}$ and integrate over (y_1, y_2), as in the derivation of (2.3), to get

$$\int \rho W^2[|F'|^2 + k^2|F|^2] - \int g\beta\,|F|^2 = 0, \tag{3.1}$$

or, separating the real and imaginary parts,

$$\int \rho[(U - c_r)^2 - c_i^2]\,[|F'|^2 + k^2|F|^2] - \int g\rho\beta\,|F|^2 = 0, \tag{3.2}$$

$$2ic_i \int \rho(U - c_r)\,[|F'|^2 + k^2|F|^2] = 0. \tag{3.3}$$

Equation (3.3) gives the result that c_r must lie in the range of U if $c_i > 0$, which goes back to Synge (1933) and has been given under various conditions by several other writers (references are given in I). However, it seems that rather more can be said. Let $Q = [|F'|^2 + k^2|F|^2]$. Then, under the assumption $c_i > 0$, (3.2) and (3.3) can be written

$$\int UQ = c_r \int Q, \tag{3.4}$$

$$\int U^2 Q = (c_r^2 + c_i^2)\int Q + \int g\rho\beta\,|F|^2. \tag{3.5}$$

Suppose now that $a \leqslant U(y) \leqslant b$. Then

$$\begin{aligned} 0 \geqslant \int (U-a)(U-b)\,Q &= \int U^2 Q - (a+b)\int UQ + ab\int Q \\ &= [c_r^2 + c_i^2 - (a+b)\,c_r + ab]\int Q + \int g\rho\beta\,|F|^2 \\ &= \{[c_r - \tfrac{1}{2}(a+b)]^2 + c_i^2 - [\tfrac{1}{2}(a-b)]^2\}\int Q + \int g\rho\beta\,|F|^2. \end{aligned}$$

Since $\beta \geqslant 0$ and $Q > 0$, this implies

$$[c_r - \tfrac{1}{2}(a+b)]^2 + c_i^2 \leqslant [\tfrac{1}{2}(a-b)]^2. \tag{3.6}$$

Thus, *the complex wave velocity c for any unstable mode must lie inside the semicircle in the upper half-plane which has the range of U for diameter.*

I have not been able to find that this rather striking extension of the familiar statement 'c_r must lie in the range of U for an unstable wave' has been pointed out before, although it is slightly reminiscent of some results of Synge; unfortunately he started with a different integral relation, which does not give so simple a result.

4. The growth rate

Miles's theorem and the semicircle theorem limit the values of the Richardson number and the complex wave velocity which are accessible to unstable modes. It is of interest to have a similar bound on the growth rate kc_i possible for an unstable wave. A very simple bound of this type can be obtained from (2.4), by observing that $|W|^{-2} \leqslant c_i^{-2}$. Thus (2.4) gives

$$k^2 \int \rho\, |G|^2 = \int \rho[\tfrac{1}{4}U'^2 - g\rho]\, |G/W|^2 - \int \rho\, |G'|^2$$
$$\leqslant \frac{1}{c_i^2} \max\,[\tfrac{1}{4}U'^2 - g\beta] \int \rho\, |G|^2,$$

and so
$$k^2 c_i^2 \leqslant \max\,[\tfrac{1}{4}U'^2 - g\beta]. \tag{4.1}$$

This estimate is not usually sharp—for example, the Couette flow, with U' constant, is known to be neutrally stable—but in most cases it will probably give the correct order of magnitude of the maximum growth rate. It is sufficient to show that c_i must approach zero as the wavelength decreases to zero, given the boundedness of U'; but there is a likelihood that in fact $kc_i \to 0$ as $k \to \infty$, and with sufficient assumptions the still stronger statement that *all* waves shorter than some critical wavelength are stable is probably true, as illustrated by the examples of Drazin and Hølmboe cited in I.

5. Connexion with Rayleigh's theorem

The proofs of Miles's theorem and the semicircle theorem given above are very similar to the ordinary proof of Rayleigh's theorem on the necessity of an inflection point for instability in homogeneous parallel inviscid flow. Synge (1933) gave a generalization of Rayleigh's theorem for the case of stratified flow. All three results can be obtained in a unified way as follows.

Assuming $c_i > 0$, let $F = W^{-n}H$, some definite branch being selected if n is not an integer. Substituting this in (2.1), one obtains

$$[\rho W^{2(1-n)} H']' - [k^2 \rho W^{2(1-n)} + n W^{1-2n} (\rho U')'$$
$$+ \rho W^{-2n} (n(1-n)\, U'^2 - g\beta)]\, H = 0. \tag{5.1}$$

Multiplication of this by $\bar{H}$ and integration over (y_1, y_2) gives the integral relation

$$\int \rho W^{2(1-n)} [|H'|^2 + k^2 |H|^2] + n \int W^{1-2n} (\rho U')'\, |H|^2$$
$$+ \int \rho W^{-2n} [n(1-n)\, U'^2 - g\beta]\, |H|^2 = 0. \tag{5.2}$$

Taking $n = 0$ in (5.2) leads to the semicircle theorem, $n = \frac{1}{2}$ gives Miles's theorem, and $n = 1$ gives Synge's generalization of Rayleigh's theorem, which states that a necessary condition for instability is that $(\rho U')' - 2\beta g \rho\, |W|^{-2} (U - c_r)$ should change sign in (y_1, y_2).

Finally, it should be mentioned that if the rigid upper boundary ($y = y_2$) of the fluid is replaced by a free surface at zero pressure, the boundary condition $H(y_2) = 0$ for (5.1) is replaced by $H' = [nW^{-1}U' + gW^{-2}]H$ at $y = y_2$. Equation (5.2) then becomes

$$\int \rho W^{2(1-n)}[|H'|^2 + k^2|H|^2] + n\int W^{1-2n}(\rho U')'|H|^2 + \int \rho W^{-2n}[n(1-n) - g\beta]|H|^2 - [\rho(nW^{1-2n}U' + gW^{-2n})|H|^2]_{y=y_2} = 0. \quad (5.3)$$

From this equation one can, however, derive Miles's theorem ($n = \frac{1}{2}$) and the semicircle theorem ($n = 0$) by very slight modifications of the arguments given above. The generalization of Rayleigh's theorem does not go through. That this is to be expected is shown by the example $\rho =$ const., $U = 0$ for $y < 0$, $U = y$ for $0 < y < 1$, $g = 0$, in which the flow is stable if the upper surface ($y = 1$) is rigid, but unstable if it is free.

REFERENCES

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