A NEW CONSERVATION-THEOREM OF HYDRODYNAMICS

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Summary — The conservation-theorem derived here can be expressed thus: Let $\vec{v}$ be the absolute velocity and $\vec{\xi} = \text{curl}\,\vec{v}$ the absolute vorticity, further $\sigma = \sigma(p)$ the specific volume of a barotropic fluid and ∇W the gradient of the action W (= HAMILTON's principal-function), then

$$\frac{D}{Dt}\left\{\sigma\,\vec{\xi}\,(\vec{v} - \nabla W)\right\} = 0\,,$$

where $\frac{D}{Dt}$ denotes differentiation following the motion of the fluid.

Zusammenfassung — Der hier abgeleitete Erhaltungs-Satz sagt Folgendes aus: Wenn $\vec{v}$ die absolute Geschwindigkeit, $\vec{\xi} = \text{curl}\,\vec{v}$ den absoluten Wirbel, $\sigma = \sigma(p)$ das spezifische Volumen einer barotropen Flüssigkeit und ∇W den Gradienten der Wirkungsfunktion bedeuten, so gilt:

$$\frac{D}{Dt}\left\{\sigma\,\vec{\xi}\,(\vec{v} - \nabla W)\right\} = 0\,,$$

wobei $\frac{D}{Dt}$ die individuelle Zeitableitung der Hydrodynamik darstellt.

I. *The new conservation-theorem and its proof from Lagrange's equations.*

In the Lagrangian method of hydrodynamics the rectangular coordinates x, y, z of a particle at time t, the velocities $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial t}$, $\frac{\partial z}{\partial t}$ and accelerations $\frac{\partial^2 x}{\partial t^2}$, $\frac{\partial^2 y}{\partial t^2}$, $\frac{\partial^2 z}{\partial t^2}$ are functions of t and of three independent parameters a, b, c which define the position of the chosen particle at a particular instant, thus a, b, c may be the rectangular coordinates of the chosen particle at the instant of time from which t is measured ($t = 0$). Using for brevity the vector notation $\vec{r} = (x, y, z)$ we can write LAGRANGE's dynamical equations referred to axes connected with the rotating earth

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$$(1) \qquad \frac{\partial}{\partial t}\left(\frac{\partial \vec{r}}{\partial t} + 2\,\vec{\omega} \times \vec{r}\right) \cdot \frac{\partial \vec{r}}{\partial a} = -\frac{\partial}{\partial a}\left(\overset{*}{\Phi} + \int \frac{dp}{\rho}\right), \text{ etc.}$$

where $\vec{\omega}$ represents the angular velocity of the earth's rotation and $\overset{*}{\Phi}$ the potential of gravity (including centrifugal potential). If the fluid is incompressible, the density ρ is a constant and $\int \frac{dp}{\rho} = \frac{p}{\rho}$, but in the following we shall consider the general case $\rho = \rho(p)$, i. e. the fluid may be compressible of such a kind that the density is a given function of the pressure p alone (barotropic fluid).

On using the transformation

$$(2) \qquad 2\,\vec{\omega} \times \frac{\partial \vec{r}}{\partial t} \cdot \frac{\partial \vec{r}}{\partial a} = \frac{\partial}{\partial t}\left(\vec{\omega} \times \vec{r} \,.\, \frac{\partial \vec{r}}{\partial a}\right) - \frac{\partial}{\partial a}(\vec{\omega} \cdot \vec{D})$$

it is possible to simplify the system of LAGRANGE's equations (1) by introduction of the moment of momentum $\vec{D}$ of the particle relative to the earth with respect to the origin $\vec{r} = 0$ (earth's center)

$$(3) \qquad \vec{D} = \vec{r} \times \frac{\partial \vec{r}}{\partial t},$$

and if we introduce further the absolute velocity

$$(4) \qquad \vec{v} = \frac{\partial \vec{r}}{\partial t} + \vec{\omega} \times \vec{r},$$

and the so-called Lagrangian function or the kinetic potential

$$(5) \qquad L = \frac{1}{2}\left(\frac{\partial \vec{r}}{\partial t}\right)^2 + \vec{\omega} \cdot \vec{D} - \left(\overset{*}{\Phi} + \int \frac{dp}{\rho}\right)$$

[cfr. H. ERTEL ([1])], we obtain the following form of LAGRANGE's equations referred to a system «fixed in space»:

$$(6) \qquad \frac{\partial}{\partial t}\left(\vec{v} \cdot \frac{\partial \vec{r}}{\partial a}\right) = \frac{\partial L}{\partial a}, \text{ etc.}$$

From this form (6) of the dynamical equations it is at once possible to obtain the first integrals

$$(7) \qquad \vec{v} \cdot \frac{\partial \vec{r}}{\partial a} = \overset{\circ}{v}_x + \frac{\partial W}{\partial a}, \text{ etc.}$$

(WEBER's transformation [cfr. H. LAMB ([2])]), where $\overset{\circ}{v}_x, \ldots\ldots$ are the initial values $(v_x)_{t=}$, ... at the moment $t = 0$ and the time-integral

$$(8) \qquad W = \int_0^t L\,dt$$

taken along the trajectory of the particle represents HAMILTON's principal-function, also called the action (German: Wirkungsintegral).

Let $x = x(a, b, c, t)$, $y = y(a, b, c, t)$, $z = z(a, b, c, t)$ be the solutions of the Lagrangian equations. The inverse transformation is given by $a = a(x, y, z, t)$, $b = b(x, y, z, t)$, $c = c(x, y, z, t)$ and we see that the principal-function $W(a, b, c, t)$

also may be regarded as a field-function $W = W\{a(x, y, z, t), b(x, y, z, t), c(x, y, z, t)\}$ from which the vector-field grad $W = \nabla W = \frac{\partial W}{\partial a}\nabla a + \frac{\partial W}{\partial b}\nabla b + \frac{\partial W}{\partial c}\nabla c$ can be obtained, appearing in the solution of the set (7):

$$\vec{v} - \nabla W = \overset{\circ}{v}_x \nabla a + \overset{\circ}{v}_y \nabla b + \overset{\circ}{v}_z \nabla c\,. \tag{9}$$

We designate by $\vec{\xi}$ the absolute vorticity

$$\vec{\xi} = \text{curl}\,\vec{v} = \text{curl}\left(\frac{\partial \vec{r}}{\partial t} + \vec{\omega} \times \vec{r}\right) = \text{curl}\left(\frac{\partial \vec{r}}{\partial t}\right) + 2\,\vec{\omega} \tag{10}$$

and this may be transformed by (9) into

$$\xi = \overset{\circ}{\xi}_x \nabla b \times \nabla c + \overset{\circ}{\xi}_y \nabla c \times \nabla a + \overset{\circ}{\xi}_z \nabla a \times \nabla b\,, \tag{11}$$

where

$$\xi_x = \frac{\partial \overset{\circ}{v}_z}{\partial b} - \frac{\partial \overset{\circ}{v}_y}{\partial c}\,,\ \text{etc.} \tag{12}$$

are the components of the initial vorticity $(\vec{\xi})_{t=0}$.

Evidently from (9) and (11) on forming the scalar product:

$$\vec{\xi}\cdot(\vec{v} - \nabla W) = (\overset{\circ}{\xi}_x \overset{\circ}{v}_x + \overset{\circ}{\xi}_y \overset{\circ}{v}_y + \overset{\circ}{\xi}_z \overset{\circ}{v}_z)\,[\nabla a,\ \nabla b,\ \nabla c] \tag{13}$$

where the triple scalar product

$$[\nabla a,\ \nabla b,\ \nabla c] = \nabla a \times \nabla b \cdot \nabla c = \rho/\overset{\circ}{\rho} = \overset{\circ}{\sigma}/\sigma \tag{14}$$

($\sigma = 1/\rho$ = specific volume) according to the Lagrangian equation of continuity

$$[\nabla a,\ \nabla b,\ \nabla c] = \frac{\partial\,(a, b, c)}{\partial\,(x, y, z)} = \frac{\rho}{\overset{\circ}{\rho}} = \frac{\overset{\circ}{\sigma}}{\sigma}\,. \tag{15}$$

Then from (13) and (14)

$$\boxed{\sigma\,\vec{\xi}\cdot(\vec{v} - \nabla W) = \overset{\circ}{\sigma}\,(\overset{\circ}{\xi}_x \overset{\circ}{v}_y + \overset{\circ}{\xi}_z \overset{\circ}{v}_x + \overset{\circ}{\xi}_y \overset{\circ}{v}_z)} \tag{16}$$

where obviously the terms on the right-hand side are functions of the a, b, c only and independent of t.

This is an important result showing that the value of the scalar product $\sigma\,\vec{\xi}\cdot(\vec{v} - \nabla W)$ is constant for all time along the trajectory of the particle (a, b, c), or in other words, $\sigma\,\vec{\xi}\cdot(\vec{v} - \nabla W)$ has the same value at each point of the trajectory starting from $x = a$, $y = b$, $z = c$ at $t = 0$, i. e.

$$\boxed{\frac{D}{Dt}\left\{\sigma\,\vec{\xi}\cdot(\vec{v} - \nabla W)\right\} = 0}\,, \tag{17}$$

where in Eulerian notation

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v\cdot\nabla \tag{18}$$

denotes differentation following the motion of the fluid. The foregoing calculations constitute the proof of the conservation-theorem given recently by ERTEL and ROSSBY [cfr. Ref. (3)].

II. *Second proof from* EULER'*s equations.*

It is our purpose now to consider a new proof follows from EULER's hydrodynamical equations referred to a system « fixed in space »

$$\frac{D\vec{v}}{Dt} = - \nabla\Phi - \sigma \nabla p = - \nabla\left(\Phi + \int \frac{dp}{\rho}\right) \tag{19}$$

where the Newtonian potential of attraction Φ is given by

$$\Phi = \overset{*}{\Phi} + \frac{1}{2} (\vec{\omega} \times \vec{r})^2 , \tag{20}$$

since the centrifugal-potential is $-\frac{1}{2}(\vec{\omega} \times \vec{r})^2$. From the equations (3, 4, 5, 8) and (20) we see that the Lagrangian function (5) can be represented as the difference of the kinetic and potential energy per unit mass

$$L = \frac{DW}{Dt} = \frac{1}{2} v^2 - \left(\Phi + \int \frac{dp}{\rho}\right) \tag{21}$$

($v = |\vec{v}|$). The following calculations necessary to prove the theorem become much simpler if we use ERTEL's general vortex-equation [cfr. Ref. (4)]

$$\frac{D}{Dt} (\sigma \vec{\xi} \cdot \nabla\psi) - (\sigma \vec{\xi} \cdot \nabla) \frac{D\psi}{Dt} = \sigma \cdot \frac{\partial(p, \sigma, \psi)}{\partial\, x, y, z)} \tag{22}$$

which contains the arbitrary function $\psi = \psi(x, y, z, t)$. If the fluid is barotropic, i. e. $\sigma = \sigma(p)$, the Jacobian $\frac{\partial(p, \sigma, \psi)}{\partial(x, y, z)}$ vanishes and the equation (22) shows that the operators $\frac{D}{Dt}$ and $\sigma \vec{\xi} \cdot \nabla$ commute. Putting for example $\psi = x$, y or z, we get in the case of a barotropic fluid the vortex equations

$$\frac{D}{Dt} (\sigma \xi_x) - (\rho \vec{\xi} \cdot \nabla) v_x = 0 \text{ , etc.}$$

or using the vector notation:

$$\frac{D}{Dt} (\sigma \vec{\xi}) - (\sigma \vec{\xi} \cdot \nabla) \vec{v} = 0 \, . \tag{23}$$

However if we put $\psi = W = \int_0^t L\, dt$, we obtain from (22) with respect to (21):

$$\frac{D}{Dt} (\sigma \vec{\xi} \cdot \nabla W) - (\sigma \vec{\xi} \cdot \nabla) \left\{ \frac{1}{2} v^2 - \left(\Phi + \int \frac{dp}{\rho}\right) \right\} = 0 \, . \tag{24}$$

Now from (19) follows

$$\sigma \vec{\xi} \cdot \frac{D\vec{v}}{Dt} = - \sigma \vec{\xi} \cdot \nabla \left(\Phi + \int \frac{dp}{\rho}\right), \tag{25}$$

and from (23):

$$\vec{v} \cdot \frac{D}{Dt}(\sigma \vec{\xi}) = \sigma \vec{\xi} \cdot \nabla\left(\frac{v^2}{2}\right). \tag{26}$$

Therefore we have, combining (25) and (26):

$$\frac{D}{Dt}(\vec{\xi} \cdot \vec{v}) = \sigma \vec{\xi} \cdot \nabla \left\{\frac{v^2}{2} - \left(\Phi + \int \frac{dp}{\rho}\right)\right\}, \tag{27}$$

and this result (27) can be written with respect to (24):

$$\frac{D}{Dt}\left\{\sigma \vec{\xi} \cdot (\vec{v} - \nabla W)\right\} = 0 \tag{28}$$

which formula is the same as (17). The foregoing calculations represents a complete proof of the new conservation-theorem from EULER's equations of hydrodynamics, because the general vortex equation (22) used above is derived by curl-differentiation from (19) combined with scalar multiplication by $\nabla\psi$ and transformation of the left-hand side with the aid of EULER's equation of continuity $\frac{D\sigma}{Dt} = \sigma \operatorname{div}(\vec{v})$. Hence the same equations are used as in the first proof from LAGRANGE's equations, namely the dynamical equations and the equation of continuity, but now in Eulerian notation.

III. *Incompressible fluids.*

For an incompressible fluid we have

$$\frac{D\sigma}{Dt} = 0 \tag{29}$$

and therefore the simplification

$$\frac{D}{Dt}\left\{\vec{\xi} \cdot (\vec{v} - \nabla W)\right\} = 0, \tag{30}$$

where

$$W = \int_0^t \left\{\frac{v^2}{2} - \left(\Phi + \frac{p}{\rho}\right)\right\} dt.$$

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