# RESEARCH NOTES

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# Stability of linear flow

T. Ellingsen and E. Palm

*Department of Mechanics, University of Oslo, Oslo, Norway*

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It is shown that a finite disturbance independent of the streamwise coordinate may lead to instability of linear flow, even though the basic velocity does not possess any inflection point.

It was shown by Rayleigh[1] that a necessary condition for instability of linear flow in an inviscid, nonstratified fluid is that the velocity profile possess an inflection point. The criterion was sharpened by Fjørtoft[2] and Høiland[3] who showed that, in addition, the numerical value of the vorticity must be a maximum at the inflection point. Rayleigh and Høiland base their analysis on a normal mode expansion in time whereas Fjørtoft avoids this assumption. The normal modes do not usually form a complete set in these problems, and Fjørtoft's proof is therefore far more general.

These results were obtained by considering infinitesimal two-dimensional perturbations. We show in this note, however, that three-dimensional disturbances may lead to another kind of instability, independent of the existence of an inflection point, which may possibly be responsible for the breakdown of laminar motion. This type of instability was originally suggested by Høiland.[4] He did not, however, draw the full conclusions from his idea.

The fluid is assumed to be inviscid, incompressible, and nonstratified, bounded by two horizontal parallel planes. The basic velocity $U(z)$ is directed along the horizontal $x$ axis and dependent on the vertical $z$ coordinate only. We consider a disturbance independent of the $x$ coordinate. The disturbance is, however, not strictly two-dimensional since it contains a velocity component along the $x$ axis. With this assumption the momentum equation for the $x$ component of the total velocity reduces to

$$Du/dt = 0 \quad . \tag{1}$$

Similarly the $x$ component of the vorticity $\xi$ is given by

$$D\xi/dt = 0 \quad . \tag{2}$$

Let $v$ and $w$ denote the velocities in the $y$ and $z$ directions, respectively. Since $\partial u/\partial x$ is zero, we may write

$$v = \partial\psi/\partial z \ , \quad w = -\partial\psi/\partial y \ ,$$

where $\psi$ is a stream function. For the moment linearizing (1) and (2), we have

$$\frac{\partial u}{\partial t} + wU' = 0 \ , \tag{3}$$

$$\frac{\partial}{\partial t}\nabla_1^2\psi = 0 \ , \tag{4}$$

where $\nabla_1^2$ is the two-dimensional Laplacian. It follows from (4) that $\nabla_1^2\psi$, and therefore $w$ is independent of time. Equation (3) may be integrated to give

$$u = u(0) - wU't \ , \tag{5}$$

showing that $u$ increases linearly with time. We therefore deduce that the basic flow $U(z)$ is unstable for this special kind of infinitesimal disturbance.

Equations (1) and (2) may, however, also be solved for finite perturbations. Equation (2) may be written

$$\frac{\partial}{\partial t}(\nabla_1^2\psi) + \frac{\partial\psi}{\partial z}\frac{\partial}{\partial y}(\nabla_1^2\psi) - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial z}(\nabla_1^2\psi) = 0 \quad . \tag{6}$$

A well-known class of solutions is that satisfying

$$\nabla_1^2\psi = f(\psi) \ , \quad \partial\psi/\partial t = 0 \ , \tag{7}$$

where $f$ is an arbitrary function. Choosing $f$ as a linear function, one solution is

$$\psi = A \sin k_1 y \sin k_2 z \quad . \tag{8}$$

Here, $A$, $k_1$, and $k_2$ are constants and $k_2$ must be chosen properly to satisfy the boundary conditions. Equation (8) represents a set of closed, steady streamlines. From (1) it follows that $u$ is conserved for the projection of the motion into these streamlines. Let us assume that initially $u$ is equal to the basic velocity $U(z)$. A particle in its orbit in the $yz$ plane will, therefore, always have a $u$ velocity equal to the value of the basic flow at the initial position of that particle. This value may, however, be very different from the local value of the basic flow. The difference is largest for streamlines with large vertical extent. The $u$ velocity thus experiences a complete redistribution, the variation of $u$ with $y$ and $t$ becoming just as dominant as the variation with $z$. Although the motion at a fixed point is periodic, the period is different for different streamlines. The entire motion is, therefore, aperiodic. This last effect must lead to large gradients in $u$, i.e., large vorticity concentrations.

We notice that this distortion of the basic profile is independent of the initial amplitude of the disturbance.

  

The time for development of the motion will, however, increase when the amplitude decreases. Since the asymptotic motion is very different from the basic flow, we conclude that this is unstable. It is possible, of course, that the developed motion is unstable. Owing to the large vorticity concentrations this indeed seems very likely so that the motion already discussed is valid only for a short span of time. Also, this possibility means instability of the basic flow.

In Eqs. (1) and (2) it is assumed that the disturbance is independent of $x$. The equations are also valid if the basic flow is oblique to the $x$ axis. For sufficiently small angles, the reasoning will be the same as in the case already discussed. For larger angles, however, the streamline field will contain a noticeable component of the basic flow, and the streamlines are usually not closed. The local changes in $u$ will, therefore, be much smaller, and an initial small disturbance will not lead to a large distortion of the mean flow, even for a large span of time. However, for sufficiently large disturbances the concentration of vorticity will be significant and may lead to local breakdowns of the basic velocity.

Equations (1) and (2) may be applied to find exact solutions of the problem. Such a solution is

$$\psi = A \sin k_1 y \sin k_2 z + B \cos \kappa z \quad , \tag{9}$$

where

$$\kappa^2 = k_1^2 + k_2^2$$

and $A$ and $B$ are arbitrary constants. $u$ is then found from the linear equation (2). Equation (9) represents a basic velocity proportional to $\sin\kappa z$, superposed on a finite disturbance. Steady solutions of (1) and (2) for $f(\psi) = \exp(-2\psi)$ have been discussed by Stuart.[5] Equation (8) also describes a finite disturbance in a channel flow, i.e., a flow bounded by vertical as well as horizontal planes. By similar reasoning we obtain the result that an inviscid channel flow is always unstable for perturbations independent of the streamwise coordinate.

The same result is also true for the flow in a circular pipe. This is seen by considering the solution

$$\psi = A J_n(kr) \cos n\theta \tag{10}$$

($n \neq 0$), where $J_n$ denotes a Bessel function and $r$ and $\theta$ are polar coordinates.

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[4]E. Høiland, Lecture Notes (1971) (in Norwegian).
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# Flow in the entrance region of noncircular ducts

C. L. Wiginton

*Department of Mathematics, The University of Houston, Houston, Texas 77004*
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A remarkably simple method of solution is given for an eigenvalue problem arising in laminar flow in ducts which uses eigenfunctions of the Helmholtz equation. The only computation involved is the determination of zeros of a series to obtain the eigenvalues.

This note should be considered as a follow-up to Refs. 1 and 2 from this journal. We propose a somewhat different and considerably simpler analysis of a boundary value problem occurring in laminar incompressible flow of a fluid in the so-called entrance region of a straight duct with arbitrary but unchanging cross section. We suppose that the fluid has constant properties and take the flow axis in the positive $z$ direction with $x$ and $y$ the cross-sectional coordinates. The flow is governed by the momentum and continuity equations. We assume the velocity to be zero at the duct wall and equal to the average at the duct entrance. The problem to be considered results from application of a linearization of the momentum equation given in Ref. 1 which is apparently the most useful and accurate analytical analysis of this problem to date. (See the references for further applications and a history of this and related methods.)

We avoid restating the problem and the transformations (stretching the axial coordinate to linearize and making variables nondimensional) and state that the final model to be solved involves writing the velocity as

$$u = u_e + u_{fd} \quad , \tag{1}$$

where $u_{fd}$ is the fully developed velocity and $u_e$ is the entrance perturbation. We seek a solution of the form

$$u_e = \sum_{i=1}^{\infty} a_i g_i \exp(-\alpha_i^2 z) \quad , \tag{2}$$

and this leads to

$$\nabla^2 g_i + \alpha_i^2 g_i = \oint_C \frac{\partial g_i}{\partial n} ds \quad , \tag{3}$$

with $g_i = 0$ on the duct boundary $C$. Solutions are pairwise orthogonal and orthogonal to unity so that the $a_i$ in (2) are easily determined using the entrance condition.

At this point we consider the vanishing or nonvanishing of the line integral in (3). In Ref. 2 it was implicit and unstated that it did not vanish. To assume that it does vanish leads to valid $g_i$'s and $\alpha_i$'s as far as (3) is concerned and these would apparently contribute to the solu-