A Numerical Method for Predicting the Perturbations of the Middle Latitude Westerlies

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Abstract

It is shown that the large-scale quasi-stationary disturbances of the middle-latitude westerlies are produced by the forced ascent of the westerly current over the continental land masses. Friction is found to have an important modifying effect on the motion. Using these results a numerical method is devised for predicting the height profile of the 500 mb pressure surface at a fixed latitude. The method involves the use of the notion of an "equivalent-barotropic atmosphere" and of the geostrophic approximation. Six actual forecasts are made for a period of one day and the results compared with observation. The accuracy obtained is thought to justify incorporation of the method into day-to-day forecast procedures.

I. Introduction

In an article by one of the co-authors,[3] a program for numerical weather prediction was outlined in which it was proposed to consider a hierarchy of atmospheric models whose study would lead to an increasing comprehension of the physical and numerical aspects of the forecast problem. The most elementary model was a barotropic atmosphere in which the motion is regarded as consisting of small perturbations on a zonal current. The problem of forecasting these perturbations constitutes the simplest non-trivial instance of a numerical forecast problem. It is the purpose of the present article to discuss this case as a step towards the realization of the general program. It is also hoped that the treatment of motions corresponding closer to reality than those previously dealt with will prove of value to the forecaster in the field.

In previous investigations of this barotropic model, kinematic constraints were always imposed in order to permit the use of simple mathematical techniques. For example, the infinite wave train on a plane earth served as a model for the upper air cyclone trough and the point source as a model for the localized action of solenoidal fields. C.-G. ROSSBY (1945), pointing out that the study of purely periodic or isolated systems does not suffice to explain phenomena associated with energy transfer, introduced the concept of dispersion into meteorology. But this concept too is applicable only to a restricted class of motions, namely motions that can be described as wave trains with slowly varying wavelengths and frequencies. In order to deal with the most general types of interaction between systems, it is clear that all kinematic constraints must be dropped. In so doing, numerical methods must replace the purely analytic

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[3] J. G. CHARNEY, *On Numerical Weather Prediction by Dynamical Methods*, to be published in the Journal of Meteorology. This article will hereafter be referred to by the symbol (NWP).

mathematical techniques, which no longer suffice to deal with the complexities of the motion. Although there is a loss in elegance of treatment in using numerical methods, the gain in generality more than compensates; it becomes possible — for the first time — to treat actual initial and boundary condition.s

A preliminary step was taken in (NWP) where a method was given for predicting the actual height profile of the 500 mb pressure surface at a prescribed latitude on the assumption that the disturbances could be regarded as perturbations of infinite lateral extent on a uniform zonal current. The agreement of an actual 24 hour forecast with observation was better than had been expected, but there was also a marked discrepancy. A deep and extensive trough in the western Pacific and a wedge in the eastern Pacific were predicted to move rapidly westward but were observed not to move at all.

The persistence of the Pacific trough and wedge system suggested the explanation that the motion is composed of a traveling "free" perturbation and a permanent "forced" perturbation produced by the action of geographically fixed perturbing forces. The authors were thus led to a study of the quasi-permanent atmospheric perturbations. It was found that the observed mean disturbances at the 500 mb level could be adequately explained as an effect of the forcing of a zonal current over the continental elevations, providing surface friction and the lateral variation of the motion were also taken into account.

Having been convinced that the main discrepancies in the forecast could be eliminated by taking into account topographical and frictional factors, the authors devised a method for incorporating these factors into the forecast equations. Applying the method, 24 hour forecasts were prepared for the period Jan. 8—13, 1946, and were compared with observation (see Figs. 8—13). In most cases the accuracy appears good enough to justify incorporating the method into routine forecasting, and in no case are the errors as great as in the original forecast (Fig. 1). The method is objective and can therefore be used to supplement forecasts of a more subjective character. In certain cases it is believed to yield a better forecast than the standard method of extrapolation, particularly where deepening or filling due to horizontal energy dispersion takes place.

II. The equivalent-barotropic atmosphere

It is a matter of experience that the large-scale perturbations in the atmosphere have the character of "external" waves in which the shape of the streamlines (or isobars) is approximately the same at all levels, and where the increase of wind with height is similar along all verticals. These properties make it possible to deal with the large-scale motion approximately as a two-dimensional problem by using a method given in (NWP). The derivation of the method will be indicated briefly here.

The vorticity equation can be written with good approximation

$$\frac{d}{dt}(f+\zeta)+\frac{f}{\varrho}\left[\frac{\partial}{\partial x}(\varrho u)+\frac{\partial}{\partial y}(\varrho v)\right]=0,$$

$$f=2\Omega\sin\varphi, \tag{1}$$

in which the operator $\frac{d}{dt}$ denotes the individual rate of change with time. ζ is the vertical vorticity component relative to the earth, f is the Coriolis parameter, u is the component of velocity in the x-direction (east), v is the component of velocity in the y-direction (north), ϱ is the density, Ω is the angular velocity of the earth and φ the latitude. Here ζ is considered as small compared to f, and the solenoid term is combined with the divergence term to give the mass-divergence. Forming the average (denoted by a bar) of the vorticity equation in the vertical direction with respect to pressure and utilizing the tendency equation, we find

$$\overline{\frac{d}{dt}(f+\zeta)}=\frac{f}{p_0}\frac{\partial p_0}{\partial t}-\frac{f}{H}w_0, \tag{2}$$

where the subscript "o" is used to denote surface values, and $H = RT_0/g$ is the height of the homogeneous atmosphere. w is the vertical velocity component, p is the pressure, T the temperature, R the specific gas constant, and g the acceleration of gravity. It should be noticed that the effect of vertical solenoids does not occur explicitly in this equation, which, so far, involves only a slight approximation. It is now assumed that the velocity field, with sufficient approximation, can be written

$$u = \bar{u}\,(x, y, t)\,A\,(p),\ v = \bar{v}\,(x, y, t)\,A\,(p) \quad (3)$$

(hence $\overline{A} = 1$), and that the vertical motion for the large-scale systems is so small that its effect can be neglected in the individual derivative of the vorticity. With these assumptions, (2) can be rewritten as

$$\frac{\partial \bar{\zeta}}{\partial t} + \left(\bar{u}\frac{\partial}{\partial x} + \bar{v}\frac{\partial}{\partial y}\right)(f + \overline{A^2}\bar{\zeta}) = = \frac{f}{p_0}\frac{\partial p_0}{\partial t} - \frac{f}{H}w_0 \quad (4)$$

According to the assumption (3), there must be a certain level $\bar{p}$ where $A(\bar{p}) = \overline{A} = 1$; at this level we have $u = \bar{u}$, $v = \bar{v}$, and $\zeta = \bar{\zeta}$, so that equation (4) may be written

$$\frac{\partial \zeta}{\partial t} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(f + \overline{A^2}\zeta) = = \frac{f}{p_0}\frac{\partial p_0}{\partial t} - \frac{f}{H}w_0 \quad (5)$$

at the level $p = \bar{p}$. In a barotropic fluid, we have $A(p) \equiv 1$, so that equation (5) (with $\overline{A^2} = 1$) holds at all levels. Equation (5) can therefore be interpreted as stating that an *equivalent-barotropic atmosphere* can be defined, whose motion corresponds to the motion of the baroclinic atmosphere at a certain level. This level ($p = \bar{p}$), where equation (5) holds, will be called the *equivalent-barotropic level.*

According to CHARNEY (NWP) a proper value of $\overline{A^2}$ in the atmosphere is 5/4 and the value of $\bar{p}$ lies between 500 mb and 600 mb; the value 500 mb will be chosen here as a matter of convenience since the data at this level is given explicitly in synoptic reports.

III. The quasi-geostrophic approximation

The large-scale perturbations in the atmosphere show a rather slow speed of propagation relative to the air. This fact indicates that these perburbations are of the type called "planetary" by ROSSBY. Besides such planetary motions, however, the equations of motion permit solutions which may be characterized as combined gravitational, inertial, and sound waves; such waves will have a speed of propagation comparable with the speed of sound.[1] The fact that the equations permit as solutions such fast-moving perturbations will cause certain difficulties for numerical integration of the equations. CHARNEY (1948) has shown, however, that by making use of the geostrophic approximation one can derive a system of equations where the fast-moving perturbations in which we are not interested are eliminated. The geostrophic approximation is applied after having eliminated the horizontal divergence in the vorticity equation by means of the equation of continuity. ELIASSEN (1949) derived virtually the same equations by utilizing the geostrophic approximation in a somewhat different manner. Using CHARNEY'S method, we simply introduce the geostrophic approximation into the vorticity equation (5) for the equivalent-barotropic level. This gives

$$\frac{\partial \zeta_g}{\partial t} + \overline{A^2}\left(u_g\frac{\partial \zeta_g}{\partial x} + v_g\frac{\partial \zeta_g}{\partial y}\right) + \beta\, v_g = = \frac{f}{p_0}\frac{\partial p_0}{\partial t} - \frac{f}{H}w_0, \quad (6)$$

where $\beta = df/dy$, u_g, v_g are the components of the geostrophic wind, and ζ_g the geostrophic vorticity, at the equivalent-barotropic level.

In the following, we shall deal with the motion in an isobaric surface and shall therefore replace the pressure as dependent variable by the height z of the isobaric surface. We then have

$$u_g = -\frac{g}{f}\frac{\partial z}{\partial y},\ v_g = \frac{g}{f}\frac{\partial z}{\partial x} \quad (7)$$

The geostrophic vorticity is, with good approximation,

$$\zeta_g = \frac{g}{f}\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) \quad (8)$$

[1] It should be emphasized that we are here concerned with very long, external waves only, and that such smaller scale phenomena as frontal waves are disregarded.

The pressure tendency is

$$\frac{\partial p}{\partial t} = g\varrho \frac{\partial z}{\partial t} \tag{9}$$

When (7), (8), and (9) are substituted into the vorticity equation (6), an equation in z alone results, except that it contains the tendency and vertical velocity at the ground on the right-hand side. In the case of motion over level ground, $w_0 = 0$, but the tendency at the ground still remains. When the computations were started, this indeterminacy was removed by making the assumption $\frac{\partial z_0}{\partial t} = \frac{\partial z}{\partial t}$. It was later recognized that it is more consistent with the assumptions made previously to write $\frac{\partial z_0}{\partial t} = \varkappa \frac{\partial z}{\partial t}$, where $\varkappa$ is the ratio of the surface geostrophic wind to the wind at the equivalent-barotropic level. However, the difference is insignificant, since the whole term may be dropped without noticeable effect on the results.

IV. A preliminary forecast attempt

The motion at the equivalent-barotropic level (500 mb) is considered as consisting of small perburbations superimposed upon a constant zonal current U. For the time being, the height perturbation z of the 500 mb surface will be considered as independent of y. Assuming motion over a level ground ($w_0 = 0$), the quasi-geostrophic vorticity equation (6) for the equivalent-barotropic level becomes

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial^2 z}{\partial x^2} + \beta\frac{\partial z}{\partial x} - \lambda^2\frac{\partial z}{\partial t} = 0 \tag{10}$$

where

$$\lambda^2 = \frac{f^2}{gH} \tag{11}$$

Strictly speaking, the factor $U' = \overline{A^2}U = 5/4\, U$ should be used in place of U in (10). However, it was not found to be worth while to make this distinction, since U is not uniquely determined from the data in any case, but depends on the method of averaging.

Equation (10) is assumed to hold at a representative latitude. It is convenient to use 1 day as the time unit and the radius of the latitude circle as the unit of length. The solution $z(x, t)$ must then be considered as periodic in x with the period 2π. By means of FOURIER analysis we find the solution in terms of the initial distribution $z(x, 0)$ in the form

$$z(x + Ut, t) =$$

$$= z(x, 0) + \int_0^{2\pi} z(\alpha, 0)\, I_{\lambda^2}(x - \alpha, t)\, d\alpha \tag{12}$$

where the influence function (or GREEN's function) $I_{\lambda^2}(x, t)$ is defined by the FOURIER series

$$I_{\lambda^2}(x, t) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\left(e^{\frac{inbt}{n^2+\lambda^2}} - 1\right)e^{inx}, \tag{13}$$

in which

$$b = \beta + \lambda^2 U \tag{14}$$

In the units used (x in radians of longitude, t in days) we find

$$\beta = 4\pi\cos^2\varphi,$$

$$\lambda^2 = \frac{\Omega^2 r^2}{gH}\sin^2 2\varphi \approx 2.5\sin^2 2\varphi, \tag{15}$$

where r denotes the earth's radius.

Taking $\varphi = 45°$ as the representative latitude,

$$\beta = 2\pi,\ \lambda^2 = 2.5 \tag{16}$$

The factor b varies with the current velocity, but this variation is very slight since U is always of the order of 1/3 radian per day. One may therefore consider b as a constant. In the computations the value $b = 7$ was adopted.

The series (13) for the influence function converges rather slowly, so that the straightforward computation by evaluating a sufficient number of terms in the series would be a very laborious task. However, the authors' associate, G. HUNT, has given a method by which the computation can be done without too much work. Equation (13) may be written

$$I_{\lambda^2}(x, t) =$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\left(e^{\frac{inbt}{n^2+\lambda^2}} - e^{\frac{ibt}{n}}\right)e^{inx} + I_0(x, t), \tag{17}$$

where

$$I_0(x, t) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \left(e^{\frac{ibt}{n}} - 1 \right) e^{inx} \quad (18)$$

is the influence function for non-divergent planetary motions. The first sum on the right-hand side of (17) converges so rapidly that it is sufficiently accurate for our purpose to sum from — 20 to + 20, as an estimate of the remainder shows. As to the function I_0, (18) may be written

$$I_0(x, t) = \frac{ibt}{2\pi} \sum_{n=-\infty}^{+\infty} \frac{e^{inx}}{n} + \frac{(ibt)^2}{2\pi \cdot 2!} \sum_{n=-\infty}^{+\infty} \frac{e^{inx}}{n^2} +$$

$$+ \frac{(ibt)^3}{2\pi \cdot 3!} \sum_{n=-\infty}^{+\infty} \frac{e^{inx}}{n^3} + \frac{(ibt)^4}{2\pi \cdot 4!} \sum_{n=-\infty}^{+\infty} \frac{e^{inx}}{n^4} +$$

$$+ \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \left[e^{\frac{ibt}{n}} - 1 - \left(\frac{ibt}{n} + \frac{(ibt)^2}{2!n^2} + \right.\right.$$

$$\left.\left. + \frac{(ibt)^3}{3!n^3} + \frac{(ibt)^4}{4!n^4} \right) \right] e^{inx} \quad (19)$$

The four first series can be summed exactly; they represent polynomials in x of the first, second, third, and fourth degree, respectively. The number of such terms to be taken is a question of economy and depends on the accuracy wanted. The convergence of the last series is so rapid that summation from — 20 to + 20 will give more than sufficient accuracy.

In this way, the functions $I_0(x, t)$ and $I_{\lambda^2}(x\ t)$, were computed for $t = 1$ day accurately to two decimal places.[1] The results are given in table I, and $I_{2.5}(x, 1)$ is graphed in Fig. 5. The prognostic formula (12) was tried out on an example from the carefully prepared and well-analysed *Historical Weather Maps* series published by the Air Weather Service of the U. S. Air Force. The height of the 500 mb surface along 45° latitude for January 12, 1946, 0400 GMT was taken as the initial distribution of z; and the corresponding profile for January 13, 1946, 0400 GMT, was computed by means of equation (12) with $U = 20°$ longitude per day. The integral in (12) was approximated by summing over intervals of 10° longitude. The profile of the 500 mb surface along 45° latitude on January 12, and the computed and observed 24 hour changes are shown graphically in Fig. 1.

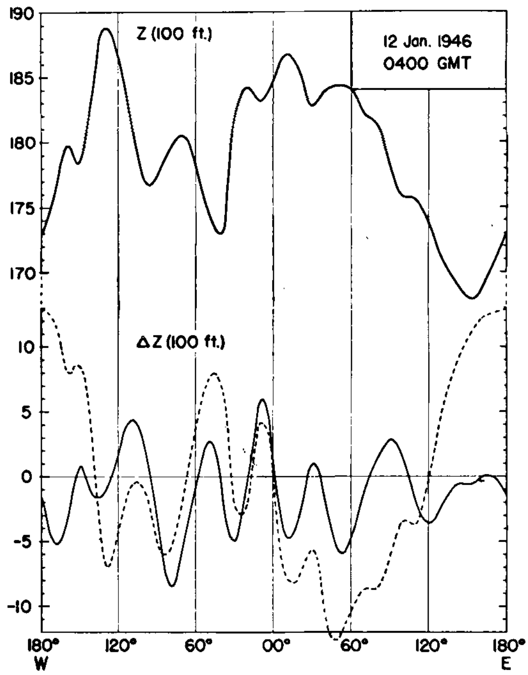

Fig. 1. The upper curve gives the observed height profile of the 500 mb surface at 45° N for January 12, 1946, 0400 GMT. The lower continuous curve gives the observed change in the following 24 hours, and the lower dashed curve the predicted change.

The zonal profile of the 500 mb surface may be described as an almost stationary, very long wave pattern, upon which is superimposed a system of shorter, migrating waves. The long wave, stationary pattern consists of two main ridges at the west coasts of Europe and North America, and of troughs over the North American east coast and the western Pacific.

The computed forecast is not bad, as far as the shorter waves are concerned; but the formula gives a rapid propagation of the very long wave pattern toward the west, in

[1] A more accurate computation of a Green's function for non-divergent planetary waves, for several different latitudes and time intervals, has been made independently by G. E. Forsythe in a paper to be published in the Journal of Research of the National Bureau of Standards.

Table I

x (degrees long.)	$I_0(x,1)$	$I_0(-x,1)$	$I_{2.5}(x,1)$	$I_{2.5}(-x,1)$	$I_6(x,1)$	$I_6(-x,1)$	$I_{10}(x,1)$	$I_{10}(-x,1)$	$I_{14}(x,1)$	$I_{14}(-x,1)$	$I_{18}(x,1)$	$I_{18}(-x,1)$
0	−6.70	0.30	−6.64	0.36	−5.78	1.22	−5.32	1.68	−5.06	1.94	−4.88	2.12
10	−3.27	0.34	−3.44	0.44	−2.65	1.10	−2.17	1.31	−1.86	1.35	−1.64	1.36
20	−1.12	0.39	−1.66	0.50	−1.17	0.98	−0.86	1.02	−0.69	0.96	−0.55	0.88
30	0.14	0.43	−0.68	0.53	−0.41	0.86	−0.30	0.76	−0.22	0.64	−0.16	0.53
40	0.78	0.47	−0.17	0.54	−0.13	0.71	−0.07	0.56	−0.04	0.44	−0.02	0.33
50	0.98	0.50	0.06	0.54	−0.02	0.58	−0.01	0.41	−0.01	0.28	0.00	0.20
60	0.94	0.52	0.16	0.53	0.02	0.47	0.02	0.29	0.01	0.19	0.02	0.12
70	0.74	0.53	0.18	0.51	0.03	0.38	0.01	0.21	0.00	0.13	0.01	0.07
80	0.48	0.53	0.17	0.48	0.03	0.30	0.01	0.15	0.00	0.08	0.00	0.04
90	0.22	0.51	0.16	0.44	0.02	0.24	0.00	0.10	−0.01	0.05	0.00	0.02
100	−0.02	0.48	0.14	0.41	0.01	0.18	0.00	0.07	−0.01	0.04	0.00	0.01
110	−0.22	0.43	0.12	0.37	0.00	0.14	−0.01	0.05	−0.01	0.02	0.00	0.00
120	−0.36	0.37	0.11	0.34	0.00	0.10	−0.01	0.03	−0.01	0.01	0.00	0.00
130	−0.45	0.28	0.10	0.30	−0.01	0.07	−0.01	0.02	0.00	0.00	0.00	0.00
140	−0.48	0.19	0.11	0.27	−0.01	0.05	−0.01	0.02	0.00	0.00	0.00	0.00
150	−0.47	0.09	0.12	0.24	−0.01	0.03	0.00	0.01	0.00	0.00	0.00	0.00
160	−0.42	−0.02	0.13	0.21	−0.01	0.01	0.00	0.01	0.00	0.00	0.00	0.00
170	−0.34	−0.14	0.14	0.18	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
180	−0.24	−0.24	0.16	0.16	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

The influence function I_{a^2} $(x, 1)$ for $a^2 = 0$, 2.5, 6, 10, 14, and 18.

contradiction to the observed persistence of this system.

As an explanation for the discrepancy, it is assumed that the very long waves are forced perturbations, set up by some geographically fixed disturbing forces. There are two conceivable ways in which the motion can be effected by the underlying surface: by thermal action, or by the forcing action of continental elevations. The former has been suggested by ROSSBY (1939) and HAURWITZ (1940 II), but is here tentatively discounted on the ground that the reversal in phase of the fixed perturbations, which should be expected to accompany the reversal of the heat and cold sources from winter to summer, is not observed to occur. The latter is therefore assumed to the dominating effect, and we are led to consider the nature of topographically forced perturbations.

V. The effect of continental elevations

Instead of putting the vertical velocity at the ground in equation (6) equal to zero, we now write $w_0 = u_{g_0}\dfrac{\partial h}{\partial x} + v_{g_0}\dfrac{\partial h}{\partial y}$, where h is the height of the ground above mean sea level.

The perturbation equation for stationary motion independent of y becomes

$$U\frac{d^3z}{dx^3} + \beta\frac{dz}{dx} = -\lambda^2 U_0\frac{dh}{dx}, \qquad (20)$$

where $h(x)$ is the mountain profile along the latitude circle in question. Assuming U_0, the geostrophic zonal current at the ground, to be a certain fraction $\varkappa$ of the velocity U at

Table II

N	A_N		
	$a^2 = 10$	$a^2 = 14$	$a^2 = 18$
−3	−0.070	−0.051	−0.037
−2	−0.100	−0.080	−0.064
−1	−0.506	−0.433	−0.382
0	−0.212	−0.182	−0.161
1	0.305	0.314	0.316
2	0.119	0.112	0.102
3	0.177	0.149	0.123
4	0.065	0.051	0.038
5	0.095	0.065	0.047
6	0.034	0.022	0.014
7	0.049	0.030	0.016
8	0.017		
9	0.023		

Numerical constants A_N in the forecast equation (45).

the 500 mb level, we get after dividing through by U and integrating:

$$\frac{d^2z}{dx^2} + \frac{\beta}{U} z = -\varkappa\lambda^2 h(x) \tag{21}$$

From this equation it follows that over the oceans, where $h = 0$, the stationary perturbation must take the form of a pure sine wave, the wave length of which is $2\pi\sqrt{U/\beta}$, ROSSBY's stationary wave length.

This result may be compared with the normal pressure profile at 20,000 feet along 45° latitude for the month of January, taken from the *Normal Weather Maps* series of the U. S. Weather Bureau. This normal profile, which is shown converted to height units at 500 mb in Fig. 4 (solid line), may be considered as approximating the stationary profile. As expected, the profile is roughly sinusoidal over the oceans; but the wave length is 140°—150° of longitude, whereas an estimate of the zonal wind speed shows that ROSSBY's stationary wave length is about 80°. We are therefore forced to the conclusion that (21) is not the correct mathematical expression of the stationary motion.

A more correct explanation of the stationary pattern follows from the consideration that the lateral variation of a topographically forced perturbation must be determined by that of the continental elevations, and these are more closely approximated in meridional section by the crest of a sine wave than by a horizontal line. Accordingly we allow for the meridional variation by assuming that the perturbation has a sine variation in the north—south direction with a half-wave length corresponding roughly to the effective lateral extent of the land profile across the zonal current, about 25° of latitude. The motion is therefore of the type studied by HAURWITZ (1940, I). Putting

$$\frac{\partial^2 z}{\partial y^2} = -m^2 z \tag{22}$$

in the expression for the relative vorticity (8), we obtain, instead of equation (21),

$$\frac{\partial^2 z}{\partial x^2} + s^2 z = -\varkappa\lambda^2 h(x) \tag{23}$$

where

$$s^2 = \frac{\beta}{U} - m^2 \tag{24}$$

In the length unit used (degrees of longitude at 45°) s represents the number of stationary waves encircling the 45° latitude circle and $m = \pi/D$, where D is the north—south half wave-length of the perturbation. The value of s must be approximately 2.5 to correspond to the stationary wave length of 140° longitude observed over the oceans (see Fig. 4). The value of U measured from the normal weather map is 0.29 radians per day or 17° longitude per day. With the value $\beta = 2\pi$ we find that m^2 must be about 15. This corresponds to a halfwave-length D of 33 degrees of latitude, which agrees well with the value 25 estimated from the shape of the continents.

The solution of the non-homogeneous equation (23) is unique when we demand that $z(x)$ and its first derivative be continuous and periodic with the period 2π. By means of FOURIER analysis, the solution may be written in the form

$$z(x) = \varkappa\lambda^2 \int_0^{2\pi} h(\alpha)\Phi_0(x-\alpha)\, d\alpha \tag{25}$$

where

$$\Phi_0(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \frac{e^{inx}}{n^2 - s^2} + \frac{1}{2\pi s^2} \tag{26}$$

Strictly speaking, $\Phi_0{}^{(x)}$ has been so chosen that the function represented by the integral (25) is not the solution to (23), but is rather the deviation of the solution from its mean valve around the latitude circle.

Equation (26) may be summed to

$$\Phi_0(x) = -\frac{\cos s(x-\pi)}{2s \sin s\pi} + \frac{1}{2\pi s^2},\ 0 \leq x \leq 2\pi;$$

$$\Phi_0(x + 2\pi) = \Phi_0(x) \tag{27}$$

With $s = 2.5$, this function is represented graphically in Fig. 2.

The formula (25) was applied to a function $h(x)$, determined from *The Oxford Advanced Atlas* (J. BARTHOLOMEW 1942) in the following way: The height along the latitude circles

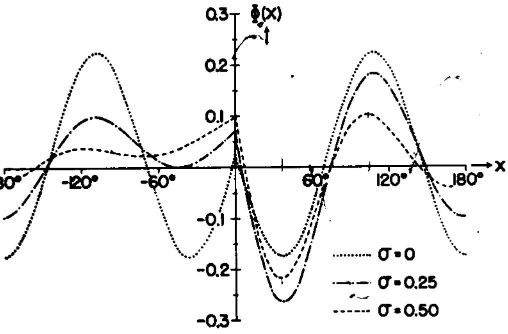

Fig. 2. The GREEN's function $\Phi_\sigma(x)$ for the stationary perturbation. The dotted curve represents the function for the case of no friction, the dash-dotted curve for the case of moderate friction, and the dashed curve for the case of strong friction.

40°, 42°, 44°, 46°, 48°, and 50° was estimated from the contour lines of the map by taking points at 10° intervals of longitude. The arithmetic mean of the six values along the meridian x was taken to represent $h(x)$. This function is represented graphically in Fig. 3. With $s = 2.5$ and $\varkappa = 0.4$ the corresponding stationary profile of the 500 mb surface was computed from (25) by approximating the integral by a sum in steps of 10° of longitude. The result is represented by the dotted curve in Fig. 4. The agreement with the solid curve, representing the normal profile, is seen to be poor, and we must again re-examine the assumptions.

We notice that equation (23) gives resonance with infinite amplitudes for the solution if s is an integer. This is a property of linear mechanical systems with no energy dissipation. In general one can say that such systems are extremely sensitive to small continuously acting forces. To remove this artificial sensitivity it is natural to consider the effect of friction.

VI. The effect of surface friction

With $U = 0.29$ radians per day, it will take an air particle three weeks to travel all around the latitude circle. It seems likely that frictional damping will be significant after such a long time, so that the greater part of the disturbance given to the air by a certain mountain will probably have disappeared when the air returns three weeks later to the same mountain.

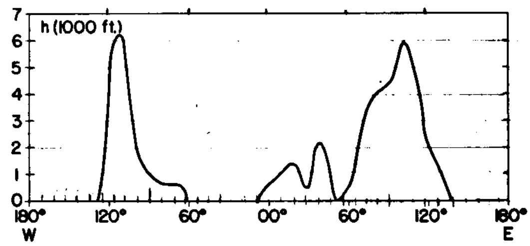

Fig. 3. Topographical profile at 45° N. The elevations represent the mean height of the earth's surface, averaged for the interval 40° N to 50° N, as a function of longitude.

It is therefore reasonable to assume that we have to take friction into account in order to explain the stationary perturbations.

One way of doing this is the following: From the theory of the motion in the friction layer near the ground, viz., the theory of the EKMAN spiral, one can compute the total air transport in the friction layer. The formulas for the transport are given by BRUNT (1941). The total mass transport along the isobars is, in the first approximation, proportional to the geostrophic transport, and will therefore be approximately non-divergent. The mass transport across the isobars towards lower pressure may be written

$$\mathbf{D} = \frac{H}{f} F \mathbf{k} \times \varrho_0 \mathbf{v}_{g_0}, \tag{28}$$

where ϱ_0 and $\mathbf{v}_{g_0}$ are the density and geo-

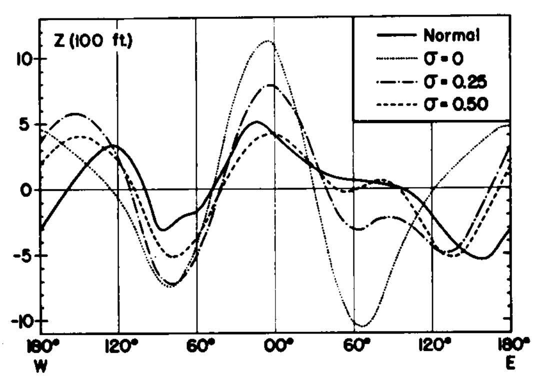

Fig. 4. Normal height profile of the 500 mb surface at 45° N for January together with computed stationary profiles for $\sigma = 0$ (no friction), for $\sigma = 0.25$ (moderate friction) and for $\sigma = 0.50$ (strong friction). For purposes of comparison the heights are represented as deviations from their respective means.

strophic wind in the friction layer, **k** is a unit vertical vector, and F is given by

$$F = \frac{\sin\, 2\alpha}{\sqrt{2}} \frac{\sqrt{Kf}}{H} \tag{29}$$

Here α is the angle between the isobars and the surface wind, and K is the "eddy-diffusivity". The transport across the isobars means that the horizontal pressure forces will do work on the air in the friction layer which will compensate for the dissipation of kinetic energy in this layer by friction. The flow across the isobars in the frictional layer will produce compensating currents across the isobars in the air above the friction layer, which in the present case will result in negative work being done by the horizontal pressure forces and a corresponding decrease in the kinetic energy. By this mechanism, the dissipating effect of surface friction can be transferred to the air above the friction layer, even though the frictional stresses in that part of the atmosphere are negligible.

As an illustrative example, we consider a barotropic linear current u along the x-axis with constant velocity above the friction layer. In the friction layer there is then a mass transport $D = \frac{H}{f} F \varrho_0 u$, in the y-direction. Assuming that the current has a finite lateral extent, this means an accumulation of mass to the left of the current and a depletion of mass to the right, so that the horizontal pressure gradient will decrease at all levels. In response to this decrease, the air above the friction layer will begin to travel across the isobars towards higher pressure. This cross-isobar current will increase in strength until it approximately compensates the mass-transport towards lower pressure in the friction layer; then the pressure field will become nearly stationary. This balance is expressed by $\int_{z_1}^{\infty} \varrho v dz = - D = - \frac{H}{f} F \varrho_0 u$, where the integral is taken upward from the top, z_1, of the friction layer. Assuming the cross-isobar component v above the friction layer to be constant, we find $v = - \frac{F}{f}\, u$. Since friction is supposed to be negligible above the friction layer, the x-component of the equation of motion is $\frac{du}{dt} = fv$. Hence we obtain

$$\frac{du}{dt} = - Fu \tag{30}$$

This equation implies an exponential decrease with time of the current velocity, the logarithmic decrement being F. Adopting the values used by BRUNT, viz., $\alpha = 22.5°$, $K = 10$ $m^2\, sec^{-1}$, and putting $f = 10^{-4}\, sec^{-1}$, $H = 8{,}000$ m, we obtain $F = 2.0 \times 10^{-6}\ sec^{-1} = 0.17$ day^{-1}. This means that the velocity decreases to $1/e$ times its original value in six days. It should be noted however that the decrease will be slower in a baroclinic current, where the current velocity increases with height. In this case, we obtain, instead of (30), $\frac{d\bar{u}}{dt} = - Fu_0$, where the bar means mean value with respect to pressure, and u_0 is the velocity at the top of the friction layer. For instance, assuming $\frac{d\bar{u}}{dt} = 2 \frac{du_0}{dt}$, one obtains a logarithmic decrement of 0.085 day^{-1}.

Similar effects of friction are dealt with by A. EINSTEIN in a suggestive essay in his book, *Mein Weltbild.* He points out that friction is responsible for the "meridional" circulation observed in a cup of tea when the contents are set in rotation. He also shows that in a curved river, the same frictional effect will cause a lateral circulation which is important for the explanation of the meandering of the river.

Since there cannot be any significant accumulation of mass in the friction layer, we have, with good approximation, $\operatorname{div} \mathbf{D} + \varrho_0 w_0 = 0$, where ϱ_0 and w_0 are the density and vertical velocity at the top of the friction layer (1,000 m above the ground, say). Hence, from (28),

$$w_0 = - \frac{1}{\varrho_0} \operatorname{div} \mathbf{D} \approx \frac{H}{f} F \zeta_{g_0} \tag{31}$$

Thus, ground friction will cause ascending motion in regions of cyclonic vorticity near the ground and descending motion in regions of anticyclonic vorticity. For instance, putting $H = 8{,}000$ m, $F = 2.0 \cdot 10^{-6}\ sec^{-1}$, and $\zeta_{g_0} = f$, which means an intense cyclonic vortex, we obtain $w_0 = 1.6$ cm sec^{-1}. Surface friction may therefore be an effect of some importance for the maintenance of the typical vertical circulations in cyclones and anticyclones.

The vertical circulation caused by friction will give rise to a damping of the perturbations in the free atmosphere in systems with little change of phase with height. For instance, in a region with relative cyclonic vorticity there will be horizontal convergence in the friction layer, a corresponding ascending motion at the top of the friction layer, and a nearly compensating divergence above the friction layer. This divergence gives a decrease of the cyclonic vorticity.[1] This effect can easily be incorporated into the equivalent-barotropic model when we assume that this model applies to the air above the friction layer. For the vertical velocity w_0 in (6) we substitute the value (31). In addition to the vertical velocity due to friction, however, we have the vertical velocity due to topography, so that the equation of the equivalent barotropic model becomes

$$\frac{\partial \zeta_g}{\partial t} + \overline{A^2}\left(u_g \frac{\partial \zeta_g}{\partial x} + v_g \frac{\partial \zeta_g}{\partial y}\right) + \beta v_g =$$

$$= \frac{f}{H}\frac{\partial z_0}{\partial t} - F\zeta_{g_0} - \frac{f}{H}\left(u_{g_0}\frac{\partial h}{\partial x} + v_{g_0}\frac{\partial h}{\partial y}\right) \quad (32)$$

This equation was derived under the assumption that the shape of the streamlines does not change with height, and that the horizontal velocity is of the form $\mathbf{v}(x, y)\, A(p)$. This assumption implies that we may write the velocity and vorticity near the ground as a certain fraction $\varkappa$ of the velocity and vorticity at the equivalent-barotropic level, i.e.,

$$u_{g_0} = \varkappa u, \; v_{g_0} = \varkappa v, \; \zeta_{g_0} = \varkappa \zeta \quad (33)$$

Hence (32) becomes

$$\left(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y} + \varkappa F\right)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) +$$

$$+ \beta \frac{\partial z}{\partial x} - \lambda^2 \frac{\partial z}{\partial t} = -\varkappa\lambda^2\left(u_g \frac{\partial h}{\partial x} + v_g \frac{\partial h}{\partial y}\right) \quad (34)$$

[1] There is no corresponding increase in vorticity in the friction layer, because there the frictional forces will cause a decrease in vorticity which will approximately compensate for the increase due to convergence.

VII. Explanation of the forced stationary perturbations

We now return to the stationary mountain perturbations. By using (34) in linearized form, we are able to take friction into account in the treatment of the motion. In deriving the perturbation equation from (34), the basic current is supposed to satisfy the corresponding homogeneous equation ($h = 0$). It should be noticed that this will not be true if the basic current is assumed to be of limited lateral extent; then the friction term will not vanish, and the basic current velocity will decrease exponentially with time. Since such a damping is not observed, one has to assume that the frictional dissipation is compensated by some mechanism not implied in (34) (e.g., probably by a slow thermally produced meridional circulation), so that as a result the westerlies are maintained. This reasoning suggests that we may neglect friction in (34) when dealing with the basic current, and consider only frictional effects due to the perturbations. In any case, this problem is formally avoided by assuming the basic current to be constant. Doing so, and assuming the perturbation to have a sine dependency on the y coordinate $\left(\frac{\partial^2 z}{\partial y^2} = -m^2 z\right)$, we arrive at the following perturbation equation for stationary flow,

$$\frac{\partial^3 z}{\partial x^3} + s^2 \frac{\partial z}{\partial x} + \sigma\left(\frac{\partial^2 z}{\partial x^2} - m^2 z\right) = -\varkappa\lambda^2 \frac{\partial h}{\partial x} \quad (35)$$

where, as in section IV, we set $\overline{A^2} = 1$. The following abbreviations are used:

$$s^2 = \frac{\beta}{U} - m^2, \; \sigma = \frac{\varkappa F}{U} \quad (36)$$

The solution is unique when it is demanded that $z(x)$ and its first and second derivatives be continuous and have the period 2π. It may be written

$$z(x) = \varkappa\lambda^2 \int_0^{2\pi} h(\alpha)\Phi_\sigma(x-\alpha)\,d\alpha \quad (37)$$

where the GREEN's function $\Phi_\sigma(x)$ is given by

$$\Phi_\sigma(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} \frac{e^{inx}}{n^2 - s^2 - i\sigma\left(n + \frac{m^2}{n}\right)} \quad (38)$$

To evaluate this series, we first determine its sum, Φ^0_σ for $m = 0$. We obtain

$$\Phi^0_\sigma(x) = -\frac{e^{-\frac{\sigma}{2}(x-\pi)}}{2s'M}\cos\left[s'(x-\pi)-\delta\right],$$

$$0 \leq x \leq 2\pi,$$

$$\Phi^0_\sigma(x+2\pi) = \Phi^0_\sigma(x),$$

where

$$s' = \sqrt{s^2 - \frac{\sigma^2}{4}},$$

$$M = \sqrt{\sinh^2\left(\frac{\sigma}{2}\pi\right)\cos^2(s'\pi) + \cosh^2\left(\frac{\sigma}{2}\pi\right)\sin^2(s'\pi)}$$

$$\sin\delta = \frac{1}{M}\sinh\left(\frac{\sigma}{2}\pi\right)\cos(s'\pi),$$

$$\cos\delta = \frac{1}{M}\cosh\left(\frac{\sigma}{2}\pi\right)\sin(s'\pi)$$

After having determined $\Phi^0_\sigma(x)$, $\Phi_\sigma(x)$ was calculated by taking 7 terms in the series for $\Phi_\sigma(x) - \Phi^0_\sigma(x)$, which converges rapidly. The calculation was performed for $s' = 2.5$ and for $\sigma = 0.25$ and 0.50, corresponding to values of K, the eddy diffusivity, in the observed range. The associated GREEN's functions are graphed in Fig. 2 together with Φ_0, the GREEN's function without friction ($\sigma = 0$). We remark that the curves can be interpreted as the stationary height profile caused by an infinitely narrow mountain range located at $x = 0$. With this interpretation we may conclude from the figure that a narrow mountain range will set up a sinusoidal wave with a sharp ridge at the mountain. The waves will be damped downstream in the case of friction and undamped in the case of no friction. Since a continent of arbitrary profile may be regarded as a sum of infinitesimal "point" mountains, we may interpret (37) as expressing the general stationary disturbance as the sum of the infinitesimal disturbances produced by these "point" mountains.

Using the GREEN's functions for $\sigma = 0.25$ and $\sigma = 0.50$, the stationary profile of the 500 mb surface corresponding to the mountain profile shown in Fig. 3 was computed from (37). The value 0.4 was chosen for the reduction factor $\varkappa$; this value seems reasonable from wind observations. The resulting stationary profiles are shown in Fig. 4. A glance at the figure shows that the agreement between the computed and observed profiles is very much better when friction is taken into account than when it is not. Indeed, the agreement in the former cases, especially for $\sigma = 0.50$, is as good as can be expected from a linear theory. It may therefore be considered established that the stationary perturbations in the westerlies are forced perturbations created by the forced ascent of the westerly current over the continents and modified by friction.

The value $\sigma = 0.25$ corresponds to BRUNT's value $K = 10$ m^2 sec^{-1}, whereas $\sigma = 0.50$ corresponds to $K = 40$ m^2 sec^{-1}. It is known, however, that the probable error in the determinations of K is so high, and the geographical variation of K so great, that correspondence of K with observed values in order of magnitude is all that can be demanded from the theory.

VIII. The revised forecast method

Having arrived at an explanation for the persistence of the very long-wave components in the observed flow in terms of topography and friction, we proceed to incorporate these factors into the forecast equation. Accordingly we replace equation (6) by (32) or (34).

Again considering small perturbations on a zonal current U, (34) can be linearized to

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + \varkappa F\right)\left(\frac{\partial^2 z}{\partial x^2} - m^2 z\right) + + \beta\frac{\partial z}{\partial x} - \lambda^2\frac{\partial z}{\partial t} = -\varkappa\lambda^2 U\frac{\partial h}{\partial x} \quad (39)$$

where, as in the case of the stationary perturbations, we have assumed a sine variation of the perturbation with y given by the relation (22).

In the following, the term "forced perturbation" is used to denote any solution of (39) and the term "free perturbation" is reserved for a solution of the corresponding homogeneous equation ($h \equiv 0$). The most general motion satisfying (39) can be regarded as the sum of a forced stationary perturbation and a free moving perturbation. Having discussed the forced perturbation in section VII, we now turn to a consideration of the free perturbation.

If we consider an elementary wave solution of the homogeneous equation of the form $z(x, t) = Ae^{i(nx-\nu t)}$, we find

$$i\nu = \varkappa F \frac{n^2 + m^2}{n^2 + m^2 + \lambda^2} + inU \frac{n^2 - s^2}{n^2 + m^2 + \lambda^2}$$

where, as before, s is the stationary wave number (24). Thus, a consequence of (39) is that the traveling free perturbations are damped, with a logarithmic decrement approximately $\varkappa F$. Using the values of $\varkappa$ employed in the treatment of the stationary perturbation, $\varkappa F$ is found to be of the order 0.1 day^{-1}. This shows that after a few weeks the traveling perturbations would be damped out and only the stationary perturbations would remain.

This damping is, of course, not in agreement with the observations. The main reason is probably that the traveling perturbations are formed as unstable waves, which are able to grow in amplitude at the expense of the energy of the basic current.[1]

As we are here concerned only with 24 hour changes in the motion, the effect of friction will in any case be slight and may be ignored. The case is different for the permanent motions; there the entire history of an air parcel in its travel around the earth must be considered, and the effect of friction is decisive. We will therefore consider the solution of (39) with $F = 0$.

We have earlier considered the general solution as the sum of a free perturbation and a stationary forced perturbation, which together satisfy the observed initial conditions. One may also regard the solution as made up of a free perturbation, which alone satisfies the observed initial conditions, and the forced perturbation produced by the topographical forcing of an initially undisturbed zonal current. The latter point of view is preferable in principle for the motion is then treated purely as an initial value problem, or as RICHARDSON called it, a "marching problem". This method makes it unnecessary to determine the stationary perturbation, which is a problem "in the large", and enables us to look at the forecasting problem as a "local" problem, the solution of which depends only on what is happening within a limited region.

Adopting the latter point of view we obtain the solution of (39) (with $F = 0$) by putting

$$z(x, t) = \sum_{n=-\infty}^{+\infty} c_n(t) e^{inx}, \quad h(x) = \sum_{n=-\infty}^{+\infty} a_n e^{inx}$$

Substituting these expressions into the differential equation, we obtain a differential equation of the first order for $c_n(t)$. Solving this, and adjusting the constants so as to satisfy the initial condition, we finally get

$$\begin{aligned} z(x + Ut, t) &= z(x, 0) + \\ &+ \int_0^{2\pi} z(\alpha, 0)\, I_{a^2}(x - \alpha, t)\, d\alpha + \\ &+ \varkappa \lambda^2 \int_0^{2\pi} h(\alpha)\, J(x - \alpha, t)\, d\alpha, \end{aligned} \tag{40}$$

where

$$I_{a^2}(x, t) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \left(e^{\frac{inbt}{n^2 + a^2}} - 1 \right) e^{inx}, \tag{41}$$

$$J(x, t) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \left(e^{inUt} - e^{\frac{inbt}{n^2 + a^2}} \right) \frac{e^{inx}}{n^2 - s^2} \tag{42}$$

and

$$a^2 = m^2 + \lambda^2 = \frac{b}{U} - s^2, \; b = \beta + \lambda^2 U,$$
$$s^2 = \frac{\beta}{U} - m^2 \tag{43}$$

Let us consider the first integral in (40). This is the solution of the homogeneous equation satisfying the initial condition. The influence function $I_{a^2}(x, t)$ is essentially the same as dealt with in section IV, the only difference being that λ^2 is replaced by a^2, since

[1] The instability may be attributed to the horizontal shear of the current (H. L. KUO, University of Chicago, in a paper to be published in the Journal of Meteorology) or to its baroclinicity (J. G. CHARNEY, 1947). In both cases the very long wave forced perturbations are stable and are therefore subject to the full damping effect of friction.

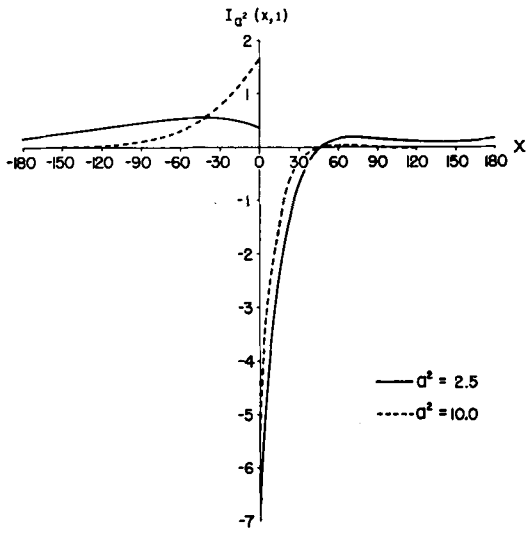

Fig. 5. The influence function $I_{a^2}(x, 1)$ for $a^2 = 2.5$ and $a^2 = 10$. This function represents the 24 hour height change of the 500 mb surface, relative to a coordinate system x traveling with the mean current, produced by a concentrated initial disturbance at $x = 0$.

we now consider perturbations with finite lateral extent. In the study of the stationary perturbations we found that m^2 should be about 15; consequently, we should expect $a^2 = 18$ to be a proper value. The influence function was computed in the same way as shown in section IV, for a time interval of one day ($t = 1$), and for $a^2 = 6$, 10, 14, and 18. The results are given in table I, and, in the case of $a^2 = 10$, also in Fig. 5. There turned out to be very little difference in the forecasts made with the influence function for $a^2 = 10$, $a^2 = 14$, and $a^2 = 18$. The value $a^2 = 10$ used in the following was based on a previous estimate of m which differs somewhat from the value given in section V. Because of the relative insensitivity of the forecast to changes in m it was not found necessary to revise the results.

It will be seen that the influence function for one day is almost zero in the greater part of the region ($-180°$ to $180°$), and has an appreciable value only in the vicinity of the origin. It is therefore not necessary in (40) to integrate all around the latitude circle; it is sufficient to take the integral over a certain influence region around the point x. This procedure is in agreement with group velocity considerations (CHARNEY, NWP) and means that it is not necessary to know the data for the entire world in order to make a 24 hour forecast in a small region. In this respect, it seems preferable to use the influence function with $a^2 = 18$ since then the influence region would be smaller.

The last term in the forecast formula (40) is a solution of the non-homogeneous equation and represents the disturbance produced by the continents after the time t in an initially undisturbed current. This part is therefore independent of the initial profile $z(x, 0)$ and can be determined once and for all as long as s and U remain the same.

In order to compute the GREEN's function $J(x, t)$[1], we write (42) in the form

$$J(x, t) = \frac{1}{2\pi}\left[\sum_{n=-\infty}^{+\infty} \frac{e^{in(x+Ut)}}{n^2 - s^2} - \sum_{n=-\infty}^{+\infty} \frac{e^{inx}}{n^2 - s^2} + \right.$$
$$\left. + \sum_{n=+\infty}^{+\infty} \left(1 - e^{\frac{inbt}{n^2+a^2}}\right) \frac{e^{inx}}{n^2 - s^2}\right].$$

The first two series can be summed exactly, and the last series converges so fast that summation from -20 to $+20$ gives sufficient accuracy. In this way, the function $J(x, 1)$ was computed for $a^2 = 10$ and $U = 0.35$ radians per day or 20° longitude per day. The result is shown in Fig. 6. The shape of the curve shows that there is a finite influence region for the mountain influence also, which means that the forecast is not influenced by mountains very far away. The width of the influence region is about the same as for the influence function $I_{10}(x, 1)$ for free perturbations. The function $J(x, 1)$ may be interpreted as the disturbance produced in an initially undisturbed current after 24 hours by an infinitely narrow mountain range situated at $x = -20°$. The reason for the two discontinuities in the curve is that the air between the discontinuities has passed over the mountain during the period,

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[1] The method given for calculating $J(x, t)$ fails when s is an integer, corresponding to resonance for the stationary perturbations. However, $J(x, t)$ is finite in this case too and can be determined by a different method. That $J(x, t)$ must be finite is obvious from energy considerations.

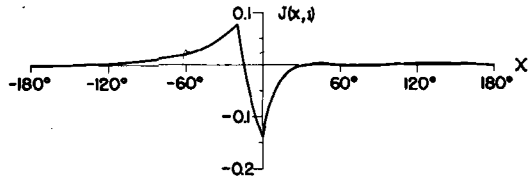

Fig. 6. The GREEN's function $J(x, 1)$ for the 24 hour height change of the 500 mb surface at latitude 45° produced by the forced motion of the current over an irregular surface.

whereas the air outside the discontinuities has not.

The corresponding 24 hour disturbance produced by the actual mountain profile is represented by the last term of (40). This term was computed with $\varkappa = 0.4$ for the mountain profile $h(x)$ shown in Fig. 3. The result is shown in Fig. 7.

It is interesting to note that the Rocky Mountains act approximately as an infinitely narrow mountain range. The computation shows that the topographical influence on a 24 hour change in the pressure field is small. In fact, in applying (40) to actual weather situations, it was found not worth while to take the topographical term into account for a 24 hour forecast, even though it would have improved the forecast slightly. It should be remembered, however, that the effect of topography will be larger for a longer forecast period.

Without the effect of topography, and with a limited influence region, the prognostic formula (40) becomes

$$z(x + Ut, t) = z(x, 0) + \int_{x-x_1}^{x+x_2} z(\alpha, 0)\, I_{a^2}\, (x - \alpha, t)\, d\alpha. \qquad (44)$$

This formula was tried out on some actual situations taken from the U. S. Air Weather Service map series *Historical Weather Maps* for January, 1946. The function $z(x, t)$ was identified with the height of the 500 mb surface at 45° latitude. The zonal wind U was taken to be 20° per day, and the value $a^2 = 10$ was used in the influence function. The integral was taken over an influence region of 120° longitude ($x_1 = 30°$, $x_2 = 90°$) to insure accuracy, although in practice it would be possible to reduce the total interval to 90° or less. It was evaluated by means of SIMPSON's formula using steps of 10° of longitude. The formula actually used was

$$z(x + 20°, 1) = z(x, 0) + \sum_{N=-3}^{+9} A_N\, z(x + N \cdot 10°, 0). \qquad (45)$$

The values of the coefficients A_N are given in table II for $a^2 = 10$, 14 and 18. In this way, a forecast for a region of 90°, say, can easily be computed in a half hour. Since the influence function has the mean value zero, we may subtract any constant from the height z in computing the sum. In order to prevent having too large a value for z, this quantity was reckoned from the level 18,000 feet.

IX. Discussion of results

The results of 6 forecasts are presented in Figs. 8—13. The upper curve in each diagram shows the initial profile of the 500 mb surface along the 45° latitude circle. This curve was obtained from the 500 mb maps by reading off the heights at every 10° of longitude to within 10 feet. The two curves in the lower part of each diagram show the computed (dashed curve) and observed (solid curve) local changes in the following 24 hours.

In judging the results, it should be borne in mind that, owing to the scarcity of aerological data, the 500 mb maps involve errors which often may be of the order of 200 feet. Since the data on which the analysis is based are most extensive in the region 130° W—80° E, the analysis will presumably be most reliable in this region. This is also the region where the method gave the best results on the average.

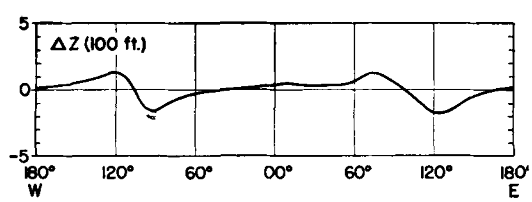

Fig. 7. The 24 hour height change of the 500 mb surface at 45° N, produced by the motion of an initially straight zonal current over the continents.

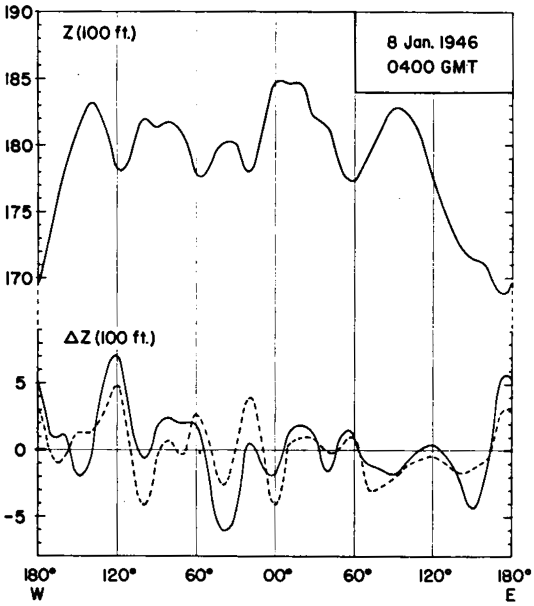

Fig. 8.

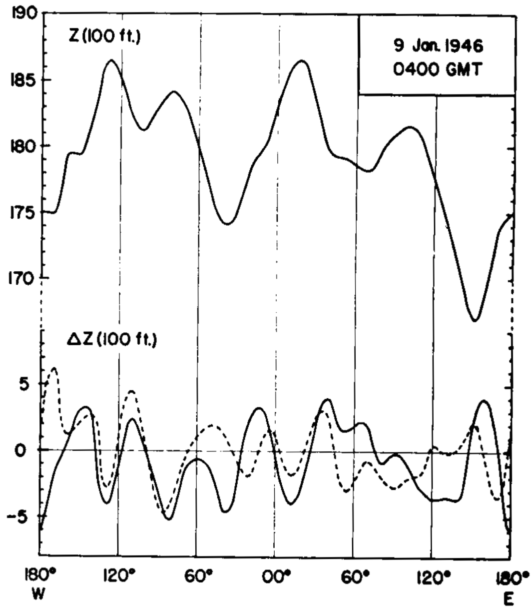

Fig. 9.

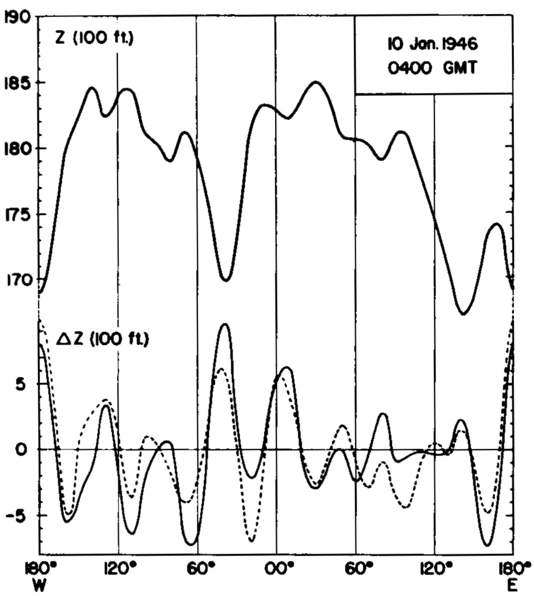

Fig. 10.

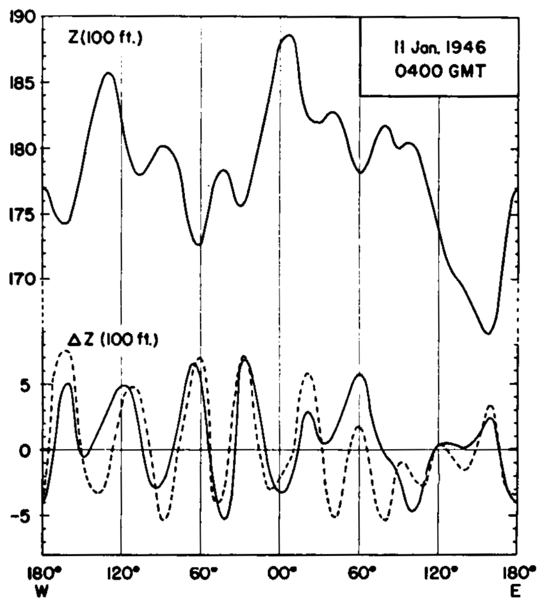

Fig. 11.

Figs. 8—13. In each figure, the upper curve represents the observed height profile of the 500 mb surface at 45° N for the date indicated in the upper right hand corner. The lower continuous curve gives the observed change in the following 24 hours, and the dashed curve the predicted change.

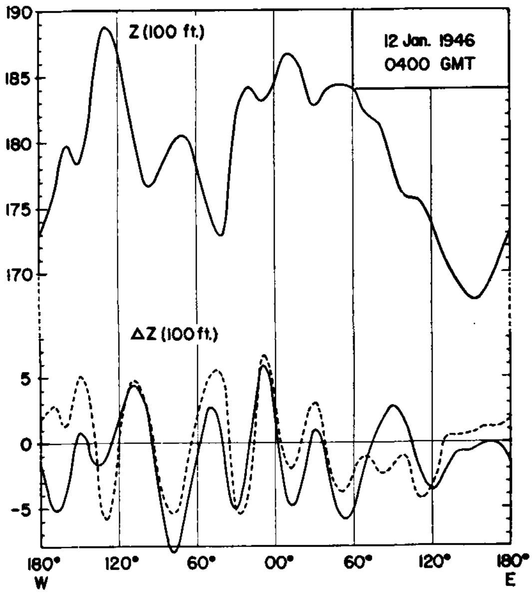

Fig. 12.

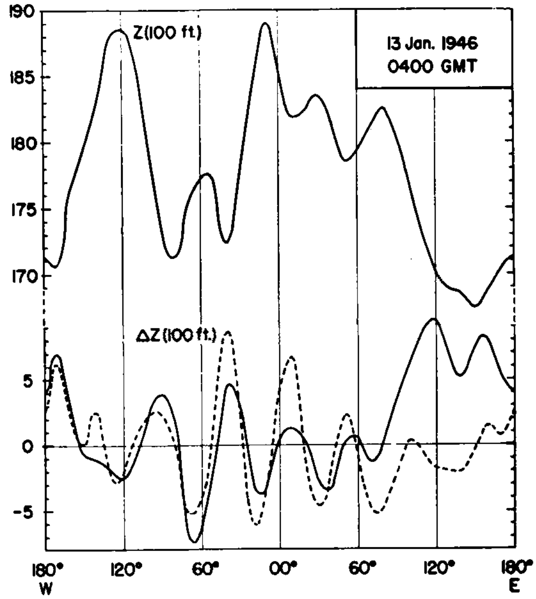

Fig. 13.

The diagram for January 13 shows an obvious example of a disagreement between the computed and observed changes due to erroneous analysis. The map for January 14 shows a tremendous filling of the entire Pacific trough, which is probably fictitious since there is very little data to substantiate it; moreover, it deepened again the next day. The large "observed" pressure rise in the region 90° E to 180° E, which is shown in the diagram for January 13, is therefore entirely wrong; in this case, the computed forecast was probably much better than the analysis for January 14.

Besides the errors due to erroneous analysis, we have to consider errors due to possible non-fulfillment of the assumptions underlying the forecast. The forecast equation was derived for small perturbations on a zonal current, and one cannot expect good results when the amplitude grows too large. Furthermore, the perturbations were assumed to vary in a certain regular manner in the y-direction; this assumption made it possible to represent the two-dimensional pressure field by a profile along one latitude circle and thereby to treat the problem as a one-dimensional one. When the actual perturbations do not show such a regular variation in the y-direction, there is always the possibility of an advection of vorticity from the north or south which cannot be predicted from the shape of the profile at one latitude. The strong deepening that took place in the Atlantic (40° W) from January 9 to 10, and which the method failed to predict, may be due, at least partly, to the advection of strong cyclonic vorticity from the north.

Errors may also result from an improper choice of the current velocity U. Even though the derivation of the forecasting formula was made for a constant U along the latitude circle, it is possible to a certain extent to account for slow variations of U with longitude. This is so because the forecast is independent of the zonal current outside the influence region. The best way to choose U has to be determined more or less empirically. The authors' choice of a constant U equal to 20° per day for all six cases is of course a very rough estimate and could probably be improved.

The errors mentioned so far are due to the assumption that the motion can be dealt with in one dimension, using the linearized equation. These errors will therefore be eliminated when it is possible to integrate the non-linear equation for two horizontal dimensions with arbitrary initial conditions. (This will probably be possible in the near future, but the computation will be very elaborate and will require high-speed and high-capacity computing machines.) It seems very likely that a computation by means of the non-linear, two-dimensional equivalent-barotropic equation would improve the forecast considerably. Since some success is obtained by using the simplified one-dimensional linearized equation, we may be justified in concluding that the quasi-geostrophic approximation and the equivalent barotropic model are legitimate approximations that render sufficient accuracy to make them useful tools in numerical forecasting.

X. General remarks on the stationary perturbations

It has for some time been recognized that the quasi-stationary perturbations of the atmosphere are caused by geographically fixed perturbing forces, but the exact nature of these forces has not been well understood. ROSSBY (1939) was able to show that their effect would be to excite a system of free perturbations, whose dimensions are in qualitative accord with the observed large-scale stationary perturbations in middle latitudes. The establishment of the continents as the primary perturbing influence enables us to fill the blanks in the picture and to determine not only the mean wave lengths of the system but also the position of the individual troughs and wedges. These are not anchored by solenoid fields along coast lines, as has been sometimes thought, but are determined by a world-wide adjustment of the zonal flow to the shape of the continents.

The relationship between zonal index and mean circulation gives the possibility of explaining the general longitudinal variation of climate as a function of the zonal index for many surface climatic factors are correlated to the upper flow, which is in turn determined by the index. Thus, because of the horizontal barotropy of the large-scale stationary perturbations, we may say that the mean surface temperature is correlated to the mean 500 mb height — except in those cases where the surface temperature is not representative of the free air temperature. Similarly, since the mean position of the polar front parallels the upper current, cyclone tracks are also determined by a knowledge of the stationary circulation pattern.

In this way, we have established a physical relationship between the zonal index and the main weather type, a relationship which has hitherto been considered mainly from a statistical point of view. This result has far reaching consequences. If it should prove possible to forecast the variations of the zonal index, it would also be possible to make at least an elementary forecast of the change in world weather. The consequences of the relationship for longrange weather forecasting, as well as for the study of climatic changes, are obvious.

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