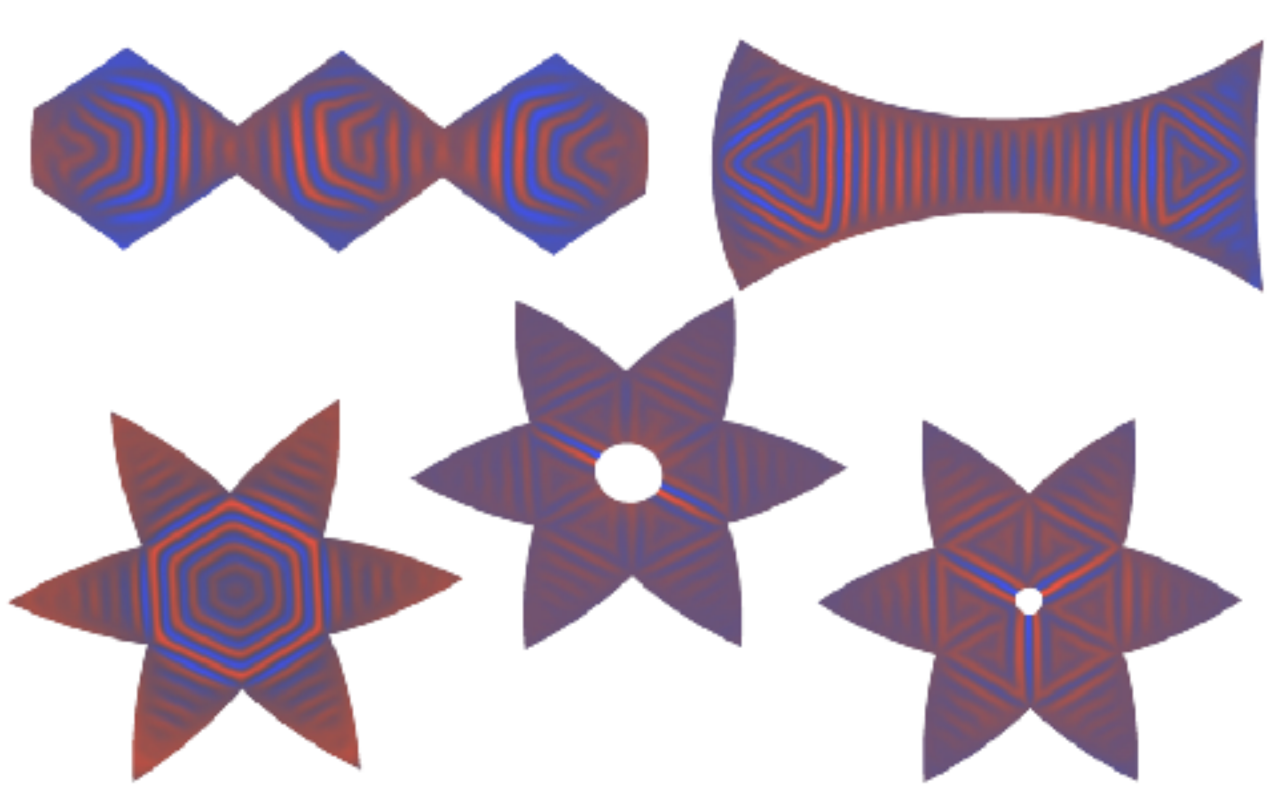
Elastic instabilities in floating shells

Eleni Katifori

Department of Physics and Astronomy, University of Pennsylvania



Acknowledgements

Collaborators

Desislava Todorova (UPenn)
Hillel Aharoni (UPenn)
Octavio Albarran (MPIDS)
Randy Kamien (UPenn)
Lucas Goehring (MPIDS)

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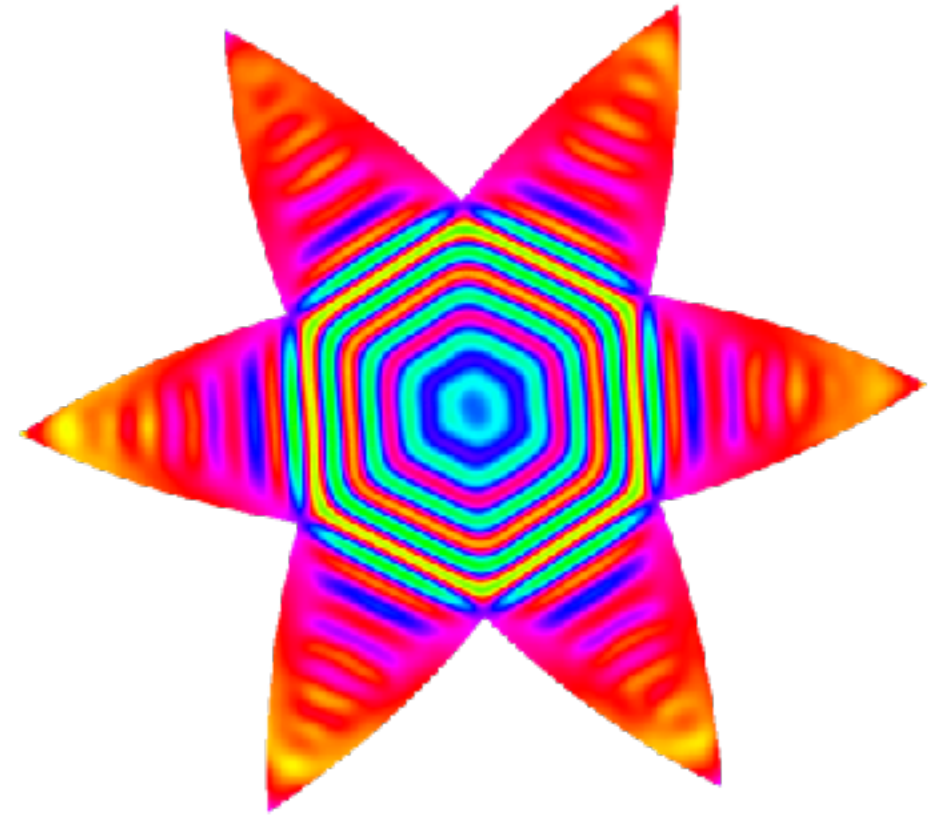
NSF

BURROUGHS WELLCOME FUND



Folding and deforming flat sheets

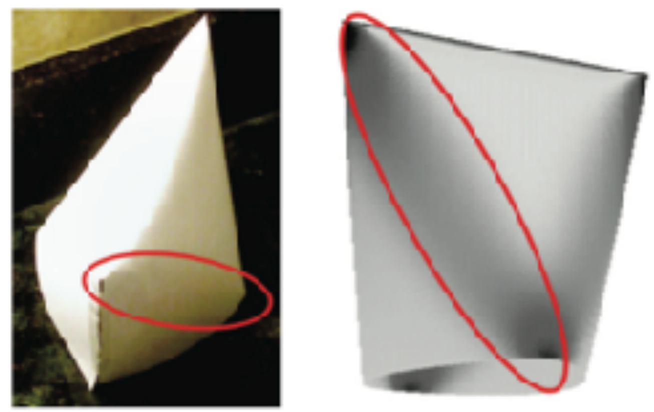
"Freestanding"

Origami

Curtain drapes

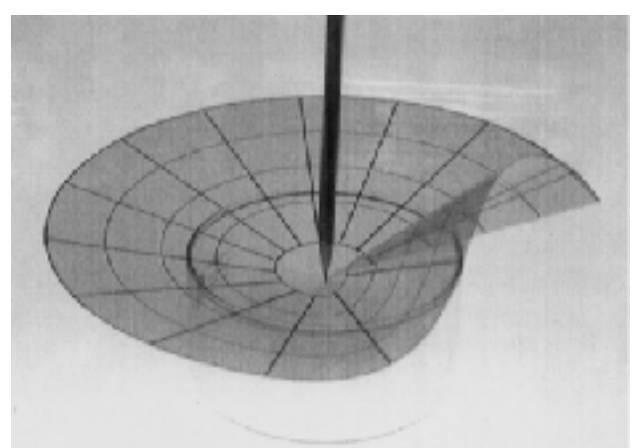
Das et al, PRL 98, 014301 (2007)

Ridges



WItten, RMP 79.2 (2007): 643

d-cones

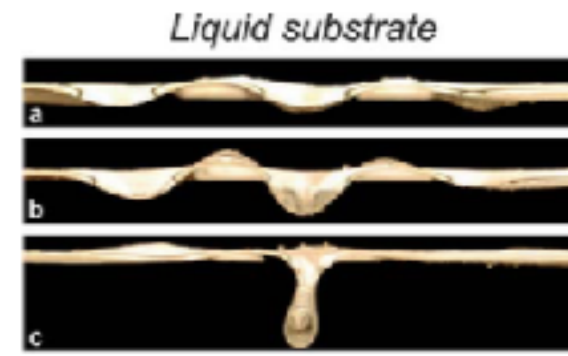


Cerda, et al, 1998, PRL. 80, 2358

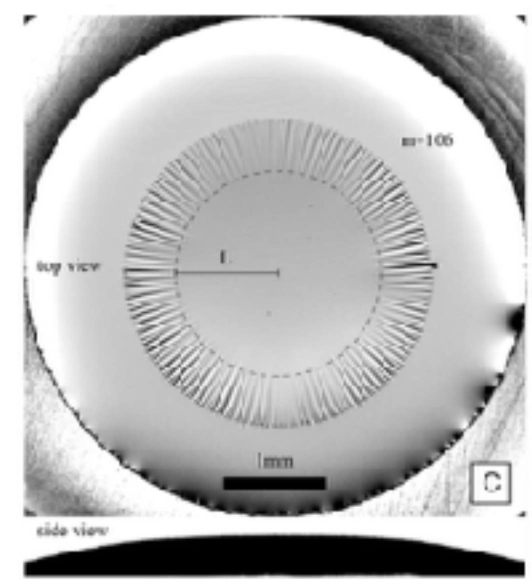
… and many more

On a substrate

Wrinkling!



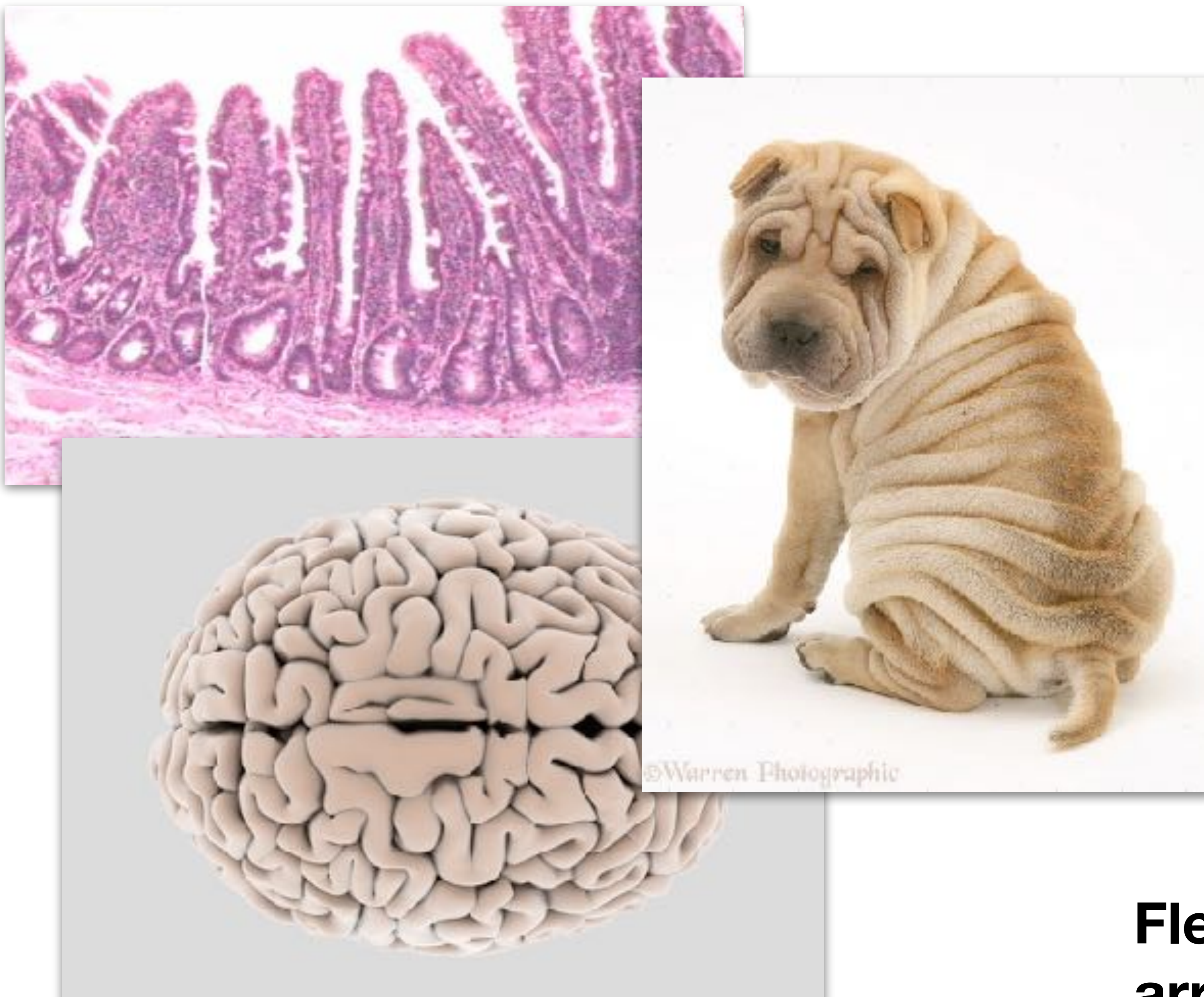
Pocivavsek et al, Science, (2008), 320, 912



King at al, PNAS 109 (2012): 9716

But why do we care about wrinkling?

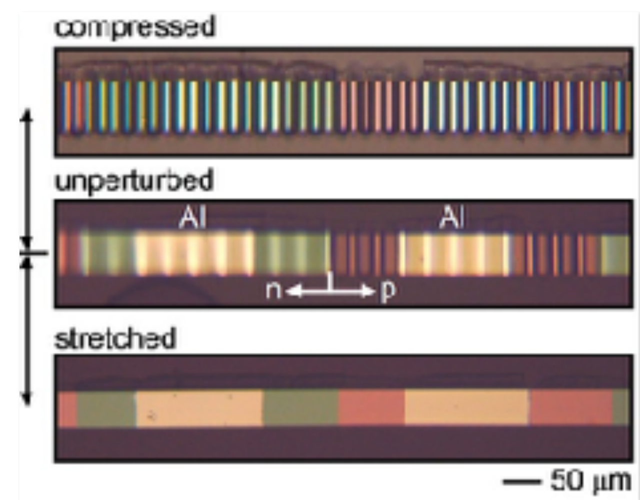
Pattern formation in nature and in the lab



A Stretchable Form of Single-Crystal Silicon for High-Performance Electronics on Rubber Substrates

Dahl-Young Khang,[1,3,4] Hanqing Jiang,[2] Young Huang,[2*] John A. Rogers[1,2,3,4*]

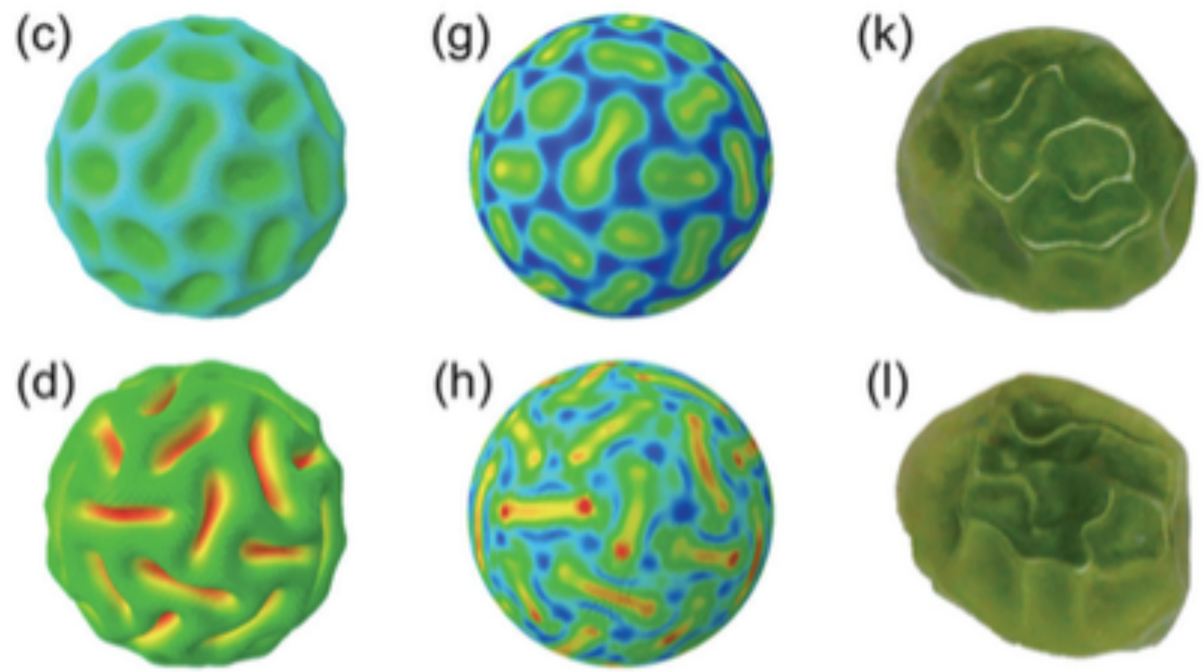
13 JANUARY 2006 VOL 311 SCIENCE



Flexible electronics, microlens arrays, micropatterning, etc

Folding and deforming curved shells

Shrinking of a soft sphere with a hard skin layer



Li et al, PRL 106, 234301 (2011)

Folding and deforming curved shells

THEORY OF ELASTICITY

L. D. LANDAU AND E. M. LIFSHITZ

§15. **Deformations of shells**

In discussing hitherto the deformations of thin plates, we have always assumed that the plate is flat in its undeformed state. However, deformations of plates which are curved in the undeformed state (called *shells*) have properties which are fundamentally different from those of the deformations of flat plates.

The stretching which accompanies the bending of a flat plate is a second-order effect in comparison with the bending deflection itself. This is seen, for example, from the fact that the strain tensor (14.1), which gives this stretching, is quadratic in ζ. The situation is entirely different in the deformation of shells: here the stretching is a first-order effect, and therefore is important even for small bending deflections. This property is most easily seen from a simple example, that of the uniform stretching of a spherical shell. If every point undergoes the same radial displacement ζ, the length

Interlude: The importance of geometry

Bending energy

$$E_{ben} \sim Eh^3 \int (c - c_0)^2 dA$$

Stretching energy

$$E_{str} \sim Eh \int \epsilon^2 dA$$

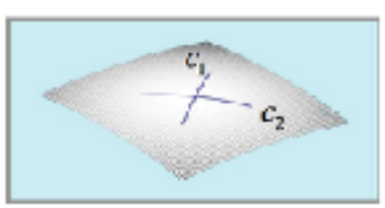
Thin shells

⇒ stretching becomes energetically unfavorable

⇒ deformation by pure bending modes

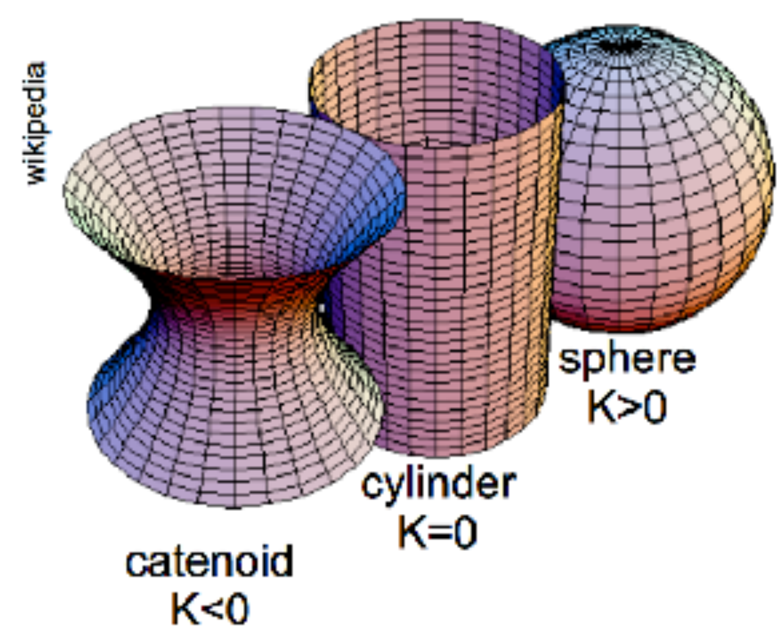
⇒ Geometry!

Isometric (inextensional) deformations



K=c1 c2

Conservation of gaussian curvature (Gauss theorem)



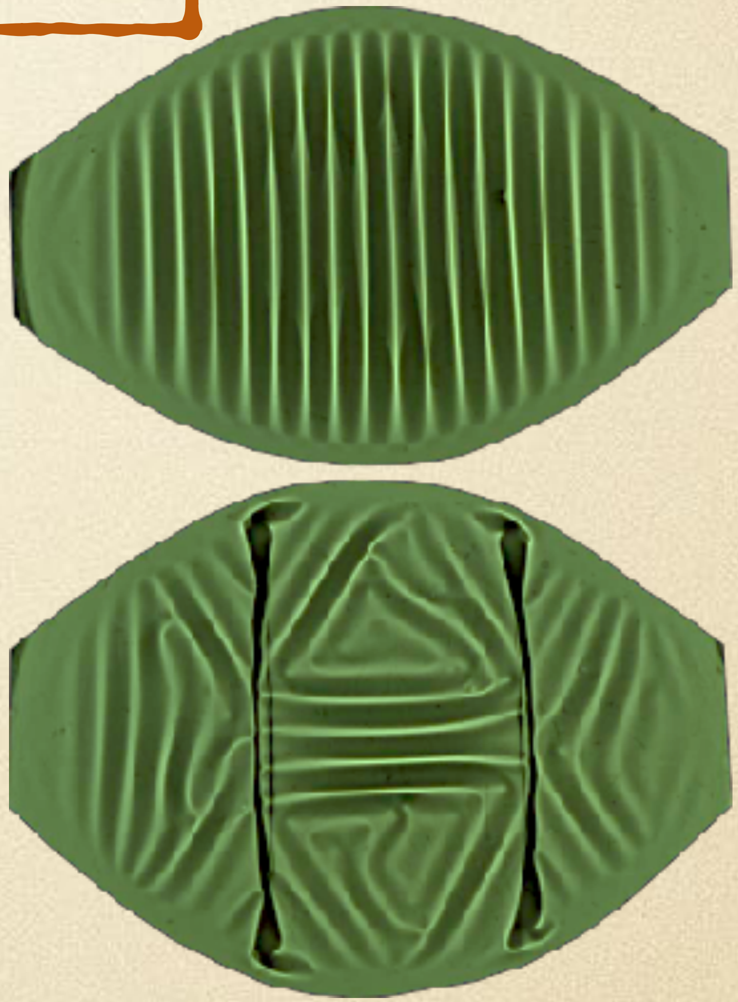
Outline

Pattern formation of curved shells supported by a liquid substrate

Wrinkling and the theory of smectics

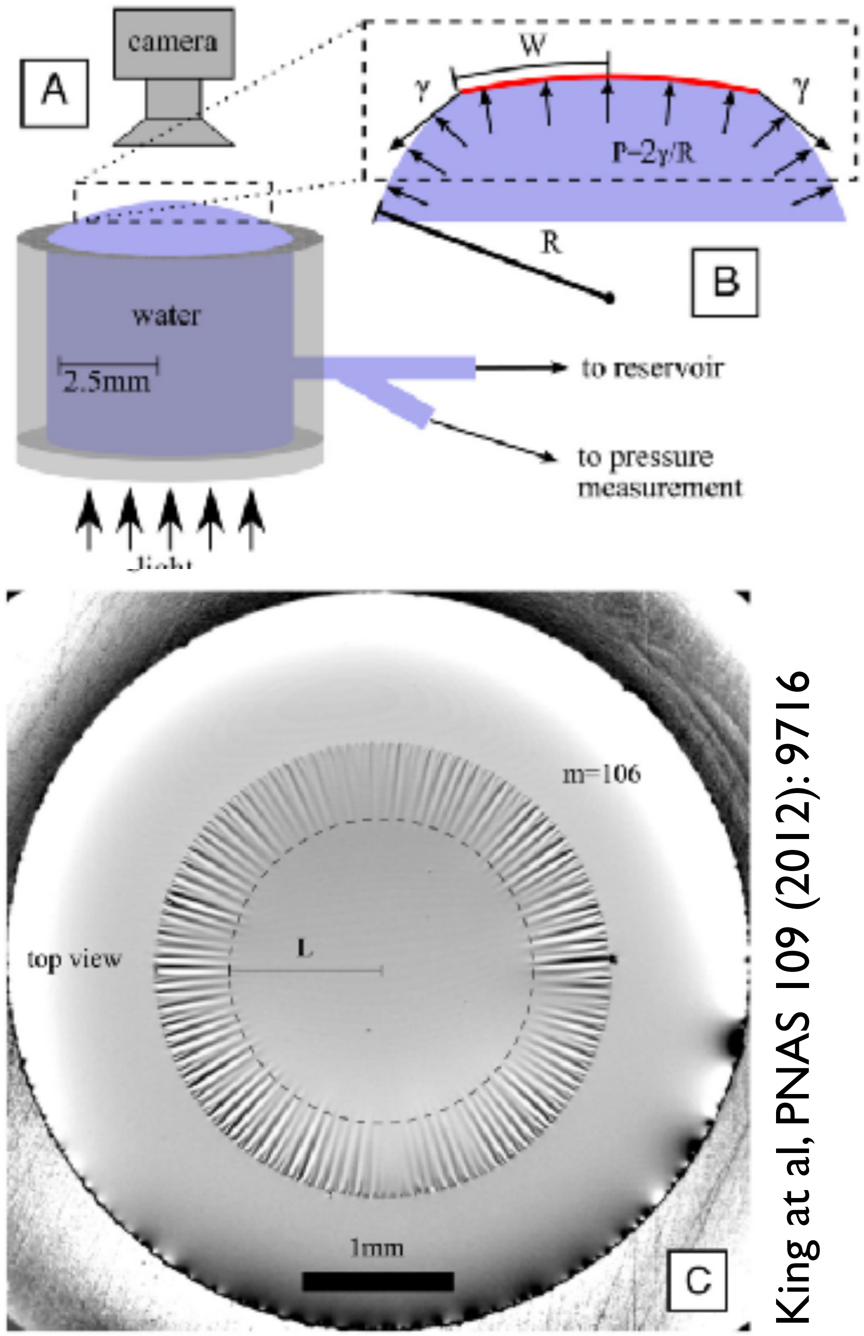
Dimples, folds and other instabilities

Engineering exotic shapes



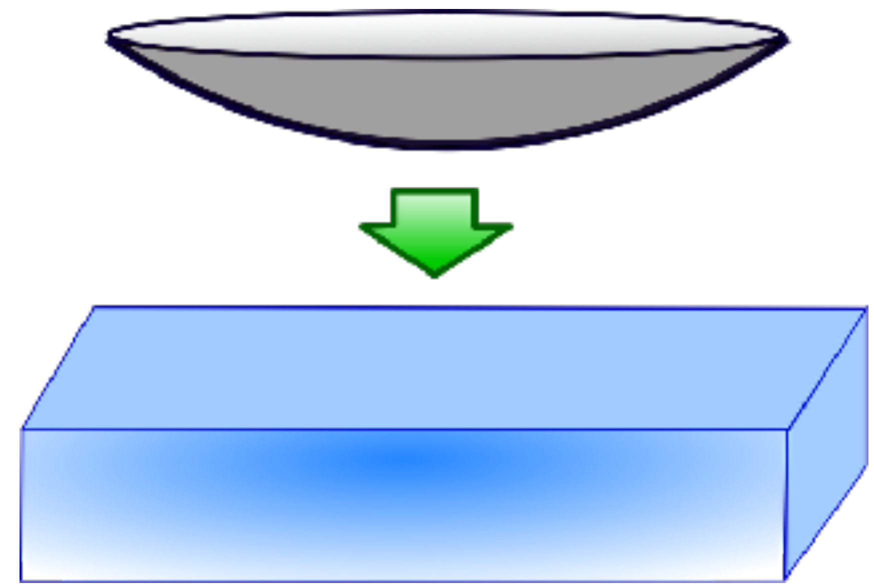
A simple problem…

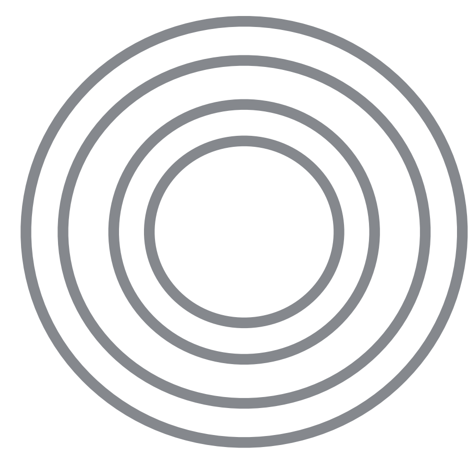
Flat sheet on *curved* substrate



King at al, PNAS 109 (2012): 9716

Curved sheet on *flat* substrate





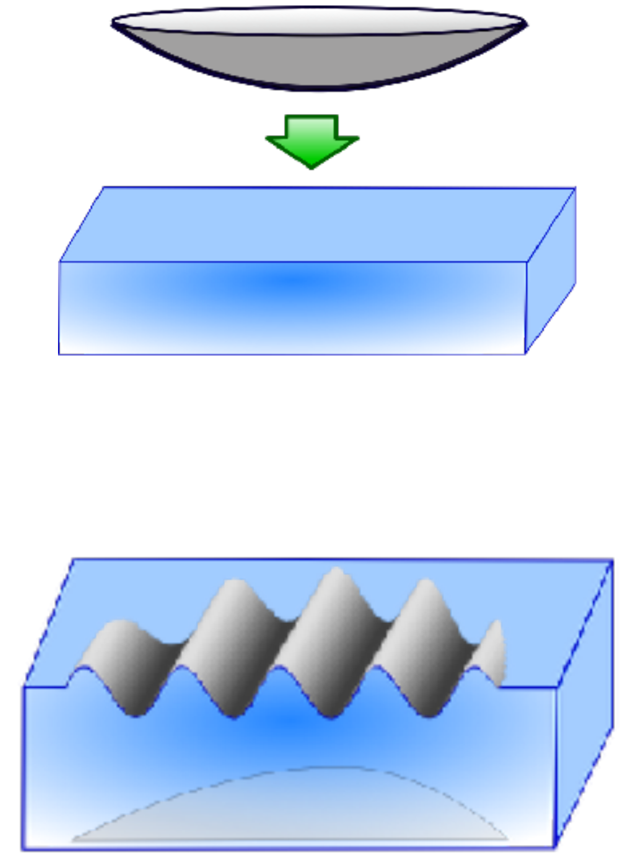
Excess area as effective confinement

Flat

Lateral compression of sheet

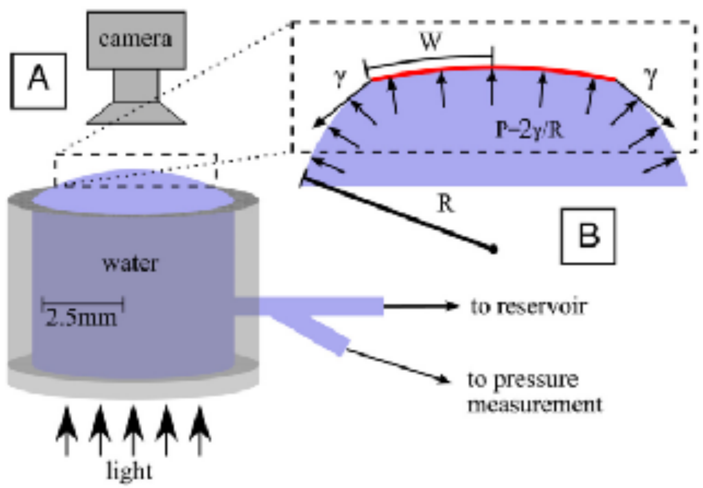
Curved

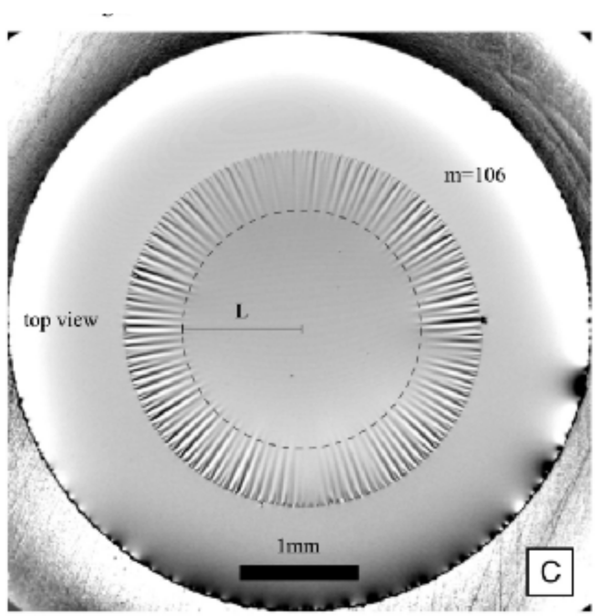
Effective lateral confinement to accommodate excess area



A simple problem turned not so simple

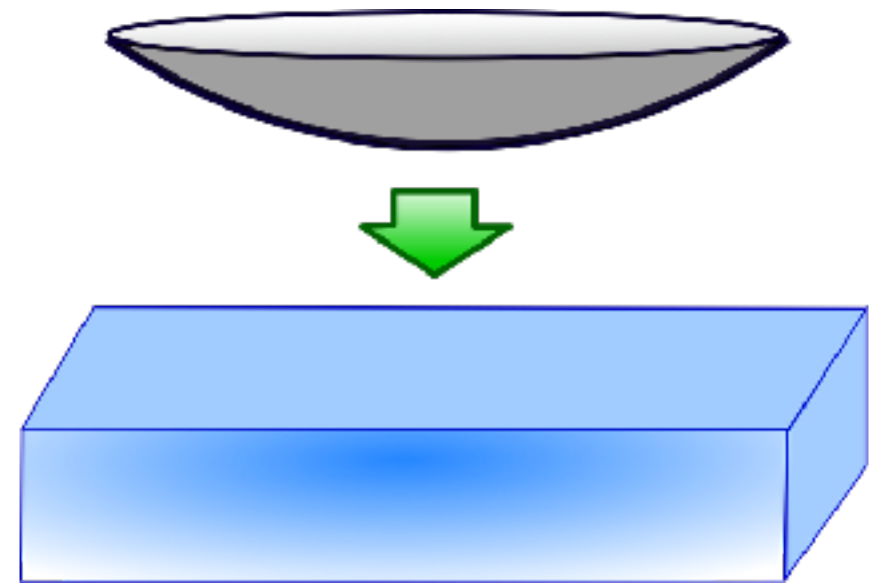
Flat sheet on *curved* substrate

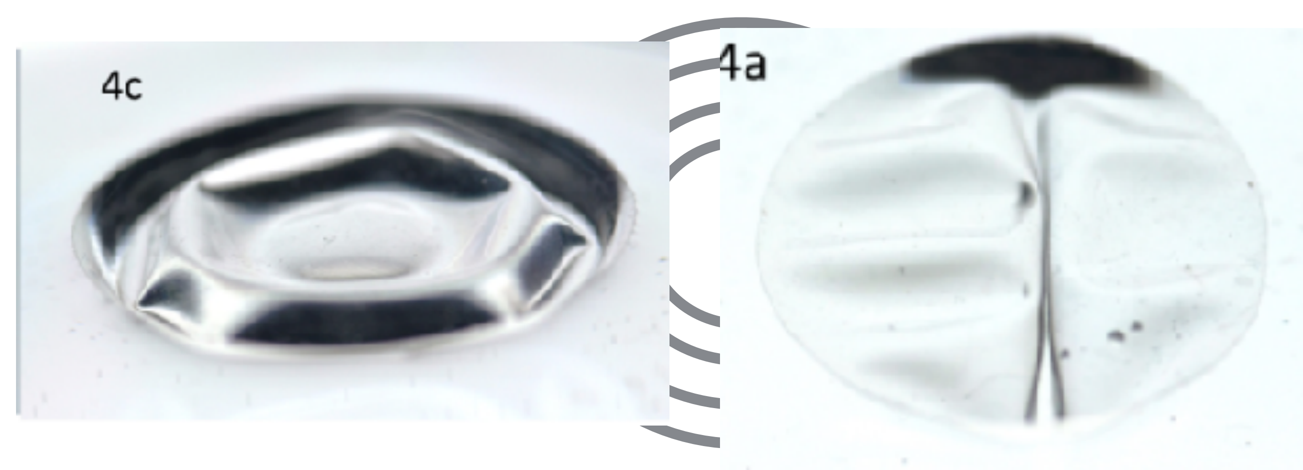




King at al, PNAS 109 (2012): 9716

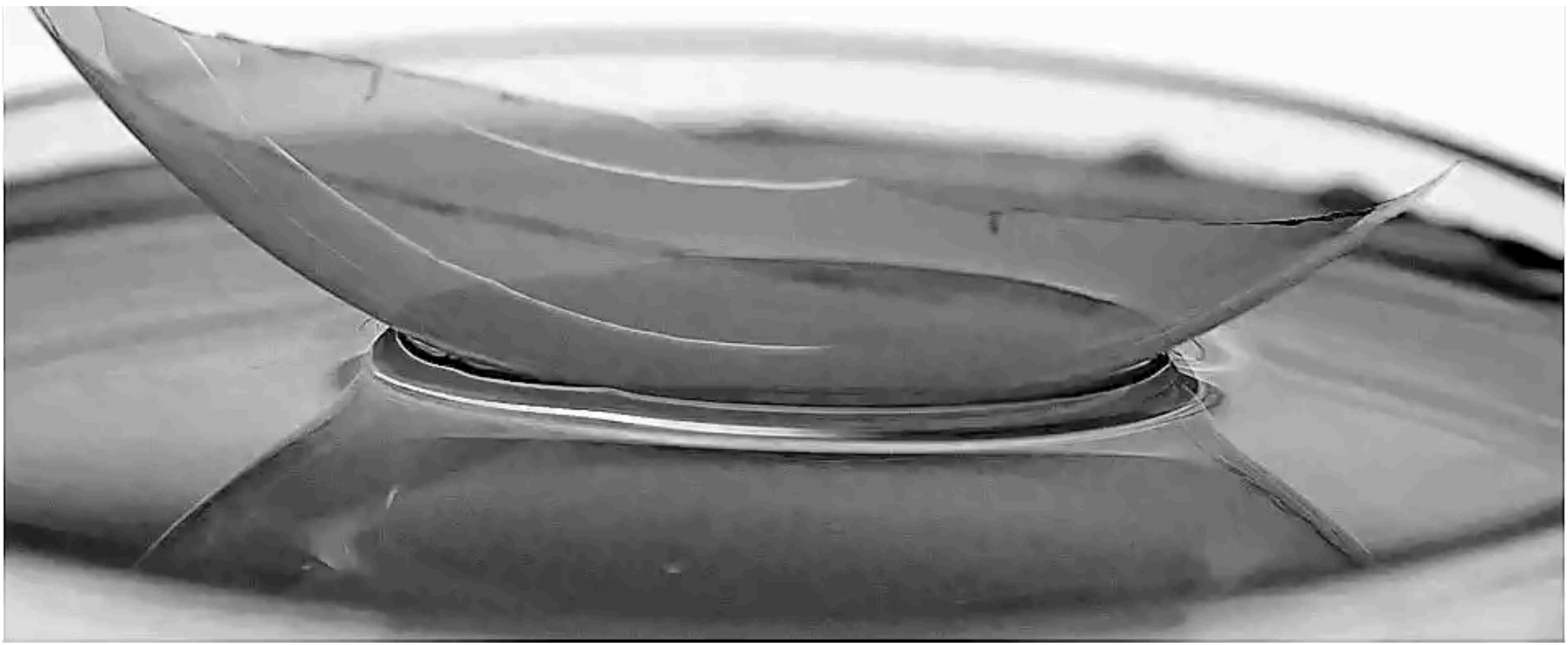
Curved sheet on *flat* substrate



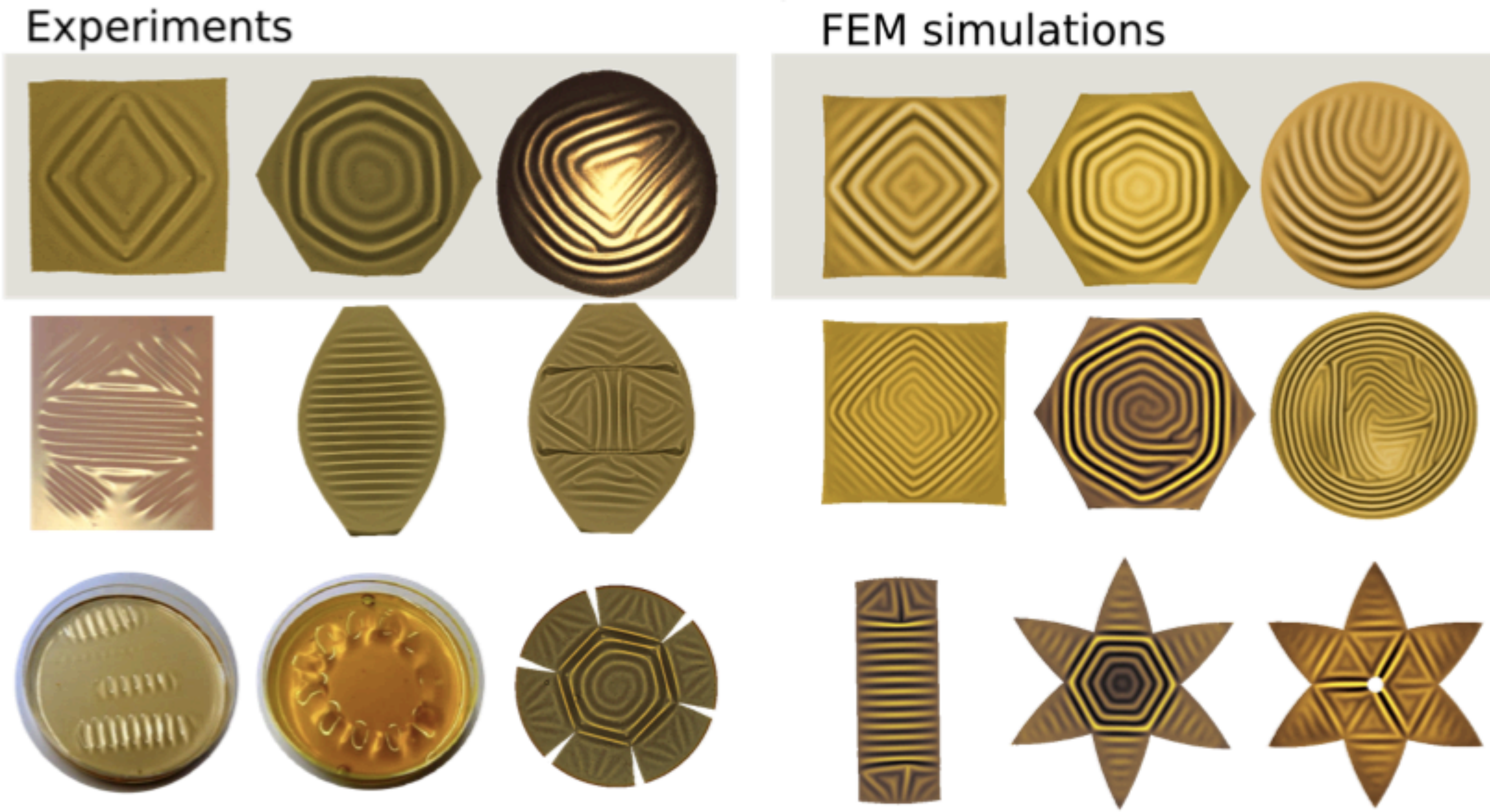


A zoo of shapes

Octavio Albarran
(MPIDS)



A zoo of shapes



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Wrinkling and the theory of smectics

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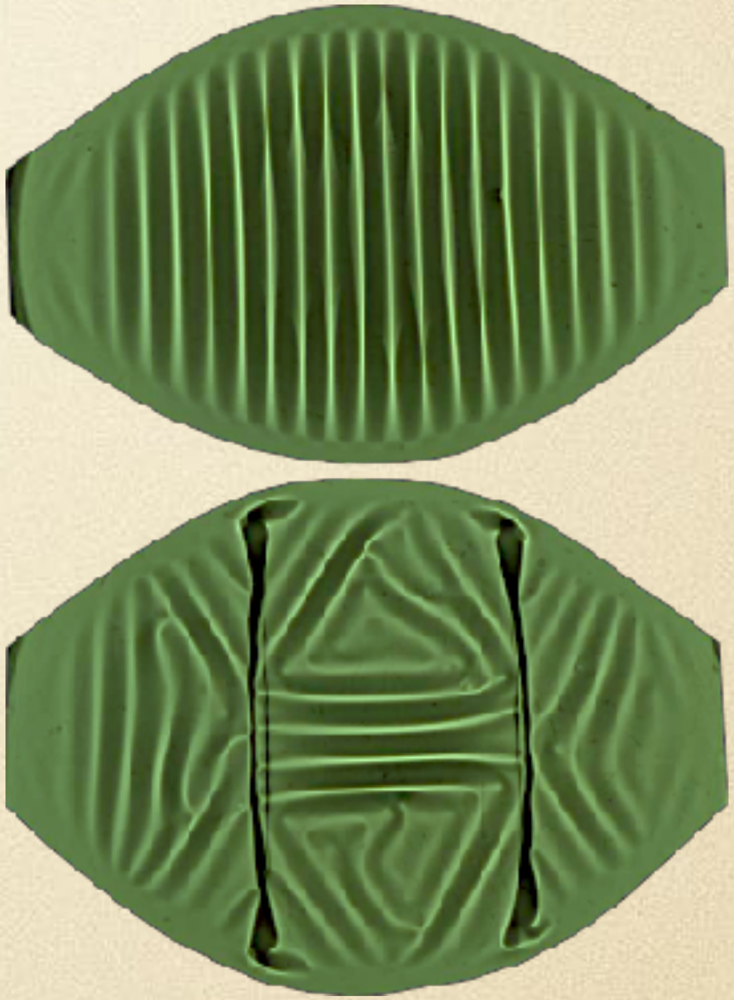
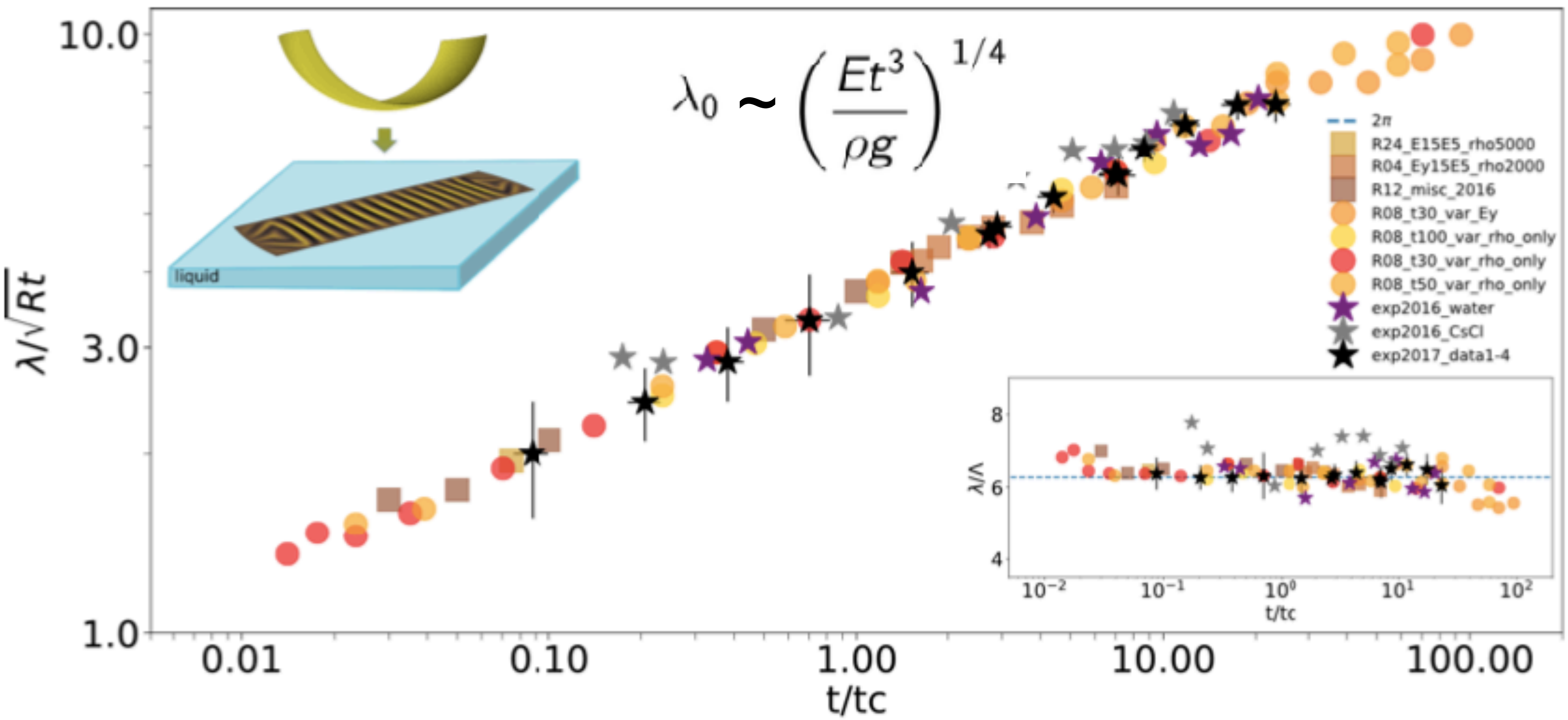
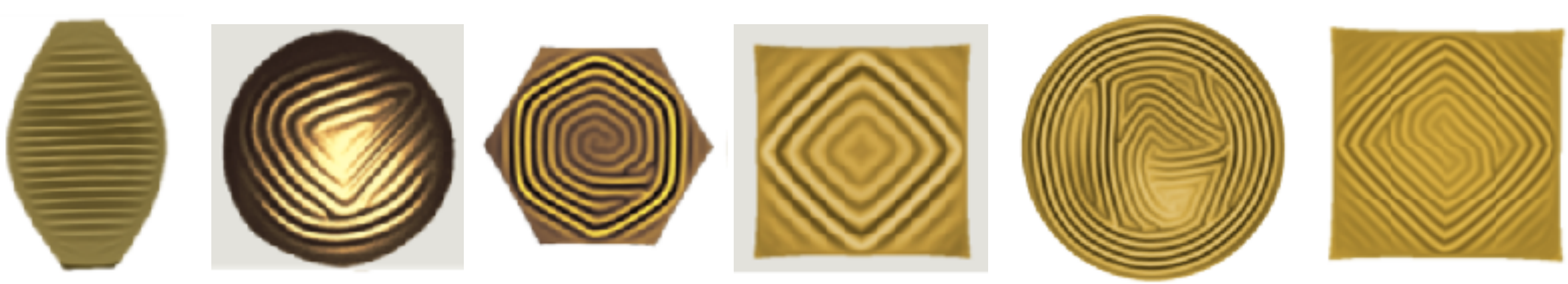


Wavelength of wrinkles

As predicted by theory for flat laterally confined sheets
- competition of bending and stretching

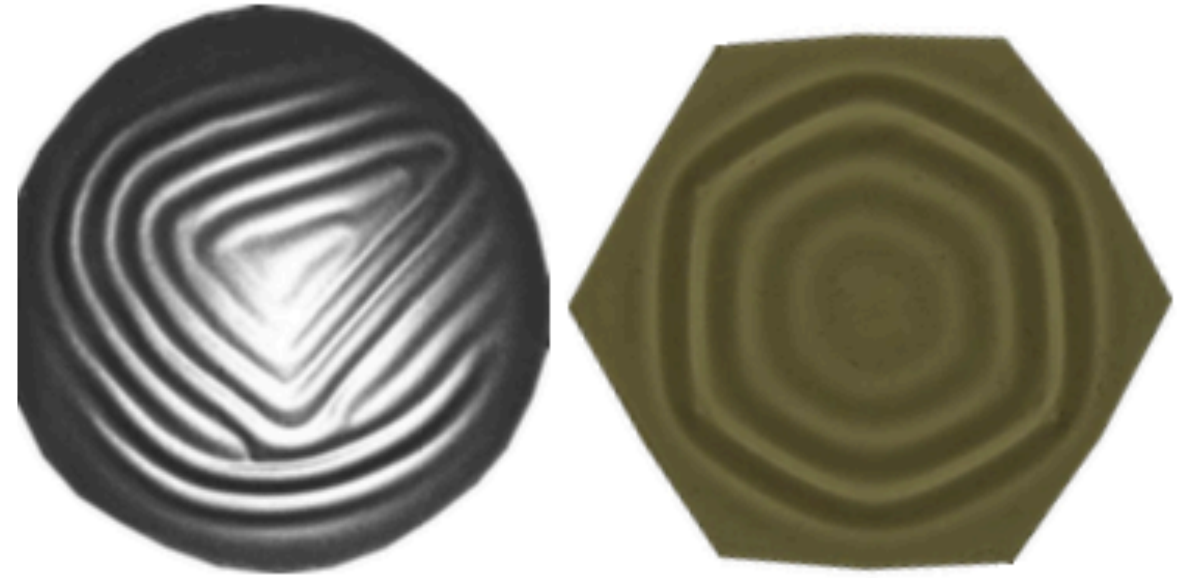


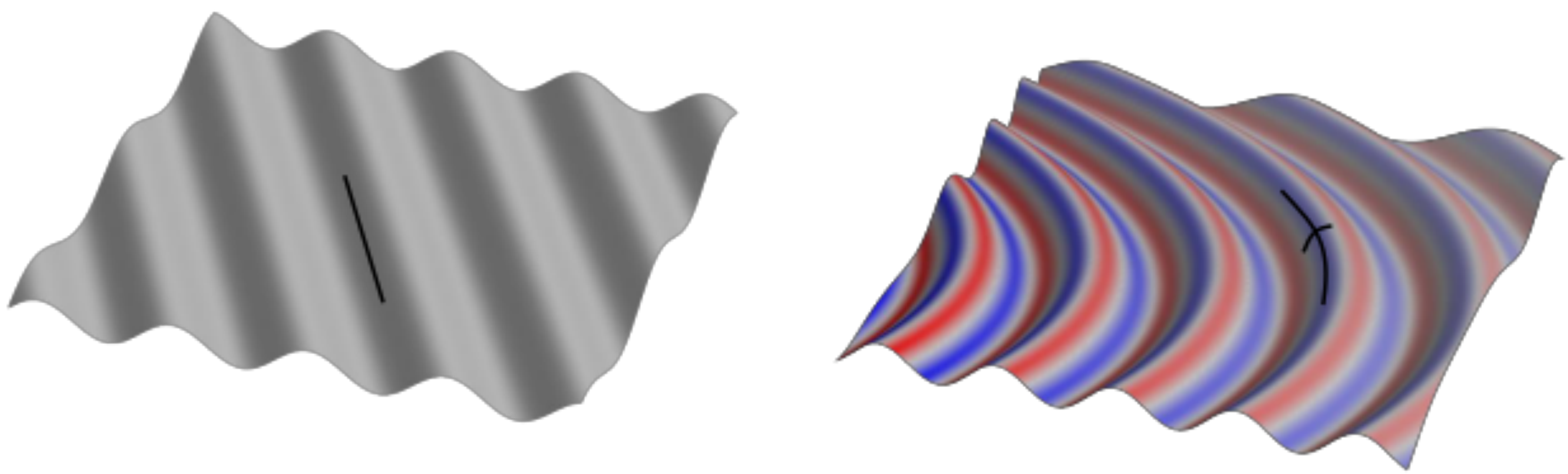


$$t_c = \frac{R_c^2 \rho g}{E}$$

Bending of wrinkles

Wrinkle bending is unfavourable

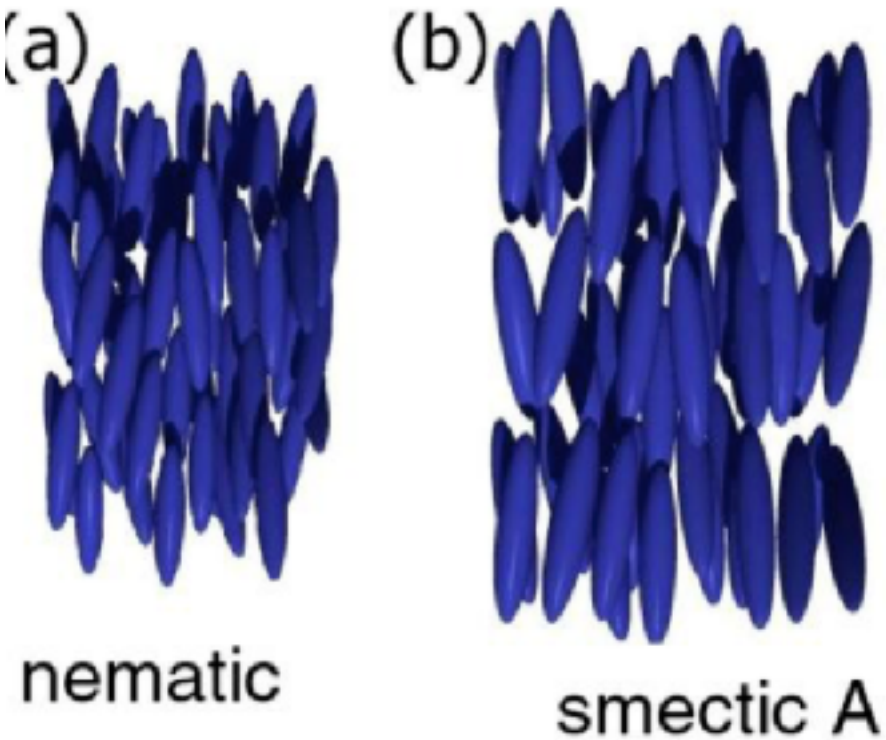




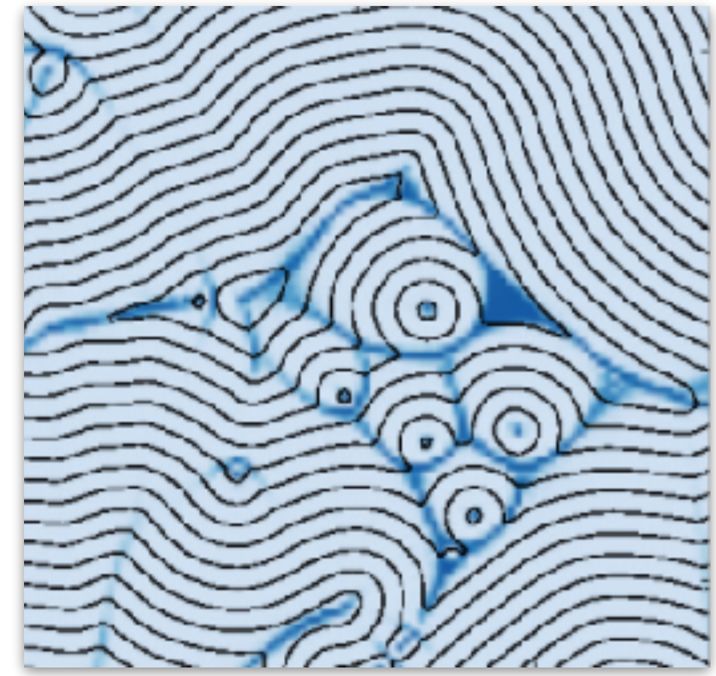
Oscillating contribution to gaussian curvature

introduces stretching and penalises bending of wrinkles

Interlude: Smectics



Level sets of a phase field φ



Smectic energy

$$F = \frac{1}{2}\int \mathrm{d}^3x\,\{B(|\nabla\phi| - |q|)^2 + K(\nabla^2\phi)^2\}$$

Compression modulus

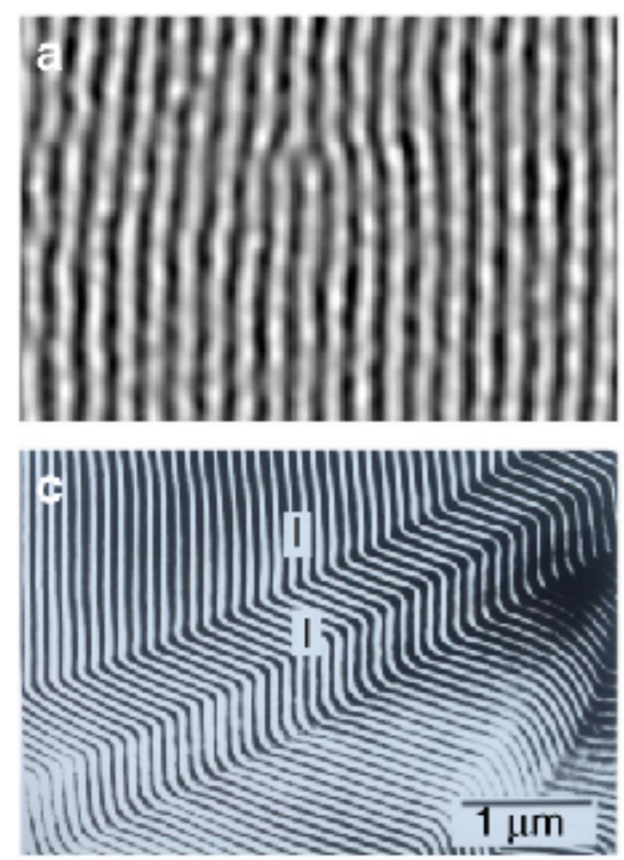
Ground state spacing

Bending modulus

Diblock copolymer melts

dislocations

Herringbone patterns



A smectic theory of wrinkles

Hillel Aharoni (UPenn)

Energy density = Stretching + Bending + Substrate

$$H = \int (U_s + U_b)\sqrt{\det \bar{\mathbf{g}}}\, du\, dv + \int U_g\, dx\, dy$$

$$U_s = \frac{Y}{8}\|\mathbf{g} - \bar{\mathbf{g}}\|^2 \qquad U_b = \frac{B}{2}\|\mathbf{b} - \bar{\mathbf{b}}\|^2 \qquad U_g = \frac{K}{2}(\mathbf{f}\cdot\mathbf{z})^2$$

Reference metric

Curvature tensor

at wrinkle scale, for small and smooth wrinkles, *can be rewritten as*

$$U_{el}/Y = \frac{1}{8}\left\|\mathbf{g}^{(cg)} - \bar{\mathbf{g}}^{(eff)}\right\|^2 + \text{const.}$$

$$\bar{\mathbf{g}}_{ij}^{(\text{eff})} = \bar{\mathbf{g}}_{ij} - 4\eta \frac{\bar{\mathbf{g}}_{ij} - \delta_{ij}}{\|\bar{\mathbf{g}}_{ij} - \delta_{ij}\|}$$

Coarse grained energy density = elastic + smectic

Excess length

$$\frac{\eta\Delta}{4}\left(\frac{|\nabla\phi|^2}{k_0{}^2} - 1\right)^2 + \frac{\Delta^2}{256}\left(\frac{\|\nabla\nabla\phi\|}{|\nabla\phi|^2}\right)^2$$

Stretchability

H. Aharoni, D. Todorova, O. Albarran, L. Goehring, R. Kamien and EK, “The smectic order of wrinkles” *(Nature Comm)*

A smectic theory of wrinkles

Hillel Aharoni
(UPenn)

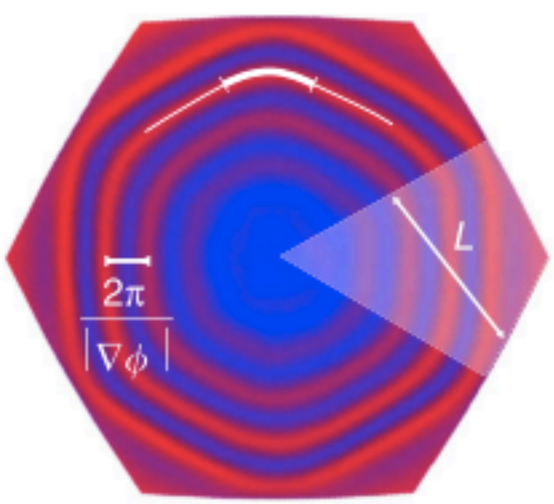
smectic

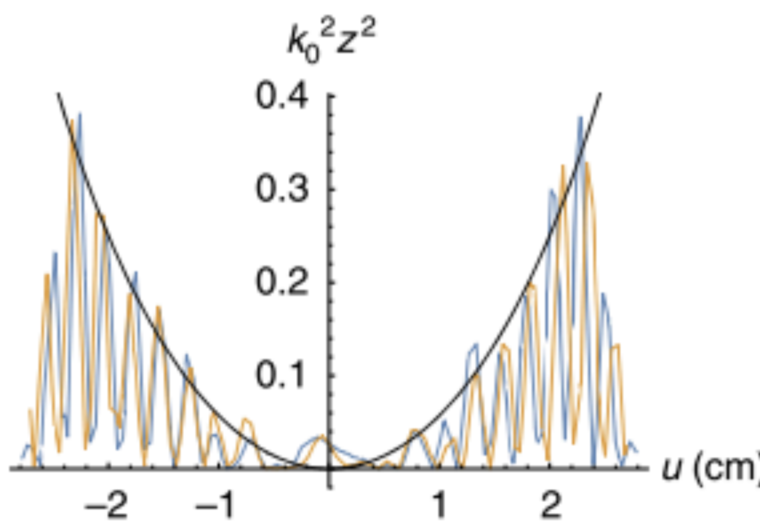
Excess length

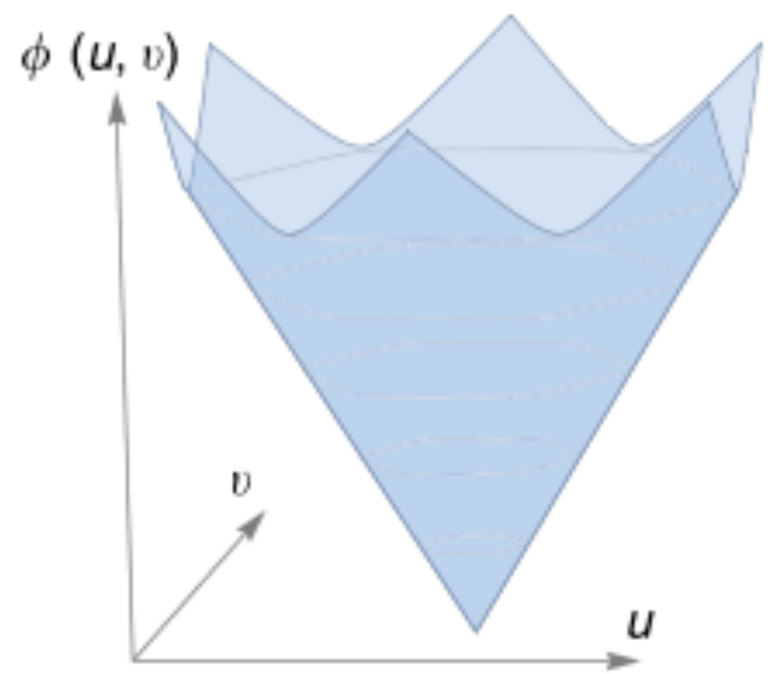
Stretchability

$$\frac{\eta\Delta}{4}\left(\frac{|\nabla\phi|^2}{k_0{}^2}-1\right)^2+\frac{\Delta^2}{256}\left(\frac{\|\nabla\nabla\phi\|}{|\nabla\phi|^2}\right)^2$$

$$F=\frac{1}{2}\int \mathrm{d}^3x\,\{B(|\nabla\phi|-|q|)^2+K(\nabla^2\phi)^2\}$$



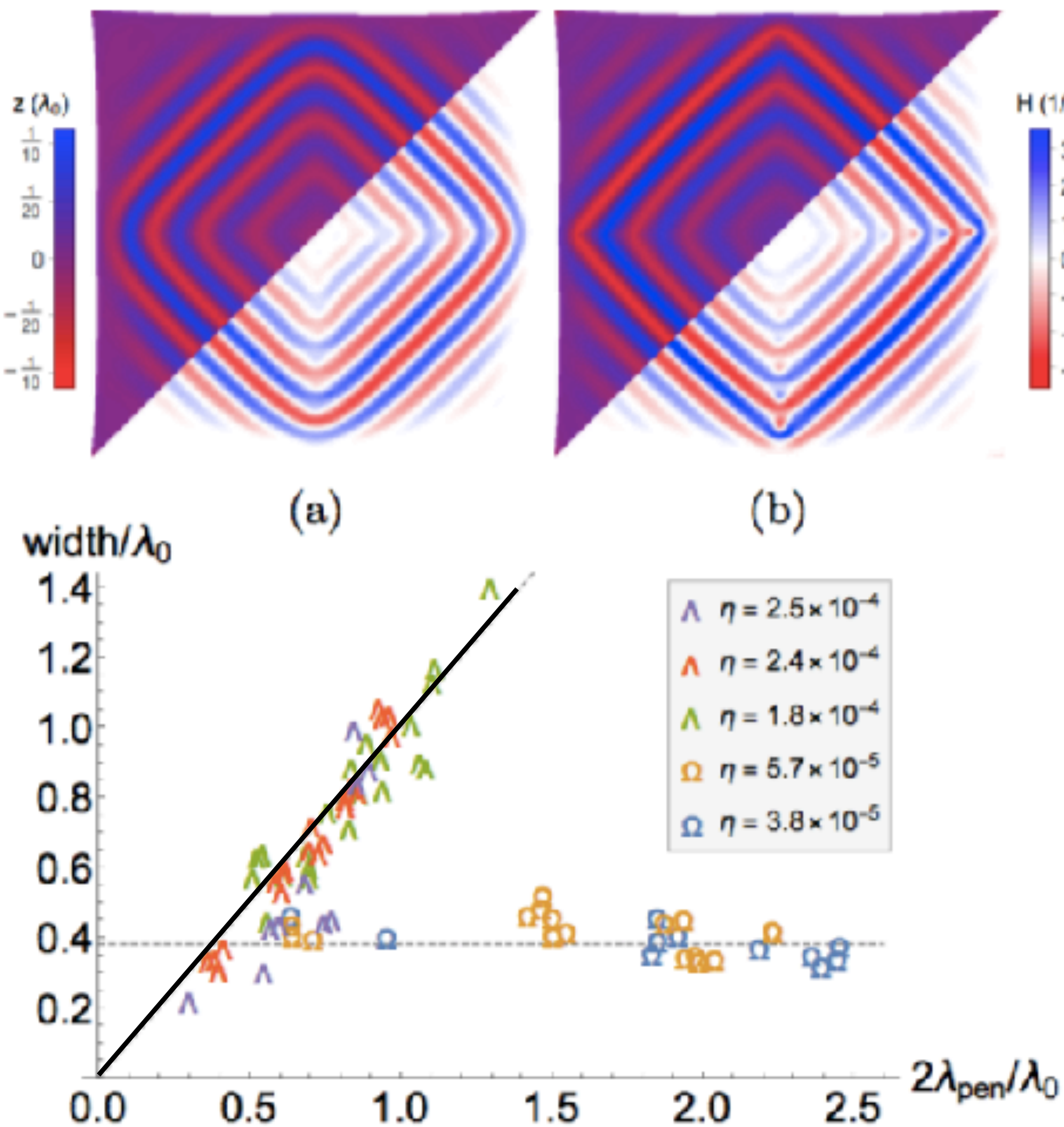


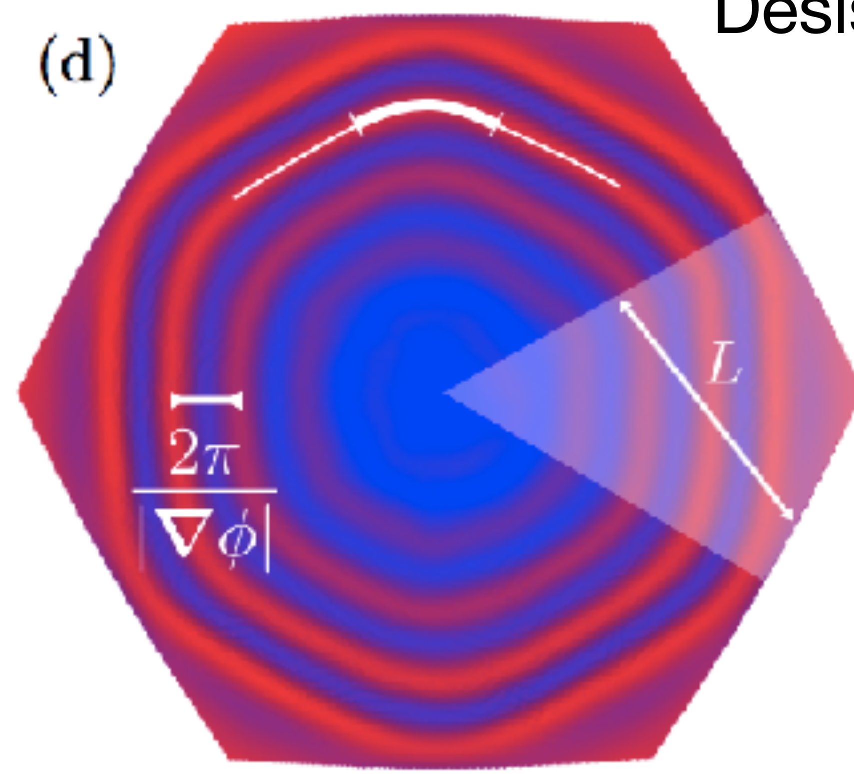


H. Aharoni, D. Todorova, O. Albarran, L. Goehring, R. Kamien and EK, “The smectic order of wrinkles” *(Nature Comm)*

Bending of wrinkles

Desislava Todorova (UPenn)



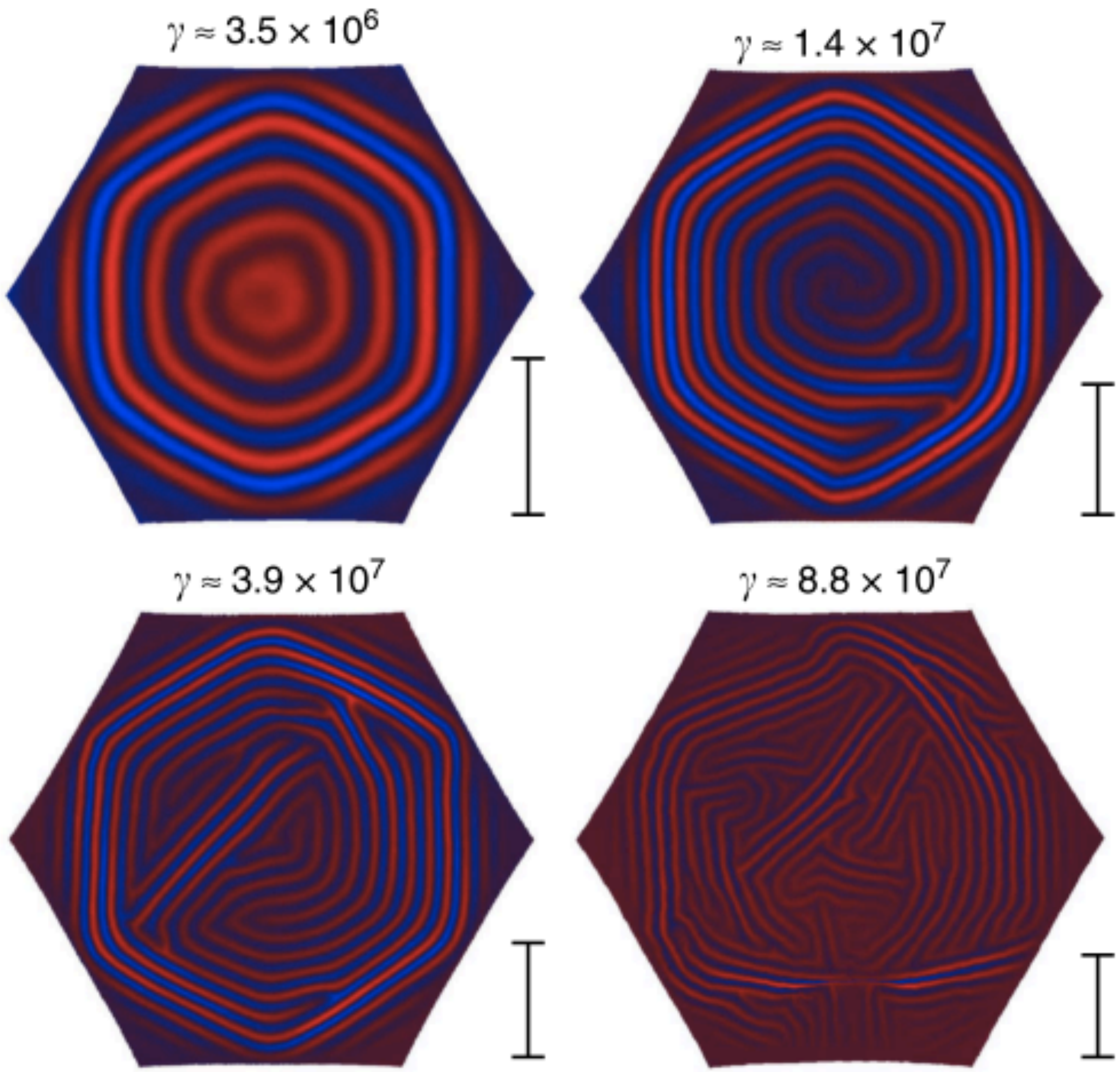


$$\lambda_{\text{pen}} = \sqrt{\frac{\text{curvature modulus}}{\text{compression modulus}}} = \frac{\sqrt{\Delta/\eta}}{16\pi}\lambda_0$$

Stretchability $\eta = \sqrt{BK}/Y$

Domain size scaling

Typical domain size L



$$\gamma = \frac{YR^2}{B}$$

energy = elastic + smectic

$$U_{\mathrm{el}} \propto \frac{D^2 L^4}{R^4}$$

Energy per unit length for domain wall=
smectic curvature modulus/penetration depth

$$U_{\mathrm{sm}} \propto \frac{tD^5}{LR^3}$$

$$L/D \propto (R/D)^{1/5} \gamma^{-1/10}$$

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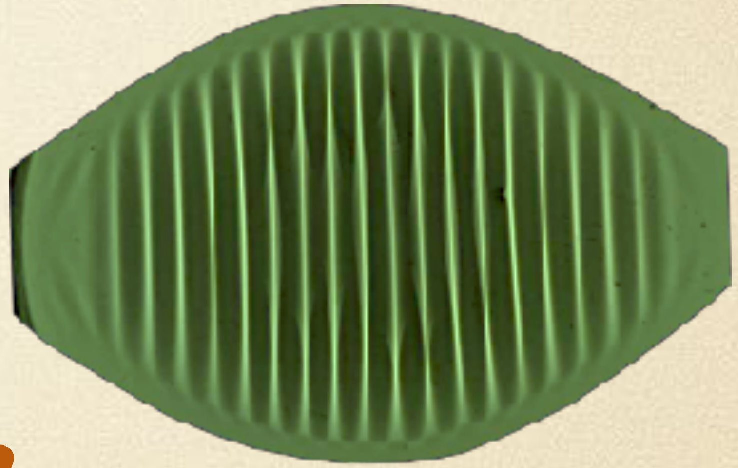

Wrinkling and the theory of smectics

Dimples, folds and other instabilities

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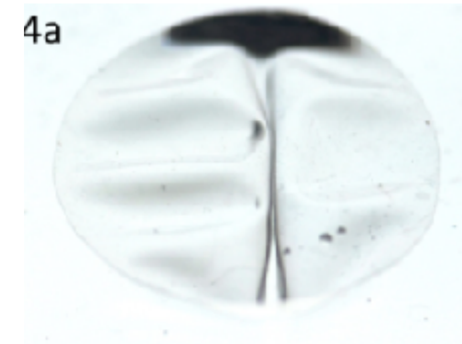
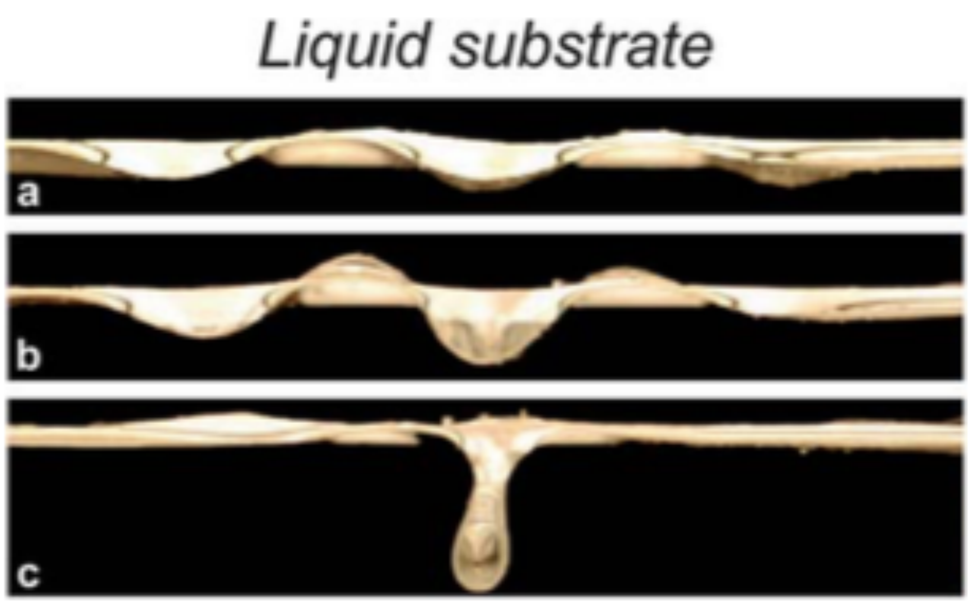
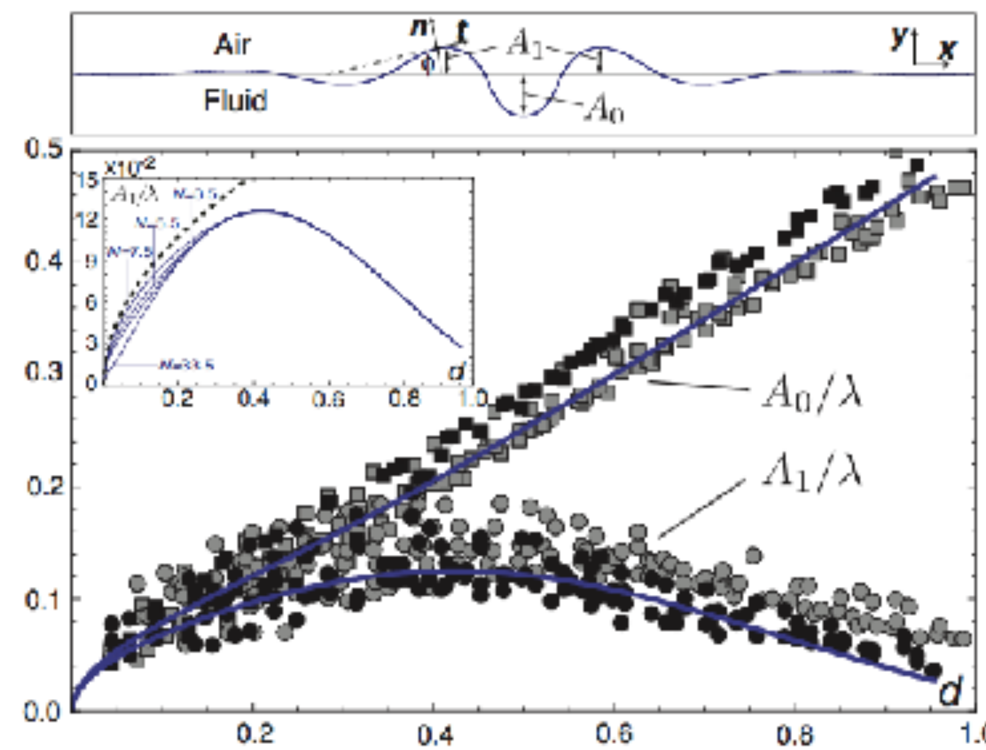
Wrinkles to folds



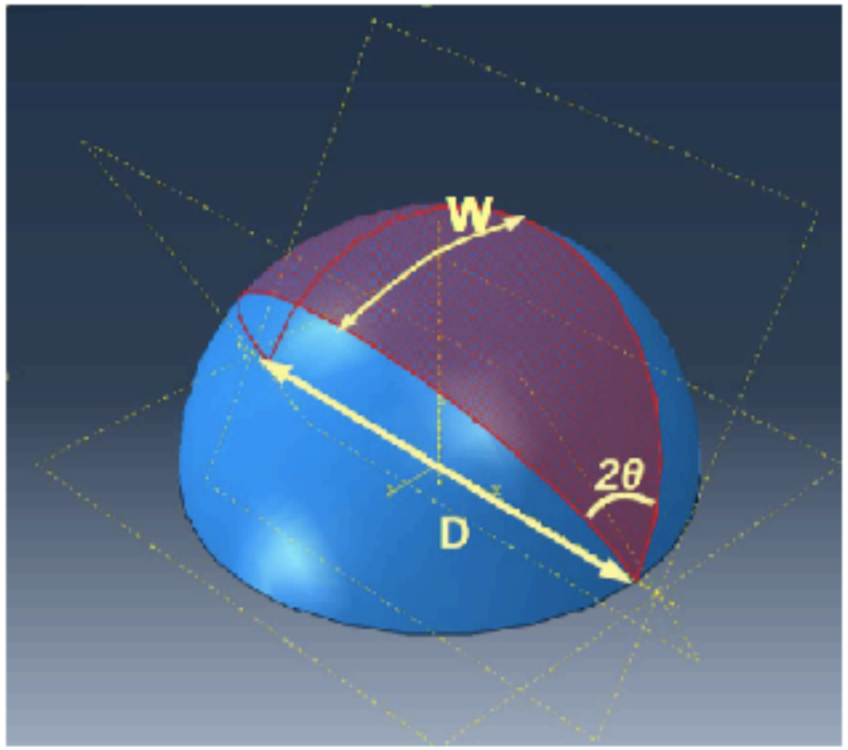


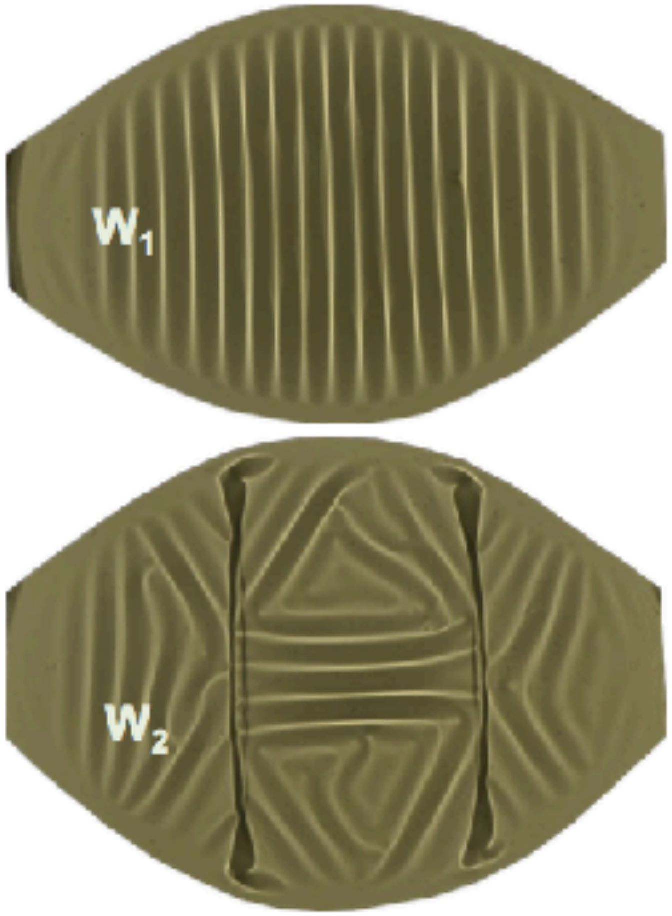
Pocivavsek et al,
Science, (2008), 320, 912



Wrinkles turn into folds

Wrinkles to folds

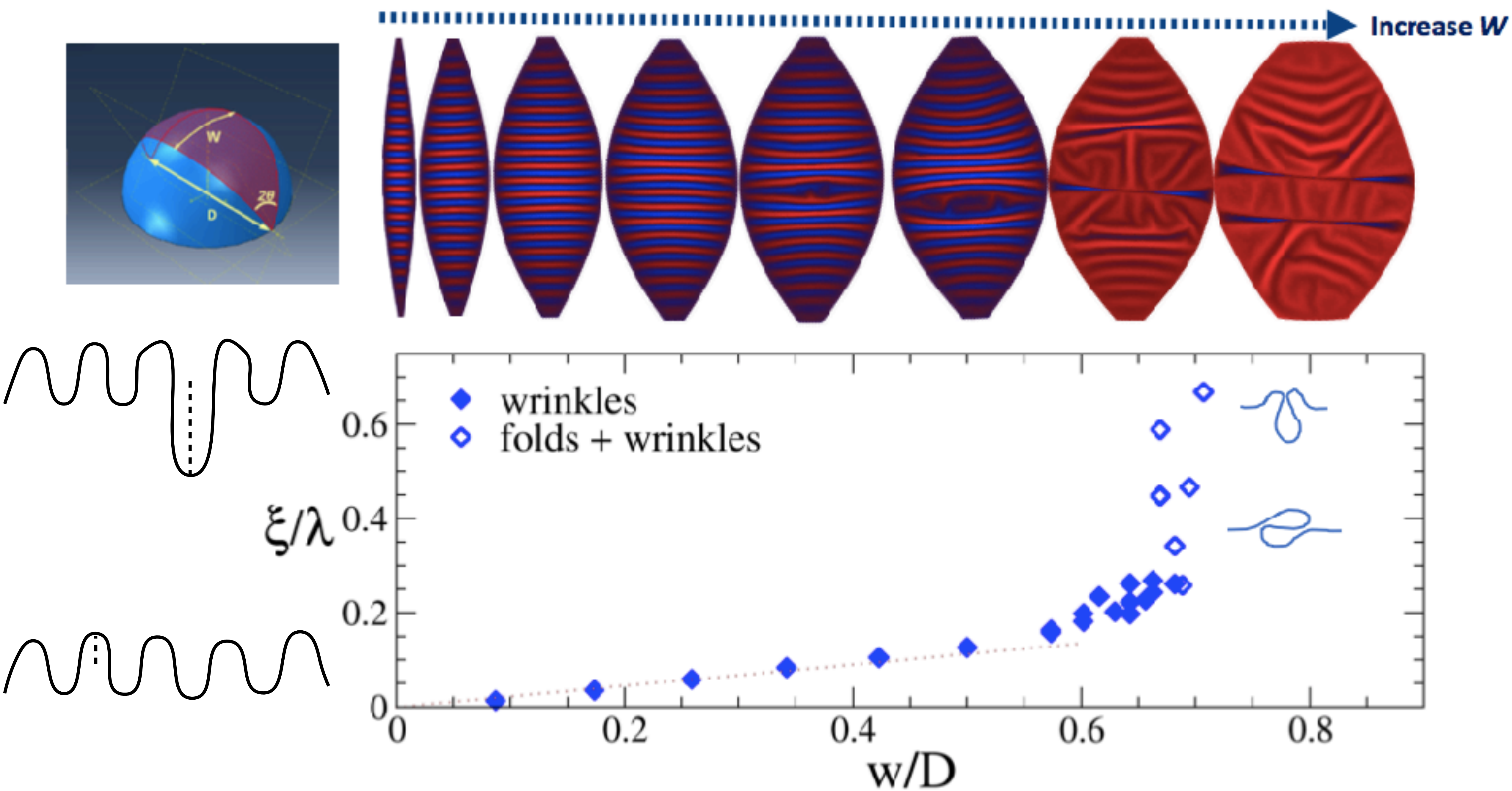




$W_1 < W_2$
Is there **a critical W**?

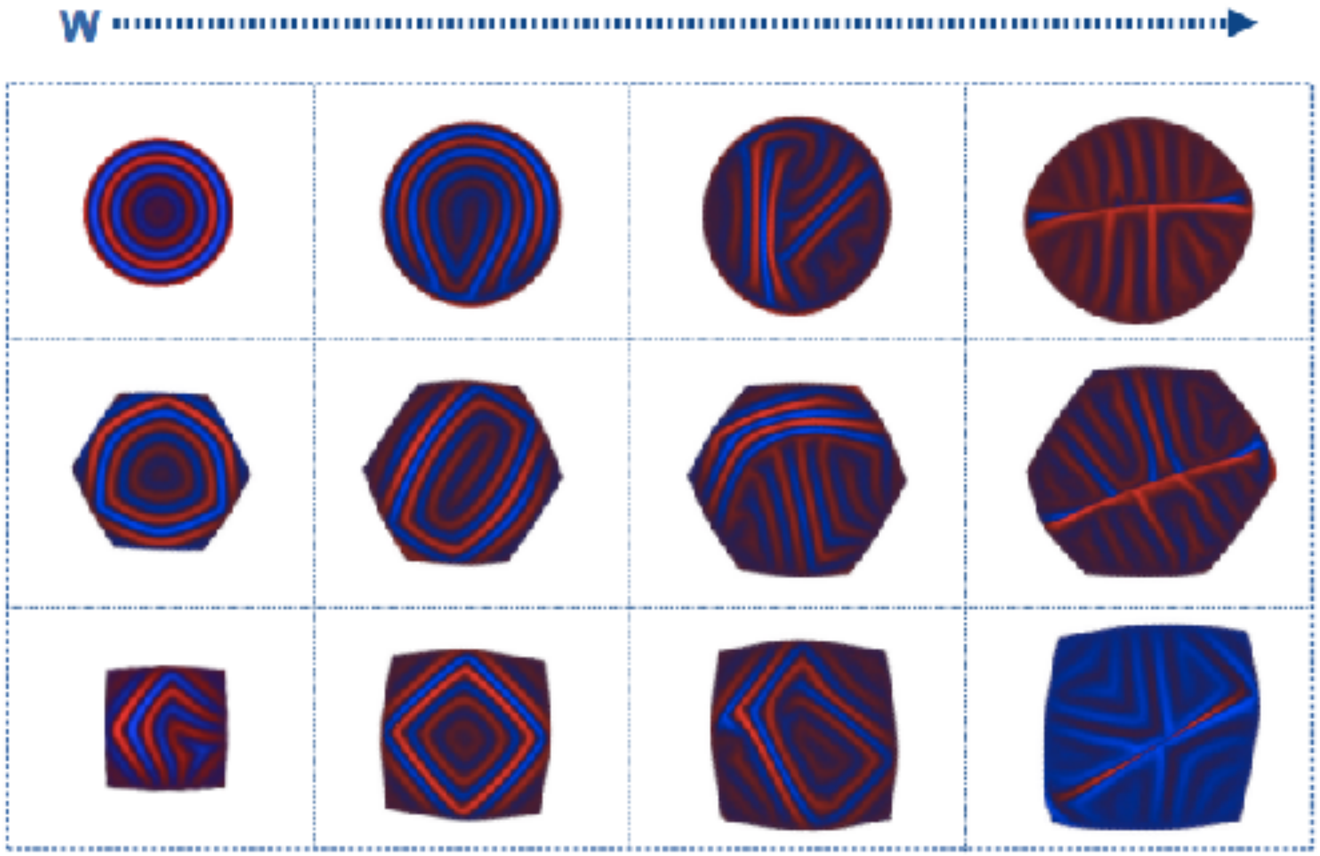
Coexistence of wrinkles and folds.

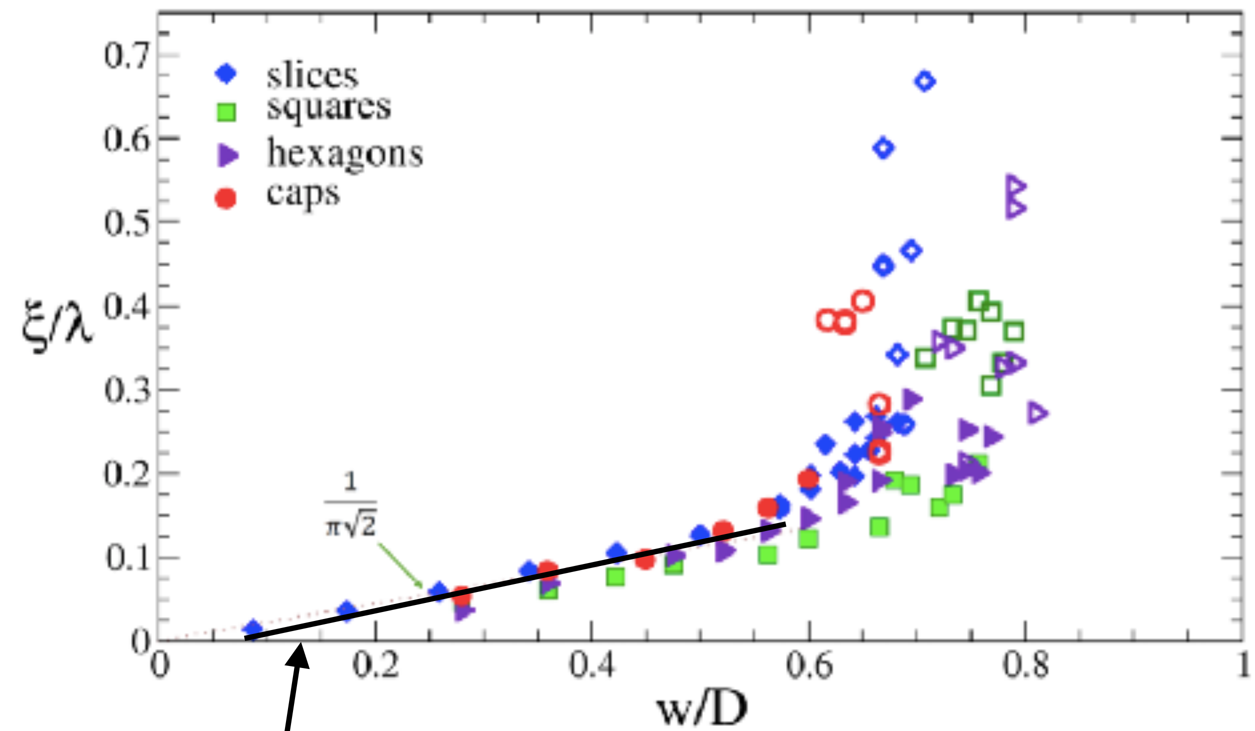
Wrinkles to folds



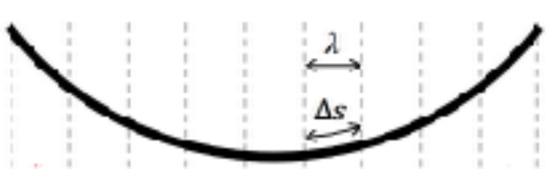
Wrinkles to folds

A wrinkle-to-fold transition in caps and polygons





excess length that needs to be accommodated in wrinkles

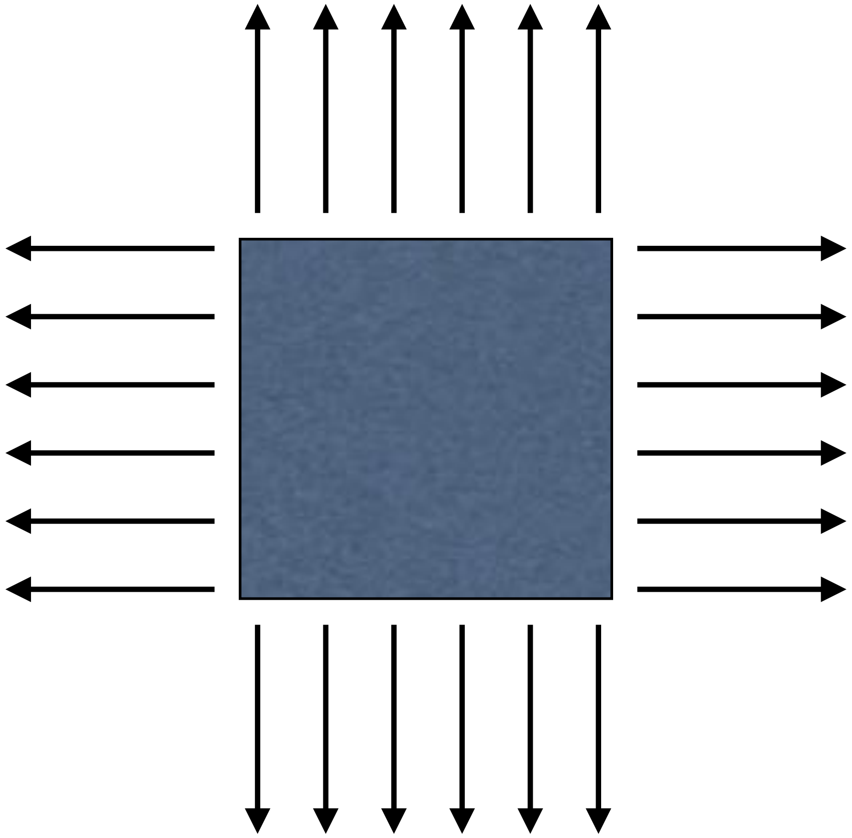


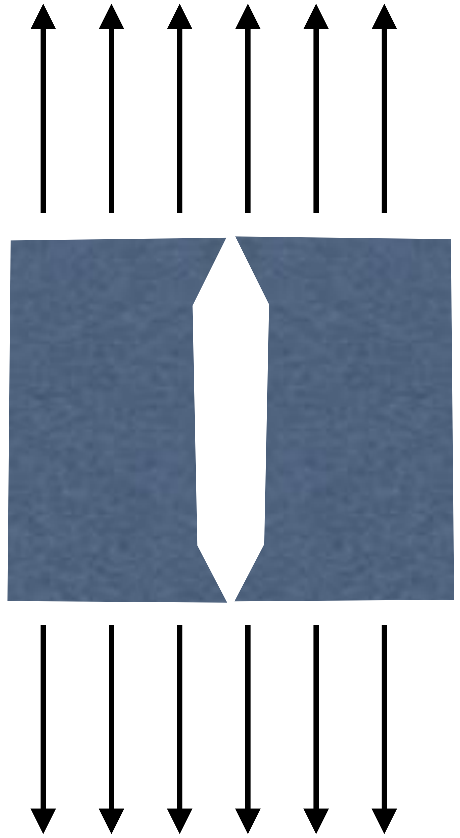
$$\epsilon = \frac{\Delta s - \lambda}{\lambda} \qquad \xi = \frac{\lambda}{\pi}\sqrt{\epsilon} \qquad \frac{\xi_m}{\lambda} = \frac{1}{\pi\sqrt{2}}\frac{W}{D}$$

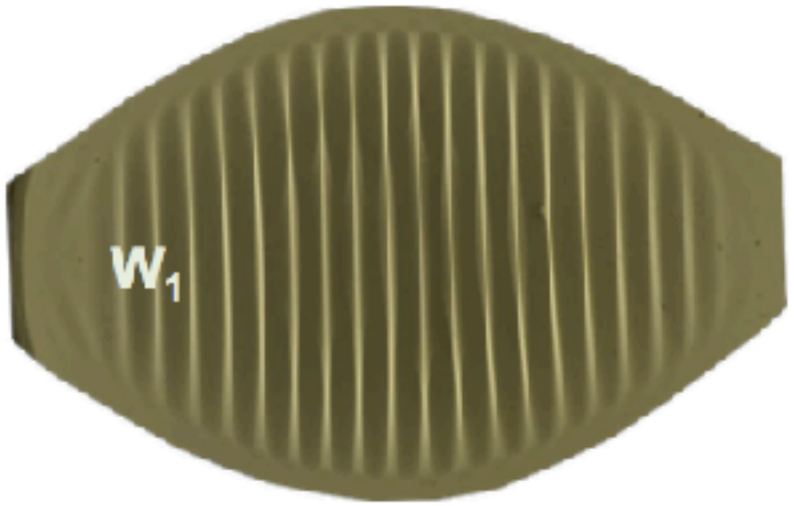
(in preparation)

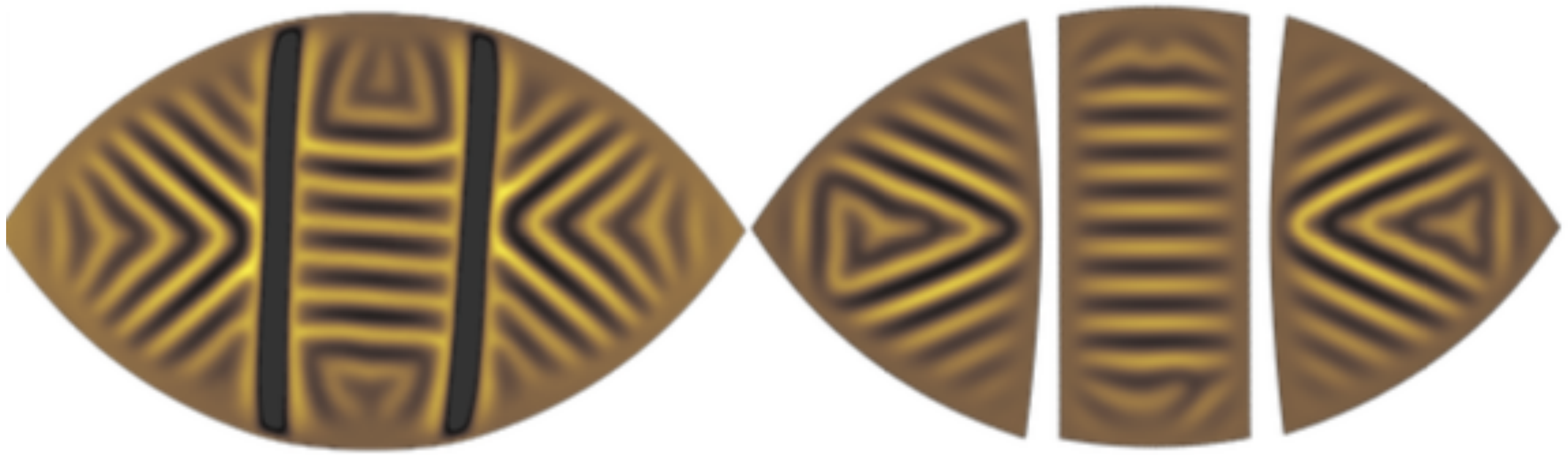
Wrinkles to folds

Folds act like anti-cracks

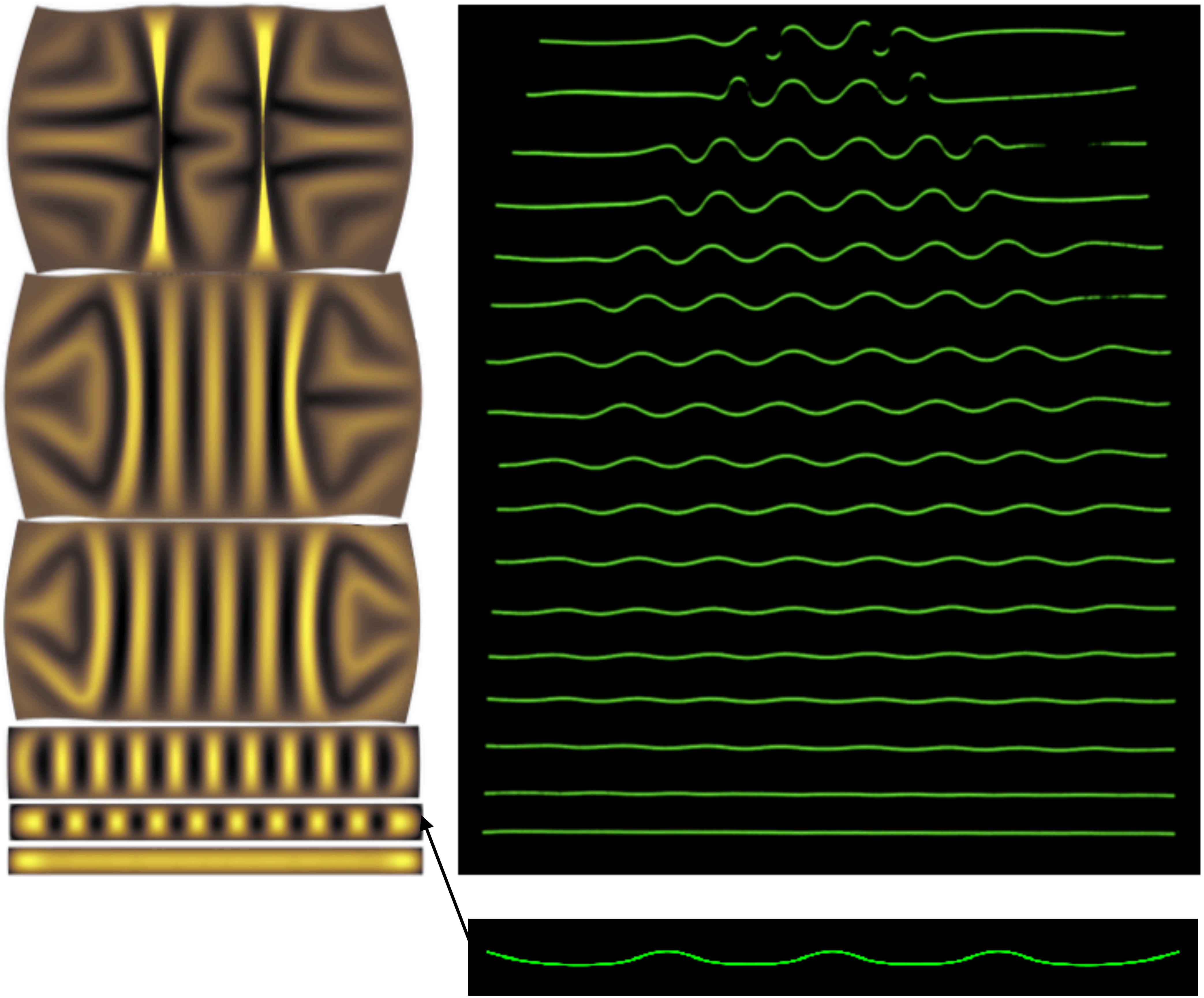








A zoo of shapes, revisited



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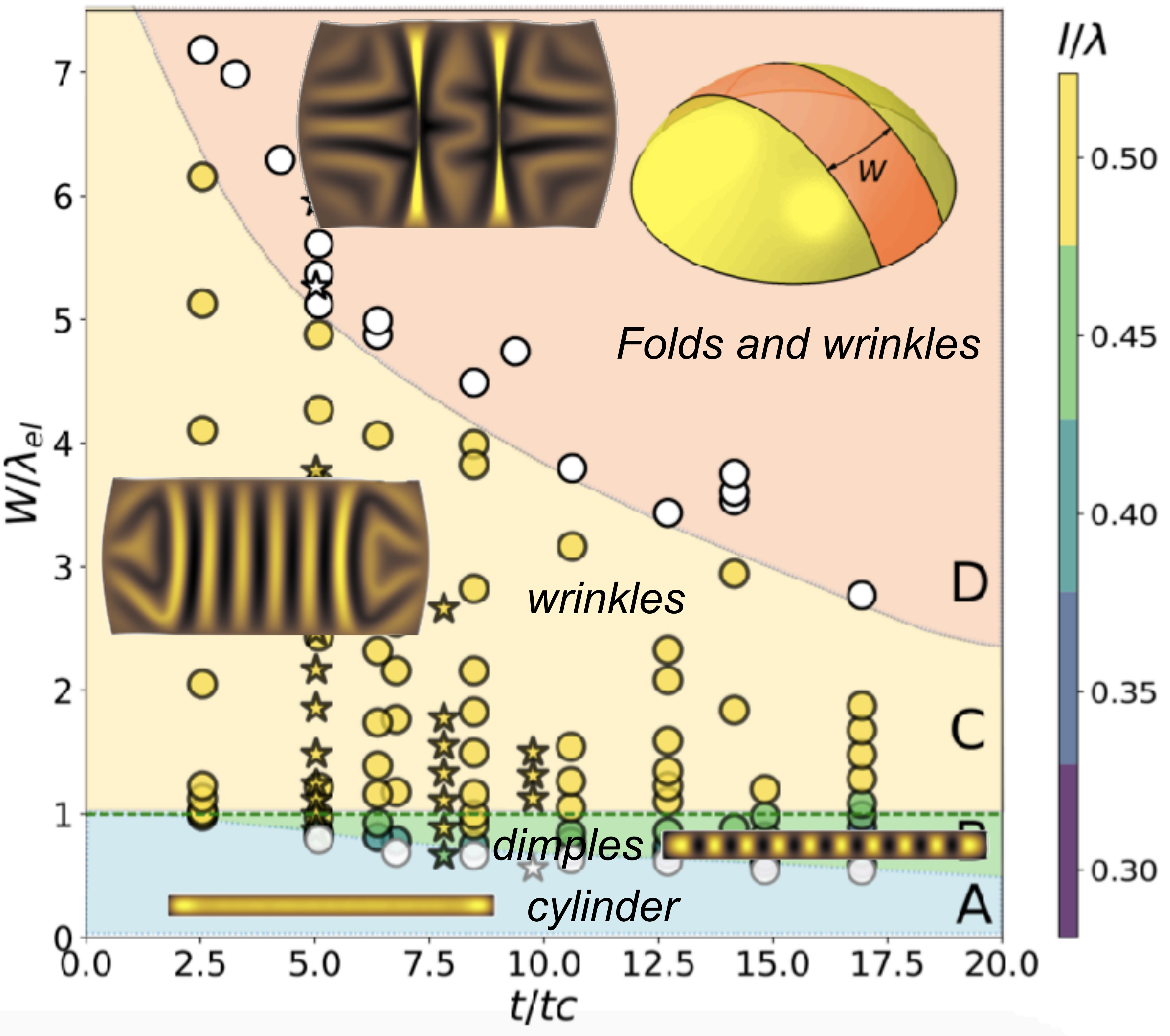

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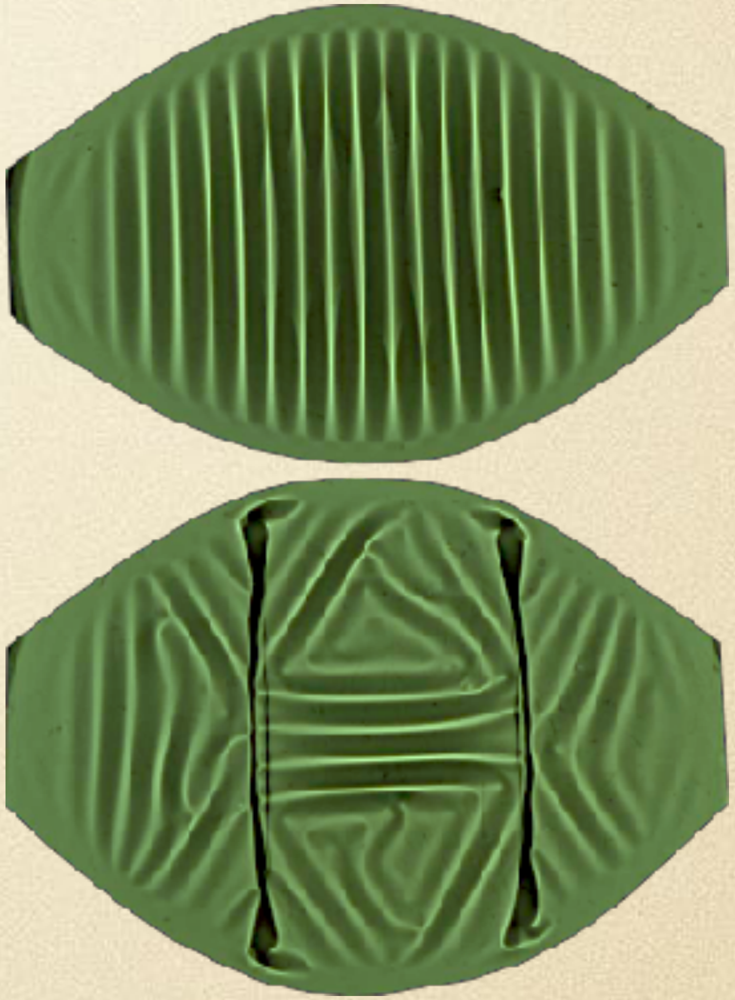

Wrinkling and the theory of smectics

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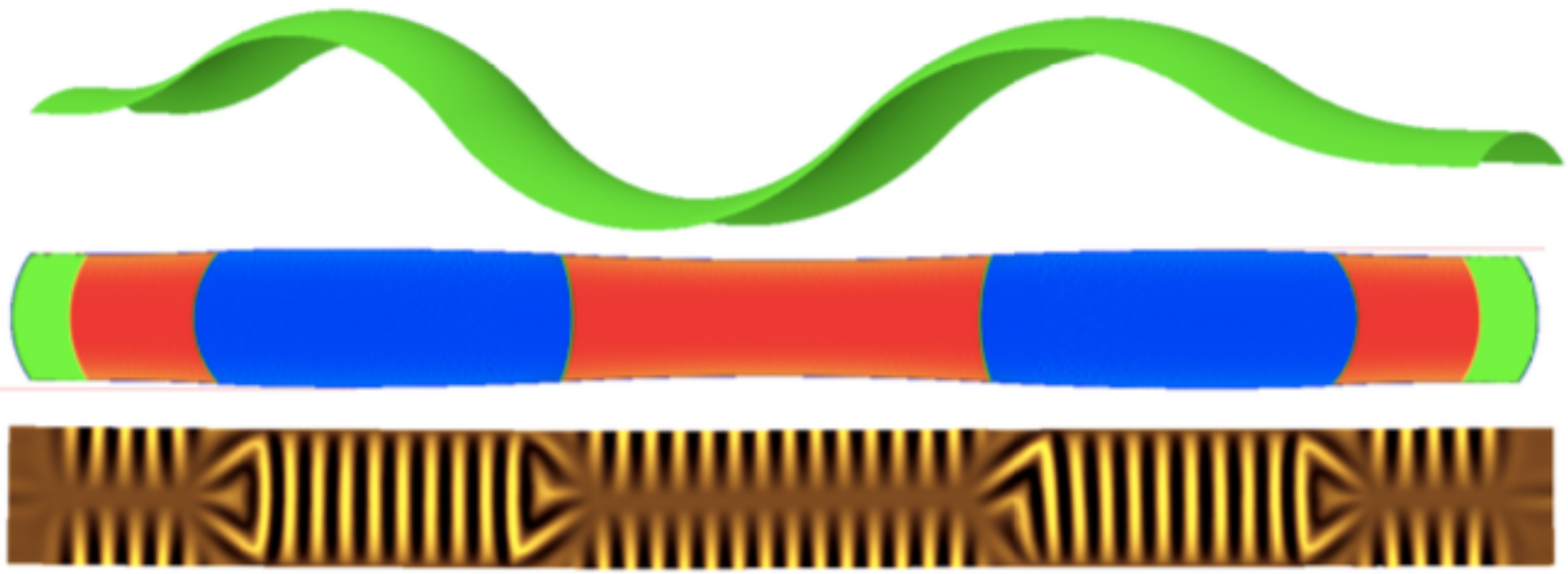
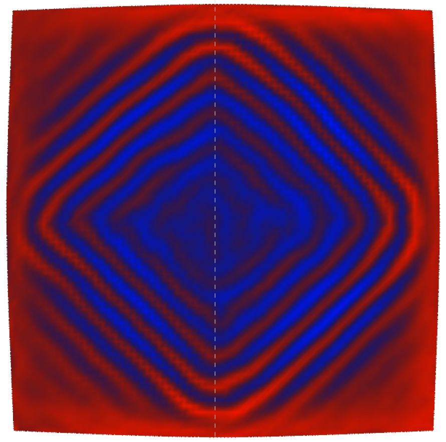
$\boxed{\text{Engineering exotic shapes}}$

So what?



Engineering patterns: stiff lines

t = 25 um



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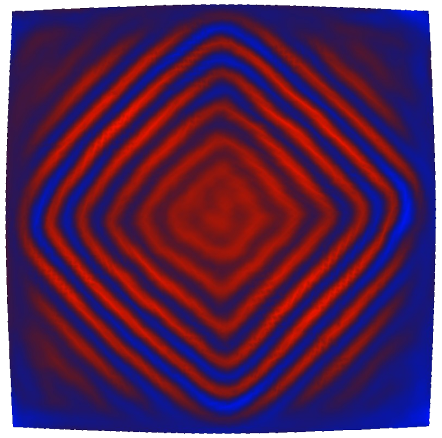
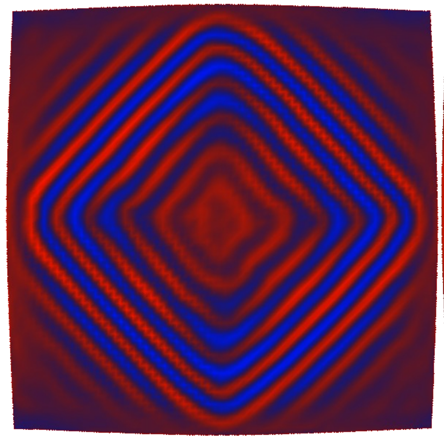


defect-free



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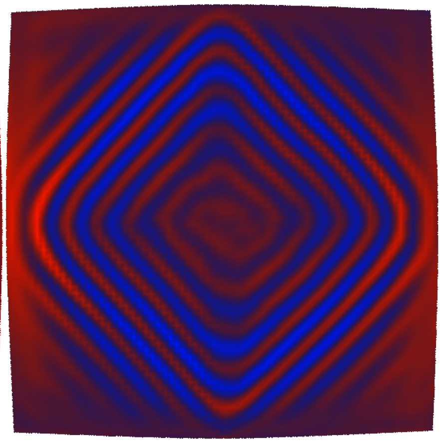

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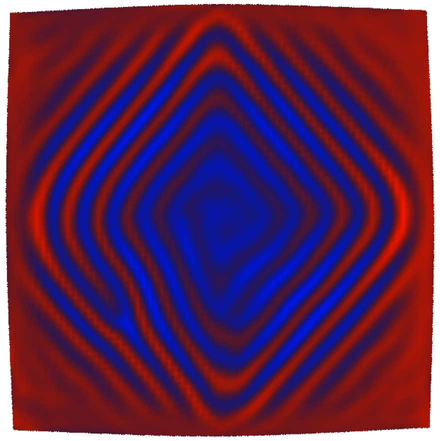

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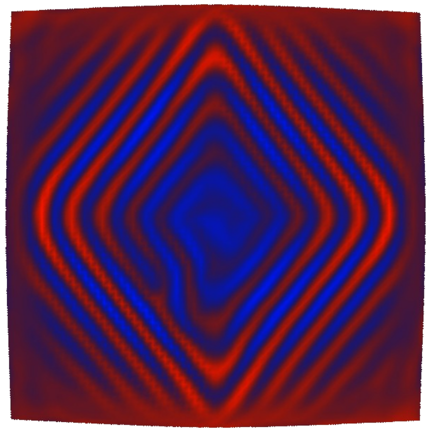

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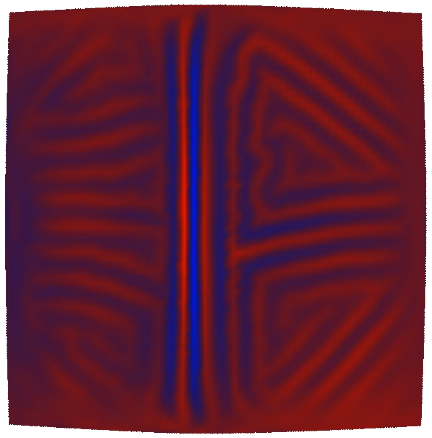
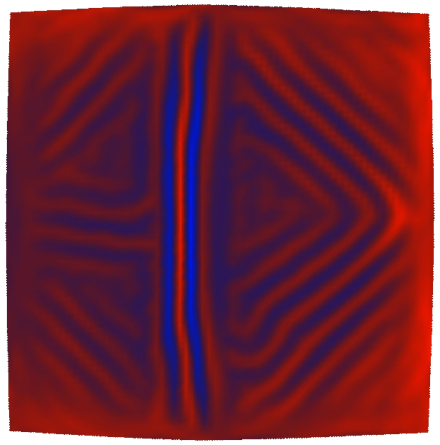


Ed/E=500



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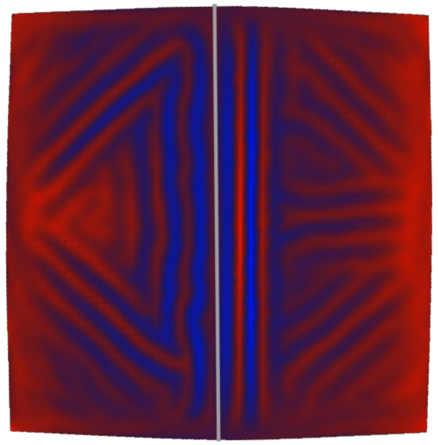

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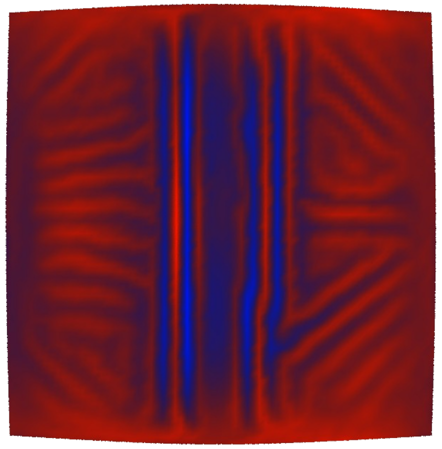
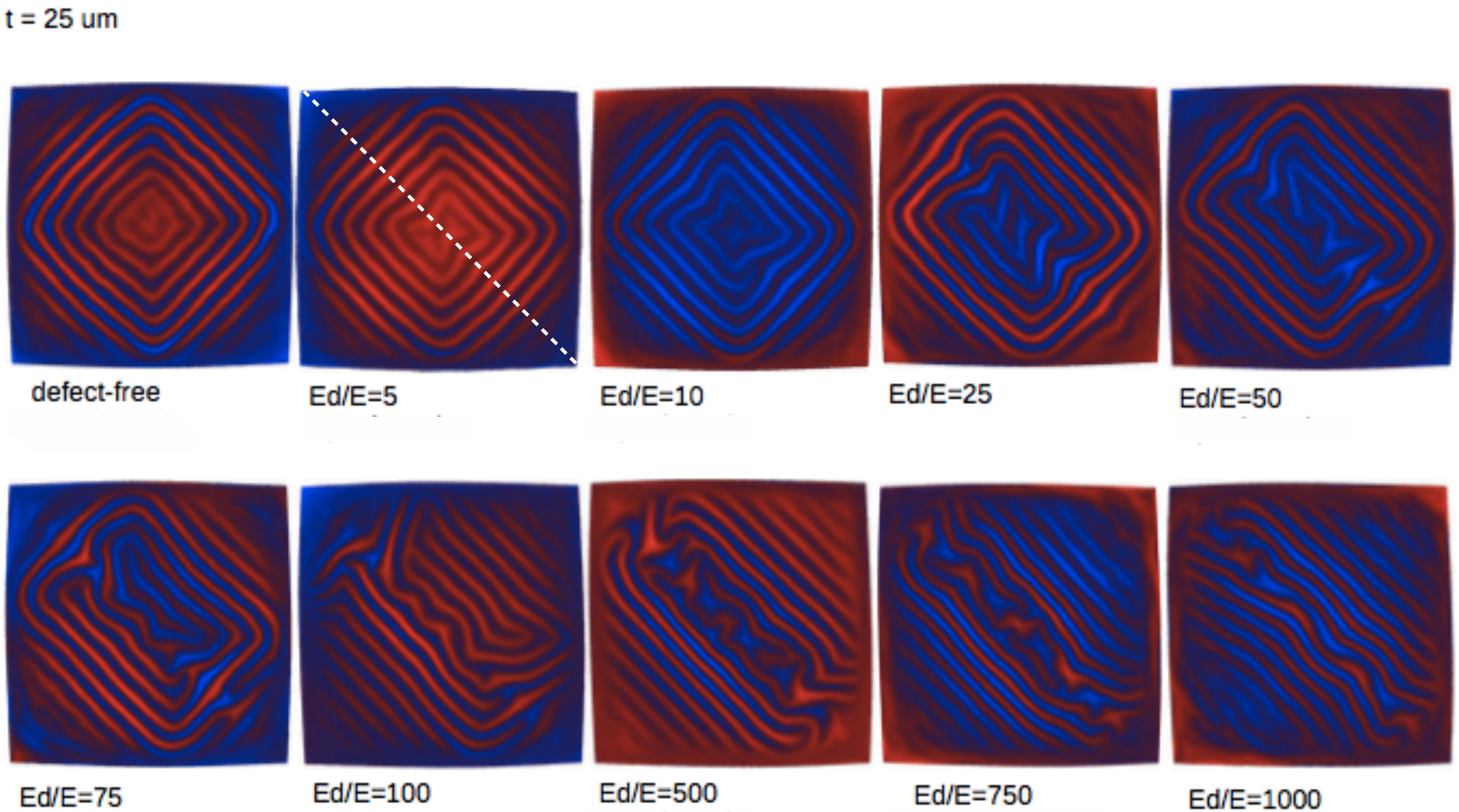


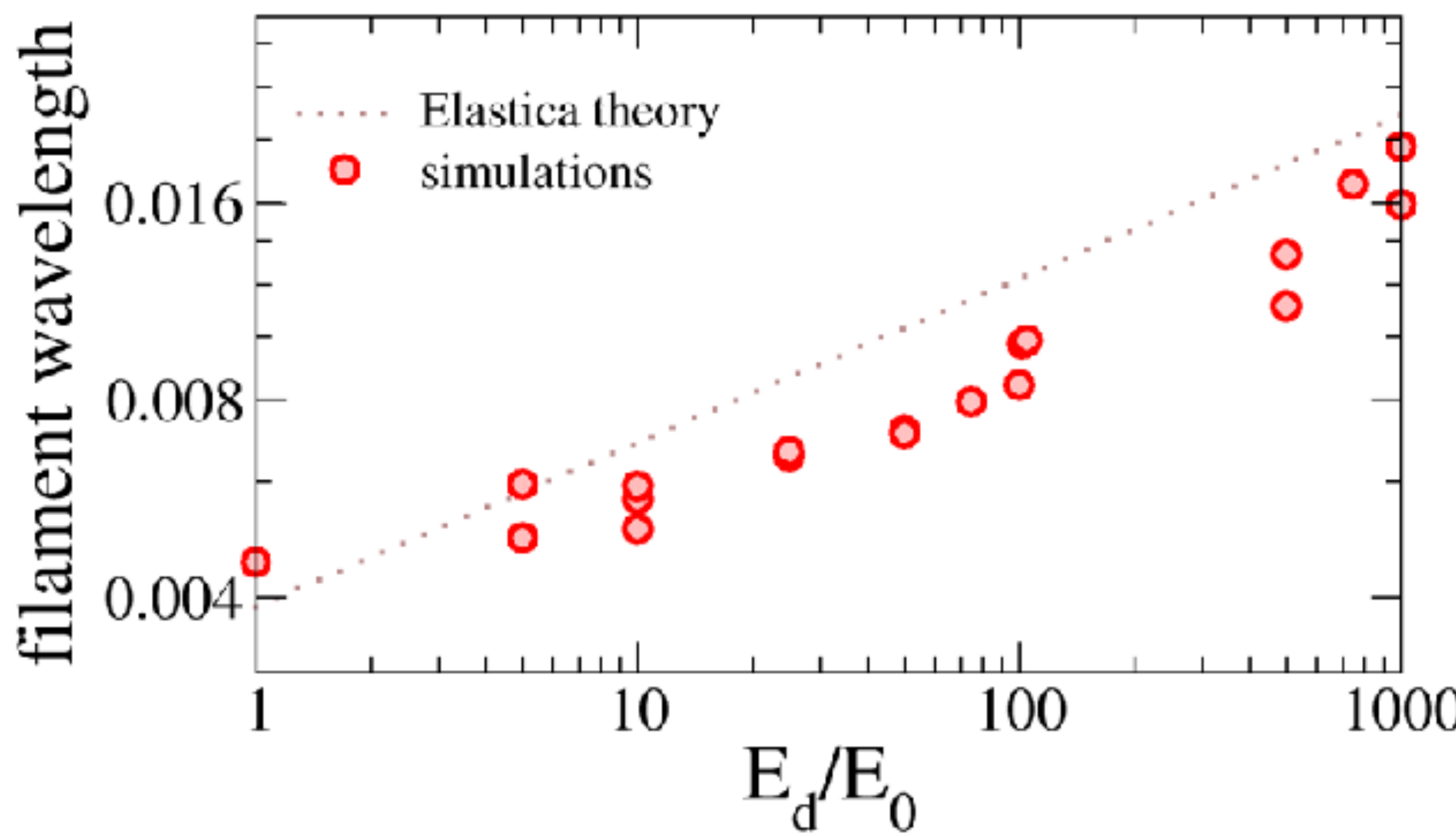
Ed/E=5000

(in preparatio

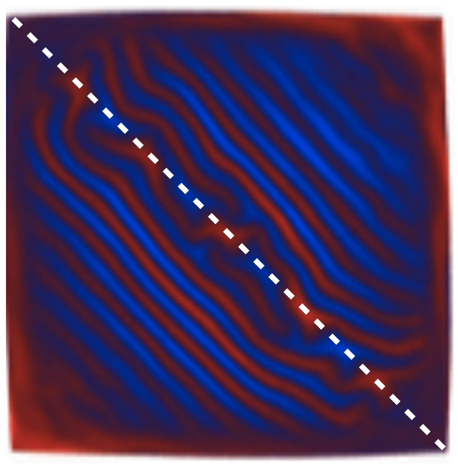
Engineering patterns: stiff lines



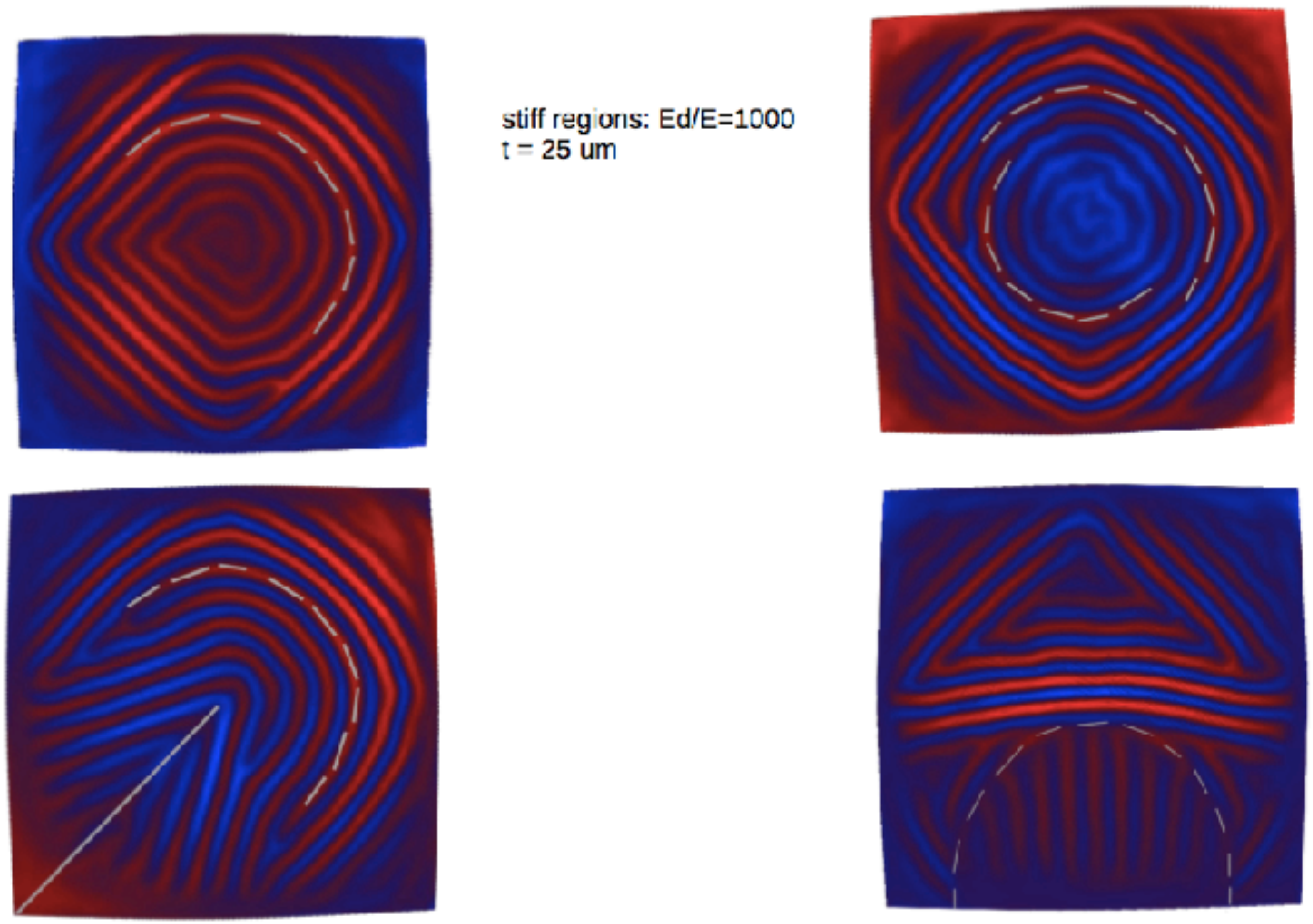
Engineering patterns: stiff lines



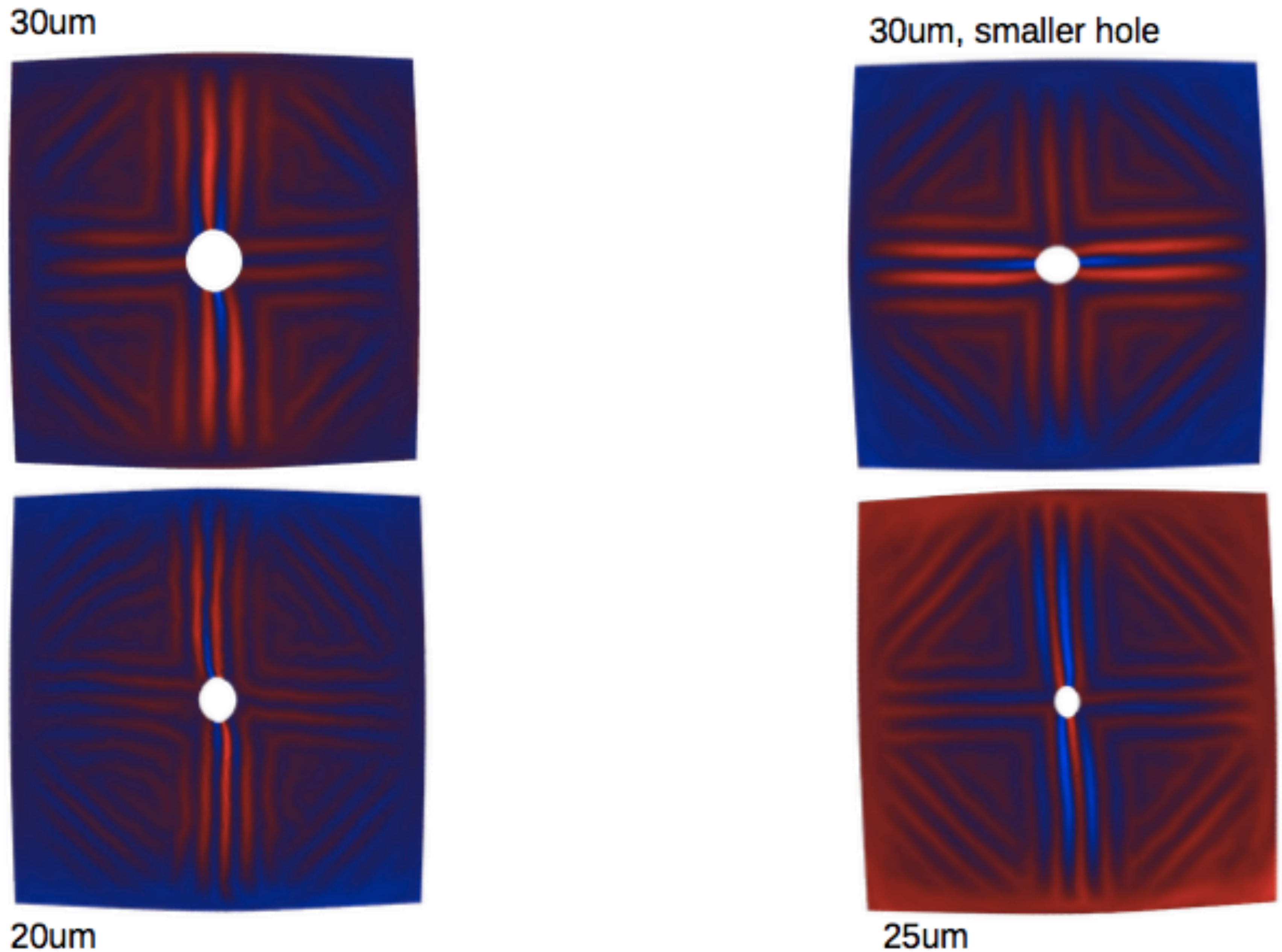
$$\lambda_0 \sim \left(\frac{Et^3}{\rho g}\right)^{1/4}$$



Engineering patterns: stiff lines

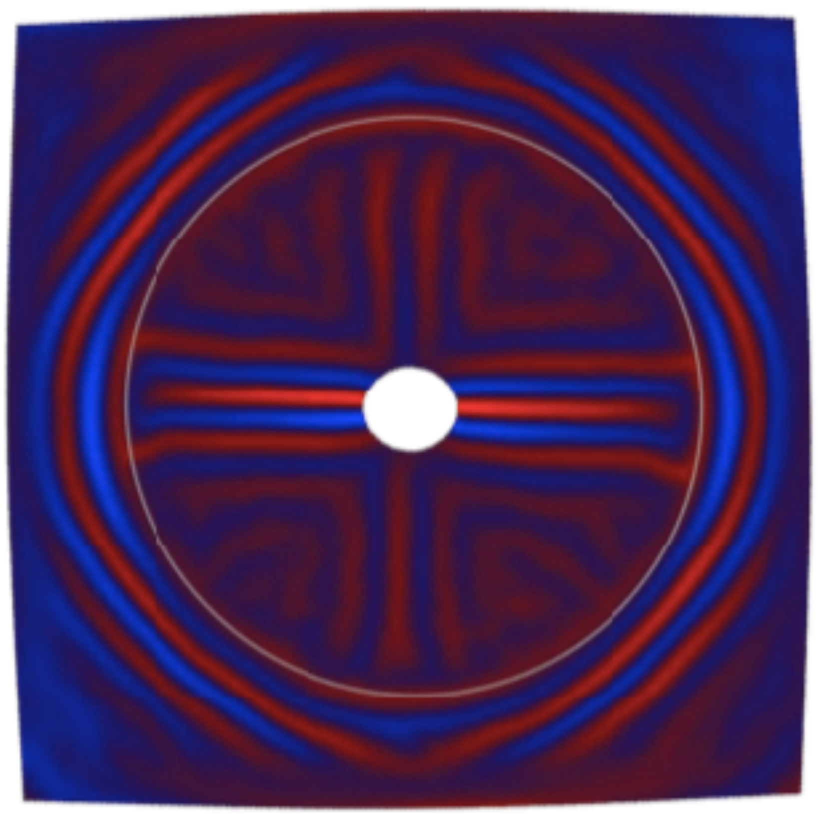


Engineering patterns: holes

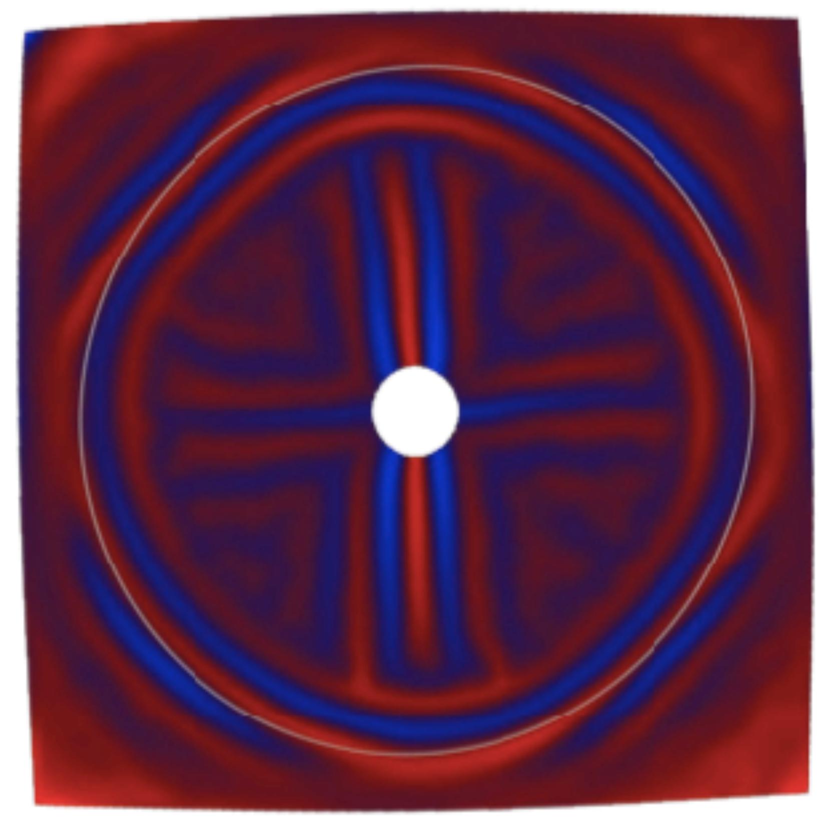


(in preparatio

Engineering patterns: stiff lines and holes

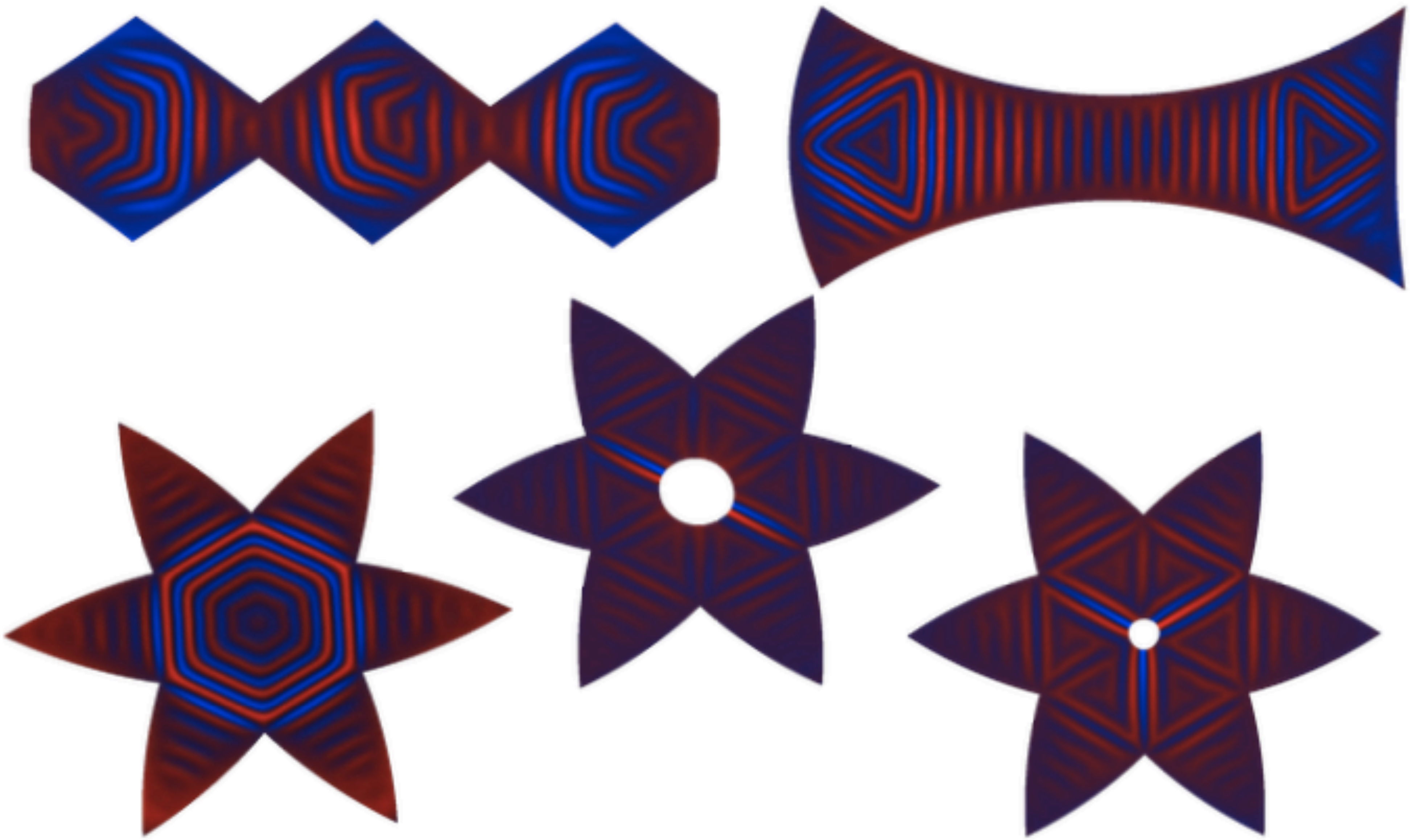


stiff lines: Ed/E=1000
t = 25 um



stiff lines: Ed/E=1000
t = 40 um

Engineering patterns: some more exotic shapes



(in preparation

Summary

Wrinkling of curved shells supported by a fluid substrate

First step towards understanding the zoology of patterns and phenomena e.g.
wrinkles and their lengthscale, domains, folds, dimpling etc

Ευχαριστώ!

H. Aharoni, D. Todorova, O. Albarran, L. Goehring, R. Kamien and EK, "The smectic order of wrinkles" (2017) *Nature communications*

O. Albarran, D. Todorova, EK, L. Goehring, "Elastic instabilities in floating shells" *(in preparation)*

Appendix