Emergence of large scale structure in planetary turbulence

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(Dated: November 28, 2012)

Turbulent flows are observed to self-organize into large scale structures such as zonal jets and coherent vortices. The simplest model that retains the relevant dynamics is a barotropic flow in a β-plane channel with turbulence sustained by random stirring. Non-linear integrations of this model show that as the energy input rate of the forcing is increased, the homogeneity of the flow is broken first by the emergence of non-zonal, coherent, westward propagating structures and second for larger energy input rates by the emergence of zonal jets. We use a non-equilibrium statistical theory, Stochastic Structural Stability Theory (SSST), to address the emergence of these coherent structures. Employing the tools of SSST, we construct a coupled dynamical system governing the joint evolution of the coherent flow and of the second order statistics of the turbulent eddies. We then treat the structural stability of a homogeneous equilibrium with no mean flow analytically. We find that non-zonal and zonal coherent structures appear as the result of structural instability and equilibrate at finite amplitude. We also find that SSST accurately predicts the critical energy input rates for the emergence of both non-zonal and zonal coherent structures in the non-linear integrations as well as the characteristics (scale, amplitude and phase speed) of these structures.

PACS numbers: 47.27.eb, 47.20.Ky, 47.27.De, 52.35.Mw, 89.75.Fb, 92.60.Bh, 92.10.A-

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Turbulence in planetary atmospheres and in plasma flows is commonly observed to be organized into large scale unidirectional (zonal) jets with long-lasting coherent eddies or vortices embedded in them [1, 2]. The jets control the transports of heat and chemical species in planetary atmospheres and separate the high temperature plasma from the cold containment vessel wall in magnetic plasma confinement devices. It is therefore important to understand the mechanisms for the emergence, equilibration and maintenance of these coherent structures. In this Letter we present a theory for the mean turbulent state and the propagation characteristics of both the zonal jets and the non-zonal coherent structures that form in the flow. We then test this theory against non-linear simulations in a simple model of forced planetary and plasma turbulence.

The simplest model that captures the turbulent dynamics and its interaction with the zonal jets and the coherent structures, is the stochastically forced barotropic vorticity equation on a plane tangent to the surface of a rotating planet:

\[ \partial_t \zeta + \psi_x \zeta_y - \psi_y \zeta_x + \beta \psi_x = -r \zeta - \nu \Delta^2 \zeta + f. \]  (1)

The relative vorticity is \( \zeta = \Delta \psi \), where \( \psi \) is the streamfunction, \( \Delta = \partial_{xx} + \partial_{yy} \) is the horizontal Laplacian, \( x \) is in the zonal (east-west) direction and \( y \) is in the meridional (north-south) direction, \( \beta = 2 \Omega \cos \phi_0 / R \) is the gradient of planetary vorticity, \( \Omega \) the rotation rate of the planet, \( \phi_0 \) the latitude of the \( \beta \)-plane and \( R \) is the radius of the planet. Equation (1) governs the dynamics of non-divergent motions at the midlatitudes of the planet. It is also the infinite effective Larmor radius limit of the Charney-Hasegawa-Mima equation that governs drift-wave turbulence in plasma flows. We are assuming linear damping with coefficient, \( r \), representing the Ekman drag induced by the horizontal boundaries and hyper-diffusion with coefficient \( \nu \) that dissipates the energy flowing into unresolved scales. The forcing term \( f \) is necessary to sustain turbulence, and may parameterize processes that have not been included in the dynamics, such as the vorticity forcing from small scale convection. In many previous studies, this exogenous excitation is taken as a temporally delta correlated and spatially homogeneous and isotropic random stirring. We will follow the same forcing protocol in this Letter and consider an isotropic ring forcing that is injecting energy at rate \( \epsilon \) in a narrow ring of wavenumbers of width \( \Delta K_f \) around the total wavenumber \( K_f \).

Equation (1) is solved in a doubly periodic domain of size \( 2\pi \times 2\pi \). The calculations presented in this Letter are for \( \beta = 10 \), \( r = 0.01 \), \( \nu = 2 \cdot 10^{-6} \), \( K_f = 8 \) and \( \Delta K_f = 1 \). The results discussed were verified to be insensitive to the parameters and the forcing protocol chosen. To illustrate some of the characteristics of the turbulent flow and the emergence of coherent structures, we consider two indices. The first is the zonal mean flow index [3] defined as the ratio of the energy of zonal jets over the total energy, \( \text{zmf} = \frac{\sum_k \hat{E}(k=0,l)}{\sum_{k,l} \hat{E}(k,l)} \), where \( \hat{E} \) is the time averaged energy power spectrum of the flow and \( k \),
The second is the non-zonal mean flow index defined in the text as a function of energy input rate $\epsilon/\epsilon_c$ for the non-linear (solid lines) and SSST (dashed lines) integrations. The critical value $\epsilon_c$ is the energy input rate at which the SSST predicts structural instability of the homogeneous turbulent state. Zonal jets emerge for $\epsilon > \epsilon_{nl}$, with $\epsilon_{nl} = 15.9\epsilon_c$ for the parameters in the simulation.

$l$ are the zonal and meridional wavenumbers respectively. The second is the non-zonal mean flow index defined as the ratio of the energy of the non-zonal modes with scales lower than the scale of the forcing over the total energy: $\text{nzmf} = \frac{\sum_{k,l < K_f} \mathcal{E}(k,l) - \sum_{k=0} \mathcal{E}(k=0,l)}{\sum_{k,l} \mathcal{E}(k,l)}$. Figure 1 shows both indices as a function of the energy input rate $\epsilon$.

For $\epsilon$ smaller than a critical value $\epsilon_c$, the turbulent flow is homogeneous and remains translationally invariant in both directions. When $\epsilon > \epsilon_c$, the translational symmetry of the flow is broken and non-zonal structures form with scales larger than the scale of the forcing. The time averaged power spectrum, shown in Fig. 2(a) for $\epsilon = 2.6\epsilon_c$, has pronounced peaks at $(k,l) = (\pm1, \pm5)$ that correspond to coherent structures propagating westward (cf. Fig. 3(a),(b)) with approximately the Rossby wave phase speed for this wave. However, at larger $\epsilon$ the propagation speed of these structures departs from that of Rossby waves. The presence and properties of such non-linear waves in similar simulations, were also reported recently [4]. For $\epsilon > \epsilon_{nl}$ indicated in Fig. 1, robust zonal jets emerge. For example the peaks at $(k,l) = (0, \pm3)$ in the spectrum of Fig. 2(b) correspond to coherent zonal jets (cf. Fig. 3(c),(d)). From Fig. 1 we see that while the jets contain over half of the total energy, substantial power remains in non-zonal structures. Previous studies, refer to the coherent non-zonal structures in this regime in which strong zonal jets dominate the flow (referred to as zonostrophic regime [5]) as satellite modes [6] or zonons [4].

The emergence of the jets has been described in previous studies in terms of an anisotropic inverse energy cascade [7–9], or in terms of inhomogeneous mixing of vorticity [10], or in terms of a direct transfer of energy from small scale waves into the zonal jets, through either non-linear interactions between finite amplitude Rossby waves [11, 12], or through shear straining of the small scale waves by the jet [13]. In addition, the mechanism for the emergence of the non-zonal structures remains elusive. Statistical equilibrium theory applied in the absence of forcing and dissipation, has been able to predict both jets and coherent vortices as maximum entropy structures [14] and a recent study has shown correspondence of the theoretical results with non-linear simulations in the limit of weak forcing and dissipation [15]. However, the relevance of these results in planetary and plasma flows that are strongly forced and dissipated and are therefore out of equilibrium remains to be shown. In this Letter we present results of an alternative, non-equilibrium statistical theory, that is termed as Stochastic Structural
The linear operator \( A_i = -U_i \partial x_i - V_i \partial y_i - (\beta + Z_y) \partial x_i \Delta^{-1} + Z_{x_i} \partial y_i \Delta^{-1} - r - \nu \Delta^2 \),

(4)

acts at the points \( x_i = (x_i, y_i) \) and governs the dynamics of linear perturbations about the instantaneous mean flow \( U = [U, V] = [-\partial_y \langle \psi \rangle, \partial_x \langle \psi \rangle] \). In (3), \( \Xi \) contains the covariance of the external forcing and terms related to third order cumulants. A second order closure is obtained if the third order cumulant is ignored and \( \Xi \) is set to be the spatial covariance of the stochastic forcing \( f \). In most earlier studies of SSST or CE2, the ensemble average was assumed to represent a zonal average. In this Letter, we adopt the more general interpretation that the ensemble average represents a Reynolds average with the ensemble mean representing coarse-graining. This interpretation has been adopted in the SSST study of turbulence in baroclinic flows [22, 23]. With this interpretation of the ensemble mean, the SSST system (2)-(3) provides the statistical dynamics of the interaction of the ensemble average field, which can be a zonal or a non-zonal coherent structure, with the fine-grained field, represented in the theory through its covariance \( C \). The SSST system defines an autonomous dynamics and its fixed points define a new type of turbulent statistical equilibria. While these equilibria formally exist only in the infinite ensemble limit, it has been shown that their characteristics manifest even in single realizations of the turbulent system. The structural stability of these turbulent equilibria can be addressed in SSST by studying their stability. Specifically, when an equilibrium of the SSST equations becomes unstable, the turbulent flow bifurcates to a different attractor. This theory therefore predicts parameters of the physical system which can lead to abrupt reorganization of the turbulent flow.

The SSST equations (2)-(3), admit for \( \nu = 0 \) the equilibrium

\[
U^E = V^E = 0, \quad C^E = \frac{\Xi}{2r},
\]

(5)

that has zero large scale flow and a homogeneous eddy field with spatial covariance dictated directly from the forcing. We now investigate the SSST stability of this equilibrium as a function of the energy input rate, \( \epsilon \), and relate the outcome of this stability analysis to the results in the non-linear simulations of (1). The stability of the homogeneous equilibrium (5) is assessed by introducing small perturbations of the form \( [\delta Z_{nm}, \delta C_{nm}] e^{in(x_1 + x_2)/2 + im(y_1 + y_2)/2.24} \) to the SSST equations (2)-(3) linearized about the equilibrium (5) and calculating the eigenvalue \( \sigma \). When \( \Re(\sigma) > 0 \), the structure with \( (x, y) \) wavenumbers \( (n, m) \) is unstable and will emerge. It can be shown that the eigenvalue \( \sigma \) satisfies the equation

\[
\epsilon = \frac{2\pi r K f \Delta K_f}{2} \sum_{k,l} \left( mk - nl \right) \left( (m^2 - n^2)K^2 - N^2 \right) + \left( 2\pi r K f \Delta K_f \right) \left( K^2 + N^2 \right)
\]

where \( K^2 = k^2 + l^2, K_f^2 = (k + n)^2 + (l + m)^2, N^2 = n^2 + m^2, k = k + n/2 \) and \( l = l + m/2 \) and the summation is over integers \( (k, l) \) satisfying \( |K - K_f| < \Delta K_f \) [24]. This stability equation reduces for \( n = 0 \) to the stability equation for the emergence of zonal flows [3, 25].

For small values of the energy input rate, the growth rate \( \Re(\sigma) \) is negative for all structures and the statistically homogeneous state is stable and persists. For a critical value \( \epsilon_c \), the homogeneous flow becomes SSST unstable, symmetry breaking occurs and coherent structures emerge. The growth rates as a function of the wavenumbers of the structure \( (n, m) \) are shown in Fig. 4. For \( \epsilon/\epsilon_c = 2.6 \), the structure with the largest growth rate, is non-zonal with \( ||(n, m)|| = (1, 5) \) and has \( \Im(\sigma) = 0.4 \), implying retrograde propagation of the eigenstructure. Note also that for this energy input rate, the zonal flows are SSST stable and are not expected to emerge. For \( \epsilon/\epsilon_c = 30 \), both zonal jets and non-zonal structures are unstable, but the zonal jets have smaller growth rates compared to the non-zonal structures [26]. The zonal jets have \( \Im(\sigma) = 0 \), implying that they grow in place, in contrast to the non-zonal coherent structures that always propagate...
Significant power at (\(\epsilon > \epsilon_c\)) dominates structures in the non-linear simulations. For equilibria obtained when the emergence of non-zonal structures in the non-linear stability analysis accurately predicts the critical \(\epsilon\) to the non-linear simulations. First of all, the SSST system (2)-(3) shows that for \(\epsilon > \epsilon_c\) structures equilibrate at larger amplitudes. However, for Fig. 1. As the energy input rate increases, the non-zonal structures typically equilibrate at finite amplitude after SSST integrations that are shown in these equilibrated structures are the zmf and nzmf indices calculated for the SSST integrations for \(\epsilon/\epsilon_c = 2.6\). The thick dashed lines in (b) show the phase speed obtained from the stability equation (6).

in the retrograde direction. Numerical integration of the SSST system (2)-(3), shows that for \(\epsilon > \epsilon_c\) the unstable structures typically equilibrate at finite amplitude after an initial period of exponential growth. As a result (2)-(3) admit in general multiple equilibria. Figure 5(a) shows the equilibrium structure with the largest domain of attraction, when \(\epsilon/\epsilon_c = 2.6\). This structure coincides in this case with the finite amplitude equilibrium of the fastest growing \((|n|, |m|) = (1, 5)\) eigenfunction and propagates as illustrated in Fig. 5(b) in the retrograde direction with a speed approximately equal to the phase speed of this unstable eigenstructure. A proxy for the amplitude of these equilibrated structures are the zmf and nzmf indices calculated for the SSST integrations that are shown in Fig. 1. As the energy input rate increases, the non-zonal structures equilibrate at larger amplitudes. However, for \(\epsilon > \epsilon_{nL}\), the equilibria with the largest domain of attraction are zonal jets and the flow is dominated by these structures (cf. Fig. 1).

The results of the SSST analysis are now compared to the non-linear simulations. First of all, the SSST stability analysis accurately predicts the critical \(\epsilon_c\) for the emergence of non-zonal structures in the non-linear simulations as shown in Fig. 1. The finite amplitude equilibria obtained when \(\epsilon > \epsilon_c\) also correspond to the dominant structures in the non-linear simulations. For \(\epsilon/\epsilon_c = 2.6\), the spectra in the nonlinear simulations show significant power at \((|n|, |m|) = (1, 5)\), the structure predicted by SSST to have the largest domain of attraction. Remarkably, the phase speed of the \((1, 5)\) waves observed in the non-linear simulations and the amplitude of these structures as illustrated by the nzmf index are approximately equal to the phase speed and amplitude of the corresponding SSST equilibrium structure (cf. Figs. 1 and 3). For \(\epsilon > \epsilon_{nL}\), in both nonlinear and SSST simulations zonal jets emerge and the power of the non-zonal structures is substantially reduced. Comparison of the number of jets and their amplitude between the SSST and the nonlinear simulations also shows good agreement. This demonstrates that the SSST system can predict the amplitude and characteristics of both the non-zonal and the zonal structures that emerge in the turbulent flow.

While the regime transition that occurs at \(\epsilon_c\) is predicted by the stability equation (6), the second transition, which is associated with the emergence of zonal flows and occurs at \(\epsilon_{nL}\), is more intriguing. The stability equation (6) predicts that the zonal structures become unstable at \(\epsilon_{zL} = 4\epsilon_c < \epsilon_{nL}\). In previous studies of SSST dynamics restricted to the interaction between zonal flows and turbulence, these initially unstable structures were found to be saddles that are stable to zonal but unstable to non-zonal perturbations. The threshold for the emergence of jets in the SSST simulations and in the nonlinear simulations is therefore determined as the energy input rate at which an SSST stable, finite amplitude zonal jet equilibrium exists. A method to correctly obtain the critical input rate \(\epsilon_{nL}\) has been recently developed [20]. It starts by recognizing that for \(\epsilon_c < \epsilon < \epsilon_{zL}\), the spectrum in the turbulent flow is modified \((C^E \neq \Xi/2r)\) and is given by the covariance \(\tilde{C}^E\) associated with the finite amplitude equilibria similar to the ones shown in Fig. 5. If this modification is taken into account, then the stability analysis around the equilibrium \([U_E, \tilde{C}^E]\) correctly predicts \(\epsilon_{nL}\).

In summary, SSST constitutes a theory that predicts the two regime transitions in the turbulent flow as the energy input rate is increased: the symmetry breaking of the homogeneous equilibrium with the emergence of coherent, propagating non-zonal structures and the emergence of zonal jets. It also predicts the characteristics of the emerging structures (their scales and their phase speed), as well as their amplitude. The relation of the non-zonal coherent structures to westward propagating vortex rings in the ocean and coherent vortices in planetary atmospheres will be the subject of future research. The fact though that these structures are more susceptible to instabilities compared to zonal jets, shows that the prevalent structures in planetary flows are zonal jets.
ACKNOWLEDGMENTS

This research was supported by the EU FP-7 under the PIRG03-GA-2008-230958 Marie Curie Grant. The authors acknowledge the hospitality of the Aspen Center for Physics supported by the NSF (under grant No. 1066293), where part of this work was done. The authors would also like to thank Navid Constantinou and Brian Farrell for fruitful discussions.

[24] Hyperdiffusion can be readily included in (6) in order to obtain correspondence with the nonlinear simulations.
[26] This result always holds when \( \beta > \beta_{\text{min}} \). For the isotropic forcing considered this is \( \beta_{\text{min}} = 4.5K_{fr} \).