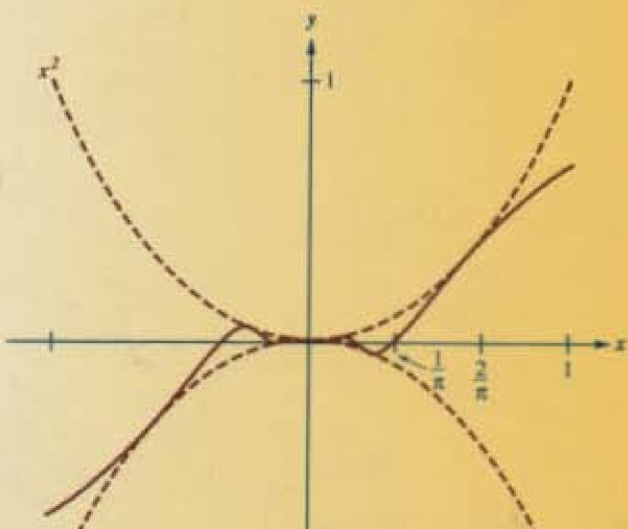
Kenneth A. Ross

ELEMENTARY ANALYSIS: THE THEORY OF CALCULUS



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(continued after index)

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Elementary Analysis

The Theory of Calculus

Springer

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Preface

Attention: Starting with the 12th printing, this book has been set in LaTeX so that the book will be more readable. In particular, there is less material on each page, so there are more pages. However, these are the only changes from previous printings except that I've updated the bibliography.

Preface to the First Edition

A study of this book, and especially the exercises, should give the reader a thorough understanding of a few basic concepts in analysis such as continuity, convergence of sequences and series of numbers, and convergence of sequences and series of functions. An ability to read and write proofs will be stressed. A precise knowledge of definitions is essential. The beginner should memorize them; such memorization will help lead to understanding.

Chapter 1 sets the scene and, except for the completeness axiom, should be more or less familiar. Accordingly, readers and instructors are urged to move quickly through this chapter and refer back to it when necessary. The most critical sections in the book are Sections 7 through 12 in Chapter 2. If these sections are thoroughly digested and understood, the remainder of the book should be smooth sailing.

The first four chapters form a unit for a short course on analysis. I cover these four chapters (except for the optional sections and Section 20) in about 38 class periods; this includes time for quizzes and examinations. For such a short course, my philosophy is that the students are relatively comfortable with derivatives and integrals but do not really understand sequences and series, much less sequences and series of functions, so Chapters 1–4 focus on these topics. On two or three occasions I draw on the Fundamental Theorem of Calculus or the Mean Value Theorem, which appear later in the book, but of course these important theorems are at least discussed in a standard calculus class.

In the early sections, especially in Chapter 2, the proofs are very detailed with careful references for even the most elementary facts. Most sophisticated readers find excessive details and references a hindrance (they break the flow of the proof and tend to obscure the main ideas) and would prefer to check the items mentally as they proceed. Accordingly, in later chapters the proofs will be somewhat less detailed, and references for the simplest facts will often be omitted. This should help prepare the reader for more advanced books which frequently give very brief arguments.

Mastery of the basic concepts in this book should make the analysis in such areas as complex variables, differential equations, numerical analysis, and statistics more meaningful. The book can also serve as a foundation for an in-depth study of real analysis given in books such as [2], [25], [26], [33], [36], and [38] listed in the bibliography.

Readers planning to teach calculus will also benefit from a careful study of analysis. Even after studying this book (or writing it) it will not be easy to handle questions such as "What is a number?", but at least this book should help give a clearer picture of the subtleties to which such questions lead.

The optional sections contain discussions of some topics that I think are important or interesting. Sometimes the topic is dealt with lightly, and suggestions for further reading are given. Though these sections are not particularly designed for classroom use, I hope that some readers will use them to broaden their horizons and see how this material fits in the general scheme of things.

I have benefitted from numerous helpful suggestions from my colleagues Robert Freeman, William Kantor, Richard Koch, and John Leahy, and from Timothy Hall, Gimli Khazad, and Jorge López. I have also had helpful conversations with my wife Lynn concerning grammar and taste. Of course, remaining errors in grammar and mathematics are the responsibility of the author.

Several users have supplied me with corrections and suggestions that I've incorporated in subsequent printings. I thank them all, including Robert Messer of Albion College who caught a subtle error in the proof of Theorem 12.1.

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37.13 Trigonometric Functions. Either approach 37.2 or 37.3 can be modified to rigorously develop the trigonometric functions. They can also be developed using power series; see [50], Chapter 8. The development of the trigonometric functions analogous to approach 37.3 can proceed as follows. Since we believe

$$\arcsin x = \int_0^x (1-t^2)^{-1/2}\,dt,$$

we can define $A(x)$ as this integral and obtain $\sin x$ from this. Then $\cos x$ and $\tan x$ are easy to obtain. In this development, the number π is defined to be $2\int_0^1 (1-t^2)^{-1/2}\,dt$.

In the exercises, use results proved in Theorem 37.4 and subsequent theorems but not the material discussed in 37.1 and 37.2.

Exercises

37.1. Prove directly that

$$\int_1^{yz} \frac{1}{t}\,dt = \int_1^{y} \frac{1}{t}\,dt + \int_1^{z} \frac{1}{t}\,dt \quad \text{for} \quad y, z \in (0, \infty).$$

37.2. Prove (iii) of Theorem 37.8.

37.3. Prove Theorem 37.8.

37.4. Consider $b > 0$, $b \neq 1$. Prove $\log_b y = \frac{L(y)}{L(b)}$ for $y \in (0, \infty)$.

37.5. Let p be any real number and define $f(x) = x^p$ for $x > 0$. Show that f is differentiable and $f'(x) = px^{p-1}$; compare Exercise 29.15. Hint: $f(x) = E(pL(x))$.

37.6. Show that $x^p y^p = (xy)^p$ for $p \in \mathbb{R}$ and positive x, y.

37.7. (a) Show that if $B(x) = b^x$, then $B'(x) = (\log_e b)b^x$.

(b) Find the derivative of $\log_b$.

37.8. For $x > 0$, let $F(x) = x^x$. Show that $F'(x) = (1 + \log_e x) \cdot x^x$.

Contents

CHAPTER 1 Introduction

The underlying space for all the analysis in this book is the set of real numbers. In this chapter we set down some basic properties of this set. These properties will serve as our axioms in the sense that it is possible to derive all the properties of the real numbers using only these axioms. However, we will avoid getting bogged down in this endeavor. Some readers may wish to refer to the appendix on set notation.

§1 The Set $\mathbb{N}$ of Natural Numbers

We denote the set $\{1, 2, 3, \ldots\}$ of all *natural numbers* by $\mathbb{N}$. Elements of $\mathbb{N}$ will also be called *positive integers*. Each natural number n has a successor, namely $n+1$. Thus the successor of 2 is 3, and 37 is the successor of 36. You will probably agree that the following properties of $\mathbb{N}$ are obvious; at least the first four are.

N1. 1 belongs to $\mathbb{N}$.
N2. If n belongs to $\mathbb{N}$, then its successor $n+1$ belongs to $\mathbb{N}$.
N3. 1 is not the successor of any element in $\mathbb{N}$.

N4. If n and m in $\mathbb{N}$ have the same successor, then $n = m$.
N5. A subset of $\mathbb{N}$ which contains 1, and which contains $n + 1$ whenever it contains n, must equal $\mathbb{N}$.

Properties N1 through N5 are known as the *Peano Axioms* or *Peano Postulates*. It turns out that most familiar properties of $\mathbb{N}$ can be proved based on these five axioms; see [3] or [28].

Let's focus our attention on axiom N5, the one axiom that may not be obvious. Here is what the axiom is saying. Consider a subset S of $\mathbb{N}$ as described in N5. Then 1 belongs to S. Since S contains $n+1$ whenever it contains n, it follows that S must contain $2 = 1 + 1$. Again, since S contains $n+1$ whenever it contains n, it follows that S must contain $3 = 2+1$. Once again, since S contains $n+1$ whenever it contains n, it follows that S must contain $4 = 3+1$. We could continue this monotonous line of reasoning to conclude that S contains any number in $\mathbb{N}$. Thus it seems reasonable to conclude that $S = \mathbb{N}$. It is this reasonable conclusion that is asserted by axiom N5.

Here is another way to view axiom N5. Assume axiom N5 is false. Then $\mathbb{N}$ contains a set S such that

(i) $1 \in S$,
(ii) if $n \in S$, then $n + 1 \in S$,

and yet $S \neq \mathbb{N}$. Consider the smallest member of the set $\{n \in \mathbb{N} : n \notin S\}$, call it n_0. Since (i) holds, it is clear that $n_0 \neq 1$. So n_0 must be a successor to some number in $\mathbb{N}$, namely $n_0 - 1$. We must have $n_0 - 1 \in S$ since n_0 is the smallest member of $\{n \in \mathbb{N} : n \notin S\}$. By (ii), the successor of $n_0 - 1$, namely n_0, must also be in S, which is a contradiction. This discussion may be plausible, but we emphasize that we have not *proved* axiom N5 using the successor notion and axioms N1 through N4, because we implicitly used two unproven facts. We assumed that every nonempty subset of $\mathbb{N}$ contains a least element and we assumed that if $n_0 \neq 1$ then n_0 is the successor to some number in $\mathbb{N}$.

Axiom N5 is the basis of *mathematical induction*. Let $P_1, P_2, P_3, \ldots$ be a list of statements or propositions that may or may not be true. The principle of mathematical induction asserts that all the statements $P_1, P_2, P_3, \ldots$ are true provided

(I_1) P_1 is true,

(I_2) P_{n+1} is true whenever P_n is true.

We will refer to (I_1), i.e., the fact that P_1 is true, as the *basis for induction* and we will refer to (I_2) as the *induction step*. For a sound proof based on mathematical induction, properties (I_1) and (I_2) must both be verified. In practice, (I_1) will be easy to check.

Example 1
Prove $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ for natural numbers n.

Solution
Our nth proposition is

$$P_n: \text{"}1 + 2 + \cdots + n = \tfrac{1}{2}n(n+1).\text{"}$$

Thus P_1 asserts that $1 = \frac{1}{2}\cdot 1(1+1)$, P_2 asserts that $1+2 = \frac{1}{2}\cdot 2(2+1)$, P_{37} asserts that $1+2+\cdots+37 = \frac{1}{2}\cdot 37(37+1) = 703$, etc. In particular, P_1 is a true assertion which serves as our basis for induction.

For the induction step, suppose that P_n is true. That is, we suppose

$$1 + 2 + \cdots + n = \tfrac{1}{2}n(n+1)$$

is true. Since we wish to prove P_{n+1} from this, we add $n+1$ to both sides to obtain

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \tfrac{1}{2}n(n+1) + (n+1) \\ &= \tfrac{1}{2}[n(n+1) + 2(n+1)] = \tfrac{1}{2}(n+1)(n+2) \\ &= \tfrac{1}{2}(n+1)((n+1)+1.) \end{aligned}$$

Thus P_{n+1} holds if P_n holds. By the principle of mathematical induction, we conclude that P_n is true for all n. □

We emphasize that prior to the last sentence of our solution we did *not* prove "P_{n+1} is true." We merely proved an implication: "if P_n is true, then P_{n+1} is true." In a sense we proved an infinite number of assertions, namely: P_1 is true; if P_1 is true then P_2 is true; if P_2 is true then P_3 is true; if P_3 is true then P_4 is true; etc. Then we applied mathematical induction to conclude P_1 is true, P_2 is true, P_3 is true, P_4 is true, etc. We also confess that formulas like the one just proved are easier to prove than to derive. It can be a tricky matter to guess such a result. Sometimes results such as this are discovered by trial and error.

Example 2
All numbers of the form $7^n - 2^n$ are divisible by 5.

Solution
More precisely, we show that $7^n - 2^n$ is divisible by 5 for each $n \in \mathbb{N}$. Our nth proposition is

$$P_n: \text{"}7^n - 2^n \quad \text{is divisible by} \quad 5.\text{"}$$

The basis for induction P_1 is clearly true, since $7^1 - 2^1 = 5$. For the induction step, suppose that P_n is true. To verify P_{n+1}, we write

$$7^{n+1} - 2^{n+1} = 7^{n+1} - 7 \cdot 2^n + 7 \cdot 2^n - 2 \cdot 2^n = 7[7^n - 2^n] + 5 \cdot 2^n.$$

Since $7^n - 2^n$ is a multiple of 5 by the induction hypothesis, it follows that $7^{n+1} - 2^{n+1}$ is also a multiple of 5. In fact, if $7^n - 2^n = 5m$, then $7^{n+1} - 2^{n+1} = 5 \cdot [7m + 2^n]$. We have shown that P_n implies P_{n+1}, so the induction step holds. An application of mathematical induction completes the proof. □

Example 3
Show that $|\sin nx| \le n|\sin x|$ for all natural numbers n and all real numbers x.

Solution
Our nth proposition is

$$P_n: \text{"}|\sin nx| \le n|\sin x| \quad \text{for all real numbers} \quad x.\text{"}$$

The basis for induction is again clear. Suppose P_n is true. We apply the addition formula for sine to obtain

$$|\sin(n+1)x| = |\sin(nx + x)| = |\sin nx \cos x + \cos nx \sin x|.$$

Now we apply the Triangle Inequality and properties of the absolute value [see 3.7 and 3.5] to obtain

$$|\sin(n+1)x| \le |\sin nx| \cdot |\cos x| + |\cos nx| \cdot |\sin x|.$$

Since $|\cos y| \le 1$ for all y we see that

$$|\sin(n+1)x| \le |\sin nx| + |\sin x|.$$

Now we apply the induction hypothesis P_n to obtain

$$|\sin(n+1)x| \le n|\sin x| + |\sin x| = (n+1)|\sin x|.$$

Thus P_{n+1} holds. Finally, the result holds for all n by mathematical induction. □

Exercises

1.1. Prove $1^2+2^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all natural numbers n.

1.2. Prove $3+11+\cdots+(8n-5) = 4n^2-n$ for all natural numbers n.

1.3. Prove $1^3+2^3+\cdots+n^3 = (1+2+\cdots+n)^2$ for all natural numbers n.

1.4. **(a)** Guess a formula for $1+3+\cdots+(2n-1)$ by evaluating the sum for $n = 1, 2, 3$, and 4. [For $n = 1$, the sum is simply 1.]

(b) Prove your formula using mathematical induction.

1.5. Prove $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n} = 2-\frac{1}{2^n}$ for all natural numbers n.

1.6. Prove that $(11)^n - 4^n$ is divisible by 7 when n is a natural number.

1.7. Prove that $7^n - 6n - 1$ is divisible by 36 for all positive integers n.

1.8. The principle of mathematical induction can be extended as follows. A list $P_m, P_{m+1}, \ldots$ of propositions is true provided (i) P_m is true, (ii) P_{n+1} is true whenever P_n is true and $n \ge m$.

(a) Prove that $n^2 > n+1$ for all integers $n \ge 2$.

(b) Prove that $n! > n^2$ for all integers $n \ge 4$. [Recall that $n! = n(n-1)\cdots 2\cdot 1$; for example, $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$.]

1.9. **(a)** Decide for which integers the inequality $2^n > n^2$ is true.

(b) Prove your claim in (a) by mathematical induction.

1.10. Prove $(2n+1)+(2n+3)+(2n+5)+\cdots+(4n-1) = 3n^2$ for all positive integers n.

1.11. For each $n \in \mathbb{N}$, let P_n denote the assertion "n^2+5n+1 is an even integer."

(a) Prove that P_{n+1} is true whenever P_n is true.

(b) For which n is P_n actually true? What is the moral of this exercise?

1.12. For $n \in \mathbb{N}$, let $n!$ [read "n factorial"] denote the product $1 \cdot 2 \cdot 3 \cdots n$. Also let $0! = 1$ and define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{for} \quad k = 0, 1, \ldots, n.$$

The *binomial theorem* asserts that

$$\begin{aligned}(a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots \\ &\quad + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n \\ &= a^n + na^{n-1}b + \tfrac{1}{2}n(n-1)a^{n-2}b^2 + \cdots + nab^{n-1} + b^n.\end{aligned}$$

(a) Verify the binomial theorem for $n = 1$, 2, and 3.

(b) Show that $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ for $k = 1, 2, \ldots, n$.

(c) Prove the binomial theorem using mathematical induction and part (b).

§2 The Set $\mathbb{Q}$ of Rational Numbers

Small children first learn to add and to multiply natural numbers. After subtraction is introduced, the need to expand the number system to include 0 and negative numbers becomes apparent. At this point the world of numbers is enlarged to include the set $\mathbb{Z}$ of all *integers*. Thus we have $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$.

Soon the space $\mathbb{Z}$ also becomes inadequate when division is introduced. The solution is to enlarge the world of numbers to include all fractions. Accordingly, we study the space $\mathbb{Q}$ of all *rational numbers*, i.e., numbers of the form $\frac{m}{n}$ where $m, n \in \mathbb{Z}$ and $n \neq 0$. Note that $\mathbb{Q}$ contains all terminating decimals such as $1.492 = \frac{1492}{1000}$. The connection between decimals and real numbers is discussed in 10.3 and §16. The space $\mathbb{Q}$ is a highly satisfactory algebraic system in which the basic operations addition, multiplication, subtraction and division can be fully studied. No system is perfect, however, and $\mathbb{Q}$

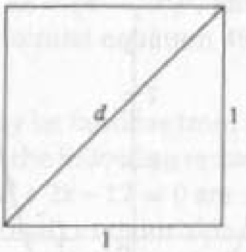

FIGURE 2.1

is inadequate in some ways. In this section we will consider the defects of Q. In the next section we will stress the good features of Q and then move on to the system of real numbers.

The set Q of rational numbers is a very nice algebraic system until one tries to solve equations like $x^2 = 2$. It turns out that no rational number satisfies this equation, and yet there are good reasons to believe that some kind of number satisfies this equation. Consider, for example, a square with sides having length one; see Figure 2.1. If d represents the length of the diagonal, then from geometry we know that $1^2 + 1^2 = d^2$, i.e., $d^2 = 2$. Apparently there is a positive length whose square is 2, which we write as $\sqrt{2}$. But $\sqrt{2}$ cannot be a rational number, as we will show in Example 2. Of course, $\sqrt{2}$ can be approximated by rational numbers. There are rational numbers whose squares are close to 2; for example, $(1.4142)^2 = 1.99996164$ and $(1.4143)^2 = 2.00024449$.

It is evident that there are lots of rational numbers and yet there are "gaps" in Q. Here is another way to view this situation. Consider the graph of the polynomial $x^2 - 2$ in Figure 2.2. Does the graph of $x^2 - 2$ cross the x-axis? We are inclined to say it does, because when we draw the x-axis we include "all" the points. We allow no "gaps." But notice that the graph of $x^2 - 2$ slips by all the rational numbers on the x-axis. The x-axis is our picture of the number line, and the set of rational numbers again appears to have significant "gaps."

There are even more exotic numbers such as π and e that are not rational numbers, but which come up naturally in mathematics. The number π is basic to the study of circles and spheres, and e arises in problems of exponential growth.

We return to $\sqrt{2}$. This is an example of what is called an algebraic number because it satisfies the equation $x^2 - 2 = 0$.

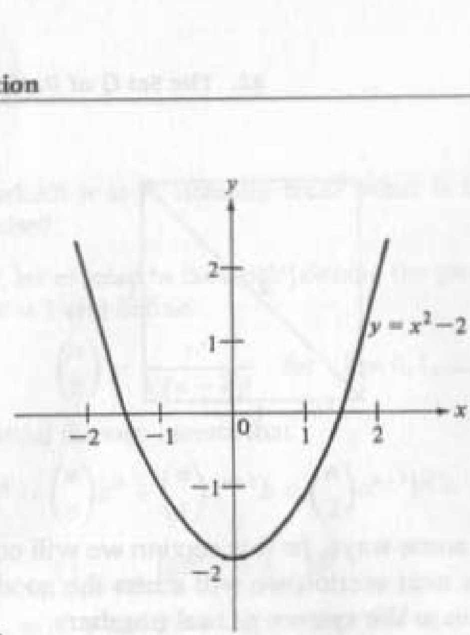

FIGURE 2.2

2.1 Definition.
A number is called an *algebraic number* if it satisfies a polynomial equation

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$$

where the coefficients $a_0, a_1, \ldots, a_n$ are integers, $a_n \neq 0$ and $n \geq 1$.

Rational numbers are always algebraic numbers. In fact, if $r = \frac{m}{n}$ is a rational number [$m, n \in \mathbb{Z}$ and $n \neq 0$], then it satisfies the equation $nx - m = 0$. Numbers defined in terms of $\sqrt{\ }$, $\sqrt[3]{\ }$, etc. [or fractional exponents, if you prefer] and ordinary algebraic operations on the rational numbers are invariably algebraic numbers.

Example 1
$\frac{4}{17}$, $3^{1/2}$, $(17)^{1/3}$, $(2 + 5^{1/3})^{1/2}$ and $((4 - 2 \cdot 3^{1/2})/7)^{1/2}$ all represent algebraic numbers. In fact, $\frac{4}{17}$ is a solution of $17x - 4 = 0$, $3^{1/2}$ represents a solution of $x^2 - 3 = 0$, and $(17)^{1/3}$ represents a solution of $x^3 - 17 = 0$. The expression $a = (2 + 5^{1/3})^{1/2}$ means $a^2 = 2 + 5^{1/3}$ or $a^2 - 2 = 5^{1/3}$ so that $(a^2 - 2)^3 = 5$. Therefore we have $a^6 - 6a^4 + 12a^2 - 13 = 0$ which shows that $a = (2 + 5^{1/3})^{1/2}$ satisfies the polynomial equation $x^6 - 6x^4 + 12x^2 - 13 = 0$. Similarly, the expression $b = ((4 - 2 \cdot 3^{1/2})/7)^{1/2}$ leads to $7b^2 = 4 - 2 \cdot 3^{1/2}$, hence

$2 \cdot 3^{1/2} = 4 - 7b^2$, hence $12 = (4 - 7b^2)^2$, hence $49b^4 - 56b^2 + 4 = 0$. Thus b satisfies the polynomial equation $49x^4 - 56x^2 + 4 = 0$.

The next theorem may be familiar from elementary algebra. It is the theorem that justifies the following remarks: the only possible rational solutions of $x^3 - 7x^2 + 2x - 12 = 0$ are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$, so the only possible (rational) monomial factors of $x^3 - 7x^2 + 2x - 12$ are $x - 1$, $x + 1$, $x - 2$, $x + 2$, $x - 3$, $x + 3$, $x - 4$, $x + 4$, $x - 6$, $x + 6$, $x - 12$, $x + 12$. We won't pursue these algebraic problems; we merely made these observations in the hope that they would be familiar.

The next theorem also allows one to prove that algebraic numbers that do not look like rational numbers are not rational numbers. Thus $\sqrt{4}$ is obviously a rational number, while $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc. turn out to be nonrational. See the examples following the theorem. Recall that an integer k is a *factor* of an integer m or *divides* m if $\frac{m}{k}$ is also an integer. An integer $p \geq 2$ is a *prime* provided the only positive factors of p are 1 and p. It can be shown that every positive integer can be written as a product of primes and that this can be done in only one way, except for the order of the factors.

2.2 Rational Zeros Theorem.
Suppose that $a_0, a_1, \ldots, a_n$ are integers and that r is a rational number satisfying the polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{1}$$

where $n \geq 1$, $a_n \neq 0$ and $a_0 \neq 0$. Write $r = \frac{p}{q}$ where p, q are integers having no common factors and $q \neq 0$. Then q divides a_n and p divides a_0.

In other words, the only rational *candidates* for solutions of (1) have the form $\frac{p}{q}$ where p divides a_0 and q divides a_n.

Proof
We are given

$$a_n \left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0.$$

We multiply through by q^n and obtain

$$a_np^n + a_{n-1}p^{n-1}q + a_{n-2}p^{n-2}q^2 + \cdots + a_2p^2q^{n-2} + a_1pq^{n-1} + a_0q^n = 0. \tag{2}$$

If we solve for a_np^n, we obtain

$$a_np^n = -q[a_{n-1}p^{n-1} + a_{n-2}p^{n-2}q + \cdots + a_2p^2q^{n-3} + a_1pq^{n-2} + a_0q^{n-1}].$$

It follows that q divides a_np^n. But p and q have no common factors, so q must divide a_n. [Here are more details: p can be written as a product of primes $p_1p_2\cdots p_k$ where the p_i's need not be distinct. Likewise q can be written as a product of primes $q_1q_2\cdots q_l$. Since q divides a_np^n, the quantity $\frac{a_np^n}{q} = \frac{a_np_1^n\cdots p_k^n}{q_1\cdots q_l}$ must be an integer. Since no p_i can equal any q_j, the unique factorization of a_n as a product of primes must include the product $q_1q_2\cdots q_l$. Thus q divides a_n.]

Now we solve (2) for a_0q^n and obtain

$$a_0q^n = -p[a_np^{n-1} + a_{n-1}p^{n-2}q + a_{n-2}p^{n-3}q^2 + \cdots + a_2pq^{n-2} + a_1q^{n-1}].$$

Thus p divides a_0q^n. Since p and q have no common factors, p must divide a_0. ∎

Example 2
$\sqrt{2}$ cannot represent a rational number.

Proof
By Theorem 2.2 the only rational numbers that could possibly be solutions of $x^2 - 2 = 0$ are $\pm1, \pm2$. [Here $n = 2$, $a_2 = 1$, $a_1 = 0$, $a_0 = -2$. So rational solutions must have the form $\frac{p}{q}$ where p divides $a_0 = -2$ and q divides $a_2 = 1$.] One can substitute each of the four numbers $\pm1, \pm2$ into the equation $x^2 - 2 = 0$ to quickly eliminate them as possible solutions of the equation. Since $\sqrt{2}$ represents a solution of $x^2 - 2 = 0$, it cannot represent a rational number. ∎

Example 3
$\sqrt{17}$ cannot represent a rational number.

Proof
The only possible rational solutions of $x^2 - 17 = 0$ are $\pm1, \pm17$ and none of these numbers are solutions. ∎

Example 4
$6^{1/3}$ cannot represent a rational number.

Proof
The only possible rational solutions of $x^3 - 6 = 0$ are $\pm 1, \pm 2, \pm 3, \pm 6$. It is easy to verify that none of these eight numbers satisfies the equation $x^3 - 6 = 0$. ■

Example 5
$a = (2 + 5^{1/3})^{1/2}$ does not represent a rational number.

Proof
In Example 1 we showed that a represents a solution of $x^6 - 6x^4 + 12x^2 - 13 = 0$. By Theorem 2.2, the only possible rational solutions are $\pm 1, \pm 13$. When $x = 1$ or -1, the left hand side of the equation is -6 and when $x = 13$ or -13, the left hand side of the equation turns out to equal 4,657,458. This last computation could be avoided by using a little common sense. Either observe that a is "obviously" bigger than 1 and less than 13, or observe that

$$13^6 - 6 \cdot 13^4 + 12 \cdot 13^2 - 13 = 13(13^5 - 6 \cdot 13^3 + 12 \cdot 13 - 1) \neq 0$$

since the term in parentheses cannot be zero: it is one less than some multiple of 13. ■

Example 6
$b = ((4 - 2\sqrt{3})/7)^{1/2}$ does not represent a rational number.

Proof
In Example 1 we showed that b is a solution of $49x^4 - 56x^2 + 4 = 0$. The only possible rational solutions are

$$\pm 1, \pm 1/7, \pm 1/49, \pm 2, \pm 2/7, \pm 2/49, \pm 4, \pm 4/7, \pm 4/49.$$

To complete our proof, all we need to do is substitute these eighteen candidates into the equation $49x^4 - 56x^2 + 4 = 0$. This prospect is so discouraging, however, that we choose to find a more clever approach. In Example 1, we also showed that $12 = (4 - 7b^2)^2$. Now if b were rational, then $4 - 7b^2$ would also be rational [Exercise 2.6], so the equation $12 = x^2$ would have a rational solution. But the only possible rational solutions to $x^2 - 12 = 0$ are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$, and these all can be eliminated by mentally substituting them into the equation. We conclude that $4-7b^2$ cannot be rational, so b cannot be rational. ■

As a practical matter, many or all of the rational candidates given by the Rational Zeros Theorem can be eliminated by approximating the quantity in question [perhaps with the aid of a calculator]. It is nearly obvious that the values in Examples 2 through 5 are not integers, while all the rational candidates are. My calculator says that b in Example 6 is approximately .2767; the nearest rational candidate is $+2/7$ which is approximately .2857.

Exercises

2.1. Show that $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{24}$, and $\sqrt{31}$ are not rational numbers.

2.2. Show that $2^{1/3}$, $5^{1/7}$, and $(13)^{1/4}$ do not represent rational numbers.

• **2.3.** Show that $(2+\sqrt{2})^{1/2}$ does not represent a rational number.

2.4. Show that $(5-\sqrt{3})^{1/3}$ does not represent a rational number.

• **2.5.** Show that $[3+\sqrt{2}]^{2/3}$ does not represent a rational number.

2.6. In connection with Example 6, discuss why $4-7b^2$ must be rational if b is rational.

1. If $r \in \mathbb{Q}$, $r \neq 0$, and x is irrational, prove that $r+x$ and rx are irrational. Hint: prove by contradiction.

§3 The Set $\mathbb{R}$ of Real Numbers

The set $\mathbb{Q}$ is probably the largest system of numbers with which you really feel comfortable. There are some subtleties but you have learned to cope with them. For example, $\mathbb{Q}$ is not simply the *set* $\{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$, since we regard some pairs of different looking fractions as equal. For example, $\frac{2}{4}$ and $\frac{3}{6}$ are regarded as the same element of $\mathbb{Q}$. A rigorous development of $\mathbb{Q}$ based on $\mathbb{Z}$, which in turn is based on $\mathbb{N}$, would require us to introduce the notion of equivalence class; see [38]. In this book we assume a familiarity with and understanding of $\mathbb{Q}$ as an algebraic system. However, in order to clarify exactly what we need to know about $\mathbb{Q}$, we set down some of its basic axioms and properties.

The basic algebraic operations in $\mathbb{Q}$ are addition and multiplication. Given a pair a, b of rational numbers, the sum $a + b$ and the product ab also represent rational numbers. Moreover, the following properties hold.

A1. $a + (b + c) = (a + b) + c$ for all a, b, c.
A2. $a + b = b + a$ for all a, b.
A3. $a + 0 = a$ for all a.
A4. For each a, there is an element $-a$ such that $a + (-a) = 0$.
M1. $a(bc) = (ab)c$ for all a, b, c.
M2. $ab = ba$ for all a, b.
M3. $a \cdot 1 = a$ for all a.
M4. For each $a \neq 0$, there is an element a^{-1} such that $aa^{-1} = 1$.
DL $a(b + c) = ab + ac$ for all a, b, c.

Properties A1 and M1 are called the *associative laws*, and properties A2 and M2 are the *commutative laws*. Property DL is the *distributive law*; this is the least obvious law and is the one that justifies "factorization" and "multiplying out" in algebra. A system that has more than one element and satisfies these nine properties is called a *field*. The basic algebraic properties of $\mathbb{Q}$ can proved solely on the basis of these field properties. We do not want to pursue this topic in any depth, but we illustrate our claim by proving some familiar properties in Theorem 3.1 below.

The set $\mathbb{Q}$ also has an order structure $\leq$ satisfying

O1. Given a and b, either $a \leq b$ or $b \leq a$.
O2. If $a \leq b$ and $b \leq a$, then $a = b$.
O3. If $a \leq b$ and $b \leq c$, then $a \leq c$.
O4. If $a \leq b$, then $a + c \leq b + c$.
O5. If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

Property O3 is called the *transitive law*. This is the characteristic property of an ordering. A field with an ordering satisfying properties O1 through O5 is called an *ordered field*. Most of the algebraic and order properties of $\mathbb{Q}$ can be established for any ordered field. We will prove a few of them in Theorem 3.2 below.

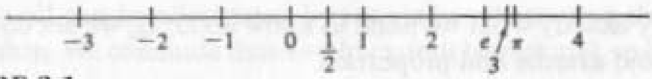

FIGURE 3.1

The mathematical system on which we will do our analysis will be the set $\mathbb{R}$ of all *real numbers*. The set $\mathbb{R}$ will include all rational numbers, all algebraic numbers, π, e, and more. It will be a set that can be drawn as the real number line; see Figure 3.1. That is, every real number will correspond to a point on the number line, and every point on the number line will correspond to a real number. In particular, unlike $\mathbb{Q}$, $\mathbb{R}$ will not have any "gaps." We will also see that real numbers have decimal expansions; see 10.3 and §16. These remarks help describe $\mathbb{R}$, but we certainly have not defined $\mathbb{R}$ as a concise mathematical object. It turns out that $\mathbb{R}$ can be defined entirely in terms of the set $\mathbb{Q}$ of rational numbers; we indicate in the optional §6 one way that this can be done. But then it is a long and tedious task to show how to add and multiply the objects defined in this way and to show that the set $\mathbb{R}$, with these operations, satisfies all the familiar algebraic and order properties that we expect to hold for $\mathbb{R}$. To develop $\mathbb{R}$ properly from $\mathbb{Q}$ in this way and to develop $\mathbb{Q}$ properly from $\mathbb{N}$ would take us several chapters. This would defeat the purpose of this book, which is to accept $\mathbb{R}$ as a mathematical system and to study some important properties of $\mathbb{R}$ and functions on $\mathbb{R}$. Nevertheless, it is desirable to specify exactly what properties of $\mathbb{R}$ we are assuming.

Real numbers, i.e., elements of $\mathbb{R}$, can be added together and multiplied together. That is, given real numbers a and b, the sum $a+b$ and the product ab also represent real numbers. Moreover, these operations satisfy the field properties A1 through A4, M1 through M4, and DL. The set $\mathbb{R}$ also has an order structure $\leq$ that satisfies properties O1 through O5. Thus, like $\mathbb{Q}$, $\mathbb{R}$ is an ordered field.

In the remainder of this section, we will obtain some results for $\mathbb{R}$ that are valid in any ordered field. In particular, these results would be equally valid if we restricted our attention to $\mathbb{Q}$. These remarks emphasize the similarities between $\mathbb{R}$ and $\mathbb{Q}$. We have not yet indicated how $\mathbb{R}$ can be distinguished from $\mathbb{Q}$ as a mathematical object, although we have asserted that $\mathbb{R}$ has no "gaps." We will make this observation much more precise in the next section, and then we will give a "gap filling" axiom that finally will distinguish $\mathbb{R}$ from $\mathbb{Q}$.

3.1 Theorem.
The following are consequences of the field properties:
- **(i)** $a + c = b + c$ *implies* $a = b$;
- **(ii)** $a \cdot 0 = 0$ *for all* a;
- **(iii)** $(-a)b = -ab$ *for all* a, b;
- **(iv)** $(-a)(-b) = ab$ *for all* a, b;
- **(v)** $ac = bc$ *and* $c \neq 0$ *imply* $a = b$;
- **(vi)** $ab = 0$ *implies either* $a = 0$ *or* $b = 0$;

for $a, b, c \in \mathbb{R}$.

Proof
- **(i)** $a+c = b+c$ implies $(a+c)+(-c) = (b+c)+(-c)$, so by A1, we have $a+[c+(-c)] = b+[c+(-c)]$. By A4, this reduces to $a+0 = b+0$, so $a = b$ by A3.
- **(ii)** We use A3 and DL to obtain $a \cdot 0 = a \cdot (0+0) = a \cdot 0 + a \cdot 0$, so $0 + a \cdot 0 = a \cdot 0 + a \cdot 0$. By (i) we conclude that $0 = a \cdot 0$.
- **(iii)** Since $a + (-a) = 0$, we have $ab + (-a)b = [a + (-a)] \cdot b = 0 \cdot b = 0 = ab + (-(ab))$. From (i) we obtain $(-a)b = -(ab)$.
- **(iv) and (v)** are left to Exercise 3.3.
- **(v)** If $ab = 0$ and $b \neq 0$, then $0 = b^{-1} \cdot 0 = 0 \cdot b^{-1} = (ab) \cdot b^{-1} = a(bb^{-1}) = a \cdot 1 = a$. ■

3.2 Theorem.
The following are consequences of the properties of an ordered field:
- **(i)** *if* $a \le b$, *then* $-b \le -a$;
- **(ii)** *if* $a \le b$ *and* $c \le 0$, *then* $bc \le ac$;
- **(iii)** *if* $0 \le a$ *and* $0 \le b$, *then* $0 \le ab$;
- **(iv)** $0 \le a^2$ *for all* a;
- **(v)** $0 < 1$;
- **(vi)** *if* $0 < a$, *then* $0 < a^{-1}$;
- **(vii)** *if* $0 < a < b$, *then* $0 < b^{-1} < a^{-1}$;

for $a, b, c \in \mathbb{R}$.

Note that $a < b$ means $a \le b$ and $a \neq b$.

Proof

(i) Suppose that $a \le b$. By O4 applied to $c = (-a) + (-b)$, we have $a + [(-a) + (-b)] \le b + [(-a) + (-b)]$. It follows that $-b \le -a$.

(ii) If $a \le b$ and $c \le 0$, then $0 \le -c$ by (i). Now by O5 we have $a(-c) \le b(-c)$, i.e., $-ac \le -bc$. From (i) again, we see that $bc \le ac$.

(iii) If we put $a = 0$ in property O5, we obtain: $0 \le b$ and $0 \le c$ imply $0 \le bc$. Except for notation, this is exactly assertion (iii).

(iv) For any a, either $a \ge 0$ or $a \le 0$ by O1. If $a \ge 0$, then $a^2 \ge 0$ by (iii). If $a \le 0$, then we have $-a \ge 0$ by (i), so $(-a)^2 \ge 0$, i.e., $a^2 \ge 0$.

(v) is left to Exercise 3.4.

(vi) Suppose that $0 < a$ but that $0 < a^{-1}$ fails. Then we must have $a^{-1} \le 0$ and $0 \le -a^{-1}$. Now by (iii) $0 \le a(-a^{-1}) = -1$, so that $1 \le 0$, contrary to (v).

(vii) is left to Exercise 3.4. ■

Another important notion that should be familiar is that of absolute value.

3.3 Definition.
We define

$$|a| = a \quad \text{if} \quad a \ge 0 \qquad \text{and} \qquad |a| = -a \quad \text{if} \quad a \le 0.$$

$|a|$ is called the *absolute value of a*.

Intuitively, the absolute value of a represents the distance between 0 and a, but in fact we will *define* the idea of "distance" in terms of the "absolute value," which in turn was defined in terms of the ordering.

3.4 Definition.
For numbers a and b we define $\text{dist}(a, b) = |a - b|$; $\text{dist}(a, b)$ represents the *distance between a and b*.

The basic properties of the absolute value are given in the next theorem.

3.5 Theorem.

(i) $|a| \geq 0$ *for all* $a \in \mathbb{R}$.

(ii) $|ab| = |a| \cdot |b|$ *for all* $a, b \in \mathbb{R}$.

(iii) $|a + b| \leq |a| + |b|$ *for all* $a, b \in \mathbb{R}$.

Proof

(i) is obvious from the definition. [The word "obvious" as used here signifies that the reader should be able to quickly see why the result is true. Certainly if $a \geq 0$, then $|a| = a \geq 0$, while $a < 0$ implies $|a| = -a > 0$. We will use expressions like "obviously" and "clearly" in place of very simple arguments, but we will not use these terms to obscure subtle points.]

(ii) There are four easy cases here. If $a \geq 0$ and $b \geq 0$, then $ab \geq 0$, so $|a| \cdot |b| = ab = |ab|$. If $a \leq 0$ and $b \leq 0$, then $-a \geq 0$, $-b \geq 0$ and $(-a)(-b) \geq 0$ so that $|a| \cdot |b| = (-a)(-b) = ab = |ab|$. If $a \geq 0$ and $b \leq 0$, then $-b \geq 0$ and $a(-b) \geq 0$ so that $|a| \cdot |b| = a(-b) = -(ab) = |ab|$. If $a \leq 0$ and $b \geq 0$, then $-a \geq 0$ and $(-a)b \geq 0$ so that $|a| \cdot |b| = (-a)b = -ab = |ab|$.

(iii) The inequalities $-|a| \leq a \leq |a|$ are obvious, since either $a = |a|$ or else $a = -|a|$. Similarly $-|b| \leq b \leq |b|$. Now four applications of O4 yield

$$-|a| + (-|b|) \leq -|a| + |b| \leq a + b \leq |a| + b \leq |a| + |b|$$

so that

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

This tells us that $a + b \leq |a| + |b|$ and also that $-(a + b) \leq |a| + |b|$. Since $|a + b|$ is equal to either $a + b$ or $-(a + b)$, we conclude that $|a + b| \leq |a| + |b|$. ■

3.6 Corollary.

$\text{dist}(a, c) \leq \text{dist}(a, b) + \text{dist}(b, c)$ *for all* $a, b, c \in \mathbb{R}$.

Proof

We can apply inequality (iii) of Theorem 3.5 to $a - b$ and $b - c$ to obtain $|(a - b) + (b - c)| \leq |a - b| + |b - c|$ or $\text{dist}(a, c) = |a - c| \leq |a - b| + |b - c| \leq \text{dist}(a, b) + \text{dist}(b, c)$. ■

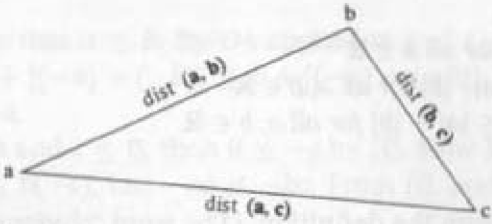

FIGURE 3.2

The inequality in Corollary 3.6 is very closely related to an inequality concerning points **a**, **b**, **c** in the plane, and the latter inequality can be interpreted as a statement about triangles: the length of a side of a triangle is less than or equal to the sum of the lengths of the other two sides. See Figure 3.2. For this reason, the inequality in Corollary 3.6 and its close relative (iii) in 3.5 are often called the *Triangle Inequality*.

3.7 Triangle Inequality.
$|a + b| \le |a| + |b|$ for all a, b.

A useful variant of the triangle inequality is given in Exercise 3.5(b).

Exercises

3.1. (a) Which of the properties A1-A4, M1-M4, DL, O1-O5 fail for $\mathbb{N}$?

(b) Which of these properties fail for $\mathbb{Z}$?

3.2. (a) The commutative law A2 was used in the proof of (ii) in Theorem 3.1. Where?

(b) The commutative law A2 was also used in the proof of (iii) in Theorem 3.1. Where?

·**3.3.** Prove (iv) and (v) of Theorem 3.1.

·**3.4.** Prove (v) and (vii) of Theorem 3.2.

3.5. (a) Show that $|b| \le a$ if and only if $-a \le b \le a$.

(b) Prove that $||a| - |b|| \le |a - b|$ for all $a, b \in \mathbb{R}$.

•3.6. (a) Prove that $|a+b+c| \leq |a|+|b|+|c|$ for all $a, b, c \in \mathbb{R}$. *Hint*: Apply the triangle inequality twice. Do *not* consider eight cases.

(b) Use induction to prove

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

for n numbers $a_1, a_2, \ldots, a_n$.

•3.7. (a) Show that $|b| < a$ if and only if $-a < b < a$.

(b) Show that $|a - b| < c$ if and only if $b - c < a < b + c$.

(c) Show that $|a - b| \leq c$ if and only if $b - c \leq a \leq b + c$.

•3.8. Let $a, b \in \mathbb{R}$. Show that if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$.

§4 The Completeness Axiom

In this section we give the completeness axiom for $\mathbb{R}$. This is the axiom that will assure us that $\mathbb{R}$ has no "gaps." It has far-reaching consequences and almost every significant result in this book relies on it. Most theorems in this book would be false if we restricted our world of numbers to the set $\mathbb{Q}$ of rational numbers.

4.1 Definition.

Let S be a nonempty subset of $\mathbb{R}$.

(a) If S contains a largest element s_0 [that is, s_0 belongs to S and $s \leq s_0$ for all $s \in S$], then we call s_0 the *maximum of* S and write $s_0 = \max S$.

(b) If S contains a smallest element, then we call the smallest element the *minimum of* S and write it as $\min S$.

Example 1

(a) Every finite nonempty subset of $\mathbb{R}$ has a maximum and a minimum. Thus

$$\max\{1,2,3,4,5\} = 5 \quad \text{and} \quad \min\{1,2,3,4,5\} = 1,$$

$$\max\{0, \pi, -7, e, 3, 4/3\} = \pi \quad \text{and} \quad \min\{0, \pi, -7, e, 3, 4/3\} = -7,$$

$$\max\{n \in \mathbb{Z} : -4 < n \leq 100\} = 100 \quad \text{and}$$

$$\min\{n \in \mathbb{Z} : -4 < n \leq 100\} = -3.$$

(b) Consider real numbers a and b where $a < b$. The following notation will be used throughout the book:

$$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}, \qquad (a,b) = \{x \in \mathbb{R} : a < x < b\},$$
$$[a,b) = \{x \in \mathbb{R} : a \leq x < b\}, \qquad (a,b] = \{x \in \mathbb{R} : a < x \leq b\}.$$

$[a, b]$ is called a *closed interval*, (a, b) is called an *open interval*, while $[a, b)$ and $(a, b]$ are called *half-open* or *semi-open intervals*. Observe that $\max[a, b] = b$ and $\min[a, b] = a$. The set (a, b) has no maximum and no minimum, since the endpoints a and b do not belong to the set. The set $[a, b)$ has no maximum, but a is its minimum.

(c) The sets $\mathbb{Z}$ and $\mathbb{Q}$ have no maximum or minimum. The set $\mathbb{N}$ has no maximum, but $\min \mathbb{N} = 1$.

(d) The set $\{r \in \mathbb{Q} : 0 \leq r \leq \sqrt{2}\}$ has a minimum, namely 0, but no maximum. This is because $\sqrt{2}$ does not belong to the set, but there are rationals in the set arbitrarily close to $\sqrt{2}$.

(e) Consider the set $\{n^{(-1)^n} : n \in \mathbb{N}\}$. This is shorthand for the set

$$\{1^{-1}, 2, 3^{-1}, 4, 5^{-1}, 6, 7^{-1}, \ldots\} = \{1, 2, \tfrac{1}{3}, 4, \tfrac{1}{5}, 6, \tfrac{1}{7}, \ldots\}.$$

The set has no maximum and no minimum.

4.2 Definition.

Let S be a nonempty subset of $\mathbb{R}$.

(a) If a real number M satisfies $s \leq M$ for all $s \in S$, then M is called an *upper bound of* S and the set S is said to be *bounded above*.

(b) If a real number m satisfies $m \leq s$ for all $s \in S$, then m is called a *lower bound of* S and the set S is said to be *bounded below*.

(c) The set S is said to be *bounded* if it is bounded above and bounded below. Thus S is bounded if there exist real numbers m and M such that $S \subseteq [m, M]$.

Example 2

(a) The maximum of a set is always an upper bound for the set. Likewise, the minimum of a set is always a lower bound for the set.

(b) Consider a, b in $\mathbb{R}$, $a < b$. The number b is an upper bound for each of the sets $[a, b]$, (a, b), $[a, b)$, $(a, b]$. Every number larger than b is also an upper bound for each of these sets, but b is the smallest or least upper bound.

(c) None of the sets $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{N}$ is bounded above. The set $\mathbb{N}$ is bounded below; 1 is a lower bound for $\mathbb{N}$ and so is any number less than 1. In fact, 1 is the largest or greatest lower bound.

(d) Any nonpositive real number is a lower bound for $\{r \in \mathbb{Q} : 0 \le r \le \sqrt{2}\}$ and 0 is the set's greatest lower bound. The least upper bound is $\sqrt{2}$.

(e) The set $\{n^{(-1)^n} : n \in \mathbb{N}\}$ is not bounded above. Among its many lower bounds, 0 is the greatest lower bound.

We now formalize two notions that have already appeared in Example 2.

4.3 Definition.

Let S be a nonempty subset of $\mathbb{R}$.

(a) If S is bounded above and S has a least upper bound, then we will call it the *supremum of* S and denote it by $\sup S$.

(b) If S is bounded below and S has a greatest lower bound, then we will call it the *infimum of* S and denote it by $\inf S$.

Note that, unlike $\max S$ and $\min S$, $\sup S$ and $\inf S$ need not belong to S. Note also that a set can have at most one maximum, minimum, supremum and infimum. Sometimes the expressions "least upper bound" and "greatest lower bound" are used instead of the Latin "supremum" and "infimum" and sometimes $\sup S$ is written lub S and $\inf S$ is written glb S. We have chosen the Latin terminology for a good reason: We will be studying the notions "lim sup" and "lim inf" and this notation is completely standard; no one writes "lim lub" for instance.

Observe that if S is bounded above, then $M = \sup S$ if and only if (i) $s \le M$ for all $s \in S$, and (ii) whenever $M_1 < M$, there exists $s_1 \in S$ such that $s_1 > M_1$.

Example 3

(a) If a set S has a maximum, then $\max S = \sup S$. A similar remark applies to sets that have minimums.

(b) If $a, b \in \mathbb{R}$ and $a < b$, then

$$\sup[a,b] = \sup(a,b) = \sup[a,b) = \sup(a,b] = b.$$

(c) As noted in Example 2, we have $\inf \mathbb{N} = 1$.
(d) If $A = \{r \in \mathbb{Q} : 0 \leq r \leq \sqrt{2}\}$, then $\sup A = \sqrt{2}$ and $\inf A = 0$.
(e) We have $\inf\{n^{(-1)^n} : n \in \mathbb{N}\} = 0$.

Notice that, in Examples 2 and 3, every set S that is bounded above possesses a least upper bound, i.e., $\sup S$ exists. This is not an accident. Otherwise there would be a "gap" between the set S and the set of its upper bounds.

4.4 Completeness Axiom.
Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. In other words, $\sup S$ *exists and is a real number.*

The completeness axiom for $\mathbb{Q}$ would assert that every nonempty subset of $\mathbb{Q}$, that is bounded above by some rational number, has a least upper bound that is a rational number. The set $A = \{r \in \mathbb{Q} : 0 \leq r \leq \sqrt{2}\}$ is a set of rational numbers and it is bounded above by some rational numbers [3/2 for example], but A has no least upper bound that is a rational number. Thus the completeness axiom does not hold for $\mathbb{Q}$! Incidentally, the set A can be described entirely in terms of rationals: $A = \{r \in \mathbb{Q} : 0 \leq r \text{ and } r^2 \leq 2\}$.

The completeness axiom for sets bounded below comes free.

4.5 Corollary.
Every nonempty subset S of $\mathbb{R}$ that is bounded below has a greatest lower bound $\inf S$.

Proof
Let $-S$ be the set $\{-s : s \in S\}$; $-S$ consists of the negatives of the numbers in S. Since S is bounded below there is an m in $\mathbb{R}$ such that $m \leq s$ for all $s \in S$. This implies that $-m \geq -s$ for all $s \in S$, so $-m \geq u$ for all u in the set $-S$. Thus $-S$ is bounded above by $-m$. The Completeness Axiom 4.4 applies to $-S$, so $\sup(-S)$ exists. Figure 4.1 suggests that we prove $\inf S = -\sup(-S)$.

Let $s_0 = -\sup(-S)$; we need to prove

$$s_0 \leq s \quad \text{for all} \quad s \in S, \tag{1}$$

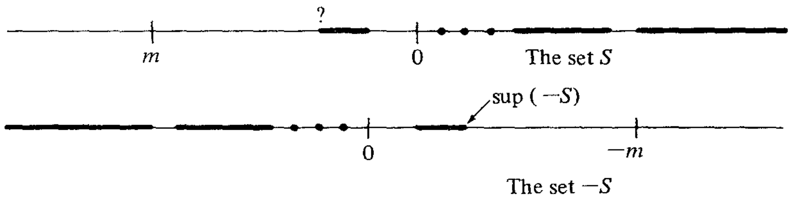

FIGURE 4.1

and

$$\text{if } t \le s \quad \text{for all} \quad s \in S, \quad \text{then} \quad t \le -s_0. \tag{2}$$

The inequality (1) will show that $-s_0$ is a lower bound for S, while (2) will show that $-s_0$ is the *greatest* lower bound, that is, $-s_0 = \inf S$. We leave the proofs of (1) and (2) to Exercise 4.9. ■

It is useful to know:

$$\text{if } a > 0, \quad \text{then} \quad \frac{1}{n} < a \quad \text{for some positive integer} \quad n, \tag{*}$$

and

$$\text{if } b > 0, \quad \text{then} \quad b < n \quad \text{for some positive integer} \quad n. \tag{**}$$

These assertions are not as obvious as they may appear. If fact, there exist ordered fields that do not have these properties. In other words, there exists a mathematical system that satisfies all the properties A1-A4, M1-M4, DL and O1-O5 in §3 and yet possesses elements $a > 0$ and $b > 0$ such that $a < 1/n$ and $n < b$ *for all* n. On the other hand, such strange elements cannot exist in $\mathbb{R}$ or $\mathbb{Q}$. We next prove this; in view of the previous remarks we *must* expect to use the Completeness Axiom.

4.6 Archimedean Property.
If $a > 0$ and $b > 0$, then for some positive integer n, we have $na > b$.

This tells us that, even if a is quite small and b is quite large, some integer multiple of a will exceed b. Or, to quote [2], given enough time, one can empty a large bathtub with a small spoon. [Note that if we set $b = 1$, we obtain assertion (*), and if we set $a = 1$, we obtain assertion (**).]

Proof

Assume the Archimedean property fails. Then there exist $a > 0$ and $b > 0$ such that $na \leq b$ for all $n \in \mathbb{N}$. In particular, b is an upper bound for the set $S = \{na : n \in \mathbb{N}\}$. Let $s_0 = \sup S$; this is where we are using the completeness axiom. Since $a > 0$, we have $s_0 < s_0 + a$, so $s_0 - a < s_0$. [To be precise, we obtain $s_0 \leq s_0 + a$ and $s_0 - a \leq s_0$ by property O4 and the fact that $a + (-a) = 0$. Then we conclude $s_0 - a < s_0$ since $s_0 - a = s_0$ implies $a = 0$ by Theorem 3.1(i).] Since s_0 is the least upper bound for S, $s_0 - a$ cannot be an upper bound for S. It follows that $s_0 - a < n_0 a$ for some $n_0 \in \mathbb{N}$. This implies that $s_0 < (n_0 + 1)a$. Since $(n_0 + 1)a$ is in S, s_0 is not an upper bound for S and we have reached a contradiction. Our assumption that the Archimedean property fails must be in error. ■

We give one more result that seems obvious from our experience with the real number line, but which cannot be proved for an arbitrary ordered field.

4.7 Denseness of $\mathbb{Q}$.

If $a, b \in \mathbb{R}$ and $a < b$, then there is a rational $r \in \mathbb{Q}$ such that $a < r < b$.

Proof

We need to show that $a < \frac{m}{n} < b$ for some integers m and n where $n > 0$, and thus we need

$$an < m < bn. \tag{1}$$

Since $b - a > 0$, the Archimedean property shows that there exists an $n \in \mathbb{N}$ such that $n(b - a) > 1$. Since $bn - an > 1$, it is fairly evident that there is an integer m between an and bn, so that (1) holds. However, the proof that such an m exists is a little delicate. We argue as follows. By the Archimedean property again, there exists an integer $k > \max\{|an|, |bn|\}$, so that

$$-k < an < bn < k.$$

Then the set $\{j \in \mathbb{Z} : -k < j \leq k \text{ and } an < j\}$ is finite and nonempty and we can set

$$m = \min\{j \in \mathbb{Z} : -k < j \leq k \text{ and } an < j\}.$$

Then $an < m$ but $m - 1 \le an$. Also, we have

$$m = (m-1) + 1 \le an + 1 < an + (bn - an) = bn,$$

so (1) holds. ■

Exercises

4.1. For each set below that is bounded above, list three upper bounds for the set. Otherwise write "NOT BOUNDED ABOVE" or "NBA."

(a) $[0, 1]$
(b) $(0, 1)$
(c) $\{2, 7\}$
(d) $\{\pi, e\}$
(e) $\{\frac{1}{n} : n \in \mathbb{N}\}$
(f) $\{0\}$
(g) $[0, 1] \cup [2, 3]$
(h) $\cup_{n=1}^{\infty} [2n, 2n + 1]$
(i) $\cap_{n=1}^{\infty} [-\frac{1}{n}, 1 + \frac{1}{n}]$
(j) $\{1 - \frac{1}{3^n} : n \in \mathbb{N}\}$
(k) $\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$
(l) $\{r \in \mathbb{Q} : r < 2\}$
(m) $\{r \in \mathbb{Q} : r^2 < 4\}$
(n) $\{r \in \mathbb{Q} : r^2 < 2\}$
(o) $\{x \in \mathbb{R} : x < 0\}$
(p) $\{1, \frac{\pi}{3}, \pi^2, 10\}$
(q) $\{0, 1, 2, 4, 8, 16\}$
(r) $\cap_{n=1}^{\infty} (1 - \frac{1}{n}, 1 + \frac{1}{n})$
(s) $\{\frac{1}{n} : n \in \mathbb{N}$ and n is prime$\}$
(t) $\{x \in \mathbb{R} : x^3 < 8\}$
(u) $\{x^2 : x \in \mathbb{R}\}$
(v) $\{\cos(\frac{n\pi}{3}) : n \in \mathbb{N}\}$
(w) $\{\sin(\frac{n\pi}{3}) : n \in \mathbb{N}\}$

4.2. Repeat Exercise 4.1 for lower bounds.

4.3. For each set in Exercise 4.1, give its supremum if it has one. Otherwise write "NO sup."

4.4. Repeat Exercise 4.3 for infima [plural of infimum].

4.5. Let S be a nonempty subset of $\mathbb{R}$ that is bounded above. Prove that if $\sup S$ belongs to S, then $\sup S = \max S$. *Hint*: Your proof should be very short.

4.6. Let S be a nonempty bounded subset of $\mathbb{R}$.

(a) Prove that $\inf S \le \sup S$. *Hint*: This is almost obvious; your proof should be short.

(b) What can you say about S if $\inf S = \sup S$?

4.7. Let S and T be nonempty bounded subsets of $\mathbb{R}$.

(a) Prove that if $S \subseteq T$, then $\inf T \le \inf S \le \sup S \le \sup T$.

(b) Prove that $\sup(S \cup T) = \max\{\sup S, \sup T\}$. *Note*: In part (b), do *not* assume $S \subseteq T$.

4.8. Let S and T be nonempty subsets of $\mathbb{R}$ with the following property: $s \le t$ for all $s \in S$ and $t \in T$.

(a) Observe that S is bounded above and that T is bounded below.

(b) Prove that $\sup S \le \inf T$.

(c) Give an example of such sets S and T where $S \cap T$ is nonempty.

(d) Give an example of sets S and T where $\sup S = \inf T$ and $S \cap T$ is the empty set.

4.9. Complete the proof that $\inf S = -\sup(-S)$ in Corollary 4.5 by proving (1) and (2).

4.10. Prove that if $a > 0$, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

4.11. Consider $a, b \in \mathbb{R}$ where $a < b$. Use Denseness of $\mathbb{Q}$ 4.7 to show that there are infinitely many rationals between a and b.

4.12. Let $\mathbb{I}$ be the set of real numbers that are not rational; elements of $\mathbb{I}$ are called *irrational numbers*. Prove that if $a < b$, then there exists $x \in \mathbb{I}$ such that $a < x < b$. *Hint*: First show $\{r + \sqrt{2} : r \in \mathbb{Q}\} \subseteq \mathbb{I}$.

4.13. Prove that the following are equivalent for real numbers a, b, c. [*Equivalent* means that either all the properties hold or none of the properties hold.]

(a) $|a - b| < c$,

(b) $b - c < a < b + c$,

(c) $a \in (b - c, b + c)$.

Hint: Use Exercise 3.7(b).

4.14. Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let S be the set of all sums $a + b$ where $a \in A$ and $b \in B$.

(a) Prove that $\sup S = \sup A + \sup B$.

(b) Prove that $\inf S = \inf A + \inf B$.

4.15. Let $a, b \in \mathbb{R}$. Show that if $a \le b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \le b$. Compare Exercise 3.8.

4.16. Show that $\sup\{r \in \mathbb{Q} : r < a\} = a$ for each $a \in \mathbb{R}$.

§5 The Symbols $+\infty$ and $-\infty$

The symbols $+\infty$ and $-\infty$ are extremely useful even though they are *not* real numbers. We will often write $+\infty$ as simply ∞. We will adjoin $+\infty$ and $-\infty$ to the set $\mathbb{R}$ and extend our ordering to the set $\mathbb{R}\cup\{-\infty, +\infty\}$. Explicitly, we will agree that $-\infty \le a \le +\infty$ for all a in $\mathbb{R}\cup\{-\infty, \infty\}$. This provides the set $\mathbb{R}\cup\{-\infty, +\infty\}$ with an ordering that satisfies properties O1, O2 and O3 of §3. We emphasize that we will *not* provide the set $\mathbb{R} \cup \{-\infty, +\infty\}$ with any algebraic structure. We may use the symbols $+\infty$ and $-\infty$, but we must continue to remember that they do not represent real numbers. Do *not* apply a theorem or exercise that is stated for real numbers to the symbols $+\infty$ or $-\infty$.

It is convenient to use the symbols $+\infty$ and $-\infty$ to extend the notation established in Example 1(b) of §4 to unbounded intervals. For real numbers $a, b \in \mathbb{R}$, we adopt the following notation:

$$[a,\infty) = \{x \in \mathbb{R} : a \le x\}, \qquad (a,\infty) = \{x \in \mathbb{R} : a < x\},$$
$$(-\infty, b] = \{x \in \mathbb{R} : x \le b\}, \qquad (-\infty, b) = \{x \in \mathbb{R} : x < b\}.$$

We occasionally also write $(-\infty, \infty)$ for $\mathbb{R}$. $[a, \infty)$ and $(-\infty, b]$ are called *closed intervals* or *unbounded closed intervals*, while (a, ∞) and $(-\infty, b)$ are called *open intervals* or *unbounded open intervals*.

Consider a nonempty subset S of $\mathbb{R}$. Recall that if S is bounded above, then $\sup S$ exists and represents a real number by the completeness axiom 4.4. We define

$$\sup S = +\infty \quad \text{if } S \text{ is not bounded above.}$$

Likewise, if S is bounded below, then $\inf S$ exists and represents a real number [Corollary 4.5]. And we define

$$\inf S = -\infty \quad \text{if } S \text{ is not bounded below.}$$

For emphasis, we recapitulate:

Let S be any nonempty subset of $\mathbb{R}$. The symbols $\sup S$ *and* $\inf S$ *always make sense. If S is bounded above, then* $\sup S$ *is a real number; otherwise* $\sup S = +\infty$. *If S is bounded below, then* $\inf S$ *is a real number; otherwise* $\inf S = -\infty$. *Moreover, we have* $\inf S \le \sup S$.

The exercises for this section clear up some loose ends. Most of them extend results in §4 to sets that are not necessarily bounded.

2 Sequences

CHAPTER

§7 Limits of Sequences

A *sequence* is a function whose domain is a set that has the form $\{n \in \mathbb{Z} : n \geq m\}$; m is usually 1 or 0. Thus a sequence is a function that has a specified value for each integer $n \geq m$. It is customary to denote a sequence by a letter such as s and to denote its value at n as s_n rather than $s(n)$. It is often convenient to write the sequence as $(s_n)_{n=m}^{\infty}$ or $(s_m, s_{m+1}, s_{m+2}, \ldots)$. If $m = 1$ we may write $(s_n)_{n \in \mathbb{N}}$ or of course $(s_1, s_2, s_3, \ldots)$. Sometimes we will write (s_n) when the domain is understood or when the results under discussion do not depend on the specific value of m. In this chapter, we will be interested in sequences whose range values are real numbers, i.e., each s_n represents a real number.

Example 1

(a) Consider the sequence $(s_n)_{n \in \mathbb{N}}$ where $s_n = \frac{1}{n^2}$. This is the sequence $(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots)$. Formally, of course, this is the function with domain $\mathbb{N}$ whose value at each n is $\frac{1}{n^2}$. The *set* of values is $\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots\}$.

(b) Consider the sequence given by $a_n = (-1)^n$ for $n \geq 0$, i.e., $(a_n)_{n=0}^{\infty}$ where $a_n = (-1)^n$. Note that the first term of the sequence is $a_0 = 1$ and the sequence is $(1, -1, 1, -1, 1, -1, 1, \ldots)$. Formally, this is a function whose domain is $\{0, 1, 2, \ldots\}$ and whose *set* of values is $\{-1, 1\}$.

It is important to distinguish between a sequence and its set of values, since the validity of many results in this book depends on whether we are working with a sequence or a set. We will always use parentheses ( ) to signify a sequence and braces { } to signify a set. The sequence given by $a_n = (-1)^n$ has an infinite number of terms even though their values are repeated over and over. On the other hand, the *set* $\{(-1)^n : n = 0, 1, 2, \ldots\}$ is exactly the set $\{-1, 1\}$ consisting of two numbers.

(c) Consider the sequence $\cos(\frac{n\pi}{3})$, $n \in \mathbb{N}$. The first term of this sequence is $\cos(\frac{\pi}{3}) = \cos 60° = \frac{1}{2}$ and the sequence looks like

$$(\tfrac{1}{2}, -\tfrac{1}{2}, -1, -\tfrac{1}{2}, \tfrac{1}{2}, 1, \tfrac{1}{2}, -\tfrac{1}{2}, -1, -\tfrac{1}{2}, \tfrac{1}{2}, 1, \tfrac{1}{2}, -\tfrac{1}{2}, -1, \ldots).$$

The set of values is $\{\cos(\frac{n\pi}{3}) : n \in \mathbb{N}\} = \{\frac{1}{2}, -\frac{1}{2}, -1, 1\}$.

(d) If $a_n = n^{1/n}$, $n \in \mathbb{N}$, the sequence is $(1, \sqrt{2}, 3^{1/3}, 4^{1/4}, \ldots)$. If we approximate values to four decimal places, the sequence looks like

$$(1, 1.4142, 1.4422, 1.4142, 1.3797, 1.3480, 1.3205, 1.2968, \ldots).$$

It turns out that a_{100} is approximately 1.0471 and that a_{1000} is approximately 1.0069.

(e) Consider the sequence $b_n = (1 + \frac{1}{n})^n$, $n \in \mathbb{N}$. This is the sequence $(2, (\frac{3}{2})^2, (\frac{4}{3})^3, (\frac{5}{4})^4, \ldots)$. If we approximate the values to four decimal places, we obtain

$$(2, 2.25, 2.3704, 2.4414, 2.4883, 2.5216, 2.5465, 2.5658, \ldots).$$

Also b_{100} is approximately 2.7048 and b_{1000} is approximately 2.7169.

The "limit" of a sequence (s_n) is a real number that the values s_n are close to for large values of n. For instance, the values of the sequence in Example 1(a) are close to 0 for large n and the values of the sequence in Example 1(d) appear to be close to 1 for large n. The sequence (a_n) given by $a_n = (-1)^n$ requires some thought. We might say that 1 is a limit because in fact $a_n = 1$ for the large values of n that are even. On the other hand, $a_n = -1$ [which is quite a distance from 1] for other large values of n. We need a concise definition in order to decide whether 1 is a limit of $a_n = (-1)^n$. It turns out that our definition will require the values to be close to the limit value for *all* large n, so 1 will *not* be a limit of the sequence $a_n = (-1)^n$.

7.1 Definition.
A sequence (s_n) of real numbers is said to *converge* to the real number s provided that

$$\begin{array}{l}\text{for each } \epsilon > 0 \text{ there exists a number } N \text{ such that} \\ n > N \text{ implies } |s_n - s| < \epsilon.\end{array} \qquad (1)$$

If (s_n) converges to s, we will write $\lim_{n\to\infty} s_n = s$, or $s_n \to s$. The number s is called the *limit* of the sequence (s_n). A sequence that does not converge to some real number is said to *diverge*.

Several comments are in order. First, in view of the Archimedean property, the number N in Definition 7.1 can be taken to be a natural number if we wish. Second, the symbol ϵ [lower case Greek epsilon] in this definition represents a positive number, not some new exotic number. However, it is traditional in mathematics to use ϵ and δ [lower case Greek delta] in situations where the interesting or challenging values are the small positive values. Third, condition (1) is an infinite number of statements, one for each positive value of ϵ. The condition states that to each $\epsilon > 0$ there corresponds a number N with a certain property, namely $n > N$ implies $|s_n - s| < \epsilon$. The value N depends on the value ϵ, and normally N must be large if ϵ is small. We illustrate these remarks in the next example.

Example 2
Consider the sequence $s_n = \frac{3n+1}{7n-4}$. If we write s_n as $\frac{3+\frac{1}{n}}{7-\frac{4}{n}}$ and note that $\frac{1}{n}$ and $\frac{4}{n}$ are very small for large n, it seems reasonable to conclude that $\lim s_n = \frac{3}{7}$. In fact, this reasoning will be completely valid after we have the limit theorems in §9:

$$\lim s_n = \lim \left[\frac{3+\frac{1}{n}}{7-\frac{4}{n}}\right] = \frac{\lim 3 + \lim(\frac{1}{n})}{\lim 7 - 4\lim(\frac{1}{n})} = \frac{3+0}{7-4\cdot 0} = \frac{3}{7}.$$

However, for now we are interested in analyzing exactly what we mean by $\lim s_n = \frac{3}{7}$. By Definition 7.1, $\lim s_n = \frac{3}{7}$ means that

$$\text{for each } \epsilon > 0 \text{ there exists a number } N \text{ such that } n > N \text{ implies } \left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < \epsilon. \tag{1}$$

As ϵ varies, N varies. In Example 2 of the next section we will show that, for this particular sequence, N can be taken to be $\frac{19}{49\epsilon} + \frac{4}{7}$. Using this observation and a calculator, we find that for ϵ equal to 1, 0.1, 0.01, 0.001 and 0.000001, respectively, N can be taken to be approximately 0.96, 4.45, 39.35, 388.33 and 387,755.67, respectively. Since we are interested only in integer values of n, we may as well drop the fractional part of N. Then we see that five of the infinitely many statements given by (1) are:

$$n > 0 \quad \text{implies} \quad \left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < 1; \tag{2}$$

$$n > 4 \quad \text{implies} \quad \left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < 0.1; \tag{3}$$

$$n > 39 \quad \text{implies} \quad \left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < 0.01; \tag{4}$$

$$n > 388 \quad \text{implies} \quad \left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < 0.001; \tag{5}$$

$$n > 387{,}755 \quad \text{implies} \quad \left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < 0.000001. \tag{6}$$

Table 7.1 partially confirms assertions (2) through (6). We could go on and on with these numerical illustrations, but it should be clear that we need a more theoretical approach if we are going to *prove* results about limits.

Table 7.1

n	$s_n = \frac{3n+1}{7n-4}$ approximately	$\lvert s_n - \frac{3}{7}\rvert$ approximately
1	1.3333	.9047
2	0.7000	.2714
3	0.5882	.1597
4	0.5417	.1131
5	0.5161	.0876
6	0.5000	.0714
40	0.4384	.0098
400	0.4295	.0010

Example 3

We return to the examples in Example 1.

(a) $\lim \frac{1}{n^2} = 0$. This will be proved in Example 1 of the next section.

(b) The sequence (a_n) where $a_n = (-1)^n$ does not converge. Thus the expression "$\lim a_n$" is meaningless in this case. We will discuss this example again in Example 4 of the next section.

(c) The sequence $\cos(\frac{n\pi}{3})$ does not converge. See Exercise 8.7.

(d) The sequence $n^{1/n}$ appears to converge to 1. We will prove $\lim n^{1/n} = 1$ in 9.7(c).

(e) The sequence (b_n) where $b_n = (1 + \frac{1}{n})^n$ converges to the number e that should be familiar from calculus. The limit $\lim b_n$ and the number e will be discussed further in the optional §37. Recall that e is approximately 2.7182818.

We conclude this section by showing that limits are unique. That is, if $\lim s_n = s$ and $\lim s_n = t$, then we must have $s = t$. In short, the values s_n cannot be getting arbitrarily close to different values for large n. To prove this, consider $\epsilon > 0$. By the definition of limit there must exist N_1 so that

$$n > N_1 \quad \text{implies} \quad |s_n - s| < \frac{\epsilon}{2}$$

and there must exist N_2 so that

$$n > N_2 \quad \text{implies} \quad |s_n - t| < \frac{\epsilon}{2}.$$

For $n > \max\{N_1, N_2\}$, the Triangle Inequality 3.7 shows that

$$|s-t| = |(s-s_n)+(s_n-t)| \leq |s-s_n| + |s_n-t| \leq \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

This shows that $|s-t| < \epsilon$ for all $\epsilon > 0$. It follows that $|s-t| = 0$, hence $s = t$.

Exercises

7.1. Write out the first five terms of the following sequences.

(a) $s_n = \frac{1}{3n+1}$ **(b)** $b_n = \frac{3n+1}{4n-1}$
(c) $c_n = \frac{n}{3^n}$ **(d)** $\sin(\frac{n\pi}{4})$

7.2. For each sequence in Exercise 7.1, determine whether it converges. If it converges, give its limit. No proofs are required.

7.3. For each sequence below, determine whether it converges and, if it converges, give its limit. No proofs are required.

(a) $a_n = \frac{n}{n+1}$ **(b)** $b_n = \frac{n^2+3}{n^2-3}$
(c) $c_n = 2^{-n}$ **(d)** $t_n = 1 + \frac{2}{n}$
(e) $x_n = 73 + (-1)^n$ **(f)** $s_n = (2)^{1/n}$
(g) $y_n = n!$ **(h)** $d_n = (-1)^n n$
(i) $\frac{(-1)^n}{n}$ **(j)** $\frac{7n^3+8n}{2n^3-31}$
(k) $\frac{9n^2-18}{6n+18}$ **(l)** $\sin(\frac{n\pi}{2})$
(m) $\sin(n\pi)$ **(n)** $\sin(\frac{2n\pi}{3})$
(o) $\frac{1}{n}\sin n$ **(p)** $\frac{2^{n+1}+5}{2^n-7}$
(q) $\frac{3^n}{n!}$ **(r)** $(1+\frac{1}{n})^2$
(s) $\frac{4n^2+3}{3n^2-2}$ **(t)** $\frac{6n+4}{9n^2+7}$

7.4. Give examples of

(a) a sequence (x_n) of irrational numbers having a limit $\lim x_n$ that is a rational number.

(b) a sequence (r_n) of rational numbers having a limit $\lim r_n$ that is an irrational number.

7.5. Determine the following limits. No proofs are required, but show any relevant algebra.

(a) $\lim s_n$ where $s_n = \sqrt{n^2+1} - n$,

(b) $\lim(\sqrt{n^2+n} - n)$,

(c) $\lim(\sqrt{4n^2+n}-2n)$.
Hint for (a): First show that $s_n = \frac{1}{\sqrt{n^2+1}+n}$.

§8 A Discussion about Proofs

In this section we give several examples of proofs using the definition of the limit of a sequence. With a little study and practice, students should be able to do proofs of this sort themselves. We will sometimes refer to a proof as a *formal proof* to emphasize that it is a rigorous mathematical proof.

Example 1
Prove that $\lim \frac{1}{n^2} = 0$.

Discussion. Our task is to consider an arbitrary $\epsilon > 0$ and show that there exists a number N [which will depend on ϵ] such that $n > N$ implies $|\frac{1}{n^2} - 0| < \epsilon$. So we expect our formal proof to begin with "Let $\epsilon > 0$" and to end with something like "Hence $n > N$ implies $|\frac{1}{n^2} - 0| < \epsilon$." In between the proof should specify an N and then verify that N has the desired property, namely that $n > N$ does indeed imply $|\frac{1}{n^2} - 0| < \epsilon$.

As is often the case with trigonometric identities, we will initially work backward from our desired conclusion, but in the formal proof we will have to be sure that our steps are reversible. In the present example, we want $|\frac{1}{n^2} - 0| < \epsilon$ and we want to know how big n must be. So we will operate on this inequality algebraically and try to "solve" for n. Thus we want $\frac{1}{n^2} < \epsilon$. By multiplying both sides by n^2 and dividing both sides by ϵ, we find that we want $\frac{1}{\epsilon} < n^2$ or $\frac{1}{\sqrt{\epsilon}} < n$. If our steps are reversible, we see that $n > \frac{1}{\sqrt{\epsilon}}$ implies $|\frac{1}{n^2} - 0| < \epsilon$. This suggests that we put $N = \frac{1}{\sqrt{\epsilon}}$.

Formal Proof
Let $\epsilon > 0$. Let $N = \frac{1}{\sqrt{\epsilon}}$. Then $n > N$ implies $n > \frac{1}{\sqrt{\epsilon}}$ which implies $n^2 > \frac{1}{\epsilon}$ and hence $\epsilon > \frac{1}{n^2}$. Thus $n > N$ implies $|\frac{1}{n^2} - 0| < \epsilon$. This proves that $\lim \frac{1}{n^2} = 0$. ■

Example 2

Prove that $\lim \frac{3n+1}{7n-4} = \frac{3}{7}$.

Discussion. For each $\epsilon > 0$, we need to decide how big n must be to guarantee that $|\frac{3n+1}{7n-4} - \frac{3}{7}| < \epsilon$. Thus we want

$$\left|\frac{21n + 7 - 21n + 12}{7(7n-4)}\right| < \epsilon \quad \text{or} \quad \left|\frac{19}{7(7n-4)}\right| < \epsilon.$$

Since $7n - 4 > 0$, we can drop the absolute value and manipulate the inequality further to "solve" for n:

$$\frac{19}{7\epsilon} < 7n - 4 \quad \text{or} \quad \frac{19}{7\epsilon} + 4 < 7n \quad \text{or} \quad \frac{19}{49\epsilon} + \frac{4}{7} < n.$$

Our steps are reversible, so we will put $N = \frac{19}{49\epsilon} + \frac{4}{7}$. Incidentally, we could have chosen N to be any number larger than $\frac{19}{49\epsilon} + \frac{4}{7}$.

Formal Proof

Let $\epsilon > 0$ and let $N = \frac{19}{49\epsilon} + \frac{4}{7}$. Then $n > N$ implies $n > \frac{19}{49\epsilon} + \frac{4}{7}$, hence $7n > \frac{19}{7\epsilon} + 4$, hence $7n - 4 > \frac{19}{7\epsilon}$, hence $\frac{19}{7(7n-4)} < \epsilon$, and hence $|\frac{3n+1}{7n-4} - \frac{3}{7}| < \epsilon$. This proves $\lim \frac{3n+1}{7n-4} = \frac{3}{7}$. ■

Example 3

Prove that $\lim \frac{4n^3+3n}{n^3-6} = 4$.

Discussion. For each $\epsilon > 0$, we need to determine how large n must be to imply

$$\left|\frac{4n^3 + 3n}{n^3 - 6} - 4\right| < \epsilon \quad \text{or} \quad \left|\frac{3n + 24}{n^3 - 6}\right| < \epsilon.$$

By considering $n > 1$, we may drop the absolute values; thus we need to find how big n must be to give $\frac{3n+24}{n^3-6} < \epsilon$. This time it would be very difficult to "solve" for or isolate n. Recall that we need to find some N such that $n > N$ implies $\frac{3n+24}{n^3-6} < \epsilon$, but we do not need to find the least such N. So we will simplify matters by making estimates. The idea is that $\frac{3n+24}{n^3-6}$ is bounded by some constant times $\frac{n}{n^3} = \frac{1}{n^2}$ for sufficiently large n. To find such a bound we will find an upper bound for the numerator and a lower bound for the denominator. For example, since $3n + 24 \leq 27n$, it suffices for us to get $\frac{27n}{n^3-6} < \epsilon$. To make the denominator smaller and yet a constant multiple of n^3, we note that $n^3 - 6 \geq \frac{n^3}{2}$ provided n is sufficiently large; in fact, all we need is $\frac{n^3}{2} \geq 6$ or $n^3 \geq 12$ or $n > 2$. So it suffices to get $\frac{27n}{n^3/2} < \epsilon$ or $\frac{54}{n^2} < \epsilon$ or $n > \sqrt{\frac{54}{\epsilon}}$, provided that $n > 2$.

Formal Proof

Let $\epsilon > 0$ and let $N = \max\{2, \sqrt{\frac{54}{\epsilon}}\}$. Then $n > N$ implies $n > \sqrt{\frac{54}{\epsilon}}$, hence $\frac{54}{n^2} < \epsilon$, hence $\frac{27n}{n^3/2} < \epsilon$. Since $n > 2$, we have $\frac{n^3}{2} \leq n^3 - 6$ and also $27n \geq 3n + 24$. Thus $n > N$ implies

$$\frac{3n+24}{n^3-6} \leq \frac{27n}{\frac{1}{2}n^3} = \frac{54}{n^2} < \epsilon,$$

and hence

$$\left| \frac{4n^3+3n}{n^3-6} - 4 \right| < \epsilon,$$

as desired. ■

Example 3 illustrates that direct proofs of even rather simple limits can get complicated. With the limit theorems of §9 we would just write

$$\lim\left[\frac{4n^3+3n}{n^3-6}\right] = \lim\left[\frac{4+\frac{3}{n^2}}{1-\frac{6}{n^3}}\right] = \frac{\lim 4 + 3 \cdot \lim(\frac{1}{n^2})}{\lim 1 - 6 \cdot \lim(\frac{1}{n^3})} = 4.$$

Example 4

Show that the sequence $a_n = (-1)^n$ does not converge.

Discussion. We will assume that $\lim(-1)^n = a$ and obtain a contradiction. No matter what a is, either 1 or -1 will have distance at least 1 from a. Thus the inequality $|(-1)^n - a| < 1$ will not hold for all large n.

Formal Proof

Assume that $\lim(-1)^n = a$ for some $a \in \mathbb{R}$. Letting $\epsilon = 1$ in the definition of the limit, we see that there exists N such that

$$n > N \quad \text{implies} \quad |(-1)^n - a| < 1.$$

By considering both an even and an odd $n > N$, we see that

$$|1 - a| < 1 \quad \text{and} \quad |-1-a| < 1.$$

Now by the Triangle Inequality 3.7

$$2 = |1-(-1)| = |1-a+a-(-1)| \le |1-a|+|a-(-1)| < 1+1 = 2.$$

This absurdity shows that our assumption that $\lim(-1)^n = a$ must be wrong, so the sequence $(-1)^n$ does not converge. ■

Example 5

Let (s_n) be a sequence of nonnegative real numbers and suppose that $s = \lim s_n$. Note that $s \ge 0$; see Exercise 8.9(a). Prove that $\lim \sqrt{s_n} = \sqrt{s}$.

Discussion. We must consider $\epsilon > 0$ and show that there exists N such that

$$n > N \quad \text{implies} \quad |\sqrt{s_n} - \sqrt{s}| < \epsilon.$$

This time we cannot expect to obtain N explicitly in terms of ϵ because of the general nature of the problem. But we can hope to show such N exists. The trick here is to violate our training in algebra and "irrationalize the denominator":

$$\sqrt{s_n} - \sqrt{s} = \frac{(\sqrt{s_n} - \sqrt{s})(\sqrt{s_n} + \sqrt{s})}{\sqrt{s_n} + \sqrt{s}} = \frac{s_n - s}{\sqrt{s_n} + \sqrt{s}}.$$

Since $s_n \to s$ we will be able to make the numerator small [for large n]. Unfortunately, if $s = 0$ the denominator will also be small. So we consider two cases. If $s > 0$, the denominator is bounded below by $\sqrt{s}$ and our trick will work:

$$|\sqrt{s_n} - \sqrt{s}| \le \frac{|s_n - s|}{\sqrt{s}},$$

so we will select N so that $|s_n - s| < \sqrt{s}\epsilon$ for $n > N$. Note that N exists, since we can apply the definition of limit to $\sqrt{s}\epsilon$ just as well as to ϵ. For $s = 0$, it can be shown directly that $\lim s_n = 0$ implies $\lim \sqrt{s_n} = 0$; the trick of "irrationalizing the denominator" is not needed in this case.

Formal Proof

Case I: $s > 0$. Let $\epsilon > 0$. Since $\lim s_n = s$, there exists N such that

$$n > N \quad \text{implies} \quad |s_n - s| < \sqrt{s}\epsilon.$$

Now $n > N$ implies

$$|\sqrt{s_n} - \sqrt{s}| = \frac{|s_n - s|}{\sqrt{s_n} + \sqrt{s}} \le \frac{|s_n - s|}{\sqrt{s}} < \frac{\sqrt{s}\epsilon}{\sqrt{s}} = \epsilon.$$

Case II: $s = 0$. This case is left to Exercise 8.3. ■

Example 6

Let (s_n) be a convergent sequence of real numbers such that $s_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim s_n = s \neq 0$. Prove that $\inf\{|s_n| : n \in \mathbb{N}\} > 0$.

Discussion. The idea is that "most" of the terms s_n are close to s and hence not close to 0. More explicitly, "most" of the terms s_n are within $\frac{1}{2}|s|$ of s, hence most s_n satisfy $|s_n| \ge \frac{1}{2}|s|$. This seems clear from Figure 8.1, but a formal proof will use the triangle inequality.

Formal Proof

Let $\epsilon = \frac{1}{2}|s| > 0$. Since $\lim s_n = s$, there exists N in $\mathbb{N}$ so that

$$n > N \quad \text{implies} \quad |s_n - s| < \frac{|s|}{2}.$$

Now

$$n > N \quad \text{implies} \quad |s_n| \ge \frac{|s|}{2}, \tag{1}$$

since otherwise the triangle inequality would imply

$$|s| = |s - s_n + s_n| \le |s - s_n| + |s_n| < \frac{|s|}{2} + \frac{|s|}{2} = |s|$$

which is absurd. If we set

$$m = \min\left\{\frac{|s|}{2}, |s_1|, |s_2|, \ldots, |s_N|\right\},$$

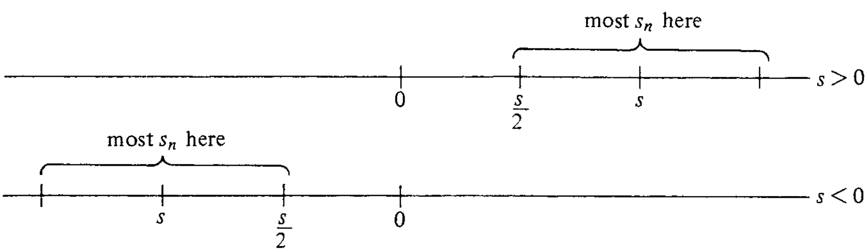

FIGURE 8.1

then we clearly have $m > 0$ and $|s_n| \geq m$ for *all* $n \in \mathbb{N}$ in view of (1). Thus $\inf\{|s_n| : n \in \mathbb{N}\} \geq m > 0$, as desired. ■

Formal proofs are required in the following exercises.

Exercises

8.1. Prove the following:
(a) $\lim \frac{(-1)^n}{n} = 0$ **(b)** $\lim \frac{1}{n^{1/3}} = 0$
(c) $\lim \frac{2n-1}{3n+2} = \frac{2}{3}$ **(d)** $\lim \frac{n+6}{n^2-6} = 0$

8.2. Determine the limits of the following sequences, and then prove your claims.
(a) $a_n = \frac{n}{n^2+1}$ **(b)** $b_n = \frac{7n-19}{3n+7}$
(c) $c_n = \frac{4n+3}{7n-5}$ **(d)** $d_n = \frac{2n+4}{5n+2}$
(e) $s_n = \frac{1}{n} \sin n$

8.3. Let (s_n) be a sequence of nonnegative real numbers, and suppose that $\lim s_n = 0$. Prove that $\lim \sqrt{s_n} = 0$. This will complete the proof for Example 5.

8.4. Let (t_n) be a bounded sequence, i.e., there exists M such that $|t_n| \leq M$ for all n, and let (s_n) be a sequence such that $\lim s_n = 0$. Prove that $\lim(s_n t_n) = 0$.

8.5. **(a)** Consider three sequences (a_n), (b_n) and (s_n) such that $a_n \leq s_n \leq b_n$ for all n and $\lim a_n = \lim b_n = s$. Prove that $\lim s_n = s$.

(b) Suppose that (s_n) and (t_n) are sequences such that $|s_n| \leq t_n$ for all n and $\lim t_n = 0$. Prove that $\lim s_n = 0$.

8.6. Let (s_n) be a sequence in $\mathbb{R}$.

(a) Prove that $\lim s_n = 0$ if and only if $\lim |s_n| = 0$.

(b) Observe that if $s_n = (-1)^n$, then $\lim |s_n|$ exists, but $\lim s_n$ does not exist.

8.7. Show that the following sequences do not converge.
(a) $\cos(\frac{n\pi}{3})$ **(b)** $s_n = (-1)^n n$
(c) $\sin(\frac{n\pi}{3})$

8.8. Prove the following [see Exercise 7.5]:
(a) $\lim[\sqrt{n^2+1} - n] = 0$ **(b)** $\lim[\sqrt{n^2+n} - n] = \frac{1}{2}$
(c) $\lim[\sqrt{4n^2+n} - 2n] = \frac{1}{4}$

8.9. Let (s_n) be a sequence that converges.

(a) Show that if $s_n \geq a$ for all but finitely many n, then $\lim s_n \geq a$.

(b) Show that if $s_n \leq b$ for all but finitely many n, then $\lim s_n \leq b$.

(c) Conclude that if all but finitely many s_n belong to $[a, b]$, then $\lim s_n$ belongs to $[a, b]$.

·8.10. Let (s_n) be a convergent sequence, and suppose that $\lim s_n > a$. Prove that there exists a number N such that $n > N$ implies $s_n > a$.

§9 Limit Theorems for Sequences

In this section we prove some basic results that are probably already familiar to the reader. First we prove that convergent sequences are bounded. A sequence (s_n) of real numbers is said to be *bounded* if the set $\{s_n : n \in \mathbb{N}\}$ is a bounded set, i.e., if there exists a constant M such that $|s_n| \leq M$ for all n.

9.1 Theorem.
Convergent sequences are bounded.

Proof
Let (s_n) be a convergent sequence, and let $s = \lim s_n$. Applying Definition 7.1 with $\epsilon = 1$ we obtain N in $\mathbb{N}$ so that

$$n > N \quad \text{implies} \quad |s_n - s| < 1.$$

From the triangle inequality we see that $n > N$ implies $|s_n| < |s| + 1$. Define $M = \max\{|s| + 1, |s_1|, |s_2|, \dots, |s_N|\}$. Then we have $|s_n| \leq M$ for all $n \in \mathbb{N}$, so (s_n) is a bounded sequence. ∎

In the proof of Theorem 9.1 we only needed to use property 7.1(1) for a single value of ϵ. Our choice of $\epsilon = 1$ was quite arbitrary.

9.2 Theorem.
If the sequence (s_n) converges to s and $k \in \mathbb{R}$, then the sequence (ks_n) converges to ks. That is, $\lim(ks_n) = k \lim s_n$.

Proof

We assume $k \neq 0$, since this result is trivial for $k = 0$. Let $\epsilon > 0$ and note that we need to show that $|ks_n - ks| < \epsilon$ for large n. Since $\lim s_n = s$, there exists N such that

$$n > N \quad \text{implies} \quad |s_n - s| < \frac{\epsilon}{|k|}.$$

Then

$$n > N \quad \text{implies} \quad |ks_n - ks| < \epsilon.$$

■

9.3 Theorem.

If (s_n) converges to s and (t_n) converges to t, then $(s_n + t_n)$ converges to $s + t$. That is,

$$\lim(s_n + t_n) = \lim s_n + \lim t_n.$$

Proof

Let $\epsilon > 0$; we need to show that

$$|s_n + t_n - (s+t)| < \epsilon \quad \text{for large} \quad n.$$

We note that $|s_n + t_n - (s+t)| \leq |s_n - s| + |t_n - t|$. Since $\lim s_n = s$, there exists N_1 such that

$$n > N_1 \quad \text{implies} \quad |s_n - s| < \frac{\epsilon}{2}.$$

Likewise, there exists N_2 such that

$$n > N_2 \quad \text{implies} \quad |t_n - t| < \frac{\epsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. Then clearly

$$n > N \quad \text{implies} \quad |s_n + t_n - (s+t)| \leq |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

■

9.4 Theorem.

If (s_n) converges to s and (t_n) converges to t, then $(s_n t_n)$ converges to st. That is,

$$\lim(s_n t_n) = (\lim s_n)(\lim t_n).$$

Discussion. The trick here is to look at the inequality

$$\begin{aligned}|s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\ &\le |s_n t_n - s_n t| + |s_n t - st| = |s_n| \cdot |t_n - t| + |t| \cdot |s_n - s|.\end{aligned}$$

For large n, $|t_n - t|$ and $|s_n - s|$ are small and $|t|$ is, of course, constant. Fortunately, Theorem 9.1 shows that $|s_n|$ is bounded, so we will be able to show that $|s_n t_n - st|$ is small.

Proof

Let $\epsilon > 0$. By Theorem 9.1 there is a constant $M > 0$ such that $|s_n| \le M$ for all n. Since $\lim t_n = t$ there exists N_1 such that

$$n > N_1 \quad \text{implies} \quad |t_n - t| < \frac{\epsilon}{2M}.$$

Also, since $\lim s_n = s$ there exists N_2 such that

$$n > N_2 \quad \text{implies} \quad |s_n - s| < \frac{\epsilon}{2(|t| + 1)}.$$

[We used $\frac{\epsilon}{2(|t|+1)}$ instead of $\frac{\epsilon}{2|t|}$, since t could be 0.] Now if $N = \max\{N_1, N_2\}$, then $n > N$ implies

$$\begin{aligned}|s_n t_n - st| &\le |s_n| \cdot |t_n - t| + |t| \cdot |s_n - s| \\ &\le M \cdot \frac{\epsilon}{2M} + |t| \cdot \frac{\epsilon}{2(|t| + 1)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

■

To handle quotients of sequences, we first deal with reciprocals.

9.5 Lemma.

If (s_n) converges to s, if $s_n \neq 0$ for all n, and if $s \neq 0$, then $(1/s_n)$ converges to $1/s$.

Discussion. We begin by considering the equality

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| = \left| \frac{s - s_n}{s_n s} \right|.$$

For large n, the numerator is small. The only possible difficulty would be if the denominator were also small for large n. This difficulty is solved in Example 6 of §8 where it is proved that $m =$

$\inf\{|s_n| : n \in \mathbb{N}\} > 0$. Thus

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| \leq \frac{|s - s_n|}{m|s|},$$

and it is clear how our proof should proceed.

Proof

Let $\epsilon > 0$. By Example 6 of §8, there exists $m > 0$ such that $|s_n| \geq m$ for all n. Since $\lim s_n = s$ there exists N such that

$$n > N \quad \text{implies} \quad |s - s_n| < \epsilon \cdot m|s|.$$

Then $n > N$ implies

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s - s_n|}{|s_n s|} \leq \frac{|s - s_n|}{m|s|} < \epsilon.$$

■

9.6 Theorem.

Suppose that (s_n) converges to s and (t_n) converges to t. If $s \neq 0$ and $s_n \neq 0$ for all n, then (t_n/s_n) converges to t/s.

Proof

By Lemma 9.5 $(1/s_n)$ converges to $1/s$, so

$$\lim \frac{t_n}{s_n} = \lim \frac{1}{s_n} \cdot t_n = \frac{1}{s} \cdot t = \frac{t}{s}$$

by Theorem 9.4. ■

The preceding limit theorems and a few standard examples allow one to easily calculate many limits.

9.7 Basic Examples.

(a) $\lim_{n\to\infty}(\frac{1}{n^p}) = 0$ for $p > 0$.
(b) $\lim_{n\to\infty} a^n = 0$ if $|a| < 1$.
(c) $\lim(n^{1/n}) = 1$.
(d) $\lim_{n\to\infty}(a^{1/n}) = 1$ for $a > 0$.

Proof

(a) Let $\epsilon > 0$ and let $N = (\frac{1}{\epsilon})^{1/p}$. Then $n > N$ implies $n^p > \frac{1}{\epsilon}$ and hence $\epsilon > \frac{1}{n^p}$. Since $\frac{1}{n^p} > 0$, this shows that $n > N$ implies

$|\frac{1}{n^p} - 0| < \epsilon$. [The meaning of n^p when p is not an integer will be discussed in §37.]

(b) We may suppose that $a \neq 0$, because $\lim_{n\to\infty} a^n = 0$ is obvious for $a = 0$. Since $|a| < 1$, we can write $|a| = \frac{1}{1+b}$ where $b > 0$. By the binomial theorem [Exercise 1.12], $(1 + b^n) \geq 1 + nb > nb$, so

$$|a^n - 0| = |a^n| = \frac{1}{(1+b)^n} < \frac{1}{nb}.$$

Now consider $\epsilon > 0$ and let $N = \frac{1}{\epsilon b}$. Then $n > N$ implies $n > \frac{1}{\epsilon b}$ and hence $|a^n - 0| < \frac{1}{nb} < \epsilon$.

(c) Let $s_n = (n^{1/n}) - 1$ and note that $s_n \geq 0$ for all n. By Theorem 9.3 it suffices to show that $\lim s_n = 0$. Since $1 + s_n = (n^{1/n})$, we have $n = (1 + s_n)^n$. For $n \geq 2$ we use the binomial expansion of $(1 + s_n)^n$ to conclude

$$n = (1 + s_n)^n \geq 1 + ns_n + \frac{1}{2}n(n-1)s_n^2 > \frac{1}{2}n(n-1)s_n^2.$$

Thus $n > \frac{1}{2}n(n-1)s_n^2$, so $s_n^2 < \frac{2}{n-1}$. Consequently, we have $s_n < \sqrt{\frac{2}{n-1}}$ for $n \geq 2$. A standard argument now shows that $\lim s_n = 0$; see Exercise 9.7.

(d) First suppose $a \geq 1$. Then for $n \geq a$ we have $1 \leq a^{1/n} \leq n^{1/n}$. Since $\lim n^{1/n} = 1$, it follows easily that $\lim(a^{1/n}) = 1$; compare Exercise 8.5(a). Suppose that $0 < a < 1$. Then $\frac{1}{a} > 1$, so $\lim(\frac{1}{a})^{1/n} = 1$ from above. Lemma 9.5 now shows that $\lim(a^{1/n}) = 1$. ∎

Example 1

Prove that $\lim s_n = \frac{1}{4}$, where

$$s_n = \frac{n^3 + 6n^2 + 7}{4n^3 + 3n - 4}.$$

Solution

We have

$$s_n = \frac{1 + \frac{6}{n} + \frac{7}{n^3}}{4 + \frac{3}{n^2} - \frac{4}{n^3}}.$$

By 9.7(a) we have $\lim \frac{1}{n} = 0$ and $\lim \frac{1}{n^3} = 0$. Hence by Theorems 9.3 and 9.2 we have

$$\lim\left(1 + \frac{6}{n} + \frac{7}{n^3}\right) = \lim(1) + 6 \cdot \lim\left(\frac{1}{n}\right) + 7 \cdot \lim\left(\frac{1}{n^3}\right) = 1.$$

Similarly, we have

$$\lim\left(4 + \frac{3}{n^2} - \frac{4}{n^3}\right) = 4.$$

Hence Theorem 9.6 implies that $\lim s_n = \frac{1}{4}$. □

Example 2
Find $\lim \frac{n-5}{n^2+7}$.

Solution
Let $s_n = \frac{n-5}{n^2+7}$. We can write s_n as $\frac{1-\frac{5}{n}}{n+\frac{7}{n}}$, but then the denominator does not converge. So we write

$$s_n = \frac{\frac{1}{n} - \frac{5}{n^2}}{1 + \frac{7}{n^2}}.$$

Now $\lim(\frac{1}{n} - \frac{5}{n^2}) = 0$ by 9.7(a) and Theorems 9.3 and 9.2. Likewise $\lim(1 + \frac{7}{n^2}) = 1$, so Theorem 9.6 tells us that $\lim s_n = \frac{0}{1} = 0$. □

Example 3
Find $\lim \frac{n^2+3}{n+1}$.

Solution
We can write $\frac{n^2+3}{n+1}$ as

$$\frac{n + \frac{3}{n}}{1 + \frac{1}{n}} \quad \text{or} \quad \frac{1 + \frac{3}{n^2}}{\frac{1}{n} + \frac{1}{n^2}}.$$

Both fractions lead to problems: either the numerator does not converge or else the denominator converges to 0. It turns out that $\frac{n^2+3}{n+1}$ does not converge and the symbol $\lim \frac{n^2+3}{n+1}$ is undefined, at least for the present; see Example 6. The reader may have the urge to use the symbol $+\infty$ here. Our next task is to make such use of the symbol $+\infty$ legitimate. For a sequence (s_n), $\lim s_n = +\infty$ will signify that the terms s_n are eventually all large. Here is the concise definition.

9.8 Definition.
For a sequence (s_n), we write $\lim s_n = +\infty$ provided

for each $M > 0$ there is a number N such that
$n > N$ implies $s_n > M$.

In this case we say that the sequence *diverges to* $+\infty$.
Similarly, we write $\lim s_n = -\infty$ provided

for each $M < 0$ there is a number N such that
$n > N$ implies $s_n < M$.

Henceforth we will say that (s_n) has a *limit* or that the *limit exists* provided (s_n) converges or diverges to $+\infty$ or diverges to $-\infty$. In the definition of $\lim s_n = +\infty$ the challenging values of M are large positive numbers: the larger M is the larger N will need to be. In the definition of $\lim s_n = -\infty$ the challenging values of M are "large" negative numbers like $-10{,}000{,}000{,}000$.

Example 4
We have $\lim n^2 = +\infty$, $\lim(-n) = -\infty$, $\lim 2^n = +\infty$ and $\lim(\sqrt{n} + 7) = +\infty$. Of course, many sequences do not have limits $+\infty$ or $-\infty$ even if they are unbounded. For example, the sequences defined by $s_n = (-1)^n n$ and $t_n = n\cos^2(\frac{n\pi}{2})$ are unbounded, but they do not diverge to $+\infty$ or $-\infty$, so the expressions $\lim[(-1)^n n]$ and $\lim[n\cos^2(\frac{n\pi}{2})]$ are meaningless. Note that $t_n = n$ when n is even and $t_n = 0$ when n is odd.

The strategy for proofs involving infinite limits is very much the same as for finite limits. We give some examples.

Example 5
Give a formal proof that $\lim(\sqrt{n} + 7) = +\infty$.

Discussion. We must consider an arbitrary $M > 0$ and show that there exists N [which will depend on M] such that

$$n > N \quad \text{implies} \quad \sqrt{n} + 7 > M.$$

To see how big N must be we "solve" for n in the inequality $\sqrt{n}+7 > M$. This inequality holds provided $\sqrt{n} > M-7$ or $n > (M-7)^2$. Thus we will take $N = (M-7)^2$.

Formal Proof

Let $M > 0$ and let $N = (M-7)^2$. Then $n > N$ implies $n > (M-7)^2$, hence $\sqrt{n} > M-7$, hence $\sqrt{n}+7 > M$. This shows that $\lim(\sqrt{n}+7) = +\infty$. ■

Example 6

Give a formal proof that $\lim \frac{n^2+3}{n+1} = +\infty$; see Example 3.

Discussion. Consider $M > 0$. We need to determine how large n must be to guarantee that $\frac{n^2+3}{n+1} > M$. The idea is to bound the fraction $\frac{n^2+3}{n+1}$ below by some multiple of $\frac{n^2}{n} = n$; compare Example 3 of §8. Since $n^2+3 > n^2$ and $n+1 \leq 2n$, we have $\frac{n^2+3}{n+1} > \frac{n^2}{2n} = \frac{1}{2}n$, and it suffices to arrange for $\frac{1}{2}n > M$.

Formal Proof

Let $M > 0$ and let $N = 2M$. Then $n > N$ implies $\frac{1}{2}n > M$, which implies

$$\frac{n^2+3}{n+1} > \frac{n^2}{2n} = \frac{1}{2}n > M.$$

Hence $\lim \frac{n^2+3}{n+1} = +\infty$. ■

The limit in Example 6 would be easier to handle if we could apply a limit theorem. But the limit theorems 9.2–9.6 do not apply.

WARNING. Do not attempt to apply the limit theorems 9.2–9.6 to infinite limits. Use Theorem 9.9 or 9.10 below or Exercises 9.9–9.12.

9.9 Theorem.

Let (s_n) and (t_n) be sequences such that $\lim s_n = +\infty$ *and* $\lim t_n > 0$ [$\lim t_n$ *can be finite or* $+\infty$]. *Then* $\lim s_n t_n = +\infty$.

Discussion. Let $M > 0$. We need to show that $s_n t_n > M$ for large n. We have $\lim s_n = +\infty$, and we need to be sure that the t_n's are bounded away from 0 for large n. We will choose a real number m

so that $0 < m < \lim t_n$ and observe that $t_n > m$ for large n. Then all we need is $s_n > \frac{M}{m}$ for large n.

Proof

Let $M > 0$. Select a real number m so that $0 < m < \lim t_n$. Whether $\lim t_n = +\infty$ or not, it is clear that there exists N_1 such that

$$n > N_1 \quad \text{implies} \quad t_n > m;$$

see Exercise 8.10. Since $\lim s_n = +\infty$, there exists N_2 so that

$$n > N_2 \quad \text{implies} \quad s_n > \frac{M}{m}.$$

Put $N = \max\{N_1, N_2\}$. Then $n > N$ implies $s_n t_n > \frac{M}{m} \cdot m = M$. ■

Example 7

Use Theorem 9.9 to prove that $\lim \frac{n^2+3}{n+1} = +\infty$; see Example 6.

Solution

We observe that $\frac{n^2+3}{n+1} = \frac{n+\frac{3}{n}}{1+\frac{1}{n}} = s_n t_n$ where $s_n = n + \frac{3}{n}$ and $t_n = \frac{1}{1+\frac{1}{n}}$. It is easy to show that $\lim s_n = +\infty$ and $\lim t_n = 1$. So by Theorem 9.9, we have $\lim s_n t_n = +\infty$. □

Here is another useful theorem.

9.10 Theorem.

For a sequence (s_n) of positive real numbers, we have $\lim s_n = +\infty$ if and only if $\lim(\frac{1}{s_n}) = 0$.

Proof

Let (s_n) be a sequence of positive real numbers. We have to show

$$\lim s_n = +\infty \quad \text{implies} \quad \lim\left(\frac{1}{s_n}\right) = 0 \tag{1}$$

and

$$\lim\left(\frac{1}{s_n}\right) = 0 \quad \text{implies} \quad \lim s_n = +\infty. \tag{2}$$

In this case the proofs will appear very similar, but the thought processes will be quite different.

To prove (1), suppose that $\lim s_n = +\infty$. Let $\epsilon > 0$ and let $M = \frac{1}{\epsilon}$. Since $\lim s_n = +\infty$, there exists N such that $n > N$ implies $s_n > M = \frac{1}{\epsilon}$. Therefore $n > N$ implies $\epsilon > \frac{1}{s_n} > 0$, so

$$n > N \quad \text{implies} \quad \left| \frac{1}{s_n} - 0 \right| < \epsilon.$$

That is, $\lim(\frac{1}{s_n}) = 0$. This proves (1).

To prove (2), we abandon the notation of the last paragraph and begin anew. Suppose that $\lim(\frac{1}{s_n}) = 0$. Let $M > 0$ and let $\epsilon = \frac{1}{M}$. Then $\epsilon > 0$, so there exists N such that $n > N$ implies $|\frac{1}{s_n} - 0| < \epsilon = \frac{1}{M}$. Since $s_n > 0$, we can write

$$n > N \quad \text{implies} \quad 0 < \frac{1}{s_n} < \frac{1}{M}$$

and hence

$$n > N \quad \text{implies} \quad M < s_n.$$

That is, $\lim s_n = +\infty$ and (2) holds. ■

Exercises

9.1. Using the limit theorems 9.2–9.6 and 9.7, prove the following. Justify all steps.

(a) $\lim \frac{n+1}{n} = 1$ **(b)** $\lim \frac{3n+7}{6n-5} = \frac{1}{2}$

(c) $\lim \frac{17n^5+73n^4-18n^2+3}{23n^5+13n^3} = \frac{17}{23}$

9.2. Suppose that $\lim x_n = 3$, $\lim y_n = 7$ and that all y_n are nonzero. Determine the following limits:

(a) $\lim(x_n + y_n)$ **(b)** $\lim \frac{3y_n - x_n}{y_n^2}$

9.3. Suppose that $\lim a_n = a$, $\lim b_n = b$, and that $s_n = \frac{a_n^3+4a_n}{b_n^2+1}$. Prove $\lim s_n = \frac{a^3+4a}{b^2+1}$ carefully, using the limit theorems.

9.4. Let $s_1 = 1$ and for $n \geq 1$ let $s_{n+1} = \sqrt{s_n + 1}$.

(a) List the first four terms of (s_n).

(b) It turns out that (s_n) converges. Assume this fact and prove that the limit is $\frac{1}{2}(1 + \sqrt{5})$.

9.5. Let $t_1 = 1$ and $t_{n+1} = \frac{t_n^2+2}{2t_n}$ for $n \geq 1$. Assume that (t_n) converges and find the limit.

9.6. Let $x_1 = 1$ and $x_{n+1} = 3x_n^2$ for $n \geq 1$.

(a) Show that if $a = \lim x_n$, then $a = \frac{1}{3}$ or $a = 0$.

(b) Does $\lim x_n$ exist? Explain.

(c) Discuss the apparent contradiction between parts (a) and (b).

9.7. Complete the proof of 9.7(c), i.e., give the standard argument needed to show that $\lim s_n = 0$.

9.8. Give the following when they exist. Otherwise assert "NOT EXIST."
(a) $\lim n^3$
(b) $\lim(-n^3)$
(c) $\lim(-n)^n$
(d) $\lim(1.01)^n$
(e) $\lim n^n$

9.9. Suppose that there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$.

(a) Prove that if $\lim s_n = +\infty$, then $\lim t_n = +\infty$.

(b) Prove that if $\lim t_n = -\infty$, then $\lim s_n = -\infty$.

(c) Prove that if $\lim s_n$ and $\lim t_n$ exist, then $\lim s_n \leq \lim t_n$.

9.10. **(a)** Show that if $\lim s_n = +\infty$ and $k > 0$, then $\lim(ks_n) = +\infty$.

(b) Show that $\lim s_n = +\infty$ if and only if $\lim(-s_n) = -\infty$.

(c) Show that if $\lim s_n = +\infty$ and $k < 0$, then $\lim(ks_n) = -\infty$.

9.11. **(a)** Show that if $\lim s_n = +\infty$ and $\inf\{t_n : n \in \mathbb{N}\} > -\infty$, then $\lim(s_n + t_n) = +\infty$.

(b) Show that if $\lim s_n = +\infty$ and $\lim t_n > -\infty$, then $\lim(s_n + t_n) = +\infty$.

(c) Show that if $\lim s_n = +\infty$ and if (t_n) is a bounded sequence, then $\lim(s_n + t_n) = +\infty$.

9.12. Assume all $s_n \neq 0$ and that the limit $L = \lim |\frac{s_{n+1}}{s_n}|$ exists.

(a) Show that if $L < 1$, then $\lim s_n = 0$. *Hint*: Select a so that $L < a < 1$ and obtain N so that $|s_{n+1}| < a|s_n|$ for $n \geq N$. Then show that $|s_n| < a^{n-N}|s_N|$ for $n > N$.

(b) Show that if $L > 1$, then $\lim |s_n| = +\infty$. *Hint*: Apply (a) to the sequence $t_n = \frac{1}{|s_n|}$; see Theorem 9.10.

9.13. Show that

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & \text{if } |a| < 1 \\ 1 & \text{if } a = 1 \\ +\infty & \text{if } a > 1 \\ \text{does not exist} & \text{if } a \le -1. \end{cases}$$

9.14. Let $p > 0$. Use Exercise 9.12 to show

$$\lim_{n\to\infty} \frac{a^n}{n^p} = \begin{cases} 0 & \text{if } |a| \le 1 \\ +\infty & \text{if } a > 1 \\ \text{does not exist} & \text{if } a < -1. \end{cases}$$

9.15. Show that $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$.

9.16. Use Theorems 9.9, 9.10 or Exercises 9.9–9.15 to prove the following:

(a) $\lim \frac{n^4+8n}{n^2+9} = +\infty$

(b) $\lim[\frac{2^n}{n^2} + (-1)^n] = +\infty$

(c) $\lim[\frac{3^n}{n^3} - \frac{3^n}{n!}] = +\infty$

9.17. Give a formal proof that $\lim n^2 = +\infty$ using only Definition 9.8.

9.18. **(a)** Verify $1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$ for $a \neq 1$.

(b) Find $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $|a| < 1$.

(c) Calculate $\lim_{n\to\infty}(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n})$.

(d) What is $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $a \ge 1$?

§10 Monotone Sequences and Cauchy Sequences

In this section we obtain two theorems [Theorems 10.2 and 10.11] that will allow us to conclude that certain sequences converge *without* knowing the limit in advance. These theorems are important because in practice the limits are not usually known in advance.

10.1 Definition.
A sequence (s_n) of real numbers is called a *nondecreasing sequence* if $s_n \le s_{n+1}$ for all n, and (s_n) is called a *nonincreasing sequence*

if $s_n \geq s_{n+1}$ for all n. Note that if (s_n) is nondecreasing, then $s_n \leq s_m$ whenever $n < m$. A sequence that is nondecreasing or nonincreasing will be called a *monotone sequence* or a *monotonic sequence*.

Example 1

The sequences defined by $a_n = 1 - \frac{1}{n}$, $b_n = n^3$ and $c_n = (1 + \frac{1}{n})^n$ are nondecreasing sequences, although this is not obvious for the sequence (c_n). The sequence $d_n = \frac{1}{n^2}$ is nonincreasing. The sequences $s_n = (-1)^n$, $t_n = \cos(\frac{n\pi}{3})$, $u_n = (-1)^n n$ and $v_n = \frac{(-1)^n}{n}$ are not monotonic sequences. Also $x_n = n^{1/n}$ is not monotonic, as can be seen by examining the first four values; see Example 1(d) in §7.

Of the sequences above, (a_n), (c_n), (d_n), (s_n), (t_n), (v_n) and (x_n) are bounded sequences. The remaining sequences, (b_n) and (u_n), are unbounded sequences.

10.2 Theorem.

All bounded monotone sequences converge.

Proof

Let (s_n) be a bounded nondecreasing sequence. Let S denote the set $\{s_n : n \in \mathbb{N}\}$, and let $u = \sup S$. Since S is bounded, u represents a real number. We show that $\lim s_n = u$. Let $\epsilon > 0$. Since $u - \epsilon$ is not an upper bound for S, there exists N such that $s_N > u - \epsilon$. Since (s_n) is nondecreasing, we have $s_N \leq s_n$ for all $n \geq N$. Of course, $s_n \leq u$ for all n, so $n > N$ implies $u - \epsilon < s_n \leq u$, which implies $|s_n - u| < \epsilon$. This shows that $\lim s_n = u$.

The proof for bounded nonincreasing sequences is left to Exercise 10.2. ■

Note that the Completeness Axiom 4.4 is a vital ingredient in the proof of Theorem 10.2.

10.3 Discussion of Decimals.

We have not given much attention to the notion that real numbers are simply decimal expansions. This notion is substantially correct, but there are subtleties to be faced. For example, different decimal expansions can represent the same real number. The somewhat more abstract developments of the set $\mathbb{R}$ of real numbers discussed in §6 turn out to be more satisfactory.

We restrict our attention to nonnegative decimal expansions and nonnegative real numbers. From our point of view, every nonnegative decimal expansion is shorthand for the limit of a bounded nondecreasing sequence of real numbers. Suppose we are given a decimal expansion $k.d_1d_2d_3d_4\cdots$ where k is a nonnegative integer and each d_j belongs to $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let

$$s_n = k + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}. \tag{1}$$

Then (s_n) is a nondecreasing sequence of real numbers, and (s_n) is bounded [by $k + 1$, in fact]. So by Theorem 10.2, (s_n) converges to a real number that we traditionally write as $k.d_1d_2d_3d_4\cdots$. For example, $3.3333\cdots$ represents

$$\lim_{n\to\infty}\left(3 + \frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^n}\right).$$

To calculate this limit, we borrow the following fact about geometric series from Example 1 in §14:

$$\lim_{n\to\infty} a(1 + r + r^2 + \cdots + r^n) = \frac{a}{1-r} \quad \text{for} \quad |r| < 1; \tag{2}$$

see also Exercise 9.18. In our case, $a = 3$ and $r = \frac{1}{10}$, so $3.3333\cdots$ represents $\frac{3}{1-\frac{1}{10}} = \frac{10}{3}$, as expected. Similarly, $0.9999\cdots$ represents

$$\lim_{n\to\infty}\left(\frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n}\right) = \frac{\frac{9}{10}}{1 - \frac{1}{10}} = 1.$$

Thus $0.9999\cdots$ and $1.0000\cdots$ are different decimal expansions that represent the same real number!

The converse of the preceding discussion also holds. That is, every nonnegative real number x has at least one decimal expansion. This will be proved, along with some related results, in the optional §16.

Unbounded monotone sequences also have limits.

10.4 Theorem.

(i) *If* (s_n) *is an unbounded nondecreasing sequence, then* $\lim s_n = +\infty$.

(ii) *If* (s_n) *is an unbounded nonincreasing sequence, then* $\lim s_n = -\infty$.

Proof

(i) Let (s_n) be an unbounded nondecreasing sequence. Let $M > 0$. Since the set $\{s_n : n \in \mathbb{N}\}$ is unbounded and it is bounded below by s_1, it must be unbounded above. Hence for some N in $\mathbb{N}$ we have $s_N > M$. Clearly $n > N$ implies $s_n \geq s_N > M$, so $\lim s_n = +\infty$.

(ii) The proof is similar and is left to Exercise 10.5. ■

10.5 Corollary.

If (s_n) *is a monotone sequence, then the sequence either converges, diverges to* $+\infty$*, or diverges to* $-\infty$*. Thus* $\lim s_n$ *is always meaningful for monotone sequences.*

Proof

Apply Theorems 10.2 and 10.4. ■

Let (s_n) be a bounded sequence in $\mathbb{R}$; it may or may not converge. It is apparent from the definition of limit in 7.1 that the limiting behavior of (s_n) depends only on sets of the form $\{s_n : n > N\}$. For example, if $\lim s_n$ exists, clearly it must lie in the interval $[u_N, v_N]$ where

$$u_N = \inf\{s_n : n > N\} \quad \text{and} \quad v_N = \sup\{s_n : n > N\};$$

see Exercise 8.9. As N increases, the sets $\{s_n : n > N\}$ get smaller, so we have

$$u_1 \leq u_2 \leq u_3 \leq \cdots \quad \text{and} \quad v_1 \geq v_2 \geq v_3 \geq \cdots;$$