# New results in Ramsey theory for trees

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# Conventions/Definitions

- All trees in this talk will be **uniquely rooted** and **finitely branching**.
- A tree $T$ will be called **homogeneous** if there exists an integer $b_T \geqslant 2$, called the **branching number** of $T$, such that every $t \in T$ has exactly $b_T$ immediate successors; e.g., every dyadic or triadic tree is homogeneous.
- A **vector tree** is a finite sequence of (possibly finite) trees having common height. The **level product** of a vector tree $\mathbf{T} = (T_1, \dots, T_d)$, denoted by $\otimes\mathbf{T}$, is defined to be the set

$$\bigcup_{n<h(\mathbf{T})} \otimes\mathbf{T}(n)$$

where $\otimes\mathbf{T}(n) = T_1(n) \times \cdots \times T_d(n)$.

# The concept of a strong subtree

A **strong subtree** of a tree $T$ is a subset $S$ of $T$ with the following properties:

(1) $S$ is uniquely rooted and balanced (that is, all maximal chains of $S$ have the same cardinality);

(2) there exists a subset $L_T(S) = \{l_n : n < h(S)\}$ of $\mathbb{N}$, called the **level set** of $S$ in $T$, such that for every $n < h(S)$ we have $S(n) \subseteq T(l_n)$;

(3) for every non-maximal $s \in S$ and every immediate successor $t$ of $s$ in $T$, there exists a *unique* immediate successor $s'$ of $s$ in $S$ such that $t \leqslant s'$.

# The Halpern–Läuchli theorem (strong subtree version)

**Theorem (Halpern & Läuchli – 1966)**

*For every integer $d \geqslant 1$ we have that $\mathrm{HL}(d)$ holds: for every $d$-tuple $(T_1, \dots, T_d)$ of uniquely rooted and finitely branching trees without maximal nodes and every finite coloring of the level product of $(T_1, \dots, T_d)$ there exist strong subtrees $(S_1, \dots, S_d)$ of $(T_1, \dots, T_d)$ of infinite height and with a common level set such that the level product of $(S_1, \dots, S_d)$ is monochromatic.*

# Some consequences

The following result is one of the earliest applications of the Halpern–Läuchli theorem.

**Theorem (Milliken – 1979 and 1981)**

*The class of strong subtrees (both finite and infinite) of a tree $T$ is partition regular.*

The reason why this result is powerful lies in the rich "geometric" properties of strong subtrees.

# The problem

(i) The natural problem whether there exists a density version of the Halpern–Läuchli theorem was first asked by Laver in the late 1960s who actually conjectured that there is such a version.

(ii) Bicker & Voigt (1983) observed that one has to restrict attention to the category of homogeneous trees. They also showed that for a single homogeneous there is a density version.

# The infinite version

**Theorem (D, Kanellopoulos & Karagiannis – 2010)**

*For every integer $d \geqslant 1$ we have that $\mathrm{DHL}(d)$ holds: for every $d$-tuple $(T_1, \dots, T_d)$ of homogeneous trees and every subset $D$ of the level product of $(T_1, \dots, T_d)$ satisfying*

$$\limsup_{n\to\infty} \frac{|D \cap \big(T_1(n) \times \cdots \times T_d(n)\big)|}{|T_1(n) \times \cdots \times T_d(n)|} > 0$$

*there exist strong subtrees $(S_1, \dots, S_d)$ of $(T_1, \dots, T_d)$ of infinite height and with a common level set such that the level product of $(S_1, \dots, S_d)$ is a subset of $D$.*

# The finite version

**Theorem (D, Kanellopoulos & Tyros – 2011)**

*For every $d \geqslant 1$, every $b_1, \ldots, b_d \geqslant 2$, every $k \geqslant 1$ and every $0 < \varepsilon \leqslant 1$ there exists an integer $N$ with the following property. If $\mathbf{T} = (T_1, \ldots, T_d)$ is a vector homogeneous tree with $b_{T_i} = b_i$ for all $i \in \{1, \ldots, d\}$, $L$ is a subset of $\mathbb{N}$ of cardinality at least $N$ and $D$ is a subset of the level product of $\mathbf{T}$ such that*

$$|D \cap \big(T_1(n) \times \cdots \times T_d(n)\big)| \geqslant \varepsilon |T_1(n) \times \cdots \times T_d(n)|$$

*for every $n \in L$, then there exist strong subtrees $(S_1, \ldots, S_d)$ of $(T_1, \ldots, T_d)$ of height $k$ and with a common level set such that the level product of $(S_1, \ldots, S_d)$ is a subset of $D$. The least integer $N$ with this property will be denoted by $\mathrm{UDHL}(b_1, \ldots, b_d | k, \varepsilon)$.*

# Comments

- The proof of the finite version is effective and gives explicit upper bounds for the numbers $\mathrm{UDHL}(b_1, \ldots, b_d|k, \varepsilon)$. These upper bounds, however, have an Ackermann-type dependence with respect to the “dimension” $d$.
- The one-dimensional case (that is, when “$d = 1$”) is due to Pach, Solymosi and Tardos (2010):

$$\mathrm{UDHL}(b|k, \varepsilon) = O_{b,\varepsilon}(k).$$

  This bound is clearly optimal.

# On the proofs

- The proof of the infinite version is based on stabilization arguments.
- The proof of the finite version is based on a density increment strategy and uses probabilistic (i.e. averaging) arguments. Following Furstenberg and Weiss (2003), for every finite vector homogeneous tree $\mathbf{T}$ define a probability measure on $\otimes\mathbf{T}$ by the rule

$$\mu_{\mathbf{T}}(A) = \mathbb{E}_{n<h(\mathbf{T})} \frac{|A \cap \otimes\mathbf{T}(n)|}{|\otimes\mathbf{T}(n)|}.$$

The crucial observation is that “lack of density increment” implies a strong concentration hypothesis for the probability measure $\mu_{\mathbf{T}}$.

# Consequences – I

## Theorem (D, Kanellopoulos & Tyros – 2011)

*For every $d \geqslant 1$, every $b_1, \ldots, b_d \geqslant 2$, every $n \geqslant 1$ and every $0 < \varepsilon \leqslant 1$ there exists a strictly positive constant $c(b_1, \ldots, b_d | n, \varepsilon)$ with the following property.*
*If $\mathbf{T} = (T_1, \ldots, T_d)$ is a vector homogeneous tree with $b_{T_i} = b_i$ for all $i \in \{1, \ldots, d\}$ and $\{A_{\mathbf{t}} : \mathbf{t} \in \otimes\mathbf{T}\}$ is a family of measurable events in a probability space $(\Omega, \Sigma, \mu)$ satisfying $\mu(A_{\mathbf{t}}) \geqslant \varepsilon$ for every $\mathbf{t} \in \otimes\mathbf{T}$, then there exist strong subtrees $(S_1, \ldots, S_d)$ of $(T_1, \ldots, T_d)$ of infinite height and with a common level set such that for every nonempty subset $F$ of the level product of $(S_1, \ldots, S_d)$ of cardinality $n$ we have*

$$\mu\Big(\bigcap_{\mathbf{t} \in F} A_{\mathbf{t}}\Big) \geqslant c(b_1, \ldots, b_d | n, \varepsilon).$$

# Consequences – II

- We obtain lower bounds of the form

$$c(b_1, \dots, b_d | n, \varepsilon) \geqslant (\varepsilon / C)^{\alpha^{2dn}}$$

where $C$ is a large constant depending on $b_1, \dots, b_d$ and $\varepsilon$ but *not* on $n$ and $\alpha = \mathrm{UDHL}(b_1, \dots, b_d | 2, \varepsilon/2)$.

# Consequences – III

- In the one-dimensional case (that is, when we deal with events indexed by a single homogeneous tree) we get significantly better lower bounds, and for certain classes of subsets of trees (such as finite chains, finite combs, doubletons and many more) *optimal* ones.
  The proofs in these cases are based, among others, on an appropriate generalization of the notion of a *Shelah line*.

# Consequences – IV

For every homogeneous tree $T$ denote by $\mathrm{Str}_2(T)$ the set of all strong subtrees of $T$ of height 2.

## Corollary (D, Kanellopoulos & Tyros – 2011)

*Let $T$ be a homogeneous tree with branching number $b$ and $0 < \varepsilon \leqslant 1$. If $\{A_S : S \in \mathrm{Str}_2(T)\}$ is a family of measurable events in a probability space $(\Omega, \Sigma, \mu)$ satisfying $\mu(A_S) \geqslant \varepsilon$ for every $S \in \mathrm{Str}_2(T)$, then there exists a strong subtree $R$ of $T$ of infinite height such that for every $n \geqslant 1$ and every $S_1, \dots, S_n \in \mathrm{Str}_2(R)$ with $S_1(0) = \cdots = S_n(0)$ we have*

$$\mu\Big(\bigcap_{i=1}^{n} A_{S_i}\Big) \geqslant c(\underbrace{b, \dots, b}_{b-\text{times}} | n, \varepsilon).$$