Applications of descriptive set theory to the geometry of Banach spaces

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Motivation – a fundamental example

Let Λ be a nonempty set.

- $\mathrm{Tr}(\Lambda)$ is the set of all trees on Λ.
- $\mathrm{WF}(\Lambda)$ is the set of all well-founded trees on Λ.
- For every $T \in \mathrm{Tr}(\Lambda)$, $\varrho(T)$ stands for the rank of T.

Fact

If $A \subseteq \mathrm{WF}(\mathbb{N})$ *is analytic, then* $\sup\{\varrho(T) : T \in A\} < \omega_1$.

First approach

- The map $T \mapsto \varrho(T)$ is a co-analytic rank on $\mathrm{WF}(\mathbb{N})$; that is, the relations

$$S \leqslant_\Sigma T \Leftrightarrow T \notin \mathrm{WF}(\mathbb{N}) \text{ or } (T \in \mathrm{WF}(\mathbb{N}) \text{ and } \varrho(S) \leqslant \varrho(T))$$

  and

$$S <_\Sigma T \Leftrightarrow T \notin \mathrm{WF}(\mathbb{N}) \text{ or } (T \in \mathrm{WF}(\mathbb{N}) \text{ and } \varrho(S) < \varrho(T))$$

  are both analytic.
- We use *boundedness*.

Second approach

- We select $F \subseteq \mathrm{Tr} \times \mathbb{N}^{\mathbb{N}}$ closed such that $\mathrm{proj}_{\mathrm{Tr}} F = A$ and we define $R \in \mathrm{Tr}(\mathbb{N} \times \mathbb{N})$ by the rule

$$(t, s) \in R \Leftrightarrow \exists (T, x) \in F \text{ with } t \in T \text{ and } s = x|\ell(s).$$

- The tree R is well-founded.
- For every $T \in A$ there exists a natural monotone map $\psi : T \to R$. Hence,

$$\sup\{\varrho(T) : T \in A\} \leqslant \varrho(R) < \omega_1.$$

Generalizing the second approach

Definition (Informal statement)

Suppose that B is a subset of a Polish space X and that $\phi : B \to \omega_1$ is an ordinal rank on B. Suppose, further, that there exists a natural notion of embedding between elements of B which is coherent with the rank ϕ in the sense that if $x, y \in B$ and x embeds into y, then $\phi(x) \leqslant \phi(y)$. We will say that B is *strongly bounded* if for every analytic subset A of B there exists $y \in B$ such that every $x \in A$ embeds into y.

Comments

- Strong boundedness implies boundedness.
- The set $\mathrm{WF}(\mathbb{N})$ is strongly bounded; but in this case strong boundedness is equivalent to boundedness.
- The first nontrivial example of a strongly bounded class was discovered by Kechris and Woodin (1991). The class was Π^1_2 (dilators).

The content of the lectures

- Several natural classes of separable Banach spaces are strongly bounded. This structural information can be used to answer a number of basic problems in the geometry of Banach spaces.
- We will review part of the mathematics involved and present some consequences.

Functional analytic background, I

Definition

A *Banach space* X is a vector space (over the real $\mathbb{R}$ or the complex $\mathbb{C}$ field) together with a norm $\|\cdot\|$ with which it is complete.

- In this talk we will consider only real Banach spaces.
- We notice, however, that a number of important tools in Banach space theory require complex structure.

Functional analytic background, II

- Examples of classical Banach spaces:

$$c_0 \ , \ \ell_p \ , \ C[0,1] \ , \ L_p.$$

- Non-classical examples but fundamental for the development of the theory:

Tsirelson space T,

James tree space JT,

Gowers–Maurey space GM,

Argyros–Haydon space AH.

Functional analytic background, III

- Problems of *subspace structure* (called by Banach as *problems of linear dimension*).
- Embedding problems.
- Problems of global structure (also discussed by Banach in his book).

Fact (Banach–Mazur)

Every separable Banach space X is isometric to a subspace of $C[0,1]$.

Functional analytic background, IV

Problem

Suppose that (P) is a property of Banach spaces and suppose that we are given a class $\mathcal{C}$ of separable Banach spaces all of which have property (P). When can we find a space Y which has property (P) and is universal for $\mathcal{C}$ (i.e. Y contains an isomorphic copy of every $X \in \mathcal{C}$)?

Example

If (P) is the property of having "separable dual" and $\mathcal{C}$ is the class of all spaces having separable dual, then this problem was asked by Banach and answered negatively by Szlenk (1968).

Functional analytic background, V

- An affirmative answer to the previous problem (for most natural properties) is related to the complexity of the given class $\mathcal{C}$ viewed as a subset of a natural coding of separable Banach spaces.

The standard Borel space $\mathcal{SB}$ of separable Banach spaces, I

- Let U be any separable Banach space universal for all separable Banach spaces. For concreteness, we will work with the space $C[0,1]$.
- Let $F(C[0,1])$ be the set of all closed subsets of $C[0,1]$. We equip $F(C[0,1])$ with the Effros–Borel structure, i.e. with the σ-algebra Σ generated by the sets

$$\{F \in F(C[0,1]) : F \cap V \neq \varnothing\}$$

where V ranges over all open subsets of $C[0,1]$.
- The measurable space $\big(F(C[0,1]), \Sigma\big)$ is *standard* (i.e. it is Borel isomorphic to the reals).

The standard Borel space $\mathbb{SB}$ of separable Banach spaces, II

Definition (Bossard)

By $\mathbb{SB}$ we shall denote the set

$$\{X \in F(C[0,1]) : X \text{ is a linear subspace}\}$$

equipped with the relative Effros–Borel structure.

Fact

The space $\mathbb{SB}$ is standard.

The standard Borel space $\mathcal{SB}$ of separable Banach spaces, III

- The space $C[0,1]$ is universal for all separable Banach spaces. Hence, we may identify every class of separable Banach spaces with a subset of $\mathcal{SB}$. With this identification properties of separable Banach spaces become sets in $\mathcal{SB}$. So, we define the following subsets of $\mathcal{SB}$ by considering classical properties of Banach spaces.

Property	Corresponding Set
being uniformly convex	UC
being reflexive	REFL
having separable dual	SD
not containing X	NC_X
being non-universal	NU

The standard Borel space $\mathcal{SB}$ of separable Banach spaces, IV

- The set UC is Borel.
- The sets REFL, SD, NC_X and NU are complete co-analytic. Moreover, natural ordinal ranks defined by Banach space theorists turned out to be co-analytic ranks.

SET	CORRESPONDING RANK
REFL	ϕ_{REFL} (Argyros – D)
SD	Sz (Szlenk)
NC_X	ϕ_{NC_X} (Bourgain)
NU	ϕ_{NU} (Bourgain)

Schauder bases

Definition

A sequence (x_n) of non-zero vectors in a Banach space X is called a *Schauder basis* of X if for every $x \in X$ there exists a unique sequence (a_n) of reals such that $x = \sum_{n\in\mathbb{N}} a_n x_n$.

- All "classical" Banach spaces have Schauder bases.
- There exists a separable Banach space without a Schauder basis (Enflo, 1973).
- Every separable Banach space which is not "too Euclidean" contains a subspace without a Schauder basis (Maurey–Pisier, Szankowski, 1978).
- On the other hand, every Banach space contains a subspace with a Schauder basis (Banach, Mazur).

The rank ϕ_{NU}, I

- We fix a normalized Schauder basis (e_n) of the space $C[0,1]$.
- Let X be a separable Banach space and $\delta \geqslant 1$. We consider the tree $T(X,(e_n),\delta)$ in X consisting of all finite sequences $(x_n)_{n=0}^k$ such that for every $a_0,\dots,a_k \in \mathbb{R}$

$$\frac{1}{\delta}\Big\|\sum_{n=0}^{k} a_n e_n\Big\| \leqslant \Big\|\sum_{n=0}^{k} a_n x_n\Big\| \leqslant \delta\Big\|\sum_{n=0}^{k} a_n e_n\Big\|.$$

Fact (Bourgain)

X is non-universal if and only if for every $\delta \geqslant 1$ the tree $T(X,(e_n),\delta)$ is well-founded.

The rank ϕ_{NU}, II

Definition (Bourgain)

For every non-universal separable Banach space X we set

$$\phi_{\mathrm{NU}}(X) = \sup\{\varrho(T(X,(e_n),\delta)) : \delta \geqslant 1\}.$$

If X is universal, then we set $\phi_{\mathrm{NU}}(X) = \omega_1$.

Theorem (Bourgain)

X is non-universal if and only if $\phi_{\mathrm{NU}}(X) < \omega_1$.

Theorem (Bossard)

The set NU is complete co-analytic and the map $\mathrm{NU} \ni X \mapsto \phi_{\mathrm{NU}}(X)$ is a co-analytic rank on NU.

Strongly bounded classes of Banach spaces, I

Definition (Argyros–D)

Let $\mathcal{C} \subseteq \mathrm{SB}$. The class $\mathcal{C}$ is said to be *strongly bounded* if for every analytic subset $\mathcal{A}$ of $\mathcal{C}$ there exists $Y \in \mathcal{C}$ that contains an isomorphic copy of every $X \in \mathcal{A}$.

Strongly bounded classes of Banach spaces, II

Theorem

The following classes are strongly bounded.

- [Argyros – D; D – Ferenczi] *The class* REFL.
- [Argyros – D; D – Ferenczi] *The class* SD.
- [D] *The class* NC_X *where* X *is a minimal Banach space not containing* ℓ_1.
- [D] *The class* NU.
- [D – Lopez Abad] *The class* US *of all unconditionally saturated separable Banach spaces.*

Consequences, I

- The problem whether the classes SD and NU are strongly bounded was posed by Alekos Kechris (mid-1980s).
- The structural information that a certain class is strongly bounded has a number of consequences.

Theorem (D)

Let $\mathcal{C} \subseteq \mathrm{SB}$. Then the following are equivalent.

(i) *There exists a non-universal separable Banach space Y that contains an isomorphic copy of every $X \in \mathcal{C}$.*

(ii) *We have $\sup\{\phi_{\mathrm{NU}}(X) : X \in \mathcal{C}\} < \omega_1$.*

(iii) *There exists an analytic subset $\mathcal{A}$ of NU such that $\mathcal{C} \subseteq \mathcal{A}$.*

Consequences, II

- The corresponding results for the classes REFL, SD and NC_X are also true. For instance:

Theorem (D–Ferenczi)

Let $\mathcal{C} \subseteq \mathrm{SB}$. Then the following are equivalent.

(i) *There exists a separable reflexive Banach space Y that contains an isomorphic copy of every $X \in \mathcal{C}$.*

(ii) *We have $\sup\{\phi_{\mathrm{REFL}}(X) : X \in \mathcal{C}\} < \omega_1$.*

(iii) *There exists an analytic subset $\mathcal{A}$ of REFL such that $\mathcal{C} \subseteq \mathcal{A}$.*

Consequences, III

- The class UC of all separable uniformly convex Banach spaces is Borel and $\mathrm{UC} \subseteq \mathrm{REFL}$. Hence:

Corollary (Odell–Schlumprecht)

There exists a separable reflexive Banach space R that contains an isomorphic copy of every separable uniformly convex Banach space.

The problem of the existence of such a space was posed by Bourgain (1980).

Consequences, IV

- For every countable ordinal ξ the set $\mathcal{S}_\xi = \{X \in \mathrm{SB} : \mathrm{Sz}(X) \leqslant \xi\}$ is a Borel subset of SD. Hence:

Corollary (D–Ferenczi)

There exists a family $\{Y_\xi : \xi < \omega_1\}$ of Banach spaces with separable dual such that for every countable ordinal ξ and every Banach space X with $\mathrm{Sz}(X) \leqslant \xi$ the space Y_ξ contains an isomorphic copy of X.

The problem of the existence of such a family was posed by Rosenthal (1979).

The “dual” property, I

- Recently, the “dual” theory was developed to treat quotients instead of embeddings.

Definition (D)

Let $\mathcal{C} \subseteq \mathrm{SB}$. We say that the class $\mathcal{C}$ is *surjectively strongly bounded* if for every analytic subset $\mathcal{A}$ of $\mathcal{C}$ there exists $Y \in \mathcal{C}$ such that every $X \in \mathcal{A}$ is a quotient of Y.

Theorem (D)

The class NC_{ℓ_1} is surjectively strongly bounded.

The "dual" property, II

Fact

Every separable Banach space is a quotient of ℓ_1*.*

Theorem (D)

Let $\mathcal{C} \subseteq \mathrm{SB}$*. Then the following are equivalent.*

(i) *There exists a separable Banach space* Y *not containing* ℓ_1 *such that every* $X \in \mathcal{C}$ *is a quotient of* Y*.*

(ii) *We have* $\sup\{\phi_{\mathrm{NC}_{\ell_1}}(X) : X \in \mathcal{C}\} < \omega_1$*.*

(iii) *There exists an analytic subset* $\mathcal{A}$ *of* NC_{ℓ_1} *such that* $\mathcal{C} \subseteq \mathcal{A}$*.*

Proof of the structural results: methodology

- Although for each class one has to develop different tools, a clear methodology has been developed which can be summarized in the following basic steps.

Step 1 One treats the case of analytic classes of separable Banach spaces with a Schauder basis.

Step 2 The general case is reduced to the previous one as follows. An appropriate embedding result is proved. That is, given a property (P) of Banach spaces and a separable Banach space X with property (P) one constructs a Banach space $Y(X)$ with a Schauder basis and with property (P) and such that $Y(X)$ contains an isomorphic copy of X.

Step 3 We parameterize the construction described in Step 2.

Analytic classes of Banach spaces with a Schauder basis, I

- Let $\mathcal{A}$ be an analytic subset of SB such that every $Y \in \mathcal{A}$ has a Schauder basis. The goal is to amalgamate the spaces in the class $\mathcal{A}$ in a very particular way.

Definition (Argyros–D)

Let X be a Banach space, T a pruned tree on a countable set Λ and $(x_t)_{t\in T}$ a normalized sequence in X indexed by the tree T. We say that $\mathfrak{X} = (X, T, \Lambda, (x_t)_{t\in T})$ is a *Schauder tree basis* if the following are satisfied.

(i) $X = \overline{\mathrm{span}}\{x_t : t \in T\}$.

(ii) For every $\sigma \in [T]$ the sequence $(x_{\sigma|n})$ is a (bi-monotone) Schauder basic sequence.

Analytic classes of Banach spaces with a Schauder basis, II

- We have the following representation result.

Lemma (Argyros–D)

Let $\mathcal{A}$ be an analytic subset of SB such that every $Y \in \mathcal{A}$ has a Schauder basis. Then there exists a Schauder tree basis $\mathfrak{X} = (X, T, \Lambda, (x_t)_{t \in T})$ such that the following are satisfied.

(i) *For every $Y \in \mathcal{A}$ there exists $\sigma \in [T]$ such that $Y \cong X_\sigma$ and*

(ii) *for every $\sigma \in [T]$ there exists $Y \in \mathcal{A}$ such that $X_\sigma \cong Y$*

where $X_\sigma = \overline{\mathrm{span}}\{x_{\sigma|n} : n \in \mathbb{N}\}$ for every $\sigma \in [T]$.

The proof uses a classical construction due to Pełczyński and "unfolding".

Analytic classes of Banach spaces with a Schauder basis, III

Definition (Argyros–D)

Let $\mathfrak{X} = (X, T, \Lambda, (x_t)_{t\in T})$ be a Schauder tree basis. The ℓ_2 *Baire sum associated with* $\mathfrak{X}$, denoted by $T_2^{\mathfrak{X}}$, is defined to be the completion of $c_{00}(T)$ equipped with the norm

$$\|z\|_{T_2^{\mathfrak{X}}} = \sup\Big\{\Big(\sum_{i=0}^{k}\|\sum_{t\in\mathfrak{s}_i} z(t)x_t\|_X^2\Big)^{1/2}\Big\}$$

where the above supremum is taken over all finite families $(\mathfrak{s}_i)_{i=0}^k$ of pairwise incomparable segments of T.

- The definition of the space $T_2^{\mathfrak{X}}$ is a variant of a classical construction due to James.

Analytic classes of Banach spaces with a Schauder basis, IV

- The standard Hamel basis $(e_t)_{t \in T}$ of $c_{00}(T)$ defines a normalized (bi-monotone) Schauder basis of $T_2^{\mathfrak{X}}$.
- For every $\sigma \in [T]$ the subspace $\mathcal{X}_\sigma = \overline{\mathrm{span}}\{e_{\sigma|n} : n \in \mathbb{N}\}$ of $T_2^{\mathfrak{X}}$ is isometric to X_σ and it is one-complemented in $T_2^{\mathfrak{X}}$ via the natural projection $P_\sigma : T_2^{\mathfrak{X}} \to \mathcal{X}_\sigma$. Therefore, the space $T_2^{\mathfrak{X}}$ contains a complemented copy of every space in the class coded by the Schauder tree basis.
- The space $T_2^{\mathfrak{X}}$, however, contains other subspaces. For instance, it always contains an isomorphic copy of c_0.

Analytic classes of Banach spaces with a Schauder basis, V

Theorem (D–Lopez Abad)

Let $\mathfrak{X} = (X, T, \Lambda, (x_t)_{t \in T})$ be a Schauder tree basis and let Y be a subspace of $T_2^{\mathfrak{X}}$. Then, either

(i) *there exist a subspace Z of Y and $\sigma \in [T]$ such that Z is isomorphic to a subspace of X_σ, or*

(ii) *Y contains a Schauder basic sequence (y_n) satisfying an upper ℓ_2 estimate (in particular, the subspace of Y spanned by the sequence (y_n) contains no ℓ_p for any $1 \leqslant p < 2$).*

- The proof of this result uses functional-analytic and combinatorial tools (mainly Ramsey theoretical).

Interpolation, I

- Interpolation is a powerful functional-analytic method to factorize operators. There are many variants. We will use the interpolation scheme invented by Davis, Fiegel, Johnson and Pełczyński.

Definition (Davis–Fiegel–Johnson–Pełczyński)

Let X be a Banach space and let W be a closed, convex, bounded and symmetric subset of X. For every $n \in \mathbb{N}$ with $n \geqslant 1$ by $\|\cdot\|_n$ we denote the equivalent norm on X induced by the Minkowski gauge of the set $2^n W + 2^{-n} B_X$. That is,

$$\|x\|_n = \inf\left\{\lambda > 0 : \frac{x}{\lambda} \in 2^n W + 2^{-n} B_X\right\}$$

for every $x \in X$.

Interpolation, II

Definition (Davis–Fiegel–Johnson–Pełczyński)

Let X be a Banach space and let W be a closed, convex, bounded and symmetric subset of X. Also let $1 < p < +\infty$. For every $x \in X$ we set

$$|x|_p = \Big(\sum_{n \geqslant 1} \|x\|_n^p \Big)^{1/p}.$$

The *p-interpolation space* of the pair (X, W) is the vector subspace of X

$$\{x \in X : |x|_p < +\infty\}$$

equipped with the $|\cdot|_p$ norm.

Amalgamated spaces, I

Definition (Argyros–D)

Let $\mathfrak{X}$ be a Schauder tree basis and consider the ℓ_2 Baire sum $T_2^{\mathfrak{X}}$ associated with $\mathfrak{X}$. We set

$$W_{\mathfrak{X}} = \overline{\mathrm{conv}}\Big\{ \bigcup_{\sigma \in [T]} B_{\mathcal{X}_\sigma} \Big\}.$$

- Notice that $W_{\mathfrak{X}}$ is closed, convex, bounded and symmetric.

Definition (Argyros–D)

Let $\mathfrak{X}$ be a Schauder tree basis and $1 < p < +\infty$. The *p-amalgamation space associated with $\mathfrak{X}$*, denoted by $A_p^{\mathfrak{X}}$, is defined to be the p-interpolation space of the pair $(T_2^{\mathfrak{X}}, W_{\mathfrak{X}})$.

Amalgamated spaces, II

- The space $A_p^{\mathfrak{X}}$ also has a Schauder basis and contains a complemented copy of every space in the class coded by the Schauder tree basis.
- However, the space $A_p^{\mathfrak{X}}$ is a much “smaller” space than $T_2^{\mathfrak{X}}$. In particular, we have the following dichotomy.

Amalgamated spaces, III

Theorem (Argyros–D)

Let $\mathfrak{X}$ be a Schauder tree basis and $1 < p < +\infty$. Let Y be a subspace of $A_p^{\mathfrak{X}}$. Then, either

(i) *the space Y contains an isomorphic copy of ℓ_p, or*

(ii) *there exist $\sigma_0, \ldots, \sigma_k \in [T]$ such that Y is isomorphic to a subspace of $X_{\sigma_0} \oplus \cdots \oplus X_{\sigma_k}$.*

Moreover, if for every $\sigma \in [T]$ the space X_σ is reflexive, then $A_p^{\mathfrak{X}}$ is reflexive too.

Amalgamated spaces, IV

- The proof of this basic dichotomy requires several steps. The arguments in its proof are also functional-analytic and combinatorial.
- Amalgamated spaces were used by Louveau, Ferenczi and Rosendal to show that the relation of isomorphism (linear and Lipschitz) between separable Banach spaces is a complete analytic equivalence relation.
- Recent progress on the equivalence relation of uniform homeomorphism between separable Banach spaces was made by Jackson, Gao and Sari.

The general case

- The machinery presented so far is very efficient for producing universal spaces for analytic classes of Banach spaces with a Schauder basis.
- As we have mentioned, for the general case we will use appropriate embedding results.

The classes REFL and SD: shrinking bases, I

Definition

Let (x_n) be a Schauder basis of a Banach space X. The sequence (x_n) is said to be *shrinking* if the bi-orthogonal functionals (x_n^*) associated with (x_n) is a Schauder basis of X^*.

- Clearly if (x_n) is a shrinking Schauder basis of X, then X^* is separable. We point out, however, that the converse is not true; that is, if (x_n) is a Schauder basis of a Banach space X with separable dual, then (x_n) is not necessarily shrinking.
- An old problem in Banach space theory asked whether every Banach space with separable dual embeds into a space with a shrinking Schauder basis.

The classes REFL and SD: Zippin's embedding theorem, II

Theorem (Zippin)

The following hold.

(i) *Every separable reflexive Banach space embeds into a reflexive space with a Schauder basis.*

(ii) *Every Banach with separable dual embeds into a space with a shrinking Schauder basis.*

The classes REFL and SD: parameterizing Zippin's theorem, III

- Based on an alternative proof of Zippin's theorem, due to Ghoussoub–Maurey–Schachermayer, Bossard parameterized Zippin's theorem as follows.

Theorem (Bossard)

Let $\mathcal{B}$ be a Borel subset of SD. Then the relation $\mathcal{Z} \subseteq \mathcal{B} \times \mathrm{SB}$ defined by

$$(X, Y) \in \mathcal{Z} \Leftrightarrow Y \text{ is isomorphic to } Z(X)$$

is analytic, where $Z(X)$ denotes the Ghoussoub–Maurey–Schachermayer space associated with X.

The classes REFL and SD: parameterizing Zippin's theorem, IV

- The proof of Zippin's theorem given by Ghoussoub, Maurey and Schachermayer uses interpolation and a powerful selection procedure (called as "dessert selection").
- I have parameterized the selection result of Ghoussoub, Maurey and Schachermayer (this yields a different proof of Bossard's result). The arguments are similar to the proof of the "strategic uniformization theorem" and use (classical) boundedness in an essential way.

The classes NU, NC_X and US: $\mathcal{L}_\infty$ spaces, I

Definition

Let X and Y be two isomorphic Banach spaces. The *Banach–Mazur distance* between X and Y is defined by

$$d(X,Y) = \inf\left\{\|T\|\cdot\|T^{-1}\| : T : X \to Y \text{ is isomorphism}\right\}.$$

Definition (Lindenstrauss–Pełczyński)

Let X be an infinite-dimensional Banach space and $\lambda \geqslant 1$. The space X is said to be a $\mathcal{L}_{\infty,\lambda}$ space if for every $\varepsilon > 0$ and every finite dimensional subspace F of X there exists a finite dimensional subspace G of X with $F \subseteq G$ and such that $d(G, \ell_\infty^k) \leqslant \lambda + \varepsilon$ where $k = \dim(G)$.
The space X is said to be $\mathcal{L}_\infty$ if it is $\mathcal{L}_{\infty,\lambda}$ for some $\lambda \geqslant 1$.

The classes NU, NC_X and US: $\mathcal{L}_\infty$ spaces, II

- The class of $\mathcal{L}_\infty$ spaces was introduced with the hope to characterize $C(K)$ spaces by local means.
- The local structure of $\mathcal{L}_\infty$ spaces imposes some global regularity properties. For instance:

Theorem (Johnson–Rosenthal–Zippin)

Every separable $\mathcal{L}_\infty$ space has a Schauder basis.

The classes NU, NC_X and US: the Bourgain–Pisier construction, III

Theorem (Bourgain–Pisier)

Let X be a separable Banach space and $\lambda > 1$. Then X embeds isometrically into a $\mathcal{L}_{\infty,\lambda}$ space, denote by $\mathcal{L}_\lambda[X]$, in such a way that the quotient $\mathcal{L}_\lambda[X]/X$ is "small" in the sense that it has the Schur and the Radon–Nikodym properties.

- A Banach space X has the Schur property if every weakly null sequence in X is automatically norm convergent. By Rosenthal's dichotomy, a Schur space is hereditarily ℓ_1.
- Roughly speaking, a space with the Radon–Nikodym property "looks like" a dual space (and, in fact, all separable dual spaces have the Radon–Nikodym property).

The classes NU, NC_X and US: the Bourgain–Pisier construction, IV

- The Bourgain–Pisier construction was the outcome of the combination of two major achievements of Banach space theory during the 1980s:
  (a) the Bourgain–Delbaen space, the first example of a $\mathcal{L}_\infty$ space not containing an isomorphic copy of c_0, and
  (b) Pisier's scheme for producing counterexamples to an old conjecture of Grothendieck.
- The building blocks of the space $\mathcal{L}_\lambda[X]$ are obtained using a method for extending operators invented by Kisliakov.

The classes NU, NC_X and US: parameterizing the Bourgain–Pisier construction, V

Theorem (D)

For every $\lambda > 1$ the relation $\mathcal{L}_\lambda \subseteq \mathrm{SB} \times \mathrm{SB}$ defined by

$$(X, Y) \in \mathcal{L}_\lambda \Leftrightarrow Y \text{ is isometric to } \mathcal{L}_\lambda[X]$$

is analytic.

- The idea of the proof is to "code" step-by-step the Bourgain–Pisier construction along the branches of a tree in an appropriate selected Polish space.

The "dual" results: the class NC_{ℓ_1}, I

Theorem (D)

For every separable Banach space X there exists a separable Banach space E_X with the following properties.

(i) *The space E_X has a Schauder basis.*

(ii) *X is a quotient of E_X.*

(iii) *Every subspace Y of E_X either contains a subspace of X or a copy of c_0.*

(iv) *The set $\mathcal{E} \subseteq \mathrm{SB} \times \mathrm{SB}$ defined by*

$$(X, Y) \in \mathcal{E} \Leftrightarrow Y \text{ is isometric to } E_X$$

is analytic.

(v) *E_X^* is separable if and only if X^* is separable.*

The "dual" results: the class NC_{ℓ_1}, II

- The proof is based on a number of different ideas and combines functional analytic tools, descriptive set theory and Ramsey theory for trees (the Halpern–Läuchli theorem and its consequences, as well as Stern's theorem).

Further consequences, I

- We know that there exist separable Banach spaces without a Schauder basis. Nevertheless:

Corollary (D)

There exists a map $f : \omega_1 \to \omega_1$ such that for every $\xi < \omega_1$ every separable Banach space X with $\phi_{\mathrm{NU}}(X) \leqslant \xi$ embeds into a Banach space Y with a Schauder basis satisfying $\phi_{\mathrm{NU}}(Y) \leqslant f(\xi)$.

- The proof is heavily based on the machinery discussed so far. No concrete bounds are known.

Further consequences, II

Corollary (D)

For every $\lambda > 1$ there exists a family $\{Y_\xi^\lambda : \xi < \omega_1\}$ of separable Banach spaces with the following properties.

(i) *For every $\xi < \omega_1$ the space Y_ξ^λ is non-universal and $\mathcal{L}_{\infty,\lambda}$.*

(ii) *If $\xi < \zeta < \omega_1$, then Y_ξ^λ embeds into Y_ζ^λ.*

(iii) *For every $\xi < \omega_1$ every separable Banach space X with $\phi_{\mathrm{NU}}(X) \leqslant \xi$ embeds into Y_ξ^λ.*

A final comment

- Beside its intrinsic functional-analytic interest, the fact that many classes of separable Banach spaces are strongly bounded makes it reasonable to expect that the phenomenon is not as rare as might be seen from a first glance and more examples should exist.