UNIFORMITY NORMS, THEIR WEAKER VERSIONS, AND APPLICATIONS

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Abstract. We show that, under some mild hypotheses, the Gowers uniformity norms (both in the additive and in the hypergraph setting) are essentially equivalent to certain weaker norms which are easier to understand. We present two applications of this equivalence: a variant of the Koopman–von Neumann decomposition, and a proof of the relative inverse theorem for the Gowers $U^s[N]$-norm using a norm-type pseudorandomness condition.

1. Introduction

1.1. This note is motivated by problems in arithmetic combinatorics and related parts of Ramsey theory, and focuses on the relation between two notions of pseudorandomness which appear in this context. The first notion is measured using the *Gowers uniformity norms* [7, 8]. These norms are very useful in order to accurately count the number of copies of certain "patterns" in subsets of discrete structures; see, e.g., [17, Lemma 11.4]. However, they are defined by estimating the correlation of a function with shifts of itself, and so their dual norms are hopelessly difficult to understand in full generality.

To compensate this problem, one adopts a functional analytic point of view. First one selects a class $\mathcal{D}$ of bounded functions (the "dual" functions), and then associates with $\mathcal{D}$ a norm defined by the rule $\|f\|_{\mathcal{D}} \coloneqq \sup\big\{|\langle f,g\rangle| : g\in\mathcal{D}\big\}$. If the set $\mathcal{D}$ is appropriately selected, then the norm $\|\cdot\|_{\mathcal{D}}$ is comparable to the Gowers uniformity norm for *bounded* functions. Unfortunately, in general, the norm $\|\cdot\|_{\mathcal{D}}$ is significantly weaker, and this apparently excludes its applicability in the study of sparse sets like the set of primes numbers.

Nevertheless, recently it was shown, first implicitly in [1] and then more explicitly in [18, 19], that the Gowers uniformity norms and their aforementioned weaker versions are essentially equivalent for a fairly large (and practically useful) family of unbounded functions.

We analyze further this phenomenon (both in the additive and in the hypergraph setting) and we show that it is more typical than anticipated. Compared with the

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results in [1, 18, 19] which rely on the "linear forms condition" (a pseudorandomness hypothesis originating from [10]), our approach is more intrinsic and is based exclusively on the properties of the Gowers uniformity norms. In a nutshell, our main results (Propositions 2.1 and 3.1) follow from the Gowers–Cauchy–Schwarz inequality and a simple decomposition method introduced in [2].

1.2. We present two applications of this equivalence. The first one is a variant of the Koopman–von Neumann decomposition (Corollary 4.4 in the main text). It asserts that a real-valued function f on a finite additive group can be approximated in a Gowers uniformity norm by a bounded function, provided that $|f|$ is majorized by a function ν which satisfies a natural norm-type pseudorandomness condition. It is important to note that, besides its separate theoretical interest, this approximation is essential for further applications. Indeed, Corollary 4.4 together with an appropriate version of the generalized von Neumann theorem—e.g., [10, Proposition 5.3]—provides yet another approach[1] to the relative Szemerédi theorem [10, Theorem 3.5], one of the main two ingredients of the Green–Tao theorem [10].

The second application (which is presented in Section 5) is a proof of the relative inverse theorem for the Gowers $U^s[N]$-norm, a result which is part of the *nilpotent Hardy–Littlewood method* invented by Green and Tao [11]. Our approach is based on Corollary 4.4 and, as such, it shows that the relative inverse theorem can also be applied under a norm-type pseudorandomness condition.

1.3. **Notation.** For every positive integer n we set $[n] \coloneqq \{1, \dots, n\}$, and for every nonempty finite set V by $|V|$ we denote its cardinality. Moreover, for every function $f \colon V \to \mathbb{R}$ by $\mathbb{E}[f(v) \,|\, v \in V]$ we denote the average of f, that is,

$$\mathbb{E}[f(v) \,|\, v \in V] \coloneqq \frac{1}{|V|} \sum_{v \in V} f(v).$$

We also write $\mathbb{E}_{v \in V} f(v)$ to denote the average of f, or simply $\mathbb{E}[f]$ if the set V is understood from the context.

We use the following $o(\cdot)$ and $O(\cdot)$ notation. If $a_1, \dots, a_k$ are parameters and η is a positive real, then we write $o_{\eta \to 0; a_1, \dots, a_k}(X)$ to denote a quantity bounded in magnitude by $X F_{a_1, \dots, a_k}(\eta)$ where $F_{a_1, \dots, a_k}$ is a function which depends on $a_1, \dots, a_k$ and goes to zero as $\eta \to 0$. Similarly, by $O_{a_1, \dots, a_k}(X)$ we denote a quantity bounded in magnitude by $X C_{a_1, \dots, a_k}$ where $C_{a_1, \dots, a_k}$ is a positive constant depending on the parameters $a_1, \dots, a_k$; we also write $Y \ll_{a_1, \dots, a_k} X$ or $X \gg_{a_1, \dots, a_k} Y$ for the estimate $|Y| = O_{a_1, \dots, a_k}(X)$.

[1] See [1, 9, 16, 20] for other proofs of the relative Szemerédi theorem.

2. The Gowers uniformity norm versus its weak version

2.1. Let Z be a finite additive group and let $s \geqslant 2$ be an integer. Also let $f\colon Z \to \mathbb{R}$ and recall that the *Gowers uniformity norm* $\|f\|_{U^s(Z)}$ *of* f is defined by the rule

$$\|f\|_{U^s(Z)} \coloneqq \mathbb{E}\Big[\prod_{\omega\in\{0,1\}^s} f(x+\omega\cdot\mathbf{h})\,\Big|\, x\in Z, \mathbf{h}\in Z^s\Big]^{1/2^s} \tag{2.1}$$

where $\omega\cdot\mathbf{h} \coloneqq \sum_{i=1}^s \omega_i\, h_i$ for every $\omega=(\omega_i)\in\{0,1\}^s$ and every $\mathbf{h}=(h_i)\in Z^s$. One can also define these norms for complex-valued functions by appropriately inserting complex conjugation operations—see [17] for details.

As we have noted, there is a natural weak version of the $U^s(Z)$-norm. Specifically, let $f\colon Z\to\mathbb{R}$ and define[2] the *weak uniformity norm* $\|f\|_{w^s(Z)}$ *of* f by setting

$$\|f\|_{w^s(Z)} \coloneqq \sup\Big\{\mathbb{E}\big[f(x)\prod_{\omega\in\{0,1\}^s\setminus\{0^s\}} h_\omega(x+\omega\cdot\mathbf{h})\,\big|\, x\in Z, \mathbf{h}\in Z^s\big]\Big\} \tag{2.2}$$

where the above supremum is taken over all families $\langle h_\omega : \omega\in\{0,1\}^s\setminus\{0^s\}\rangle$ of $[-1,1]$-valued functions on Z, and $0^s=(0,\dots,0)\in\{0,1\}^s$ denotes the sequence of length s taking the constant value 0. We observe that

$$\|f\|_{w^s(Z)} \leqslant \|f\|_{U^s(Z)} \tag{2.3}$$

as can be seen by the Gowers–Cauchy–Schwarz inequality (see, e.g., [17, (11.6)]).

2.2. **The main result.** By (2.1) and (2.2), it follows readily that for every function $f\colon Z\to[-1,1]$ we have $\|f\|_{U^s(Z)} \leqslant \|f\|_{w^s(Z)}^{1/2^s}$. The following proposition shows that the estimate (2.3) can also be *reversed* provided that f is merely bounded in magnitude by a function $\nu\colon Z\to\mathbb{R}^+$ satisfying a norm-type pseudorandomness condition.

Proposition 2.1. *Let Z be a finite additive group, let $s\geqslant 2$ be an integer and let $0<\eta\leqslant 1$. Also let $\nu\colon Z\to\mathbb{R}^+$ such that*

$$\|\nu-1\|_{U^{2s}(Z)} \leqslant \eta. \tag{2.4}$$

Finally, let $f\colon Z\to\mathbb{R}$ with $|f|\leqslant\nu$. If $\|f\|_{w^s(Z)}\leqslant\eta$, then

$$\|f\|_{U^s(Z)} = o_{\eta\to 0;s}(1). \tag{2.5}$$

Proposition 2.1 can be proved arguing as in [18, Theorem 11] and using slightly stronger pseudorandomness hypotheses (see also [19, Proposition 3.7] for a variant of this argument). We will give a proof using as a main tool the following simple consequence of the Gowers–Cauchy–Schwarz inequality for the $U^{2s}(Z)$-norm which was first observed (in a slightly less general form) in the proof of Proposition 4 in [18].

[2]There is no standard terminology for these norms.

Fact 2.2. *Let Z be a finite additive group and let $s \geqslant 2$ be an integer. Also let $g\colon Z \to \mathbb{R}$, let $\langle g^{(k)}_\omega \colon k \in \{1,2\}, \omega \in \{0,1\}^s \setminus \{0^s\}\rangle$ be a family of real-valued functions on Z, and set*

$$I \coloneqq \mathbb{E}\Big[g(x) \prod_{k\in\{1,2\}} \prod_{\omega\in\{0,1\}^s\setminus\{0^s\}} g^{(k)}_\omega(x+\omega\cdot \mathbf{h}_k)\,\Big|\, x\in Z, \mathbf{h}_1,\mathbf{h}_2\in Z^s\Big].$$

Then we have

$$|I| \leqslant \|g\|_{U^{2s}(Z)} \cdot \prod_{k\in\{1,2\}} \prod_{\omega\in\{0,1\}^s\setminus\{0^s\}} \|g^{(k)}_\omega\|_{U^{2s}(Z)}.$$

Proof. We identify $\{0,1\}^{2s}$ with $\{0,1\}^s\times\{0,1\}^s$ and we write every $\omega\in\{0,1\}^{2s}$ as $\omega=(\omega_1,\omega_2)$ where $\omega_1,\omega_2\in\{0,1\}^s$. We define a family $\langle g_\omega\colon \omega\in\{0,1\}^{2s}\rangle$ of real-valued functions on Z by setting: (i) $g_{(0^s,0^s)}=g$, (ii) $g_{(\omega,0^s)}=g^{(1)}_\omega$ and $g_{(0^s,\omega)}=g^{(2)}_\omega$ if $\omega\in\{0,1\}^s\setminus\{0^s\}$, and (iii) $g_{(\omega_1,\omega_2)}=1$ if $\omega_1,\omega_2\in\{0,1\}^s\setminus\{0^s\}$. Noticing that

$$I=\mathbb{E}\Big[\prod_{\omega\in\{0,1\}^{2s}} g_\omega(x+\omega\cdot\mathbf{h})\,\Big|\, x\in Z,\, \mathbf{h}\in Z^{2s}\Big],$$

the result follows from the Gowers–Cauchy–Schwarz inequality. □

We proceed to the proof of Proposition 2.1.

Proof of Proposition 2.1. We will show that for every nonempty subset Ω of $\{0,1\}^s$ and for every (possibly empty) family $\langle h_\omega : \omega\in\{0,1\}^s\setminus\Omega\rangle$ of $[-1,1]$-valued functions on Z we have[3]

$$\mathbb{E}\Big[\prod_{\omega\in\Omega} f(x+\omega\cdot\mathbf{h}) \prod_{\omega\in\{0,1\}^s\setminus\Omega} h_\omega(x+\omega\cdot\mathbf{h})\,\Big|\, x\in Z,\mathbf{h}\in Z^s\Big] = o_{\eta\to0;s}(1). \tag{2.6}$$

Clearly, this is enough to complete the proof.

We proceed by induction on the cardinality of Ω. Since the left-hand side of (2.6) is invariant under permutations of the cube, the initial case $|\Omega|=1$ follows from our assumption that $\|f\|_{w^s(Z)}\leqslant\eta$. Next, let $m\in\{1,\dots,2^s-1\}$ and assume that (2.6) has been proved for every $\Omega\subseteq\{0,1\}^s$ with $|\Omega|=m$. Fix $\Omega'\subseteq\{0,1\}^s$ with $|\Omega'|=m+1$. By permuting the cube if necessary, we may assume that $0^s\in\Omega'$. Set $\Omega\coloneqq\Omega'\setminus\{0^s\}$ and notice that $|\Omega|=m$. Also let $\langle h_\omega:\omega\in\{0,1\}^s\setminus\Omega'\rangle$ be an arbitrary family of $[-1,1]$-valued function on Z. We have to show that

$$\mathbb{E}\Big[f(x)\prod_{\omega\in\Omega} f(x+\omega\cdot\mathbf{h}) \prod_{\omega\in\{0,1\}^s\setminus\Omega'} h_\omega(x+\omega\cdot\mathbf{h})\,\Big|\, x\in Z,\mathbf{h}\in Z^s\Big] = o_{\eta\to0;s}(1)$$

or, equivalently,

$$\mathbb{E}[f(x)G(x)\,|\,x\in Z] = o_{\eta\to0;s}(1) \tag{2.7}$$

[3]In (2.6) we follow the convention that the product of an empty family of functions is equal to the constant function 1.

where $G\colon Z\to\mathbb{R}$ is the marginal defined by the rule

$$G(x)=\mathbb{E}\Big[\prod_{\omega\in\Omega} f(x+\omega\cdot\mathbf{h})\prod_{\omega\in\{0,1\}^s\setminus\Omega'} h_\omega(x+\omega\cdot\mathbf{h})\,\Big|\,\mathbf{h}\in Z^s\Big]. \tag{2.8}$$

Since $|f|\leqslant\nu$ and $\mathbb{E}[\nu]\leqslant\|\nu\|_{U^{2s}(Z)}\leqslant 1+\eta$, by the Cauchy–Schwarz inequality, it is enough to prove that

$$\mathbb{E}[(\nu-1)G^2]=o_{\eta\to0;s}(1)\ \text{ and }\ \mathbb{E}[G^2]=o_{\eta\to0;s}(1). \tag{2.9}$$

The first estimate in (2.9) follows from Fact 2.2 and the fact that $\|\nu-1\|_{U^{2s}(Z)}\leqslant\eta$; indeed, observe that

$$|\mathbb{E}[(\nu-1)G^2]|\leqslant\|\nu-1\|_{U^{2s}(Z)}\cdot\|f\|^{2|\Omega|}_{U^{2s}(Z)}\cdot\prod_{\omega\in\{0,1\}^s\setminus\Omega'}\|h_\omega\|^2_{U^{2s}(Z)}\ll_s\eta.$$

For the second estimate, as in [2, Theorem 7.1], we will use a simple decomposition. Specifically, let $\beta>0$ be a cut-off parameter and write $G^2=\mathbf{1}_{[|G|\leqslant\beta]}G^2+\mathbf{1}_{[|G|>\beta]}G^2$. As we shall see, any value of β greater than 1 would suffice for the proof; for concreteness we will use the value $\beta=2$. By linearity of expectation, it is enough to show that

$$\mathbb{E}[\mathbf{1}_{[|G|\leqslant2]}G^2]=o_{\eta\to0;s}(1)\ \text{ and }\ \mathbb{E}[\mathbf{1}_{[|G|>2]}G^2]=o_{\eta\to0;s}(1). \tag{2.10}$$

The first part of (2.10) can be handled easily by our inductive assumptions. Indeed, set $h_{0^s}=\mathbf{1}_{[|G|\leqslant2]}(G/2)$ and notice that

$$\mathbb{E}[\mathbf{1}_{[G|\leqslant2]}G^2]=2\cdot\mathbb{E}\Big[h_{0^s}(x)\prod_{\omega\in\Omega} f(x+\omega\cdot\mathbf{h})\prod_{\omega\in\{0,1\}^s\setminus\Omega'} h_\omega(x+\omega\cdot\mathbf{h})\,\Big|\,x\in Z,\mathbf{h}\in Z^s\Big]$$

which is $o_{\eta\to0;s}(1)$ since $|h_{0^s}|\leqslant1$ and $\Omega\subseteq\{0,1\}^s$ satisfies $|\Omega|=m$. For the second part of (2.10), observe that

$$\mathbb{E}[\mathbf{1}_{[|G|>2]}G^2]\leqslant\mathbb{E}[\mathbf{1}_{[\mathcal{N}>2]}\mathcal{N}^2] \tag{2.11}$$

where $\mathcal{N}\colon Z\to\mathbb{R}$ is defined by $\mathcal{N}(x)=\mathbb{E}[\prod_{\omega\in\Omega}\nu(x+\omega\cdot\mathbf{h})\,|\,\mathbf{h}\in Z^s]$. The function $\mathcal{N}$ satisfies the following moment estimate: for every $A\subseteq Z$ and every $k\in\{1,2\}$ we have

$$\mathbb{E}[\mathbf{1}_A\mathcal{N}^k]=\mathbf{P}(A)+o_{\eta\to0;s}(1) \tag{2.12}$$

where $\mathbf{P}(A)=\mathbb{E}[\mathbf{1}_A]=|A|/|Z|$ is the probability of A with respect to the uniform probability measure $\mathbf{P}$ on Z. Indeed, since $|\mathbb{E}[\mathbf{1}_A\mathcal{N}^k]-\mathbf{P}(A)|=|\mathbb{E}[\mathbf{1}_A(\mathcal{N}^k-1)]|$, the estimate in (2.12) follows from Fact 2.2, a telescopic argument and the fact that $\|\nu-1\|_{U^{2s}(Z)}\leqslant\eta$. Now, combining (2.11) and (2.12) for $k=2$ and invoking Markov's inequality, we have

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{[|G|>2]}G^2] &\leqslant \mathbf{P}\big([\mathcal{N}>2]\big)+o_{\eta\to0;s}(1)\leqslant\mathbf{P}\big([|\mathcal{N}-1|>1]\big)+o_{\eta\to0;s}(1)\\ &\leqslant \mathbb{E}\big[|\mathcal{N}-1|\big]+o_{\eta\to0;s}(1).\end{aligned}$$

On the other hand, by (2.12) for $k=1$, we see that

$$\mathbb{E}\big[|\mathcal{N}-1|\big]=\mathbb{E}[\mathbf{1}_{[\mathcal{N}\geqslant1]}(\mathcal{N}-1)]+\mathbb{E}[\mathbf{1}_{[\mathcal{N}<1]}(1-\mathcal{N})]=o_{\eta\to0;s}(1).$$

Therefore, $\mathbb{E}[\mathbf{1}_{[|G|>2]}G^2] = o_{\eta\to 0;s}(1)$ as desired. □

Remark 1. It is not hard to see that the proof of Proposition 2.1 in fact yields that for every $0<\varepsilon\leqslant 1$, if $\nu\colon Z\to\mathbb{R}^+$ satisfies $\|\nu-1\|_{U^{2s}(Z)}\leqslant\eta$ for some $0<\eta\leqslant\varepsilon$ and $f\colon Z\to\mathbb{R}$ is such that $|f|\leqslant\nu$ and $\|f\|_{w^s(Z)}\leqslant\varepsilon$, then we have $\|f\|_{U^s(Z)}\ll_s \varepsilon^C + o_{\eta\to 0}(1)$ where $C=(2^s\cdot 2^{2^s-1})^{-1}$.

Remark 2. By appropriately modifying the proof of Proposition 2.1, one can establish the equivalence between the $U^s(Z)$-norm and its weak version using more general pseudorandomness hypotheses. In particular, we have the following proposition which is related to [2, Theorem 7.1].

Proposition 2.3. *Let Z be a finite additive group, let $s\geqslant 2$ be an integer and let $0<\eta\leqslant 1$. Let $1<p\leqslant\infty$, let q denote the conjugate exponent of p and set $\ell:=\min\{2n: n\in\mathbb{N}$ and $2n\geqslant 2q\}$. Also let $\nu\colon Z\to\mathbb{R}^+$ such that*

$$\|\nu-\psi\|_{U^{\ell s}(Z)}\leqslant\eta \tag{2.13}$$

where $\psi\colon Z\to\mathbb{R}$ satisfies[4] *$\|\psi\|_{L_p}\leqslant 1$ and $\|\psi\|_{U^{\ell s}(Z)}\leqslant 1$. Finally, let $f\colon Z\to\mathbb{R}$ with $|f|\leqslant\nu$. If $\|f\|_{w^s(Z)}\leqslant\eta$, then*

$$\|f\|_{U^s(Z)} = o_{\eta\to 0;s,p}(1). \tag{2.14}$$

Observe that Proposition 2.1 corresponds to the case "$p=\infty$" and "$\psi=1$". Also note that, by Hölder's inequality, if p is sufficiently large, then the estimate $\|\psi\|_{U^{\ell s}(Z)}\leqslant 1$ follows from the estimate $\|\psi\|_{L_p}\leqslant 1$.

3. The box norm versus the cut norm

3.1. Let V be a nonempty finite set and let $s\geqslant 2$ be an integer. Also let $F\colon V^s\to\mathbb{R}$ and recall that the *box norm* $\|F\|_{\square(V^s)}$ *of* F is defined by the rule

$$\|F\|_{\square(V^s)} := \mathbb{E}\Big[\prod_{\omega\in[2]^s} F\big(\pi_\omega(x)\big)\,\Big|\, x\in V^{s\times 2}\Big]^{1/2^s} \tag{3.1}$$

where for every $\omega=(\omega_i)\in[2]^s$ by $\pi_\omega\colon V^{s\times 2}\to V^s$ we denote the projection $\pi_\omega\big((x_{ij})\big)=(x_{i\,\omega_i})_{i=1}^s$. These norms are the abstract versions of the Gowers uniformity norms; indeed, notice that for every finite additive group Z and every $f\colon Z\to\mathbb{R}$ we have

$$\|f\|_{U^s(Z)} = \|f(x_1+\dots+x_s)\|_{\square(Z^s)}. \tag{3.2}$$

We will also work with the following slight variants of the box norms which first appeared in [13]: for every even integer $\ell\geqslant 2$ we define the *ℓ-box norm* $\|F\|_{\square_\ell(V^s)}$ *of* F by setting

$$\|F\|_{\square_\ell(V^s)} := \mathbb{E}\Big[\prod_{\omega\in[\ell]^s} F\big(\pi_\omega(x)\big)\,\Big|\, x\in V^{s\times \ell}\Big]^{1/\ell^s} \tag{3.3}$$

[4]Here, the L_p-norm of ψ is computed using the uniform probability measure on Z, that is, $\|\psi\|_{L_p}:=\mathbb{E}[\,|\psi(x)|^p\,|\,x\in Z]^{1/p}$.

where, as above, for every $\omega = (\omega_i) \in [\ell]^s$ by $\pi_\omega \colon V^{s\times \ell} \to V^s$ we denote the projection $\pi_\omega\big((x_{ij})\big) = (x_{i\,\omega_i})_{i=1}^s$. Clearly, the $\Box_2(V^s)$-norm coincides with the $\Box(V^s)$-norm. As the parameter ℓ increases, the quantity $\|F\|_{\Box_\ell(V^s)}$ also increases and measures the integrability of F. In particular, for bounded functions all these norms are essentially equivalent. This fact, together with some basic properties of the ℓ-box norms, are discussed in the appendix.

The box norm also has a natural weak version which is known as the *cut norm* and originates from [6]. Specifically, let V, s and F be as above, and define[5] the *cut norm* $\|F\|_{\mathrm{cut}(V^s)}$ *of* F by the rule

$$\|F\|_{\mathrm{cut}(V^s)} \coloneqq \sup\Big\{\mathbb{E}\big[F\big(\pi_{1^s}(x)\big) \prod_{\omega\in[2]^s\setminus\{1^s\}} H_\omega\big(\pi_\omega(x)\big) \,\big|\, x\in V^{s\times 2}\big]\Big\} \tag{3.4}$$

where the above supremum is taken over all families $\langle H_\omega : \omega\in[2]^s\setminus\{1^s\}\rangle$ of $[-1,1]$-valued functions on V^s, and $1^s = (1,\dots,1)\in[2]^s$ denotes the sequence of length s taking the constant value 1. By the Gowers–Cauchy–Schwarz inequality for the $\Box(V^s)$-norm, we see that

$$\|F\|_{\mathrm{cut}(V^s)} \leqslant \|F\|_{\Box(V^s)}. \tag{3.5}$$

Also observe that if F is $[-1,1]$-valued, then $\|F\|_{\Box(V^s)} \leqslant \|F\|_{\mathrm{cut}(V^s)}^{1/2^s}$.

3.2. **The main result.** The following proposition is the analogue of Proposition 2.1 and establishes the equivalence of the box norm with the cut norm.

Proposition 3.1. *Let V be a nonempty finite set, let $s\geqslant 2$ be an integer and let $0<\eta\leqslant 1$. Also let $\nu\colon V^s\to\mathbb{R}^+$ such that*

$$\|\nu-1\|_{\Box_4(V^s)} \leqslant \eta. \tag{3.6}$$

Finally, let $F\colon V^s\to\mathbb{R}$ with $|F|\leqslant\nu$. If $\|F\|_{\mathrm{cut}(V^s)}\leqslant\eta$, then

$$\|F\|_{\Box(V^s)} = o_{\eta\to 0;s}(1). \tag{3.7}$$

It is possible to prove Proposition 3.1 arguing as in [1, Theorem 2.17]. However, as the reader has probably already noticed, Proposition 3.1 can be proved arguing precisely as in Proposition 2.1, using instead of Fact 2.2 the following elementary consequence of the Gowers–Cauchy–Schwarz inequality for the $\Box_4(V^s)$-norm.

Fact 3.2. *Let V be a nonempty finite set and let $s\geqslant 2$ be an integer. Also let $G\colon V^s\to\mathbb{R}$, let $\langle G_\omega^{(k)} : k\in\{1,2\}, \omega\in[2]^s\setminus\{1^s\}\rangle$ be a family of real-valued functions on V^s, and set*[6]

$$I \coloneqq \mathbb{E}\Big[G(y)\prod_{k\in\{1,2\}}\ \prod_{\omega\in[2]^s\setminus\{1^s\}} G_\omega^{(k)}\big(\pi_\omega(y,z_k)\big)\,\Big|\, y,z_1,z_2\in V^s\Big].$$

[5] In several places in the literature, the cut norm is defined by taking the supremum in (3.4) over all families $\langle H_\omega : \omega\in[2]^s\setminus\{1^s\}\rangle$ of $[0,1]$-valued functions on V^s. However, it is clear that this more restrictive definition yields an equivalent norm.

[6] Here, we identify $V^{s\times 2}$ with $V^s\times V^s$ via the bijection $V^{s\times 2}\ni(x_{ij})\mapsto\big((x_{i1}),(x_{i2})\big)\in V^s\times V^s$. In particular, we write uniquely every $x\in V^{s\times 2}$ as $x=(y,z)\in V^s\times V^s$.

Then we have

$$|I| \leqslant \|G\|_{\square_4(V^s)} \cdot \prod_{k\in\{1,2\}} \prod_{\omega\in[2]^s\setminus\{1^s\}} \|G^{(k)}_\omega\|_{\square_4(V^s)}.$$

Proof. Define a map $\{1,3\}^s \setminus \{1^s\} \ni \omega = (\omega_i) \mapsto \omega' = (\omega'_i) \in [2]^s \setminus \{1^s\}$ by setting $\omega'_i = 1$ if $\omega_i = 1$, and $\omega'_i = 2$ if $\omega_i = 3$. Then we may write

$$I = \mathbb{E}\Big[G_{1^s}\big(\pi_{1^s}(x)\big) \prod_{\omega\in[2]^s\setminus\{1^s\}} G_\omega\big(\pi_\omega(x)\big) \prod_{\omega\in\{1,3\}^s\setminus\{1^s\}} G_\omega\big(\pi_\omega(x)\big) \,\Big|\, x \in V^{s\times 4}\Big]$$

where $G_{1^s} = G$, $G_\omega = G^{(1)}_\omega$ for every $\omega \in [2]^s \setminus \{1^s\}$, and $G_\omega = G^{(2)}_{\omega'}$ for every $\omega \in \{1,3\}^s \setminus \{1^s\}$. Thus, setting $G_\omega = 1$ for all other $\omega \in [4]^s$, we see that

$$I = \mathbb{E}\Big[\prod_{\omega\in[4]^s} G_\omega\big(\pi_\omega(x)\big) \,\Big|\, x \in V^{s\times 4}\Big]$$

and the result follows from the Gowers–Cauchy–Schwarz inequality. □

Remark 3. We point out that Proposition 2.3 can also be extended in the hypergraph setting. Specifically, we have the following proposition; see [2, Section 7] for further results in this direction.

Proposition 3.3. *Let V be a nonempty finite set, let $s \geqslant 2$ be an integer and let $0 < \eta \leqslant 1$. Let $1 < p \leqslant \infty$, let q denote the conjugate exponent of p and set $\ell \coloneqq \min\{2n : n \in \mathbb{N}$ and $2n \geqslant 2q+2\}$. Also let $\nu\colon V^s \to \mathbb{R}^+$ such that*

$$\|\nu - \psi\|_{\square_\ell(V^s)} \leqslant \eta \tag{3.8}$$

where $\psi\colon V^s \to \mathbb{R}$ satisfies[7] $\|\psi\|_{L_p} \leqslant 1$ *and* $\|\psi\|_{\square_\ell(V^s)} \leqslant 1$. *Finally, let $F\colon V^s \to \mathbb{R}$ with $|F| \leqslant \nu$. If $\|F\|_{\mathrm{cut}(V^s)} \leqslant \eta$, then*

$$\|F\|_{\square(V^s)} = o_{\eta\to 0;s,p}(1). \tag{3.9}$$

3.3. **Transferring Proposition 3.1 to the additive setting.** There is an additive version of Proposition 3.1 which is somewhat distinct from Proposition 2.1 and is obtained by transferring the ℓ-box norms and the cut norm in the additive setting via formula (3.2). Specifically, let Z be a finite additive group, let $s \geqslant 2$ be an integer and let $f\colon Z \to \mathbb{R}$. For every even integer $\ell \geqslant 2$ we define the (s,ℓ)*-uniformity norm* $\|f\|_{U^s_\ell(Z)}$ *of f* by

$$\|f\|_{U^s_\ell(Z)} \coloneqq \|f(x_1 + \cdots + x_s)\|_{\square_\ell(Z^s)}. \tag{3.10}$$

Respectively, we define the *s-additive cut norm* $\|f\|_{\mathrm{cut}^s(Z)}$ *of f* by the rule

$$\|f\|_{\mathrm{cut}^s(Z)} \coloneqq \|f(x_1 + \cdots + x_s)\|_{\mathrm{cut}(Z^s)}. \tag{3.11}$$

(Notice that the additive cut norm is slightly stronger than the weak uniformity norm; in particular, we have $\|f\|_{w^s(Z)} \leqslant \|f\|_{\mathrm{cut}^s(Z)}$.) Taking into account (3.10) and (3.11), we see that Proposition 3.1 can be reformulated as follows.

[7]Here, as in Proposition 2.3, the L_p-norm of ψ is computed using the uniform probability measure on V^s, that is, $\|\psi\|_{L_p} \coloneqq \mathbb{E}[\,|\psi(x)|^p \mid x \in V^s]^{1/p}$.

Corollary 3.4. *Let Z be a finite additive group, let $s \geqslant 2$ be an integer and let $0 < \eta \leqslant 1$. Also let $\nu\colon Z \to \mathbb{R}^+$ such that*

$$\|\nu - 1\|_{U_4^s(Z)} \leqslant \eta. \tag{3.12}$$

Finally, let $f\colon Z \to \mathbb{R}$ with $|f| \leqslant \nu$. If $\|f\|_{\mathrm{cut}^s(Z)} \leqslant \eta$, then

$$\|f\|_{U^s(Z)} = o_{\eta\to 0;s}(1). \tag{3.13}$$

4. A variant of the Koopman–von Neumann decomposition

4.1. **Overview.** The *Koopman–von Neumann decomposition* is a circle of results asserting that, under certain circumstances, one can decompose a function f as $f = f_{\mathrm{bnd}} + f_{\mathrm{err}}$ where f_{bnd} is bounded in magnitude by 1 and f_{err} has small uniformity norm[8]. To see the relevance in this context of the equivalence between the uniformity norms and their weaker versions, note that one can first approximate f by a bounded function f_{bnd} such that the difference $f - f_{\mathrm{bnd}}$ is small in a weaker norm, and then upgrade this information using the results in the previous sections. This strategy (also used in [18, 19]) is quite effective partly because the aforementioned weaker approximation can be achieved relatively easily using various methods. We will use one of these methods, the so-called *dense model theorem*.

4.2. **Consequences of the dense model theorem.** We begin by recalling the dense model theorem; we will state the formulation which is closest to the purposes of this note (see [15, Theorem 1.1] or [19, Theorem 3.5]).

Proposition 4.1. *Let X be a finite set and let $\mathcal{F}$ be a family of $[-1,1]$-valued functions on X. Also let $0 < \eta \leqslant 1$, and let $\nu\colon X \to \mathbb{R}^+$ such that $\mathbb{E}[\nu] \leqslant 1+\eta$ and satisfying*

$$\Big|\mathbb{E}\Big[(\nu - 1)\prod_{i=1}^{k} F_i\Big]\Big| \leqslant \eta \tag{4.1}$$

for every $F_1, \dots, F_k \in \mathcal{F}$. Then for every $g\colon X \to \mathbb{R}$ with $0 \leqslant g \leqslant \nu$ there exists $w\colon X \to [0,1]$ such that

$$\sup\big\{|\mathbb{E}\big[(g-w)F\big]| : F \in \mathcal{F}\big\} = o_{\eta\to 0}(1). \tag{4.2}$$

We will need two consequences of Proposition 4.1. The first one concerns functions defined on a finite additive group Z. Recall that by $\|\cdot\|_{\mathrm{cut}^s(Z)}$ we denote the additive cut norm defined in (3.11).

Corollary 4.2. *Let Z be a finite additive group and let $s \geqslant 2$ be an integer. Also let $0 < \eta \leqslant 1$ and $\nu\colon Z \to \mathbb{R}^+$ such that $\|\nu - 1\|_{\mathrm{cut}^s(Z)} \leqslant \eta$. Then for every $g\colon Z \to \mathbb{R}$ with $0 \leqslant g \leqslant \nu$ there exists $w\colon Z \to [0,1]$ such that $\|g - w\|_{\mathrm{cut}^s(Z)} = o_{\eta\to 0}(1)$.*

[8] As we have already noted in the introduction, in applications it is not enough to control the error-term f_{err} using a weaker norm.

Consequently, for every $f\colon Z \to \mathbb{R}$ *with* $|f| \leqslant \nu$ *there exists* $h\colon Z \to [-1,1]$ *such that* $\|f-h\|_{\mathrm{cut}^s(Z)} = o_{\eta\to 0}(1)$.

The second consequence is the analogue of Corollary 4.2 for hypergraphs.

Corollary 4.3. *Let* V *be a nonempty finite set and let* $s \geqslant 2$ *be an integer. Also let* $0<\eta\leqslant 1$ *and* $\nu\colon V^s \to \mathbb{R}^+$ *such that* $\|\nu-1\|_{\mathrm{cut}(V^s)} \leqslant \eta$. *Then for every* $G\colon V^s\to\mathbb{R}$ *with* $0\leqslant G\leqslant \nu$ *there exists* $W\colon V^s \to [0,1]$ *such that* $\|G-W\|_{\mathrm{cut}(V^s)} = o_{\eta\to 0}(1)$. *Consequently, for every* $F\colon V^s\to\mathbb{R}$ *with* $|F|\leqslant\nu$ *there exists* $H\colon V^s\to[-1,1]$ *such that* $\|F-H\|_{\mathrm{cut}(V^s)} = o_{\eta\to 0}(1)$.

Corollary 4.3 is a straightforward consequence of Proposition 4.1. On the other hand, Corollary 4.2 follows by applying Proposition 4.1 for the family $\mathcal{F}$ of all convex combinations[9] of functions $D\colon Z\to\mathbb{R}$ of the form

$$D(z) = \mathbb{E}\Big[\prod_{\omega\in[2]^s\setminus\{1^s\}} H_\omega\big(\pi_\omega(x)\big)\,\Big|\, x=(x_{ij})\in Z^{s\times 2} \text{ with } \sum_{i=1}^{s} x_{i1} = z\Big]$$

where $H_\omega\colon Z^s \to [-1,1]$ for every $\omega\in[2]^s\setminus\{1^s\}$. Indeed, it is not hard to see that this family $\mathcal{F}$ is closed under multiplication (see, e.g., the proof of Lemma 3.3 in [20]).

4.3. **The main results.** We are ready to state our first result in this section. It is a variant of [10, Proposition 8.1] (see also [11, Proposition 10.3]).

Corollary 4.4. *Let* Z *be a finite additive group, let* $s\geqslant 2$ *be an integer and let* $0<\eta\leqslant 1$. *Also let* $\nu\colon Z\to\mathbb{R}^+$ *such that*

$$\|\nu-1\|_{U^{2s}(Z)} \leqslant \eta. \tag{4.3}$$

Then for every $f\colon Z\to\mathbb{R}$ *with* $|f|\leqslant\nu$ *there exists* $h\colon Z\to[-1,1]$ *such that*

$$\|f-h\|_{U^s(Z)} = o_{\eta\to 0;s}(1). \tag{4.4}$$

Moreover, if f *is nonnegative, then* h *is also nonnegative.*

Note that if Z is a finite additive group and $f\colon Z\to\mathbb{R}^+$ is a function which is approximated by a $[0,1]$-valued function on Z in the sense of (4.4)—that is, there exists $h\colon Z\to[0,1]$ such that $\|f-h\|_{U^s(Z)} = o(1)$—then f is majorized by a function $\nu\colon Z\to\mathbb{R}^+$ satisfying $\|\nu-1\|_{U^s(Z)} = o(1)$; indeed, simply take $\nu \coloneqq f+(1-h)$. Thus we see that the pseudorandomness hypothesis (4.3) is nearly optimal.

Proof of Corollary 4.4. We first observe that, by (4.3), the monotonicity of the Gowers norms $\|\cdot\|_{U^s(Z)} \leqslant \|\cdot\|_{U^{2s}(Z)}$, the identity (3.2) and (3.5), we have that $\|\nu-1\|_{\mathrm{cut}^s(Z)}\leqslant\eta$. Therefore, by Corollary 4.2, there exists $h\colon Z\to[-1,1]$ such

[9] The need to convexify the set of "dual" functions is very natural from a functional analytic perspective; see, e.g., [9].

that $\|f-h\|_{\mathrm{cut}^s(Z)} = o_{\eta\to 0}(1)$. Set $\nu' := (\nu+1)/2$ and notice that $|f-h|/2 \leqslant \nu'$ and $\|\nu'-1\|_{U^{2s}(Z)} \leqslant \eta$. By Proposition 2.1, the result follows. □

Our second result is a variant of [16, Theorem 3.9].

Corollary 4.5. *Let V be a nonempty finite set, let $s \geqslant 2$ be an integer and let $0<\eta\leqslant 1$. Also let $\nu\colon V^s \to \mathbb{R}^+$ such that*

$$\|\nu - 1\|_{\square_4(V^s)} \leqslant \eta. \tag{4.5}$$

Then for every $F\colon V^s \to \mathbb{R}$ with $|F|\leqslant \nu$ there exists $H\colon Z\to[-1,1]$ such that

$$\|F-H\|_{\square(V^s)} = o_{\eta\to 0;s}(1). \tag{4.6}$$

Moreover, if F is nonnegative, then H is also nonnegative.

Proof. It is identical to the proof of Corollary 4.4. Indeed, by (4.5) and Corollary 4.3, there exists $H\colon V^s\to[-1,1]$ such that $\|F-H\|_{\mathrm{cut}(V^s)} = o_{\eta\to 0}(1)$. By Proposition 3.1, the result follows. □

5. On the relative inverse theorem for the Gowers $U^s[N]$-norm

5.1. **Overview.** In order to put the main result of this section in a proper context, we begin with a brief discussion on the *nilpotent Hardy–Littlewood method* invented by Green and Tao [11]. It is a powerful method for obtaining precise asymptotic estimates (as $N\to+\infty$) for expressions of the form

$$\sum_{n\in K\cap\mathbb{Z}^d}\prod_{i=1}^{t} f_i\big(\psi_i(n)\big) \tag{5.1}$$

where $f_1,\dots,f_t\colon\mathbb{Z}\to\mathbb{R}$ are arithmetic functions supported on the set of positive integers, $K\subseteq[-N,N]^d$ is a convex body and $\psi_1,\dots,\psi_t\colon\mathbb{Z}^d\to\mathbb{Z}$ are affine linear forms no two of which are affinely dependent. The first step of the method relies on the *generalized von Neumann theorem*—see [11, Proposition 7.1]—which reduces the estimation of the quantity in (5.1) to a norm estimate

$$\|f_i - 1\|_{U^s[N]} = o_s(1) \ \text{ for every } s\geqslant 2 \text{ and every } i\in\{1,\dots,t\} \tag{5.2}$$

where $\|\cdot\|_{U^s[N]}$ stands for the s-th Gowers uniformity norm on the interval $[N]$ which we will shortly recall. This reduction can be performed provided that $|f_1|,\dots,|f_t|$ are simultaneously majorized by a function ν satisfying the "linear forms condition" (see [11, Definition 6.2]). The second (and more substantial) step of the method reduces the estimate (5.2) to a non-correlation estimate

$$\mathbb{E}_{n\in[N]}\,(f_i(n)-1)F(g^n\cdot x) = o_{s,G/\Gamma,M}(1) \tag{5.3}$$

where G/Γ is an $(s-1)$-step nilmanifold equipped with a smooth Riemannian metric $d_{G/\Gamma}$, $F\colon G/\Gamma\to[-1,1]$ is a function with Lipschitz constant at most M, $g\in G$ and $x\in G/\Gamma$. (We recall the notion of an $(s-1)$-step nilmanifold below.) For *bounded* functions, the equivalence between (5.2) and (5.3) is a deep result which is known as the *inverse theorem for the Gowers $U^s[N]$-norm* and is due to Green, Tao and Ziegler [12]. One of the main steps in [11] was to transfer the inverse theorem to the unbounded setting. This was achieved with the *relative inverse theorem for the Gowers $U^s[N]$-norm*—see [11, Proposition 10.1]—which can be applied provided that $|f_i|$ is majorized by a function ν satisfying the aforementioned linear forms condition and an additional pseudorandomness condition known as the "correlation condition" (see [11, Definition 6.3]).

Recently, a part of the proof of [11, Proposition 10.1] was revisited in [18]. One pleasant consequence of the approach in [18] is that the relative inverse theorem (and, consequently, the whole nilpotent Hardy–Littlewood method) can be applied assuming that the majorant ν satisfies only the linear forms condition[10].

Our aim in this section is to give yet another proof of the relative inverse theorem using a norm-type pseudorandomness condition. To this end, it is convenient at this point to properly introduce the concepts discussed so far.

5.1.1. *Uniformity norms on intervals.* Let $N \geqslant 1$ be an integer and let $f\colon [N] \to \mathbb{R}$ be a function. We select an integer $N' > 2N$ and we identify (in the obvious way) the discrete interval $[N]$ with a subset of the cyclic group $\mathbb{Z}_{N'} \coloneqq \mathbb{Z}/N'\mathbb{Z}$. The *Gowers uniformity norm* $\|f\|_{U^s[N]}$ *of* f *on the interval* $[N]$ is defined by setting

$$\|f\|_{U^s[N]} \coloneqq \|f\mathbf{1}_{[N]}\|_{U^s(\mathbb{Z}_{N'})}/\|\mathbf{1}_{[N]}\|_{U^s(\mathbb{Z}_{N'})} \tag{5.4}$$

where $\mathbf{1}_{[N]}\colon \mathbb{Z}_{N'} \to \{0,1\}$ stands for the indicator function of $[N]$. We note that the quantity $\|f\|_{U^s[N]}$ is, in fact, intrinsic and is independent of the choice of N'—see [11, Appendix B] for more details.

5.1.2. *Nilmanifolds.* Let $s \geqslant 2$ be an integer and recall that an $(s-1)$-*step nilmanifold* is a homogeneous space $X \coloneqq G/\Gamma$ where G is an $(s-1)$-step nilpotent, connected, simply connected Lie group, and Γ is a discrete cocompact subgroup of G. The group G acts on G/Γ by left multiplication and this action will be denoted by $(g,x) \mapsto g \cdot x$. As in [11], we will assume that each nilmanifold G/Γ is equipped with a smooth Riemannian metric $d_{G/\Gamma}$; in particular, if $F\colon G/\Gamma \to \mathbb{R}$ is a function, then its Lipschitz constant is computed using the metric $d_{G/\Gamma}$.

5.2. **The main result.** We are ready to state the main result in this section. As we have indicated, it is a refinement of [11, Proposition 10.1].

Theorem 5.1. *For every integer $s \geqslant 2$, every $C \geqslant 20$ and every $0 < \delta \leqslant 1$ there exist $\eta > 0$, a constant $M > 0$, a finite collection $\mathcal{M}$ of $(s-1)$-step nilmanifolds (each equipped with a smooth Riemannian metric), and a constant $c > 0$ with the*

[10]The possibility that one could dispense with the need for the correlation condition entirely, was also noted in [4, Appendix A].

following property. Let N be a positive integer and let $N' \in [CN, 2CN]$ be a prime. Also let $\nu\colon \mathbb{Z}_{N'} \to \mathbb{R}^+$ satisfying

$$\|\nu - 1\|_{U^{2s}(\mathbb{Z}_{N'})} \leqslant \eta. \tag{5.5}$$

Finally, let $f\colon [N] \to \mathbb{R}$ with $|f(n)| \leqslant \nu(n)$ for every $n \in [N]$. If $\|f\|_{U^s[N]} \geqslant \delta$, then there exist $(G/\Gamma, d_{G/\Gamma}) \in \mathcal{M}$, a function $F\colon G/\Gamma \to [-1,1]$ with Lipschitz constant at most M, $g \in G$ and $x \in G/\Gamma$ such that

$$|\mathbb{E}_{n\in[N]} f(n)F(g^n \cdot x)| \geqslant c. \tag{5.6}$$

We notice that the estimate in (5.5) follows if we assume that the function ν satisfies the $(4^s, 4s, 1)$-linear forms condition in the sense of [11, Definition 6.2], but (5.5) is certainly easier to grasp. It is likely that one can follow a similar approach in other instances of the transfer method, and replace the linear forms condition with a norm estimate of the form (5.5) for a suitable uniformity norm[11].

Remark 4. Using Corollary 3.4 instead of Proposition 2.1, it is easy to verify that Theorem 5.1 also holds if the majorant ν satisfies $\|\nu - 1\|_{U^s_4(\mathbb{Z}_{N'})} \leqslant \eta$, a condition which is slightly different from (5.5). However, the use of the $U^{2s}(\mathbb{Z}_{N'})$-norm in Theorem 5.1 is conceptually more natural in the present arithmetic context.

5.3. **Preliminary tools.** As in [11], the proof of Theorem 5.1 is based on three ingredients. The first one is the inverse theorem for the Gowers $U^s[N]$-norm [12]. It gives a criterion for checking that a bounded arithmetic function has non-negligible uniformity norm.

Theorem 5.2. *For every integer $s \geqslant 2$ and every $0 < \delta \leqslant 1$ there exist a constant $M > 0$, a finite collection $\mathcal{M}$ of $(s-1)$-step nilmanifolds (each equipped with a smooth Riemannian metric), and a constant $c > 0$ with the following property. Let N be a positive integer, and let $f\colon [N] \to [-1,1]$ be a function with $\|f\|_{U^s[N]} \geqslant \delta$. Then there exist $(G/\Gamma, d_{G/\Gamma}) \in \mathcal{M}$, a function $F\colon G/\Gamma \to [-1,1]$ with Lipschitz constant at most M, $g \in G$ and $x \in G/\Gamma$ such that*

$$|\mathbb{E}_{n\in[N]} f(n)F(g^n \cdot x)| \geqslant c. \tag{5.7}$$

It is more natural to formulate Theorem 5.2 for complex-valued functions which are bounded in magnitude by 1; however, we will not need the complex version of Theorem 5.2 for the proof of Theorem 5.1.

To state the second ingredient, we first recall some definitions. Let $s \geqslant 2$ be an integer. Also let N be a positive integer, let $F\colon [N] \to \mathbb{R}$ be a function, and define the *dual uniformity norm* $\|F\|_{U^s[N]^*}$ *of* F by the rule

$$\|F\|_{U^s[N]^*} \coloneqq \sup\big\{|\mathbb{E}_{n\in[N]} f(n)F(n)| : \|f\|_{U^s[N]} \leqslant 1\big\}. \tag{5.8}$$

[11] In this direction we recall (see also [1]) that it is not known whether for every integer $k \geqslant 3$ there exists an integer $s \geqslant k-1$ such that the relative Szemerédi theorem for k-term arithmetic progressions holds true under the condition $\|\nu - 1\|_{U^s(\mathbb{Z}_N)} = o(1)$.

We will need the following result which follows from [11, Proposition 11.2].

Proposition 5.3. *Let $s \geqslant 2$ be an integer, let $(G/\Gamma, d_{G/\Gamma})$ be an $(s-1)$-step nilmanifold, and let $M > 0$. Also let $F\colon G/\Gamma \to [-1,1]$ be a function with Lipschitz constant at most M, $g \in G$ and $x \in G/\Gamma$. Finally, let N be a positive integer and let $0 < \varepsilon \leqslant 1$. Then there exists a decomposition*

$$F(g^n \cdot x) = F_1(n) + F_2(n) \quad \textit{for every } n \in [N] \tag{5.9}$$

where the functions $F_1, F_2\colon [N] \to \mathbb{R}$ obey the estimates

$$\|F_1\|_{\ell_\infty} = O(\varepsilon) \quad \textit{and} \quad \|F_2\|_{U^s[N]^*} = O_{s,M,\varepsilon,G/\Gamma}(1). \tag{5.10}$$

We point out that, by [11, Proposition 11.2], one can additionally ensure that the function F_2 in the above decomposition is an "averaged nilsequence" in the sense of [11, Definition 11.1]. We also note that the proof of [11, Proposition 11.2] is non-effective and yields no estimate for the dual uniformity norm of F_2. However, explicit estimates can be obtained by combining [14, Lemmas A.2 and A.3]—see [14, Appendix A] for more details on this approach.

The last ingredient needed for the proof of Theorem 5.1 is the following version of Corollary 4.4 which concerns functions defined on intervals of $\mathbb{Z}$.

Corollary 5.4. *For every integer $s \geqslant 2$, every $C \geqslant 20$ and every $0 < \varepsilon \leqslant 1$ there exist a positive integer N_0 and $\eta > 0$ with the following property. Let $N \geqslant N_0$ be an integer and let $N' \in [CN, 2CN]$ be a prime. Also let $\nu\colon \mathbb{Z}_{N'} \to \mathbb{R}^+$ satisfying*

$$\|\nu - 1\|_{U^{2s}(\mathbb{Z}_{N'})} \leqslant \eta. \tag{5.11}$$

Finally, let $f\colon [N] \to \mathbb{R}$ with $|f(n)| \leqslant \nu(n)$ for every $n \in [N]$. Then there exists a function $h\colon [N] \to [-1,1]$ such that

$$\|f - h\|_{U^s[N]} \leqslant \varepsilon. \tag{5.12}$$

Moreover, if f is nonnegative, then h is also nonnegative.

Proof. It is a consequence of Corollary 4.4 and a standard truncation argument. Specifically, fix the parameters s, C and ε, and set

$$\alpha = \Big(\frac{\varepsilon}{32C}\Big)^{2^s} \quad \text{and} \quad N_0 = \lceil 2/\alpha \rceil. \tag{5.13}$$

Moreover, by Corollary 4.4, we select $0 < \eta \leqslant 1$ such that for every finite additive group Z, every $\nu'\colon Z \to \mathbb{R}^+$ satisfying $\|\nu' - 1\|_{U^{2s}(Z)} \leqslant \eta$ and every $g\colon Z \to \mathbb{R}$ with $|g| \leqslant \nu'$ there exists $w\colon Z \to [-1,1]$ such that $\|g - w\|_{U^s(Z)} \leqslant \varepsilon\alpha/(32C)$. We will show that N_0 and η are as desired.

So, let N, N', ν and f be as in the statement of the corollary, and let $\tilde{f}\colon \mathbb{Z}_{N'} \to \mathbb{R}$ be the extension of f obtained by setting $\tilde{f}(n) = 0$ if $n \notin [N]$. By the choice of η, there exists $H\colon \mathbb{Z}_{N'} \to [-1,1]$ satisfying

$$\|\tilde{f} - H\|_{U^s(\mathbb{Z}_{N'})} \leqslant \frac{\varepsilon\alpha}{32C}. \tag{5.14}$$

We claim that $\|f-h\|_{U^s[N]} \leqslant \varepsilon$ where $h\colon [N] \to [-1,1]$ is the restriction of H on $[N]$. Indeed, set $l = \lfloor \alpha N \rfloor$ and let $2L$ be the least even integer greater than or equal to N; notice that $N \geqslant L \geqslant l \geqslant 2$ and $\alpha/2 \leqslant l/N \leqslant \alpha$. Next, write $N' = 2k+1$ and identify $\mathbb{Z}_{N'}$ with the interval $\{-k,\dots,k\}$. Let $\varphi\colon \mathbb{Z}_{N'} \to [0,1]$ be the cut-off function which is nonzero on the set $\{-l+2,\dots,2L+l-1\}$, increases linearly from 0 to 1 between $-l+1$ and 1, is equal to 1 on $[2L]$, and decreases linearly from 1 to 0 between $2L$ and $2L+l$. Observe that $\tilde{f}\varphi = \tilde{f}$ and so, setting $\tilde{h} \coloneqq H\mathbf{1}_{[N]}$, we have

$$\tilde{f} - \tilde{h} = (\tilde{f} - H)\varphi + H(\varphi - \mathbf{1}_{[N]}). \tag{5.15}$$

Also note that the Fourier transform $\widehat{\varphi}$ of φ satisfies the estimate $\|\widehat{\varphi}\|_{\ell_1(\mathbb{Z}_{N'})} \leqslant 4L/l$ (see, e.g., the proof of Lemma A.1 in [5] where this is explained in some detail). Hence, by the triangle inequality and [17, (11.11)], we have[12]

$$\|(\tilde{f}-H)\varphi\|_{U^s(\mathbb{Z}_{N'})} \leqslant \|\widehat{\varphi}\|_{\ell_1(\mathbb{Z}_{N'})} \cdot \|\tilde{f}-H\|_{U^s(\mathbb{Z}_{N'})} \leqslant \frac{4N}{l}\, \|\tilde{f}-H\|_{U^s(Z_{N'})}. \tag{5.16}$$

On the other hand, since $H(\varphi - \mathbf{1}_{[N]})$ is bounded in magnitude by 1 and is supported on a subset of $\mathbb{Z}_{N'}$ of cardinality at most $2l+1$, we obtain that

$$\|H(\varphi - \mathbf{1}_{[N]})\|_{U^s(Z_{N'})} \leqslant \Big(\frac{2l+1}{N'}\Big)^{1/2^s} \leqslant \Big(\frac{3l}{CN}\Big)^{1/2^s}. \tag{5.17}$$

Finally, note that $\|\mathbf{1}_{[N]}\|_{U^s(Z_{N'})} \geqslant \mathbb{E}[\mathbf{1}_{[N]}] = N/N' \geqslant 1/2C$. Thus, by (5.15)–(5.17), the triangle inequality and the definition of the $U^s[N]$-norm, we see that

$$\|f-h\|_{U^s[N]} \leqslant 2C\Big(\frac{4N}{l}\, \|\tilde{f}-H\|_{U^s(Z_{N'})} + \Big(\frac{3l}{CN}\Big)^{1/2^s}\Big).$$

By the previous inequality and taking into account the choice of α, l and the estimate (5.14), we conclude that $\|f-h\|_{U^s[N]} \leqslant \varepsilon$. □

5.4. **Proof of Theorem 5.1.** We follow the proof from [11, Proposition 10.1] quite closely[13]. We first observe that, by compactness, for every positive integer d there exists a constant $D \geqslant 1$ such that for every $N \in [d]$ and every $f\colon [N] \to \mathbb{R}$ we have $\|f\|_{U^s[N]} \leqslant D\|\widehat{f}\|_{\ell_\infty}$. (Here, we identify $[N]$ with $\mathbb{Z}_N$.) Therefore, if $N \in [d]$, then Theorem 5.1 follows using as nilmanifold the torus $\mathbb{R}/\mathbb{Z}$. Thus, at the cost of worsening the constants, it is enough to prove Theorem 5.1 for every sufficiently large positive integer N.

So, fix the parameters s, C and δ, and let $M, \mathcal{M}$ and c be as in Theorem 5.2 when applied for $\delta/2$. Next, by Proposition 5.3, we select $K \geqslant 1$ such that for every $(G/\Gamma, d_{G/\Gamma}) \in \mathcal{M}$, every function $F\colon G/\Gamma \to [-1,1]$ with Lipschitz constant at most M, every $g \in G$, every $x \in G/\Gamma$ and every positive integer N we have the decomposition (5.9) with $\|F_1\|_{\ell_\infty} \leqslant c/12$ and $\|F_2\|_{U^s[N]^*} \leqslant K$. Finally, let N_0

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[12] Note that here we work with the complex version of the Gowers uniformity norm.

[13] Actually, there is a minor oversight in the proof of [11, Proposition 10.1] which is fixed in the present paper. Specifically, the appeal to Proposition 8.2 at the top of [11, page 1796] is invalid without appeal to the material from [11, Section 11].

and η be as in Corollary 5.4 when applied for $\varepsilon := \min\{\delta/2, c/(4K)\}$. We claim that Theorem 5.1 holds true for $\eta, M, \mathcal{M}$ and $c/2$ provided that $N \geqslant N_0$. Indeed, let N be an arbitrary positive integer with $N \geqslant N_0$, and let N', ν and f be as in the statement of the theorem. By Corollary 5.4, there exists $h\colon [N] \to [-1,1]$ such that $\|f-h\|_{U^s[N]} \leqslant \varepsilon$; in particular, we have $\|h\|_{U^s[N]} \geqslant \delta/2$ and so, by Theorem 5.2, there exist $(G/\Gamma, d_{G/\Gamma}) \in \mathcal{M}$, a function $F\colon G/\Gamma \to [-1,1]$ with Lipschitz constant at most M, $g \in G$ and $x \in G/\Gamma$ such that

$$|\mathbb{E}_{n\in[N]}\, h(n)F(g^n\cdot x)| \geqslant c. \tag{5.18}$$

Write $F(g^n\cdot x) = F_1(n) + F_2(n)$ with $\|F_1\|_{\ell_\infty} \leqslant c/12$ and $\|F_2\|_{U^s[N]^*} \leqslant K$, and notice that, by (5.18) and the triangle inequality, it suffices to show that

$$|\mathbb{E}_{n\in[N]}\big(f(n)-h(n)\big)F_1(n)| \leqslant \frac{c}{4} \ \text{ and } \ |\mathbb{E}_{n\in[N]}\big(f(n)-h(n)\big)F_2(n)| \leqslant \frac{c}{4}. \tag{5.19}$$

The first part of (5.19) follows from the fact that $\mathbb{E}\big[|f-h|\big] \leqslant \mathbb{E}[\nu+1] \leqslant 3$ and $\|F_1\|_{\ell_\infty} \leqslant c/12$. On the other hand, by the choice of ε and h, we have

$$|\mathbb{E}_{n\in[N]}\big(f(n)-h(n)\big)F_2(n)| \leqslant \|f-h\|_{U^s[N]}\cdot\|F_2\|_{U^s[N]^*} \leqslant \varepsilon K \leqslant \frac{c}{4}$$

and the proof is completed.

Appendix A. Basic properties of uniformity norms

Proposition A.1. *Let V be a nonempty finite set and let $s \geqslant 2$ be an integer.*

(a) (Gowers–Cauchy–Schwarz inequality) *Let $\ell \geqslant 2$ be an even integer and for every $\omega \in [\ell]^s$ let $F_\omega\colon V^s \to \mathbb{R}$. Then we have*

$$\Big|\mathbb{E}\Big[\prod_{\omega\in[\ell]^s} F_\omega\big(\pi_\omega(x)\big)\,\Big|\, x \in V^{s\times\ell}\Big]\Big| \leqslant \prod_{\omega\in[\ell]^s} \|F_\omega\|_{\square_\ell(V^s)}. \tag{A.1}$$

In particular, if Z is a finite additive group, then we have

$$\Big|\mathbb{E}\Big[\prod_{\omega\in\{0,1\}^s} f_\omega(x+\omega\cdot\mathbf{h})\,\Big|\, x\in Z, \mathbf{h}\in Z^s\Big]\Big| \leqslant \prod_{\omega\in\{0,1\}^s} \|f_\omega\|_{U^s(Z)} \tag{A.2}$$

for every family $\langle f_\omega : \omega\in\{0,1\}^s\rangle$ of real-valued functions on Z.

(b) *For every even integer $\ell \geqslant 2$ the quantity $\|\cdot\|_{\square_\ell(V^s)}$ is a norm on $\mathbb{R}^{V^s}$. Moreover, if $\ell_1 \leqslant \ell_2$ are even positive integers, then for every $F\colon V^s \to \mathbb{R}$ we have $\|F\|_{\square_{\ell_1}(V^s)} \leqslant \|F\|_{\square_{\ell_2}(V^s)}$.*

(c) *Let $\ell \geqslant 2$ be an even integer, let $0 < \eta \leqslant 1$, and let $\nu\colon V^s \to \mathbb{R}^+$ satisfying $\|\nu - 1\|_{\square_{\ell+2}(V^s)} \leqslant \eta$. Then for every $F\colon V^s \to \mathbb{R}$ with $|F| \leqslant \nu$ we have*

$$\|F\|_{\square_\ell(V^s)} \leqslant \|F\|_{\square(V^s)}^{1/\ell^s} + o_{\eta\to 0;s,\ell}(1). \tag{A.3}$$

In particular, for every $F\colon V^s \to [-1,1]$ we have $\|F\|_{\square_\ell(V^s)} \leqslant \|F\|_{\square(V^s)}^{1/\ell^s}$.

Proof. Part (a) for $\ell = 2$ is well-known (see [11, Lemma B.2] or [17, Section 11.1]). The general case can be proved with similar arguments—see [3, Proposition 2.1] for details. Part (b) is an easy consequence of the Gowers–Cauchy–Schwarz inequality. Part (c) is a special (but more informative) case of [11, Proposition 7.1]. For the convenience of the reader we will sketch a proof.

We begin by introducing some pieces of notation. For every $\omega = (\omega_i) \in [\ell]^s$ we set $S(\omega) = \{i \in [s] : \omega_i = \ell\}$, and for every (possibly empty) $d \subseteq [s]$ let $\Omega'_{\omega,d}$ denote the set of all $\omega' = (\omega'_i) \in [\ell+1]^s$ such that $\omega'_i \in \{\ell, \ell+1\}$ if $i \in S(\omega) \cap d$, and $\omega'_i = \omega_i$ otherwise. Next, for every $d \subseteq [s]$ let $I_d = ([s] \times [\ell]) \cup (d \times \{\ell+1\})$ and define[14] $F_d, G_d \colon V^{I_d} \to \mathbb{R}$ by the rule

$$F_d(x') = \prod_{\omega' \in \Omega'_{c,d}} F\big(\pi_{\omega'}(x')\big) \quad \text{and} \quad G_d(x') = \prod_{\omega \in A_d} \prod_{\omega' \in \Omega'_{\omega,d}} F\big(\pi_{\omega'}(x')\big) \prod_{\omega \in B_d} \prod_{\omega' \in \Omega'_{\omega,d}} \nu\big(\pi_{\omega'}(x')\big)$$

where $c = (\ell, \dots, \ell) \in [\ell]^s$ denotes the sequence of length s taking the constant value ℓ, $A_d = \{\omega \in [\ell]^s \setminus \{c\} : d \subseteq S(\omega)\}$, $B_d = \{\omega \in [\ell]^s \setminus \{c\} : d \nsubseteq S(\omega)\}$ and $\pi_{\omega'}(x') = (x'_{i\,\omega'_i})_{i=1}^s$ for every $x' \in V^{I_d}$ and every $\omega' = (\omega'_i) \in [\ell+1]^s$ such that $\{i \in [s] : \omega'_i = \ell+1\} \subseteq d$. Finally, we set $Q_d = \mathbb{E}[F_d\, G_d]$.

Now observe that $Q_\emptyset = \mathbb{E}\big[\prod_{\omega \in [\ell]^s} F\big(\pi_\omega(x)\big) \,|\, x \in V^{s \times \ell}\big] = \|F\|^{\ell^s}_{\square_\ell(V^s)}$. Moreover,

$$\begin{aligned} Q_{[s]} &= \mathbb{E}\Big[\prod_{\omega' \in \{\ell,\ell+1\}^s} F\big(\pi_{\omega'}(x')\big) \prod_{\omega' \in [\ell+1]^s \setminus \{\ell,\ell+1\}^s} \nu\big(\pi_{\omega'}(x')\big) \,\Big|\, x' \in V^{s \times (\ell+1)} \Big] \\ &= \|F\|^{2^s}_{\square(V^s)} + o_{\eta\to 0;s,\ell}(1). \end{aligned}$$

Indeed, write $Q_{[s]} = Q^{(1)}_{[s]} + Q^{(2)}_{[s]}$ where $Q^{(1)}_{[s]} = \mathbb{E}\big[\prod_{\omega' \in \{\ell,\ell+1\}^s} F\big(\pi_{\omega'}(x')\big)\big]$ and

$$Q^{(2)}_{[s]} = \mathbb{E}\Big[\prod_{\omega' \in \{\ell,\ell+1\}^s} F\big(\pi_{\omega'}(x')\big) \cdot \Big(\prod_{\omega' \in [\ell+1]^s \setminus \{\ell,\ell+1\}^s} \nu\big(\pi_{\omega'}(x')\big) - 1 \Big) \Big].$$

(Here, the first expectation is taken over all $x' \in V^{s \times \{\ell,\ell+1\}}$ and the second expectation is taken over all $x' \in V^{s \times (\ell+1)}$.) Notice that $Q^{(1)}_{[s]} = \|F\|^{2^s}_{\square(V^s)}$. On the other hand, by a telescopic argument, the Gowers–Cauchy–Schwarz inequality for the $\square_{\ell+2}(V^s)$-norm and the fact that $|F| \leqslant \nu$ and $\|\nu - 1\|_{\square_{\ell+2}(V^s)} \leqslant \eta$, we obtain

$$|Q^{(2)}_{[s]}| \leqslant \sum_{k=2^s+1}^{(\ell+1)^s} \|F\|^{2^s}_{\square_{\ell+2}(V^s)} \cdot \|\nu - 1\|_{\square_{\ell+2}(V^s)} \cdot \|\nu\|^{(\ell+1)^s - k}_{\square_{\ell+2}(V^s)} = o_{\eta\to 0;s,\ell}(1).$$

Finally, by repeated applications of the Cauchy–Schwarz inequality, we see that $Q_d^2 \leqslant \big(1 + o_{\eta\to 0;s,\ell}(1)\big) \cdot Q_{d \cup \{i\}}$ for every (possibly empty) $d \subsetneq [s]$ and every $i \in [s] \setminus d$. In particular, we have $Q_\emptyset^{2^s} \leqslant \big(1 + o_{\eta\to 0;s,\ell}(1)\big) \cdot Q_{[s]}$. Since $Q_\emptyset = \|F\|^{\ell^s}_{\square_\ell(V^s)}$, $Q_{[s]} = \|F\|^{2^s}_{\square(V^s)} + o_{\eta\to 0;s,\ell}(1)$ and

$$\|F\|_{\square(V^s)} \leqslant \|\nu\|_{\square(V^s)} \leqslant \|\nu\|_{\square_{\ell+2}(V^s)} \leqslant 1 + \eta$$

the result follows. $\square$

[14] In this definition, as in the proof of Proposition 2.1, we follow the convention that the product of an empty family of functions is equal to the constant function 1.

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