A SIMPLE PROOF OF THE DENSITY HALES–JEWETT THEOREM

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Abstract. We give a purely combinatorial proof of the density Hales–Jewett theorem that is modeled after Polymath's proof but is significantly simpler. In particular, we avoid the use of the equal-slices measure and work exclusively with the uniform measure.

1. Introduction

We begin by introducing some pieces of notation and some terminology. For every pair k, n of positive integers let $[k]^n$ be the set of all sequences of length n having values in $[k] \coloneqq \{1, \dots, k\}$. The elements of $[k]^n$ will be referred to as *words*. Also fix a letter v. A *variable word* is a finite sequence of length n having values in $[k] \cup \{v\}$ where the letter v appears at least once. If ℓ is a variable word and $i \in [k]$, then $\ell(i)$ is the word obtained by substituting all appearances of the letter v in ℓ by i. A *combinatorial line* of $[k]^n$ is a set of the form $\{\ell(i) : i \in [k]\}$ where ℓ is a variable word. If A is a subset of $[k]^n$, then its *density* is the quantity $|A|/k^n$ where $|A|$ stands for the cardinality of the set A.

The following result is known as the *density Hales–Jewett theorem* and is due to Furstenberg and Katznelson [6].

Theorem 1. *For every integer $k \geqslant 2$ and every $0 < \delta \leqslant 1$ there exists an integer N with the following property. If $n \geqslant N$ and A is a subset of $[k]^n$ of density δ, then A contains a combinatorial line of $[k]^n$. The least integer N with this property will be denoted by* $\mathrm{DHJ}(k, \delta)$.

The density Hales–Jewett theorem is a fundamental result of Ramsey theory. It has several strong results as consequences, most notably the famous Szemerédi theorem on arithmetic progressions [14] and its multidimensional version [5].

Because of its significance the density Hales–Jewett theorem has received considerable attention and there are, by now, several different proofs [2, 10, 15]. Our goal in this paper is to give yet another proof of the density Hales–Jewett theorem that is modeled after Polymath's proof [10] but places one of its crucial parts in a general conceptual framework. In fact, the argument was found in the course of obtaining a density version of the Carlson–Simpson theorem [3] and we decided

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to present it also within the context of the density Hales–Jewett theorem since it simplifies the method in [10].

To proceed with our discussion it is useful at this point to recall the strategy of Polymath's proof. It is based on the density increment method. Specifically, one argues that if a subset A of $[k]^n$ of density δ fails to contain a combinatorial line, then A has density $\delta+\gamma$ inside a large subspace of $[k]^n$ where γ is a positive constant that depends only on δ; once this is done Theorem 1 follows by a standard iteration. To achieve this goal, one proceeds in two steps: firstly one shows that A must correlate with a "structured" set B more than expected, and then argues that the "structured" set B can be partitioned in subspaces.

The proof of the second step given in [10] is a non-trivial modification of an argument due to Ajtai and Szemerédi [1]. It is essentially a "greedy" algorithm with an elegant proof that appears to be optimal, and we offer no new insight.

To execute the first step it is necessary to have a "probabilistic" version of Theorem 1. This means that a dense subset of $[k]^n$ not only will contain a combinatorial line but, actually, a non-trivial portion of them. Unfortunately, such a naive "probabilistic" version is false. To overcome this problem the participants of the polymath project introduced the *equal-slices* measure, a probability measure on $[k]^n$, and argued that for the equal-slices measure Theorem 1 does have a density version. While the idea of changing the measure is an important one, it necessitates a number of tools whose relevance to Theorem 1 can be justified only a posteriori.

We propose a different way to obtain such a "probabilistic" version that enables us to work exclusively with the uniform measure on $[k]^n$. Our approach is based on an old paper of Erdős and Hajnal [4] that initiated the study of the following general problem in Ramsey theory. Suppose that we are given a Ramsey space $\mathbb{S}$; for concreteness the reader may think of $[k]^n$ for some large n. Suppose, further, that we are given a family $\{A_s : s\in\mathbb{S}\}$ of measurable events in a probability space (Ω,Σ,μ) satisfying $\mu(A_s)\geqslant\delta>0$ for every $s\in\mathbb{S}$. The goal is then to find a "substructure" $\mathbb{S}'$ of $\mathbb{S}$ (in the case of the density Hales–Jewett theorem, $\mathbb{S}'$ is a combinatorial line of $[k]^n$) such that the events in the family $\{A_s : s\in\mathbb{S}'\}$ are highly correlated. Many density results in Ramsey theory can be formulated in this way and so does the density Hales–Jewett theorem; see, [6, Proposition 2.1]. It is precisely this form that we are taking advantage of, together with some simple coloristic and averaging arguments, and execute the first step.

Some final remarks about how this paper is written. We have made no attempt to optimize the argument. Instead, we tried to make the exposition as clear as possible. The bounds we get have an Ackermann-type dependence with respect to k and coincide, essentially, with the bounds from Polymath's proof for all sufficiently large values of k. The fundamental problem whether there exist primitive recursive bounds for the numbers $\mathrm{DHJ}(k,\delta)$ is open and is likely to require a more sophisticated approach.

2. Background material

By $\mathbb{N} = \{0, 1, 2, \dots\}$ we denote the natural numbers. As we have already mentioned, the cardinality of a set X will be denoted by $|X|$. For every nonempty finite set X by $\mathbb{E}_{x\in X}$ we shall denote the average $\frac{1}{|X|}\sum_{x\in X}$. If it is clear which set X we are referring to, then this average will be denoted simply by $\mathbb{E}_x$.

We recall some definitions related to the Hales–Jewett theorem [8]. Specifically, let $k, m, n \in \mathbb{N}$ with $k \geqslant 2$ and $n \geqslant m \geqslant 1$ and fix an m-tuple $v_1, \dots, v_m$ of distinct letters. An *m-variable word* of $[k]^n$ is a finite sequence of length n having values in $[k] \cup \{v_1, \dots, v_m\}$ where, for each $j \in [m]$, the letter v_j appears at least once. If z is an m-variable word and $a_1, \dots, a_m \in [k]$, then $z(a_1, \dots, a_m)$ is the word obtained by substituting in z the letter v_j with a_j for every $j \in [m]$. An *m-dimensional subspace* of $[k]^n$ is a set of the form $\{z(a_1, \dots, a_m) : a_1, \dots, a_m \in [k]\}$ where z is an m-variable word. Observe that an 1-dimensional subspace of $[k]^n$ is just a combinatorial line. If V is an m-dimensional subspace of $[k]^n$, then by $\mathrm{Lines}(V)$ we shall denote the set of all combinatorial lines of $[k]^n$ that are contained in V. Moreover, for every subset A of $[k]^n$ the *density of A in V*, denoted by $\mathrm{dens}_V(A)$, is the quantity $|A \cap V|/|V|$. The density of A in $[k]^n$ will be denoted simply by $\mathrm{dens}(A)$.

Let V be an m-dimensional subspace of $[k]^n$ and let z be the m-variable word that generates it. Notice that z induces a natural "isomorphism" between $[k]^m$ and V defined by $[k]^m \ni (a_1, \dots, a_m) \mapsto z(a_1, \dots, a_m) \in V$. Thus, in practice, we may identify m-dimensional subspaces of $[k]^n$ with "copies" of $[k]^m$ inside $[k]^n$. Having this identification in mind, for every $k' \in \mathbb{N}$ with $2 \leqslant k' \leqslant k$ we set

$$V \upharpoonright k' = \{z(a_1, \dots, a_m) : a_1, \dots, a_m \in [k']\}.$$

Now let $n, l \in \mathbb{N}$ with $n, l \geqslant 1$. For every $x \in [k]^n$ and every $y \in [k]^l$ by $x^\frown y$ we shall denote the concatenation of x and y. Notice that $x^\frown y \in [k]^{n+l}$. More generally, if $A \subseteq [k]^n$ and $B \subseteq [k]^l$ then we set $A^\frown B = \{x^\frown y : x \in A \text{ and } y \in B\}$.

Finally we record, for future use, the following consequence of the Graham–Rothschild theorem [7].

Proposition 2. *For every integer $k \geqslant 2$ and every integer $m \geqslant 1$ there exists an integer N with the following property. For every integer $n \geqslant N$ and every set $\mathcal{L}$ of combinatorial lines of $[k]^n$ there exists an m-dimensional subspace V of $[k]^n$ such that either* $\mathrm{Lines}(V) \subseteq \mathcal{L}$ *or* $\mathrm{Lines}(V) \cap \mathcal{L} = \emptyset$. *The least integer N with this property will be denoted by* $\mathrm{GR}(k, m)$.

Proposition 2 can be proved by repeated applications of the Hales–Jewett theorem much in the spirit of Ramsey's classical theorem; see, e.g., [9, Theorem 2.4.1]. Another excellent and short proof can be found in [12, §4]. Also we notice that there exist reasonable upper bounds for the numbers $\mathrm{GR}(k, m)$. Specifically, it follows from the work of Shelah [11] that there exists a primitive recursive function

$\phi\colon \mathbb{N}^2 \to \mathbb{N}$ belonging to the class $\mathcal{E}^6$ of Grzegorczyk's hierarchy such that for every integer $k \geqslant 2$ and every integer $m \geqslant 1$ we have $\mathrm{GR}(k,m) \leqslant \phi(k,m)$.

3. Preliminary tools

In this section we will gather some preliminary tools which are needed for the proof of Theorem 1 but are not directly related to the main argument. To simplify the exposition, below and in the rest of the paper, we will write "DHJ_k" to denote the proposition that for every $0<\delta\leqslant 1$ the number $\mathrm{DHJ}(k,\delta)$ is finite.

The first result, taken from [6], asserts that the density Hales–Jewett theorem implies its multidimensional version.

Proposition 3. *Let $k\in\mathbb{N}$ with $k\geqslant 2$ and assume DHJ_k. Then for every integer $m\geqslant 1$ and every $0<\delta\leqslant 1$ there exists an integer $\mathrm{MDHJ}(k,m,\delta)$ with the following property. If $n\geqslant \mathrm{MDHJ}(k,m,\delta)$, then every subset A of $[k]^n$ of density at least δ contains an m-dimensional subspace of $[k]^n$.*

Proof. By induction on m. The case "$m=1$" is the content of DHJ_k. Let $m\in\mathbb{N}$ with $m\geqslant 1$ and assume that the result has been proved up to m. For every $0<\delta\leqslant 1$ we set $\mathrm{MDHJ}(k,m+1,\delta)=M+\mathrm{MDHJ}(k,m,\delta 2^{-1}(k+1)^{-M})$ where $M=\mathrm{DHJ}(k,\delta/2)$. We claim that with this choice the result follows. Indeed, let $n\geqslant \mathrm{MDHJ}(k,m+1,\delta)$ and fix a subset A of $[k]^n$ with $\mathrm{dens}(A)\geqslant\delta$. For every $x\in[k]^{n-M}$ let $A_x=\{y\in[k]^M : x^\frown y\in A\}$. Notice that $\mathbb{E}_x\mathrm{dens}(A_x)\geqslant\delta$. Therefore, there exists a subset B of $[k]^{n-M}$ with $\mathrm{dens}(B)\geqslant\delta/2$ such that for every $x\in B$ we have $\mathrm{dens}(A_x)\geqslant\delta/2$. By the choice of M, for every $x\in B$ there exists a combinatorial line ℓ_x of $[k]^M$ such that $\ell_x\subseteq A_x$. The number of combinatorial lines of $[k]^M$ is less than $(k+1)^M$. Therefore, there exist a combinatorial line ℓ of $[k]^M$ and a subset C of B with $\mathrm{dens}(C)\geqslant\delta 2^{-1}(k+1)^{-M}$ such that $\ell\subseteq A_x$ for every $x\in C$. Since $n-M\geqslant\mathrm{MDHJ}(k,m,\delta 2^{-1}(k+1)^{-M})$ there exists an m-dimensional subspace W of $[k]^{n-M}$ with $W\subseteq C$. We set $V=W^\frown\ell$. Then V is an $(m+1)$-dimensional subspace of $[k]^n$ and clearly $V\subseteq A$. The proof is completed. □

The second result asserts that every dense subset of $[k]^n$ becomes extremely uniformly distributed when restricted to a suitable subspace of $[k]^n$.

Lemma 4. *Let $k,m\in\mathbb{N}$ with $k\geqslant 2$ and $m\geqslant 1$. Also let $0<\varepsilon<1$. If $n\geqslant\varepsilon^{-1}k^m m$, then for every subset A of $[k]^n$ with $\mathrm{dens}(A)>\varepsilon$ there exist some $l<n$ and an m-dimensional subspace V of $[k]^l$ such that for every $x\in V$ we have $\mathrm{dens}(A_x)\geqslant\mathrm{dens}(A)-\varepsilon$ where $A_x=\{y\in[k]^{n-l} : x^\frown y\in A\}$.*

Proof. We set $V_1=[k]^m$ and we observe that $\mathbb{E}_{x\in V_1}\mathrm{dens}(A_x)=\mathrm{dens}(A)$. Also let $\varrho=\varepsilon(k^m-1)^{-1}$. If V_1 does not satisfy the requirements of the lemma, then there exists $x_1\in V_1$ such that $\mathrm{dens}(A_{x_1})\geqslant\mathrm{dens}(A)+\varrho$. Next we set $V_2=x_1^\frown[k]^m$ and we notice that $\mathbb{E}_{x\in V_2}\mathrm{dens}(A_x)\geqslant\mathrm{dens}(A)+\varrho$. If V_2 does not satisfy the requirements of the lemma, then there exists $x_2 \in V_2$ such that $\mathrm{dens}(A_{x_2}) \geqslant \mathrm{dens}(A) + 2\varrho$. This process must, of course, terminate after at most $\lfloor \varrho^{-1} \rfloor$ iterations. Noticing that $(\lfloor \varrho^{-1} \rfloor + 1)m < n$ the result follows. □

Combining Proposition 3 and Lemma 4 we obtain the following corollary.

Corollary 5. *Let $k \in \mathbb{N}$ with $k \geqslant 2$ and assume DHJ_k. Then for every integer $m \geqslant 1$ and every $0 < \delta \leqslant 1$ there exists an integer $\mathrm{MDHJ}^*(k,m,\delta)$ with the following property. If $n \geqslant \mathrm{MDHJ}^*(k,m,\delta)$, then for every subset A of $[k+1]^n$ of density at least δ there exists an m-dimensional subspace V of $[k+1]^n$ such that $V \restriction k$ is contained in A.*

Proof. We set $\mathrm{MDHJ}^*(k,m,\delta) = (\delta/2)^{-1}(k+1)^M M$ where $M = \mathrm{MDHJ}(k,m,\delta/2)$. Let $n \geqslant \mathrm{MDHJ}^*(k,m,\delta)$ and fix a subset A of $[k+1]^n$ with $\mathrm{dens}(A) \geqslant \delta$. By Lemma 4, there exist some $l < n$ and an M-dimensional subspace W of $[k+1]^l$ such that $\mathrm{dens}(A_x) \geqslant \delta/2$ for every $x \in W$. We set $Z = W \restriction k$. On the one hand, we have $|A \cap (Z^\frown[k+1]^{n-l})| \geqslant (\delta/2)|Z^\frown[k+1]^{n-l}|$ since $\mathrm{dens}(A_x) \geqslant \delta/2$ for every $x \in Z$. On the other hand, the family $\{Z^\frown y : y \in [k+1]^{n-l}\}$ forms a partition of $Z^\frown[k+1]^{n-l}$ into sets of equal size. Hence, there exists $y_0 \in [k+1]^{n-l}$ such that $|A \cap (Z^\frown y_0)| \geqslant (\delta/2)|Z^\frown y_0|$. Observe that $Z^\frown y_0$ is isomorphic to $[k]^M$. Thus, by the choice of M, there exists an m-dimensional subspace $\tilde{V}$ of $Z^\frown y_0$ such that $\tilde{V} \subseteq A$. Let V be the unique m-dimensional subspace of $[k+1]^n$ with $V \restriction k = \tilde{V}$. Then V is as desired. □

4. Proof of Theorem 1

The proof proceeds by induction on k. The case "$k = 2$" follows from the classical Sperner theorem [13]. So let $k \in \mathbb{N}$ with $k \geqslant 2$ and assume DHJ_k. First we introduce some numerical invariants. Specifically, for every $0 < \delta \leqslant 1$ we set

$$(1) \qquad m_0 = \mathrm{DHJ}(k,\delta/4), \quad \theta = \frac{\delta/4}{(k+1)^{m_0} - k^{m_0}}, \quad \eta = \frac{\delta\theta}{48} \quad \text{and} \quad \gamma = \frac{\delta\eta^2}{k}.$$

The main step of the proof of DHJ_{k+1} is the following dichotomy.

Proposition 6. *Let $k \in \mathbb{N}$ with $k \geqslant 2$ and assume DHJ_k. Then for every $0 < \delta \leqslant 1$ and every integer $d \geqslant 1$ there exists an integer $N(k,d,\delta)$ with the following property. If $n \geqslant N(k,d,\delta)$ and A is a subset of $[k+1]^n$ with $\mathrm{dens}(A) \geqslant \delta$, then either A contains a combinatorial line of $[k+1]^n$, or there exists a d-dimensional subspace V of $[k+1]^n$ such that $\mathrm{dens}_V(A) \geqslant \delta + \gamma/2$ where γ is as in (1).*

Using Proposition 6 the numbers $\mathrm{DHJ}(k+1,\delta)$ can be estimated easily via a standard iteration. And, of course, this is enough to complete the proof of the density Hales–Jewett Theorem.

It remains to prove Proposition 6. This is our goal in the following subsection.

4.1. **Proof of Proposition 6.** The proof is based on a series of lemmas. *We emphasize that, in what follows, we will assume* DHJ_k. Also, for every integer $m \geqslant 1$ and every $0 < \varepsilon \leqslant 1$ we set

$$n(m,\varepsilon) = \varepsilon^{-1}(k+1)^m m. \tag{2}$$

We start with the following lemma.

Lemma 7. *Let $0 < \delta \leqslant 1$ and $m \in \mathbb{N}$ with $m \geqslant m_0$. If $n \geqslant n(\mathrm{GR}(k,m), \eta^2/2)$, then for every subset A of $[k+1]^n$ with $\mathrm{dens}(A) \geqslant \delta$ there exist some $l < n$ and an m-dimensional subspace U of $[k+1]^l$ such that*

(a) *for every $u \in U$ we have $\mathrm{dens}(A_u) \geqslant \delta - \eta^2/2$, and*
(b) *for every $\ell \in \mathrm{Lines}(U \upharpoonright k)$ we have $\mathrm{dens}\big(\bigcap_{u\in\ell} A_u\big) \geqslant \theta$,*

where, as in Lemma 4, $A_u = \{y \in [k+1]^{n-l} : u^\frown y \in A\}$ for every $u \in U$.

Proof. We apply Lemma 4 and we obtain some $l < n$ and a subspace V of $[k+1]^l$ of dimension $\mathrm{GR}(k,m)$ such that $\mathrm{dens}(A_v) \geqslant \delta - \eta^2/2$ for every $v \in V$. We set

$$\mathcal{L} = \Big\{\ell \in \mathrm{Lines}(V \upharpoonright k) : \mathrm{dens}\Big(\bigcap_{v\in\ell} A_v\Big) \geqslant \theta\Big\}.$$

By Proposition 2, there exists an m-dimensional subspace Y of $V \upharpoonright k$ such that either $\mathrm{Lines}(Y) \subseteq \mathcal{L}$ or $\mathrm{Lines}(Y) \cap \mathcal{L} = \emptyset$. If $\mathrm{Lines}(Y) \subseteq \mathcal{L}$, then let U be the unique subspace of $[k+1]^l$ such that $U \upharpoonright k = Y$. It is easily checked that U satisfies the requirements of the lemma.

Therefore the proof will be completed once we show that $\mathrm{Lines}(Y) \cap \mathcal{L} \neq \emptyset$. To this end, first, we select an m_0-dimensional subspace Z of Y. By the choice of η in (1) and the fact that $Z \subseteq V$, we have $\mathrm{dens}(A_z) \geqslant \delta/2$ for every $z \in Z$. Hence there exists $B \subseteq [k+1]^{n-l}$ with $\mathrm{dens}(B) \geqslant \delta/4$ such that $|A \cap (Z^\frown y)| \geqslant (\delta/4)|Z^\frown y|$ for every $y \in B$. Let $y \in B$ be arbitrary. By the previous discussion and the choice of m_0 in (1), there exists $\ell_y \in \mathrm{Lines}(Z)$ such that $\ell_y^\frown y \subseteq A$. The number of combinatorial lines of Z is $(k+1)^{m_0} - k^{m_0}$. It follows that there exist $\ell_0 \in \mathrm{Lines}(Z)$ and a subset C of B with $\mathrm{dens}(C) \geqslant \theta$ such that $\ell_0^\frown y \subseteq A$ for every $y \in C$. This implies that $\ell_0 \in \mathrm{Lines}(Y) \cap \mathcal{L}$ and the proof is completed. □

The next result asserts that if we have "lack of density increment", then we can find a subspace W of $[k+1]^n$ of sufficiently large dimension satisfying two properties. Firstly the density of A inside W is essentially the same as the density of A in $[k+1]^n$ and, secondly, with plenty of lines contained in $A \cap (W \upharpoonright k)$.

Lemma 8. *Let $0 < \delta \leqslant 1$ and $m \in \mathbb{N}$ with $m \geqslant m_0$. Also let $n \geqslant n(\mathrm{GR}(k,m), \eta^2/2)$ and let A be a subset of $[k+1]^n$ with $\mathrm{dens}(A) \geqslant \delta$. Then either there exists an m-dimensional subspace X of $[k+1]^n$ such that $\mathrm{dens}_X(A) \geqslant \delta + \eta^2/2$, or there exists an m-dimensional subspace W of $[k+1]^n$ such that $\mathrm{dens}_W(A) \geqslant \delta - 2\eta$ and*

$$|\{\ell \in \mathrm{Lines}(W \upharpoonright k) : \ell \subseteq A\}| \geqslant (\theta/2)|\mathrm{Lines}(W \upharpoonright k)|. \tag{3}$$

Proof. Clearly we may assume that $\mathrm{dens}_X(A) < \delta + \eta^2/2$ for every m-dimensional subspace X of $[k+1]^n$. By Lemma 7, there exist some $l < n$ and an m-dimensional subspace U of $[k+1]^l$ such that $\mathrm{dens}(A_u) \geqslant \delta - \eta^2/2$ for every $u \in U$ and $\mathrm{dens}\big(\bigcap_{u\in\ell} A_u\big) \geqslant \theta$ for every $\ell \in \mathrm{Lines}(U \upharpoonright k)$.

The first property implies, in particular, that $\mathbb{E}_{y\in[k+1]^{n-l}}\mathrm{dens}_{U^\frown y}(A) \geqslant \delta - \eta^2/2$. For every $y \in [k+1]^{n-l}$ the set $U^\frown y$ is an m-dimensional subspace of $[k+1]^n$. Thus, by our assumptions, we have $\mathrm{dens}_{U^\frown y}(A) < \delta + \eta^2/2$ for every $y \in [k+1]^{n-l}$. It follows that there exists a subset H_1 of $[k+1]^{n-l}$ with $\mathrm{dens}(H_1) \geqslant 1 - \eta$ such that $\mathrm{dens}_{U^\frown y}(A) \geqslant \delta - 2\eta$ for every $y \in H_1$.

Now for every $y \in [k+1]^{n-l}$ let $\mathcal{L}_y = \{\ell \in \mathrm{Lines}(U \upharpoonright k) : y \in \bigcap_{u\in\ell} A_u\}$. Since $\mathrm{dens}\big(\bigcap_{u\in\ell} A_u\big) \geqslant \theta$ for every $\ell \in \mathrm{Lines}(U \upharpoonright k)$ we have

$$\mathbb{E}_{y\in[k+1]^{n-l}} \frac{|\mathcal{L}_y|}{|\mathrm{Lines}(U \upharpoonright k)|} = \mathbb{E}_{\ell\in\mathrm{Lines}(U\upharpoonright k)}\mathrm{dens}\Big(\bigcap_{u\in\ell} A_u\Big) \geqslant \theta.$$

Hence, there exists a subset H_2 of $[k+1]^{n-l}$ with $\mathrm{dens}(H_2) \geqslant \theta/2$ such that $|\mathcal{L}_y| \geqslant (\theta/2)|\mathrm{Lines}(U \upharpoonright k)|$ for every $y \in H_2$.

By the choice of θ and η in (1), we have $\eta < \theta/2$. It follows that the set $H_1 \cap H_2$ is nonempty. We select $y_0 \in H_1 \cap H_2$ and we set $W = U^\frown y_0$. It is easy to check that W is as desired. $\square$

From this point on the proof follows the steps of Polymath's proof. A crucial ingredient (perhaps the single most important one) is the notion of an insensitive set which we are about to recall. To this end, we will need the following terminology. Let $x, y \in [k+1]^n$ and write $x = (x_r)_{r=1}^n$ and $y = (y_r)_{r=1}^n$. Also let $i, j \in [k+1]$ with $i \neq j$. We say that x and y are (i,j)-*equivalent* if for every $s \in [k+1] \setminus \{i,j\}$ we have $\{r \in [n] : x_r = s\} = \{r \in [n] : y_r = s\}$.

Definition 9. *Let $i, j \in [k+1]$ with $i \neq j$ and let A be a subset of $[k+1]^n$. The set A is said to be* (i,j)-insensitive *provided that for every $x \in A$ and every $y \in [k+1]^n$ if x and y are (i,j)-equivalent, then $y \in A$.*

If V is an m-dimensional subspace of $[k+1]^n$ and A is a subset of V, then A is said to be (i,j)-insensitive in V *if, identifying V with $[k+1]^m$, A becomes an (i,j)-insensitive subset of $[k+1]^m$.*

It is easy to see that the family of all (i,j)-insensitive subsets of $[k+1]^n$ is closed under intersections, unions and complements. The same remark, of course, applies to the family of all (i,j)-insensitive sets of a subspace V of $[k+1]^n$.

Also we need to introduce some more numerical invariants. Precisely, for every $0 < \delta \leqslant 1$ let m_0 and η be as in (1) and set

$$\lambda = \frac{k+1}{k} \quad \text{and} \quad M_0 = \max\Big\{m_0, \frac{\log \eta^{-1}}{\log \lambda}\Big\}. \tag{4}$$

We proceed with the following lemma.

Lemma 10. *Let $0<\delta\leqslant 1$ and $m\in\mathbb{N}$ with $m\geqslant M_0$. Let $n\geqslant n(\mathrm{GR}(k,m),\eta^2/2)$ and let A be a subset of $[k+1]^n$ with $\mathrm{dens}(A)\geqslant\delta$. Assume that A contains no combinatorial line of $[k+1]^n$ and $\mathrm{dens}_X(A)<\delta+\eta^2/2$ for every m-dimensional subspace X of $[k+1]^n$. Then there exist an m-dimensional subspace W of $[k+1]^n$ and a subset C of W satisfying the following properties.*

(a) *We have $\mathrm{dens}_W(C)\geqslant\theta/4$ and $C=\bigcap_{i=1}^k C_i$ where C_i is $(i,k+1)$-insensitive in W for every $i\in[k]$.*
(b) *We have $\mathrm{dens}_W\big(A\cap(W\setminus C)\big)\geqslant(\delta+6\eta)\mathrm{dens}_W(W\setminus C)$ and, moreover, $\mathrm{dens}_W\big(A\cap(W\setminus C)\big)\geqslant\delta-3\eta$.*

Proof. By our assumptions, we apply Lemma 8 and we obtain an m-dimensional subspace W of $[k+1]^n$ such that $\mathrm{dens}_W(A)\geqslant\delta-2\eta$ and satisfying inequality (3). For every $\ell\in\mathrm{Lines}(W\upharpoonright k)$ let $\bar{\ell}$ be the unique combinatorial line of W such that $\bar{\ell}\upharpoonright k=\ell$. Let $B=\{\bar{\ell}(k+1):\ell\in\mathrm{Lines}(W\upharpoonright k)\text{ with }\ell\subseteq A\}$ and set $C=B\cup\big(A\cap(W\upharpoonright k)\big)$. We will show that W and C are as desired. First we argue for (a). Identifying W with $[k+1]^m$, for every $x\in W$ let $x^{k+1\to i}$ be the unique element of W obtained by replacing all appearances of $k+1$ in x by i. Setting $C_i=\{x\in W:x^{k+1\to i}\in A\cap(W\upharpoonright k)\}$ for every $i\in[k]$, we see that C_i is $(i,k+1)$-insensitive in W and $C=C_1\cap\dots\cap C_k$. Next observe that the map $\mathrm{Lines}(W\upharpoonright k)\ni\ell\mapsto\bar{\ell}(k+1)\in W$ is one-to-one. Hence,

$$\begin{aligned}|C| &\geqslant |B|=|\{\ell\in\mathrm{Lines}(W\upharpoonright k):\ell\subseteq A\}|\overset{(3)}{\geqslant}(\theta/2)|\mathrm{Lines}(W\upharpoonright k)|\\ &= (\theta/2)((k+1)^m-k^m)\overset{(4)}{\geqslant}(\theta(1-\eta)/2)(k+1)^m\overset{(1)}{\geqslant}(\theta/4)|W|.\end{aligned}$$

This shows that part (a) is satisfied. For part (b), notice first that our assumption that A contains no combinatorial line of $[k+1]^n$ implies that $A\cap C\subseteq W\upharpoonright k$. Therefore, $\mathrm{dens}_W(A\cap C)\leqslant\lambda^{-m}\leqslant\lambda^{-M_0}\leqslant\eta$. Since $\mathrm{dens}_W(A)\geqslant\delta-2\eta$ we see that $\mathrm{dens}_W\big(A\cap(W\setminus C)\big)\geqslant\delta-3\eta$. Moreover,

$$\frac{\mathrm{dens}_W\big(A\cap(W\setminus C)\big)}{\mathrm{dens}_W(W\setminus C)}\geqslant\frac{\delta-3\eta}{1-\theta/4}\geqslant(\delta-3\eta)(1+\theta/4)\overset{(1)}{\geqslant}\delta+6\eta$$

and the proof is completed. □

The following corollary completes the first part of the proof of Proposition 6. It shows that if A contains no combinatorial line, then it must correlate significantly with a "structured" subset of $[k+1]^n$.

Corollary 11. *Let $0<\delta\leqslant 1$ and $m\in\mathbb{N}$ with $m\geqslant M_0$. Let $n\geqslant n(\mathrm{GR}(k,m),\eta^2/2)$ and let A be subset of $[k+1]^n$ with $\mathrm{dens}(A)\geqslant\delta$. Assume that A contains no combinatorial line of $[k+1]^n$. Then there exist an m-dimensional subspace W of $[k+1]^n$ and a family $\{D_1,\dots,D_k\}$ of subsets of W such that D_i is $(i,k+1)$-insensitive in W for every $i\in[k]$ and, moreover, setting $D=D_1\cap\dots\cap D_k$ we have $\mathrm{dens}_W(D)\geqslant\gamma$ and $\mathrm{dens}_W(A\cap D)\geqslant(\delta+\gamma)\mathrm{dens}_W(D)$.*

Proof. First assume that there exist an m-dimensional subspace X of $[k+1]^n$ such that $\mathrm{dens}_X(A) \geqslant \delta + \eta^2/2$. Then we set $W = X$ and $D_i = X$ for every $i \in [k]$. Since $\eta^2/2 \geqslant \gamma$, it is clear that with these choices the result follows. Otherwise, by Lemma 10, there exist an m-dimensional subspace W of $[k+1]^n$ and a set $C = C_1 \cap \cdots \cap C_k$, where C_i is $(i, k+1)$-insensitive in W for every $i \in [k]$, such that $\mathrm{dens}_W\big(A \cap (W \setminus C)\big) \geqslant (\delta + 6\eta)\mathrm{dens}_W(W \setminus C)$ and $\mathrm{dens}_W\big(A \cap (W \setminus C)\big) \geqslant \delta - 3\eta$.

We set $P_1 = W \setminus C_1$ and $P_i = (W \setminus C_i) \cap C_1 \cap \cdots \cap C_{i-1}$ if $i \in \{2, \dots, k\}$. Also, for every $i \in [k]$ let $\lambda_i = \mathrm{dens}_W(P_i)/\mathrm{dens}_W(W \setminus C)$ and $\delta_i = \mathrm{dens}_W(A \cap P_i)/\mathrm{dens}_W(P_i)$ with the convention that $\delta_i = 0$ if P_i happens to be empty. The family $\{P_1, \dots, P_k\}$ is a partition of $W \setminus C$ and so $\sum_{i=1}^k \lambda_i \delta_i = \mathrm{dens}_W\big(A \cap (W \setminus C)\big)/\mathrm{dens}_W(W \setminus C) \geqslant \delta + 6\eta$. Hence, there exists $i_0 \in [k]$ such that $\lambda_{i_0} \geqslant 3\eta/k$ and $\delta_{i_0} \geqslant \delta + 3\eta$. We set $D_i = C_i$ if $i < i_0$, $D_{i_0} = W \setminus C_{i_0}$ and $D_i = W$ if $i > i_0$. Clearly D_i is $(i, k+1)$-insensitive in W for every $i \in [k]$. Moreover, we have $D_1 \cap \cdots \cap D_k = P_{i_0}$ and so $\mathrm{dens}_W(P_{i_0}) = \lambda_{i_0}\mathrm{dens}_W(W \setminus C) \geqslant (3\eta/k)(\delta - 3\eta) \geqslant \gamma$ and $\mathrm{dens}_W(A \cap P_{i_0}) = \delta_{i_0}\mathrm{dens}_W(P_{i_0}) \geqslant (\delta + 3\eta)\mathrm{dens}_W(P_{i_0}) \geqslant (\delta + \gamma)\mathrm{dens}_W(P_{i_0})$ as desired. □

The second part of the proof of Proposition 6 is a tiling procedure that enables us to partition any "structured" subset of $[k+1]^n$ (that is, any subset of $[k+1]^n$ of the form $D_1 \cap \cdots \cap D_k$ where D_i is $(i, k+1)$-insensitive for every $i \in [k]$) into subspaces of sufficiently large dimension. First one treats the case of insensitive sets. To this end, for every $0 < \beta \leqslant 1$ and every integer $m \geqslant 1$ we set

$$(5) \quad M_1 = \mathrm{MDHJ}^*(k, m, \beta) \quad \text{and} \quad F(m, \beta) = \lceil \beta^{-1}(k+1+m)^{M_1}(k+1)^{M_1 - m} M_1 \rceil$$

where $\mathrm{MDHJ}^*(k, m, \beta)$ is as defined in Corollary 5. We have the following lemma.

Lemma 12. *Let $0 < \beta \leqslant 1$ and $m \in \mathbb{N}$ with $m \geqslant 1$. Also let $i \in [k]$. If $n \geqslant F(m, \beta)$, then for every $(i, k+1)$-insensitive subset D of $[k+1]^n$ with $\mathrm{dens}(D) \geqslant 2\beta$ there exists a family $\mathcal{V}$ of pairwise disjoint m-dimensional subspaces of $[k+1]^n$ which are all contained in D and are such that $\mathrm{dens}(D \setminus \cup\mathcal{V}) < 2\beta$.*

Proof. We set $\Theta = \beta(k+1+m)^{-M_1}(k+1)^{m-M_1}$. For every $x \in [k+1]^{n-M_1}$ let $D_x = \{y \in [k+1]^{M_1} : x^\frown y \in D\}$. Since $\mathbb{E}_{x \in [k+1]^{n-M_1}} \mathrm{dens}(D_x) = \mathrm{dens}(D) \geqslant 2\beta$, there exists a subset T_1 of $[k+1]^{n-M_1}$ with $\mathrm{dens}(T_1) \geqslant \beta$ such that $\mathrm{dens}(D_x) \geqslant \beta$ for every $x \in T_1$. Let $x \in T_1$ be arbitrary. By the choice of M_1 in (5) and Corollary 5, there exists a subspace V_x of $[k+1]^{M_1}$ of dimension m such that $V_x \upharpoonright k \subseteq D_x$. It follows that $x^\frown(V_x \upharpoonright k) \subseteq D$, and so, $x^\frown V_x \subseteq D$ since D is $(i, k+1)$-insensitive. The number of m-dimensional subspaces of $[k+1]^{M_1}$ is less than $(k+1+m)^{M_1}$. Therefore there exists a subspace V_1 of $[k+1]^{M_1}$ such that the set $S_1 = \{x \in [k+1]^{n-M_1} : x^\frown V_1 \subseteq D\}$ has density at least $\beta(k+1+m)^{-M_1}$. Notice that S_1 is $(i, k+1)$-insensitive. We set $\mathcal{V}_1 = \{x^\frown V_1 : x \in S_1\}$. It is clear that $\mathcal{V}_1$ is a family of pairwise disjoint m-dimensional subspaces of $[k+1]^n$ such that $\cup\mathcal{V}_1 \subseteq D$. Moreover, by the choice of Θ, we have $\mathrm{dens}(\cup\mathcal{V}_1) \geqslant \Theta$.

If $\mathrm{dens}(D \setminus \cup\mathcal{V}_1) < 2\beta$, then we are done. Otherwise let $D_1 = D \setminus \cup\mathcal{V}_1$. The set D_1 is not $(i, k+1)$-insensitive but is "almost" insensitive in the following sense. For every $y \in [k+1]^{M_1}$ if we set $D_1^y = \{x \in [k+1]^{n-M_1} : x^\frown y \in D_1\}$, then D_1^y is $(i, k+1)$-insensitive. This is clear if $y \notin V_1$. On the other hand if $y \in V_1$, then $D_1^y = \{x \in [k+1]^{n-M_1} : x^\frown y \in D\} \setminus S_1$ and the claim follows since both D and S_1 are $(i, k+1)$-insensitive. Now for every pair $(x, y) \in [k+1]^{n-2M_1} \times [k+1]^{M_1}$ let $D_1^{(x,y)} = \{z \in [k+1]^{M_1} : x^\frown z^\frown y \in D_1\}$. Using the previous remarks it is easily seen that the set $D_1^{(x,y)}$ is $(i, k+1)$-insensitive. Moreover, $\mathbb{E}_{(x,y)}\mathrm{dens}(D_1^{(x,y)}) = \mathrm{dens}(D_1) \geqslant 2\beta$. Arguing precisely as before, it is possible to select an m-dimensional subspace V_2 of $[k+1]^{M_1}$ such that the set $S_2 = \{(x, y) \in [k+1]^{n-2M_1} \times [k+1]^{M_1} : x^\frown V_2^\frown y \subseteq D_1\}$ has density at least $\beta(k+1+m)^{-M_1}$. Also observe that for every $y \in [k+1]^{M_1}$ the set $S_2^y = \{x \in [k+1]^{n-2M_1} : x^\frown V_2^\frown y \in D_1\}$ is $(i, k+1)$-insensitive. We set $\mathcal{V}_2 = \mathcal{V}_1 \cup \{x^\frown V_2^\frown y : (x, y) \in S_2\}$. Then $\mathcal{V}_2$ is a new family of pairwise disjoint m-dimensional subspaces of $[k+1]^n$ with $\cup\mathcal{V}_2 \subseteq D$ and $\mathrm{dens}(\cup\mathcal{V}_2) \geqslant \mathrm{dens}(\cup\mathcal{V}_1) + \Theta$.

We continue similarly. At each step the density of the union of the members of the new collection of subspaces is increased by Θ. So this process must stop after at most $\lfloor \Theta^{-1} \rfloor$ iterations. Since $n \geqslant \beta^{-1}(k+1+m)^{M_1}(k+1)^{M_1-m}M_1 = M_1/\Theta$ the above algorithm will eventually terminate and the proof is completed. $\square$

By recursion on $r \in [k]$, for every $0 < \beta \leqslant 1$ and every $m \in \mathbb{N}$ with $m \geqslant 1$ we define the integer $F^{(r)}(m, \beta)$ by the rule

$$F^{(1)}(m, \beta) = F(m, \beta) \quad \text{and} \quad F^{(r+1)}(m, \beta) = F^{(r)}\big(F(m, \beta), \beta\big). \tag{6}$$

The following corollary completes the second part of the proof of Proposition 6.

Corollary 13. *Let $0 < \beta \leqslant 1$, $m \in \mathbb{N}$ with $m \geqslant 1$ and $r \in [k]$. Let $n \geqslant F^{(r)}(m, \beta)$ and for every $i \in [r]$ let D_i be an $(i, k+1)$-insensitive subset of $[k+1]^n$. We set $D = D_1 \cap \cdots \cap D_r$. If $\mathrm{dens}(D) \geqslant 2r\beta$, then there exists a family $\mathcal{V}$ of pairwise disjoint m-dimensional subspaces of $[k+1]^n$ which are all contained in D and are such that $\mathrm{dens}(D \setminus \cup\mathcal{V}) < 2r\beta$.*

Proof. By induction on r. The case "$r = 1$" follows from Lemma 12. Assume that the result has been proved up to $r \in [k-1]$. Fix $n \geqslant F^{(r+1)}(m, \beta)$ and let $D_1, \dots, D_{r+1}$ be a family of subsets of $[k+1]^n$ as described above. By our inductive hypothesis, there exists a family $\mathcal{V}_1$ of pairwise disjoint $F(m, \beta)$-dimensional subspaces of $[k+1]^n$ which are all contained in $D' \coloneqq D_1 \cap \cdots \cap D_r$ and are such that $\mathrm{dens}(D' \setminus \cup\mathcal{V}_1) < 2r\beta$. Let $\mathcal{V}_2 = \{V \in \mathcal{V}_1 : \mathrm{dens}_V(D_{r+1}) \geqslant 2\beta\}$. For every $V \in \mathcal{V}_2$ let $\mathcal{B}_V$ be the collection of m-dimensional subspaces of V resulting by Lemma 12 when applied to the set $V \cap D_{r+1}$. We set $\mathcal{V} = \{W : V \in \mathcal{V}_2 \text{ and } W \in \mathcal{B}_V\}$. Then $\mathcal{V}$ is as desired. $\square$

We are now ready to give the proof of Proposition 6.

Proof of Proposition 6. For every $0 < \delta \leqslant 1$ and every $d \in \mathbb{N}$ with $d \geqslant 1$ let $\beta = \gamma^2/4k$ and $m(d) = \max\{M_0, F^{(k)}(d, \beta)\}$. We define

$$N(k,d,\delta) = n\big(\mathrm{GR}\big(k, m(d)\big), \eta^2/2\big). \tag{7}$$

Fix $n \geqslant N(k,d,\delta)$ and a subset A of $[k+1]^n$ with $\mathrm{dens}(A) \geqslant \delta$. Assume that A contains no combinatorial line of $[k+1]^n$. By Corollary 11, there exist a subspace W of $[k+1]^n$ of dimension $m(d)$ and a family $\{D_1, \dots, D_k\}$ of subsets of W such that D_i is $(i, k+1)$-insensitive in W for every $i \in [k]$ and, setting $D = D_1 \cap \cdots \cap D_k$, we have $\mathrm{dens}_W(D) \geqslant \gamma$ and $\mathrm{dens}_W(A \cap D) \geqslant (\delta + \gamma)\mathrm{dens}_W(D)$. By Corollary 13, there exists a family $\mathcal{V}$ of pairwise disjoint d-dimensional subspaces such that $\cup\mathcal{V} \subseteq D$ and $\mathrm{dens}_W(D \setminus \cup\mathcal{V}) < 2k\beta = \gamma^2/2$. Combining the previous estimates, we see that $\mathrm{dens}_W(A \cap \cup\mathcal{V}) \geqslant (\delta + \gamma/2)\mathrm{dens}_W(\cup\mathcal{V})$. Hence, there exists $V \in \mathcal{V}$ such that $\mathrm{dens}_W(A \cap V) \geqslant (\delta + \gamma/2)\mathrm{dens}_W(V)$ or equivalently $\mathrm{dens}_V(A) \geqslant \delta + \gamma/2$. □

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