

DENSE SUBSETS OF PRODUCTS OF FINITE TREES

PANDELIS DODOS, VASSILIS KANELLOPOULOS AND KONSTANTINOS TYROS

ABSTRACT. We prove a “uniform” version of the finite density Halpern–Läuchli theorem. Specifically, we say that a tree T is homogeneous if it is uniquely rooted and there is an integer $b \geq 2$, called the branching number of T , such that every $t \in T$ has exactly b immediate successors. We show the following.

For every integer $d \geq 1$, every $b_1, \dots, b_d \in \mathbb{N}$ with $b_i \geq 2$ for all $i \in \{1, \dots, d\}$, every integer $k \geq 1$ and every real $0 < \varepsilon \leq 1$ there exists an integer N with the following property. If (T_1, \dots, T_d) are homogeneous trees such that the branching number of T_i is b_i for all $i \in \{1, \dots, d\}$, L is a finite subset of \mathbb{N} of cardinality at least N and D is a subset of the level product of (T_1, \dots, T_d) satisfying

$$|D \cap (T_1(n) \times \dots \times T_d(n))| \geq \varepsilon |T_1(n) \times \dots \times T_d(n)|$$

for every $n \in L$, then there exist strong subtrees (S_1, \dots, S_d) of (T_1, \dots, T_d) of height k and with a common level set such that the level product of (S_1, \dots, S_d) is contained in D . The least integer N with this property will be denoted by $\text{UDHL}(b_1, \dots, b_d|k, \varepsilon)$.

The main point is that the result is independent of the position of the finite set L . The proof is based on a density increment strategy and gives explicit upper bounds for the numbers $\text{UDHL}(b_1, \dots, b_d|k, \varepsilon)$.

1. INTRODUCTION

1.1. Statement of the problem and the main result. It is well-known that several results in Ramsey theory have an infinite and a finite version. While the proofs of the infinite versions are usually conceptually “cleaner” and yield formally stronger results (see, e.g., [22]), an analysis of the corresponding finite versions gives explicit and non-trivial estimates for certain numerical invariants commonly known as *Ramsey numbers*. These invariants are of fundamental importance and are central for the development of Ramsey theory (see [13]).

The main goal of the present paper is to give an effective proof of a “uniform” version of the finite density Halpern–Läuchli theorem and to obtain quantitative information on the corresponding “density Halpern–Läuchli” numbers. To proceed with our discussion it is useful at this point to recall the Halpern–Läuchli theorem [15]. It is a rather deep pigeonhole principle for trees. It has several equivalent

2000 *Mathematics Subject Classification*: 05D10, 05C05.

Key words: homogeneous trees, vector trees, strong subtrees, level product, density.

The first named author was supported by NSF grant DMS-0903558.

forms which are discussed in great detail in [33, §3.1]. We will state the “strong subtree version” which is the most important one from a combinatorial perspective.

Theorem 1. *For every integer $d \geq 1$, every tuple (T_1, \dots, T_d) of uniquely rooted and finitely branching trees without maximal nodes and every finite coloring of the level product*

$$\bigcup_{n \in \mathbb{N}} T_1(n) \times \dots \times T_d(n)$$

of (T_1, \dots, T_d) there exist strong subtrees (S_1, \dots, S_d) of (T_1, \dots, T_d) of infinite height and with a common level set such that the level product of (S_1, \dots, S_d) is monochromatic.

We recall that a subtree S of a tree $(T, <)$ is said to be *strong* if: (a) S is uniquely rooted and balanced (that is, all maximal chains of S have the same cardinality), (b) every level of S is a subset of some level of T , and (c) for every non-maximal node $s \in S$ and every immediate successor t of s in T there exists a unique immediate successor s' of s in S with $t \leq s'$. The *level set* of a strong subtree S of a tree T is the set of levels of T containing a node of S .

Although the notion of a strong subtree was isolated in the 1960s, it was highlighted with the work of Milliken in [19, 20] who used Theorem 1 to show that the family of strong subtrees of a uniquely rooted and finitely branching tree is partition regular. The Halpern–Läuchli theorem and Milliken’s theorem can be considered as the starting point of Ramsey theory for trees, a rich area of combinatorics with significant applications, most notably in the geometry of Banach spaces (see, for instance, [3, 4, 5, 11, 16, 17, 18, 33] and [1, 6, 30, 32] for applications).

Theorem 1 has a density version that was conjectured by Laver in the late 1960s and obtained, recently, in [7]. To state it let us recall that a tree T is said to be *homogeneous* if it is uniquely rooted and there exists an integer $b \geq 2$, called the *branching number* of T , such that every $t \in T$ has exactly b immediate successors.

Theorem 2. *For every integer $d \geq 1$, every tuple (T_1, \dots, T_d) of homogeneous trees and every subset D of the level product of (T_1, \dots, T_d) satisfying*

$$\limsup_{n \rightarrow \infty} \frac{|D \cap (T_1(n) \times \dots \times T_d(n))|}{|T_1(n) \times \dots \times T_d(n)|} > 0$$

there exist strong subtrees (S_1, \dots, S_d) of (T_1, \dots, T_d) of infinite height and with a common level set such that the level product of (S_1, \dots, S_d) is contained in D .

We should point out that the assumption in Theorem 2 that the trees (T_1, \dots, T_d) are homogeneous is not redundant. On the contrary, various examples given in [2] show that it is essentially optimal.

While Theorem 2 is infinite-dimensional, it has a finite counterpart which is obtained via a standard compactness argument. Precisely, it follows by Theorem 2 that for every integer $d \geq 1$, every $b_1, \dots, b_d \in \mathbb{N}$ with $b_i \geq 2$ for all $i \in \{1, \dots, d\}$,

every integer $k \geq 1$, every real $0 < \varepsilon \leq 1$ and every $M = \{m_0 < m_1 < \dots\}$ infinite subset of \mathbb{N} , there exists an integer N with the following property. If (T_1, \dots, T_d) is a tuple of homogeneous trees such that the branching number of T_i is b_i for all $i \in \{1, \dots, d\}$ and D is a subset of the level product of (T_1, \dots, T_d) satisfying

$$|D \cap (T_1(m_n) \times \dots \times T_d(m_n))| \geq \varepsilon |T_1(m_n) \times \dots \times T_d(m_n)|$$

for every $n \leq N$, then there exist strong subtrees (S_1, \dots, S_d) of (T_1, \dots, T_d) of height k and with a common level set such that the level product of (S_1, \dots, S_d) is contained in D . We shall denote the least integer N with this property by $\text{DHL}(b_1, \dots, b_d | k, \varepsilon, M)$. We emphasize that the compactness argument yields that the integer $\text{DHL}(b_1, \dots, b_d | k, \varepsilon, M)$ depends on the choice of the infinite set M .

There are two basic problems left open by the previous approach. The first one is to provide explicit upper bounds for the numbers $\text{DHL}(b_1, \dots, b_d | k, \varepsilon, M)$. This is, of course, related to the ineffectiveness of the compactness method. The second problem lies deeper and concerns the uniform boundedness of the numbers $\text{DHL}(b_1, \dots, b_d | k, \varepsilon, M)$ with respect to the last parameter. Precisely, if the parameters b_1, \dots, b_d, k and ε are fixed, then does there exist an integer j such that $\text{DHL}(b_1, \dots, b_d | k, \varepsilon, M) \leq j$ for every infinite subset M of \mathbb{N} ? In other words, is it true that a finite subset of the level product of a fixed tuple of homogeneous trees will necessarily contain a “substructure” as long as it is dense in sufficiently many levels? This information is needed in various applications ([8]) and constitutes the proper finite analog of Theorem 2.

Our main result answers the above questions. Precisely, we show the following theorem.

Theorem 3. *For every integer $d \geq 1$, every $b_1, \dots, b_d \in \mathbb{N}$ with $b_i \geq 2$ for all $i \in \{1, \dots, d\}$, every integer $k \geq 1$ and every real $0 < \varepsilon \leq 1$ there exists an integer N with the following property. If (T_1, \dots, T_d) are homogeneous trees such that the branching number of T_i is b_i for all $i \in \{1, \dots, d\}$, L is a finite subset of \mathbb{N} of cardinality at least N and D is a subset of the level product of (T_1, \dots, T_d) satisfying*

$$|D \cap (T_1(n) \times \dots \times T_d(n))| \geq \varepsilon |T_1(n) \times \dots \times T_d(n)|$$

for every $n \in L$, then there exist strong subtrees (S_1, \dots, S_d) of (T_1, \dots, T_d) of height k and with a common level set such that the level product of (S_1, \dots, S_d) is contained in D . The least integer N with this property will be denoted by $\text{UDHL}(b_1, \dots, b_d | k, \varepsilon)$.

As we have already mentioned, the proof of Theorem 3 is effective and gives explicit upper bounds for the numbers $\text{UDHL}(b_1, \dots, b_d | k, \varepsilon)$. These upper bounds are admittedly rather weak; they have an Ackermann-type dependence with respect to the “dimension” d . However, they are in line with several other bounds obtained recently in the area; see [12, 23, 25].

1.2. Related work. There are several results in the literature closely related to the one-dimensional case of Theorem 3, namely when we deal with a single homogeneous tree. The earliest reference we are aware of is the paper [2] by Bicker and Voigt, though related problems have been circulated among experts much earlier. The first significant progress, however, was made by Furstenberg and Weiss in [10] who obtained a “parameterized” version of Szémerdi’s theorem on arithmetic progressions [31]. Specifically, it was shown in [10] that for every integer $b \geq 2$, every integer $k \geq 1$ and every real $0 < \varepsilon \leq 1$ there exists an integer N with the following property. If T is a finite homogeneous tree with branching number b and of height at least N , L is a subset of $\{0, \dots, h(T) - 1\}$ of cardinality at least $\varepsilon h(T)$ and D is a subset of T satisfying

$$|D \cap T(n)| \geq \varepsilon |T(n)|$$

for every $n \in L$, then D contains a strong subtree of T of height k whose level set is an arithmetic progression. We shall denote by $\text{FW}(b|k, \varepsilon)$ the least integer N with this property. The method in [10] was qualitative in nature, and as such, could not provide explicit estimates for the numbers $\text{FW}(b|k, \varepsilon)$. The work of Furstenberg and Weiss was revisited, recently, in [21] by Pach, Solymosi and Tardos who effectively reduced the aforementioned result to Szémerdi’s theorem with an elegant combinatorial argument. It follows, in particular, from the analysis in [21] that $\text{UDHL}(b|k, \varepsilon) = O_{b, \varepsilon}(k)$, an upper bound which is essentially optimal.

The proof, however, of the higher-dimensional case of Theorem 3 follows quite different arguments and is closer in spirit to Polymath’s proof [23] of the density Hales–Jewett theorem [9]. It proceeds by induction on the “dimension” d and is based on a density increment strategy, a powerful and fruitful method pioneered by Roth [27].

1.3. Organization of the paper. The paper is organized as follows. In §2 we gather some background material. In §3 we introduce the concept of a *level selection*. It is a natural notion permitting us to reduce the proof of our main result to the study of certain dense subsets with special properties. In §4 we give a detailed outline of the proof of Theorem 3 emphasizing, in particular, its main features. In the next three sections, §5–§7, we prove several preparatory results. We notice that these sections are largely independent of each other and can be read separately. This material is used in §8 which contains the last step of the argument. Finally, the proof of Theorem 3 is given in §9.

To facilitate the interested reader we have also included, in an appendix, a sketch of the proof of the multidimensional version of Milliken’s theorem [20]. This result and the corresponding bounds are needed for the proof of Theorem 3. The arguments are essentially borrowed from [29] and are included for completeness.

2. BACKGROUND MATERIAL

By $\mathbb{N} = \{0, 1, 2, \dots\}$ we shall denote the natural numbers. The cardinality of a set X will be denoted by $|X|$ while its powerset will be denoted by $\mathcal{P}(X)$. If X is a nonempty finite set, then by $\mathbb{E}_{x \in X}$ we shall denote the average $\frac{1}{|X|} \sum_{x \in X}$. If it is clear from the context which set X we are referring to, then this average will be denoted simply by \mathbb{E}_x . For every function $f: \mathbb{N} \rightarrow \mathbb{N}$ and every $k \in \mathbb{N}$ by $f^{(k)}: \mathbb{N} \rightarrow \mathbb{N}$ we shall denote the k -th iteration of f defined recursively by the rule $f^{(0)}(n) = n$ and $f^{(k+1)}(n) = f(f^{(k)}(n))$ for every $n \in \mathbb{N}$.

2.1. Trees. By the term *tree* we mean a nonempty partially ordered set $(T, <)$ such that the set $\{s \in T : s < t\}$ is finite and linearly ordered under $<$ for every $t \in T$. The cardinality of this set is defined to be the *length* of t in T and will be denoted by $\ell_T(t)$. For every $n \in \mathbb{N}$ the n -level of T , denoted by $T(n)$, is defined to be the set $\{t \in T : \ell_T(t) = n\}$. The *height* of T , denoted by $h(T)$, is defined as follows. If there exists $k \in \mathbb{N}$ with $T(k) = \emptyset$, then we set $h(T) = \max\{n \in \mathbb{N} : T(n) \neq \emptyset\} + 1$; otherwise, we set $h(T) = \infty$.

For every node t of a tree T the set of *successors* of t in T is defined by

$$(1) \quad \text{Succ}_T(t) = \{s \in T : t \leq s\}.$$

The set of *immediate successors* of t in T is the subset of $\text{Succ}_T(t)$ defined by $\text{ImmSucc}_T(t) = \{s \in T : t \leq s \text{ and } \ell_T(s) = \ell_T(t) + 1\}$. A node $t \in T$ is said to be *maximal* if the set $\text{ImmSucc}_T(t)$ is empty.

Let $n \in \mathbb{N}$ with $n < h(T)$ and $F \subseteq T(n)$. The *density* of F is defined by

$$(2) \quad \text{dens}(F) := \frac{|F|}{|T(n)|}.$$

More generally, for every $m \in \mathbb{N}$ with $m \leq n$ and every $t \in T(m)$ the *density of F relative to the node t* is defined by

$$(3) \quad \text{dens}(F | t) := \frac{|F \cap \text{Succ}_T(t)|}{|T(n) \cap \text{Succ}_T(t)|}.$$

A *subtree* S of a tree $(T, <)$ is a subset of T viewed as a tree equipped with the induced partial ordering. For every $n \in \mathbb{N}$ with $n < h(T)$ we set

$$(4) \quad T \upharpoonright n = T(0) \cup \dots \cup T(n).$$

Notice that $h(T \upharpoonright n) = n + 1$. An *initial subtree* of T is a subtree of T of the form $T \upharpoonright n$ for some $n \in \mathbb{N}$.

A tree T is said to be *balanced* if all maximal chains of T have the same cardinality. It is said to be *uniquely rooted* if $|T(0)| = 1$; the *root* of a uniquely rooted tree T is defined to be the node $T(0)$.

2.2. Vector trees. A *vector tree* \mathbf{T} is a nonempty finite sequence of trees having a common height; this common height is defined to be the *height* of \mathbf{T} and will be denoted by $h(\mathbf{T})$. We notice that, throughout the paper, we will start the enumeration of vector trees with 1 instead of 0.

For every vector tree $\mathbf{T} = (T_1, \dots, T_d)$ and every $n \in \mathbb{N}$ with $n < h(\mathbf{T})$ we set

$$(5) \quad \mathbf{T} \upharpoonright n = (T_1 \upharpoonright n, \dots, T_d \upharpoonright n).$$

A vector tree of this form is called a *vector initial subtree* of \mathbf{T} . Also let

$$(6) \quad \mathbf{T}(n) = (T_1(n), \dots, T_d(n))$$

and

$$(7) \quad \otimes \mathbf{T}(n) = T_1(n) \times \dots \times T_d(n).$$

The *level product* of \mathbf{T} , denoted by $\otimes \mathbf{T}$, is defined to be the set

$$(8) \quad \bigcup_{n < h(\mathbf{T})} \otimes \mathbf{T}(n).$$

For every $\mathbf{t} = (t_1, \dots, t_d) \in \otimes \mathbf{T}$ we set

$$(9) \quad \text{Succ}_{\mathbf{T}}(\mathbf{t}) = (\text{Succ}_{T_1}(t_1), \dots, \text{Succ}_{T_d}(t_d)).$$

Finally, we say that a vector tree $\mathbf{T} = (T_1, \dots, T_d)$ is *uniquely rooted* if for every $i \in \{1, \dots, d\}$ the tree T_i is uniquely rooted. Notice that if \mathbf{T} is uniquely rooted, then $\mathbf{T}(0) = \otimes \mathbf{T}(0)$; the element $\mathbf{T}(0)$ will be called the *root* of \mathbf{T} .

2.3. Strong subtrees and vector strong subtrees. A subtree S of a uniquely rooted tree T is said to be *strong* provided that: (a) S is uniquely rooted and balanced, (b) every level of S is a subset of some level of T , and (c) for every non-maximal node $s \in S$ and every $t \in \text{ImmSucc}_T(s)$ there exists a unique node $s' \in \text{ImmSucc}_S(s)$ such that $t \leq s'$. The *level set* of a strong subtree S of T is defined to be the set

$$(10) \quad L_T(S) = \{m \in \mathbb{N} : \text{exists } n < h(S) \text{ with } S(n) \subseteq T(m)\}.$$

The concept of a strong subtree is naturally extended to vector trees. Specifically, a *vector strong subtree* of a uniquely rooted vector tree $\mathbf{T} = (T_1, \dots, T_d)$ is a vector tree $\mathbf{S} = (S_1, \dots, S_d)$ such that S_i is a strong subtree of T_i for every $i \in \{1, \dots, d\}$ and $L_{T_1}(S_1) = \dots = L_{T_d}(S_d)$.

2.4. Homogeneous trees and vector homogeneous trees. Let $b \in \mathbb{N}$ with $b \geq 2$. By $b^{<\mathbb{N}}$ we shall denote the set of all finite sequences having values in $\{0, \dots, b-1\}$. The empty sequence is denoted by \emptyset and is included in $b^{<\mathbb{N}}$. We view $b^{<\mathbb{N}}$ as a tree equipped with the (strict) partial order \sqsubset of end-extension. Notice that $b^{<\mathbb{N}}$ is a homogeneous tree with branching number b . If $n \geq 1$, then $b^{<n}$ stands for the initial subtree of $b^{<\mathbb{N}}$ of height n . For every $t, s \in b^{<\mathbb{N}}$ by $t \frown s$ we shall denote the concatenation of t and s .

For technical reasons we will not work with abstract homogeneous trees but with a concrete subclass. Observe that all homogeneous trees with the same branching number are pairwise isomorphic, and so, such a restriction will have no effect in the generality of our results.

Convention. *In the rest of this paper by the term “homogeneous tree” (respectively, “finite homogeneous tree”) we will always mean a strong subtree of $b^{<\mathbb{N}}$ of infinite (respectively, finite) height for some integer $b \geq 2$. For every, possibly finite, homogeneous tree T by b_T we shall denote the branching number of T . We follow the same conventions for vector trees. In particular, by the term “vector homogeneous tree” (respectively, “finite vector homogeneous tree”) we will always mean a vector strong subtree of $(b_1^{<\mathbb{N}}, \dots, b_d^{<\mathbb{N}})$ of infinite (respectively, finite) height for some integers b_1, \dots, b_d with $b_i \geq 2$ for all $i \in \{1, \dots, d\}$. For every, possibly finite, vector homogeneous tree $\mathbf{T} = (T_1, \dots, T_d)$ we set $b_{\mathbf{T}} = (b_{T_1}, \dots, b_{T_d})$.*

The above convention enables us to effectively enumerate the set of immediate successors of a given non-maximal node of a, possibly finite, homogeneous tree T . Specifically, for every non-maximal $t \in T$ and every $p \in \{0, \dots, b_T - 1\}$ let

$$(11) \quad t^{\wedge^T} p = \text{ImmSucc}_T(t) \cap \text{Succ}_{b_T^{<\mathbb{N}}}(t^{\wedge} p)$$

and notice that

$$(12) \quad \text{ImmSucc}_T(t) = \{t^{\wedge^T} p : p \in \{0, \dots, b_T - 1\}\}.$$

2.5. Canonical isomorphisms and vector canonical isomorphisms. Let T and S be two, possibly finite, homogeneous trees with the same branching number and the same height. The *canonical isomorphism* between T and S is defined to be the unique bijection $\mathbf{I}: T \rightarrow S$ satisfying: (a) $\ell_T(t) = \ell_S(\mathbf{I}(t))$ for every $t \in T$, and (b) $\mathbf{I}(t^{\wedge^T} p) = \mathbf{I}(t)^{\wedge^S} p$ for every non-maximal $t \in T$ and every $p \in \{0, \dots, b_T - 1\}$. Observe that if R is a strong subtree of T , then the image $\mathbf{I}(R)$ of R under the canonical isomorphism is a strong subtree of S and satisfies $L_T(R) = L_S(\mathbf{I}(R))$.

Respectively, let $\mathbf{T} = (T_1, \dots, T_d)$ and $\mathbf{S} = (S_1, \dots, S_d)$ be two, possibly finite, vector homogeneous trees with $b_{\mathbf{T}} = b_{\mathbf{S}}$ and $h(\mathbf{T}) = h(\mathbf{S})$. For every $i \in \{1, \dots, d\}$ let \mathbf{I}_i be the canonical isomorphism between T_i and S_i . The *vector canonical isomorphism* between $\otimes \mathbf{T}$ and $\otimes \mathbf{S}$ is the map $\mathbf{I}: \otimes \mathbf{T} \rightarrow \otimes \mathbf{S}$ defined by the rule

$$(13) \quad \mathbf{I}((t_1, \dots, t_d)) = (\mathbf{I}_1(t_1), \dots, \mathbf{I}_d(t_d)).$$

Notice that the vector canonical isomorphism \mathbf{I} is a bijection.

2.6. Milliken’s theorem. For every finite vector homogeneous tree \mathbf{T} and every integer $1 \leq k \leq h(\mathbf{T})$ by $\text{Str}_k(\mathbf{T})$ we shall denote the set of all vector strong subtrees of \mathbf{T} of height k . We will need the following elementary fact.

Fact 4. Let $d \in \mathbb{N}$ with $d \geq 1$ and $b_1, \dots, b_d \in \mathbb{N}$ with $b_i \geq 2$ for all $i \in \{1, \dots, d\}$. Also let $m \in \mathbb{N}$ and define

$$(14) \quad q(b_1, \dots, b_d, m) = \frac{(\prod_{i=1}^d b_i^{b_i})^{m+1} - (\prod_{i=1}^d b_i)^{m+1}}{\prod_{i=1}^d b_i^{b_i} - \prod_{i=1}^d b_i}.$$

If \mathbf{T} is a finite vector homogeneous tree with $b_{\mathbf{T}} = (b_1, \dots, b_d)$ and of height at least $m+2$, then the cardinality of the set

$$(15) \quad \text{Str}_2(\mathbf{T}, m+1) = \{\mathbf{F} \in \text{Str}_2(\mathbf{T}) : \otimes \mathbf{F}(1) \subseteq \otimes \mathbf{T}(m+1)\}$$

is $q(b_1, \dots, b_d, m)$.

The following partition result is due to Milliken (see [20, Theorem 2.1]).

Theorem 5. For every integer $d \geq 1$, every $b_1, \dots, b_d \in \mathbb{N}$ with $b_i \geq 2$ for all $i \in \{1, \dots, d\}$, every pair of integers $m \geq k \geq 1$ and every integer $r \geq 2$ there exists an integer M with the following property. For every finite vector homogeneous tree \mathbf{T} with $b_{\mathbf{T}} = (b_1, \dots, b_d)$ and of height at least M and every r -coloring of the set $\text{Str}_k(\mathbf{T})$ there exists $\mathbf{S} \in \text{Str}_m(\mathbf{T})$ such that the set $\text{Str}_k(\mathbf{S})$ is monochromatic. The least integer M with this property will be denoted by $\text{Mil}(b_1, \dots, b_d | m, k, r)$.

The original proof of Theorem 5 was ineffective, and as such, could not provide quantitative information on the numbers $\text{Mil}(b_1, \dots, b_d | m, k, r)$. An analysis of the finite version of Milliken's theorem has been carried out recently by Sokić in [29] yielding explicit and reasonable upper bounds. In particular, we have the following refinement of Theorem 5.

Theorem 6. For every integer $k \geq 1$ there exists a primitive recursive function $\phi_k: \mathbb{N}^3 \rightarrow \mathbb{N}$ belonging to the class \mathcal{E}^{5+k} of Grzegorzczuk's hierarchy such that for every integer $d \geq 1$, every $b_1, \dots, b_d \in \mathbb{N}$ with $b_i \geq 2$ for all $i \in \{1, \dots, d\}$, every integer $m \geq k$ and every integer $r \geq 2$ we have

$$(16) \quad \text{Mil}(b_1, \dots, b_d | m, k, r) \leq \phi_k\left(\prod_{i=1}^d b_i^{b_i}, m, r\right).$$

Theorem 6 was not explicitly isolated in [29]. For the convenience of the reader and for completeness, we will sketch the proof in the appendix.

We will also need a certain consequence of Theorem 5. To state it we need, first, to introduce some notation. Specifically, for every finite vector homogeneous tree \mathbf{T} and every integer $1 \leq k \leq h(\mathbf{T})$ we set

$$(17) \quad \text{Str}_k^0(\mathbf{T}) = \{\mathbf{S} \in \text{Str}_k(\mathbf{T}) : \mathbf{S}(0) = \mathbf{T}(0)\}.$$

Corollary 7. Let $d \in \mathbb{N}$ with $d \geq 1$ and $b_1, \dots, b_d \in \mathbb{N}$ with $b_i \geq 2$ for every $i \in \{1, \dots, d\}$. Also let $m, k, r \in \mathbb{N}$ with $m \geq k \geq 1$ and $r \geq 2$. If \mathbf{T} is finite vector homogeneous tree with $b_{\mathbf{T}} = (b_1, \dots, b_d)$ and

$$(18) \quad h(\mathbf{T}) \geq \text{Mil}\left(\underbrace{b_1, \dots, b_1}_{b_1\text{-times}}, \dots, \underbrace{b_d, \dots, b_d}_{b_d\text{-times}} | m, k, r\right) + 1$$

then for every r -coloring of $\text{Str}_{k+1}^0(\mathbf{T})$ there exists $\mathbf{R} \in \text{Str}_{m+1}^0(\mathbf{T})$ such that the set $\text{Str}_{k+1}^0(\mathbf{R})$ is monochromatic.

The reduction of Corollary 7 to Theorem 5 is standard; see, e.g., [19, 20, 33].

2.7. The signature of a subset of a finite homogeneous tree. Let T be a finite homogeneous tree and let D be a subset of T . Following [21], we define the *signature* of D in T to be the set

$$(19) \quad S_T(D) = \{L_T(S) : S \text{ is a strong subtree of } T \text{ with } S \subseteq D\}.$$

Also let

$$(20) \quad w_T(D) = \sum_{n < h(T)} \text{dens}(D \cap T(n)).$$

The following result is due to Pach, Solymosi and Tardos and relates the above defined quantities (see [21, Lemma 3']).

Lemma 8. *Let T be a finite homogeneous tree. Then for every $D \subseteq T$ we have*

$$(21) \quad |S_T(D)| \geq \left(\frac{b_T}{b_T - 1} \right)^{w_T(D)}.$$

Notice that, by Lemma 8, we have

$$(22) \quad \text{UDHL}(b|k, \varepsilon) = O_{b, \varepsilon}(k)$$

for every integer $b \geq 2$, every integer $k \geq 1$ and every real $0 < \varepsilon \leq 1$. The implied constant in (22) can, of course, be estimated efficiently using Chernoff's bound.

2.8. Two Markov-type inequalities. Throughout the paper we will use two elementary variants of Markov's inequality. We isolate them, below, for the convenience of the reader.

Fact 9. *Let $0 < \varepsilon \leq 1$, $N \in \mathbb{N}$ with $N \geq 1$ and a_1, \dots, a_N in $[0, 1]$. Assume that $\mathbb{E}_i a_i \geq \varepsilon$. Then for every $0 < \varepsilon' < \varepsilon$ we have $|\{i \in \{1, \dots, N\} : a_i \geq \varepsilon'\}| \geq (\varepsilon - \varepsilon')N$.*

Fact 10. *Let $0 < \varepsilon \leq 1$, $N \in \mathbb{N}$ with $N \geq 1$ and a_1, \dots, a_N in $[0, 1]$ such that $\mathbb{E}_i a_i \geq \varepsilon$. Also let $\delta > 0$ and assume that $|\{i \in \{1, \dots, N\} : a_i \geq \varepsilon + \delta^2\}| \leq \delta^3 N$. Then $|\{i \in \{1, \dots, N\} : a_i \geq \varepsilon - \delta\}| \geq (1 - \delta)N$.*

3. LEVEL SELECTIONS

We start with the following definition.

Definition 11. *Let \mathbf{T} be a finite vector homogeneous tree and W a homogeneous tree. We say that a map $D: \otimes \mathbf{T} \rightarrow \mathcal{P}(W)$ is a level selection if there exists a subset $L(D) = \{l_0 < \dots < l_{h(\mathbf{T})-1}\}$ of \mathbb{N} , called the level set of D , such that for every integer $n < h(\mathbf{T})$ and every $\mathbf{t} \in \otimes \mathbf{T}(n)$ we have that $D(\mathbf{t}) \subseteq W(l_n)$.*

For every level selection $D: \otimes \mathbf{T} \rightarrow \mathcal{P}(W)$ the height $h(D)$ of D is defined to be the height $h(\mathbf{T})$ of the finite vector homogeneous tree \mathbf{T} . The density $\delta(D)$ of D is the quantity defined by

$$(23) \quad \delta(D) = \min \{ \text{dens}(D(\mathbf{t})) : \mathbf{t} \in \otimes \mathbf{T} \}.$$

Finally, if \mathbf{S} is a vector strong subtree of \mathbf{T} , then by $D \upharpoonright \mathbf{S}$ we shall denote the restriction of the level selection D on $\otimes \mathbf{S}$.

We are ready to state our main result concerning the structure of level selections.

Theorem 12. *For every integer $d \geq 1$, every $b_1, \dots, b_d, b_{d+1} \in \mathbb{N}$ with $b_i \geq 2$ for all $i \in \{1, \dots, d+1\}$, every integer $k \geq 1$ and every real $0 < \varepsilon \leq 1$ there exists an integer N with the following property. If $\mathbf{T} = (T_1, \dots, T_d)$ is a finite vector homogeneous tree with $b_{\mathbf{T}} = (b_1, \dots, b_d)$, W is a homogeneous tree with $b_W = b_{d+1}$ and $D: \otimes \mathbf{T} \rightarrow \mathcal{P}(W)$ is a level selection with $\delta(D) \geq \varepsilon$ and of height at least N , then there exist a vector strong subtree \mathbf{S} of \mathbf{T} and a strong subtree R of W with $h(\mathbf{S}) = h(R) = k$ and such that for every $n \in \{0, \dots, k-1\}$ we have*

$$(24) \quad R(n) \subseteq \bigcap_{\mathbf{s} \in \otimes \mathbf{S}(n)} D(\mathbf{s}).$$

The least integer N with this property will be denoted by $\text{LS}(b_1, \dots, b_{d+1}|k, \varepsilon)$.

Theorem 12 is the main ingredient of the proof of Theorem 3. Its proof will occupy the bulk of this paper and will be given in §9. Let us mention, however, at this point the following simple fact which provides the link between the “uniform density Halpern–Läuchli” numbers and the “level selection” numbers.

Fact 13. *For every integer $d \geq 1$, every $b_1, \dots, b_d, b_{d+1} \in \mathbb{N}$ with $b_i \geq 2$ for all $i \in \{1, \dots, d+1\}$, every integer $k \geq 1$ and every real $0 < \varepsilon \leq 1$ we have*

$$(25) \quad \text{UDHL}(b_1, \dots, b_{d+1}|k, \varepsilon) \leq \text{UDHL}(b_1, \dots, b_d|\text{LS}(b_1, \dots, b_{d+1}|k, \varepsilon/2), \varepsilon/2).$$

Proof. For notational convenience we set $m = \text{LS}(b_1, \dots, b_{d+1}|k, \varepsilon/2)$. We fix a vector homogeneous tree (T_1, \dots, T_d, W) with $b_{T_i} = b_i$ for all $i \in \{1, \dots, d\}$ and $b_W = b_{d+1}$, and we set $\mathbf{T} = (T_1, \dots, T_d)$. Also let L be a finite subset of \mathbb{N} with

$$(26) \quad |L| \geq \text{UDHL}(b_1, \dots, b_d|m, \varepsilon/2)$$

and let D be a subset of the level product of (T_1, \dots, T_d, W) such that

$$(27) \quad |D \cap (T_1(n) \times \dots \times T_d(n) \times W(n))| \geq \varepsilon |T_1(n) \times \dots \times T_d(n) \times W(n)|$$

for every $n \in L$. We need to find a vector strong subtree of (T_1, \dots, T_d, W) of height k whose level product is contained in D .

To this end we argue as follows. For every $n \in L$ we define a subset C_n of $\otimes \mathbf{T}(n)$ by the rule

$$(28) \quad \mathbf{t} \in C_n \Leftrightarrow \text{dens}(\{w \in W(n) : (\mathbf{t}, w) \in D\}) \geq \varepsilon/2$$

and we observe that

$$(29) \quad |C_n| \geq (\varepsilon/2) |\otimes \mathbf{T}(n)|.$$

By (26) and (29), there exists a vector strong subtree \mathbf{Z} of \mathbf{T} with $h(\mathbf{Z}) = m$ and such that $\otimes \mathbf{Z} \subseteq \bigcup_{n \in L} C_n$. Therefore, the “section map” $D: \otimes \mathbf{Z} \rightarrow \mathcal{P}(W)$, defined by $D(\mathbf{z}) = \{w \in W : (\mathbf{z}, w) \in D\}$ for every $\mathbf{z} \in \otimes \mathbf{Z}$, is a level selection of height m and with $\delta(D) \geq \varepsilon/2$. By the choice of m , it is possible to find a vector strong subtree $\mathbf{S} = (S_1, \dots, S_d)$ of \mathbf{Z} and a strong subtree R of W with $h(\mathbf{S}) = h(R) = k$ and satisfying the inclusion in (24) for every $n \in \{0, \dots, k-1\}$. It follows that (S_1, \dots, S_d, R) is a vector strong subtree of (T_1, \dots, T_d, W) of height k whose level product is contained in D . Thus, the proof is completed. \square

4. OUTLINE OF THE ARGUMENT

In this section we will give a detailed outline of the proof of Theorem 3. The first step is given in Fact 13. Indeed, by Fact 13, the task of estimating the “uniform density Halpern–Läuchli” numbers reduces to that of estimating the “level selection” numbers. To achieve this goal, we will follow an inductive procedure which can be schematically described as follows:

$$(30) \quad \left. \begin{array}{ll} \text{UDHL}(b_1, \dots, b_d | \ell, \eta) & \text{for every } \ell \text{ and } \eta \\ \text{LS}(b_1, \dots, b_{d+1} | k, \eta) & \text{for every } \eta \end{array} \right\} \Rightarrow \text{LS}(b_1, \dots, b_{d+1} | k+1, \varepsilon).$$

Precisely, in order to estimate the number $\text{LS}(b_1, \dots, b_{d+1} | k+1, \varepsilon)$ we need to have at our disposal the numbers $\text{UDHL}(b_1, \dots, b_d | \ell, \eta)$ as well as the numbers $\text{LS}(b_1, \dots, b_{d+1} | k, \eta)$ for every integer $\ell \leq \ell_0$ and every $\eta \in [\theta_0, 1]$ where ℓ_0 is a large enough integer and θ_0 is an appropriately chosen positive constant which is very small compared with the given density ε .

Before we proceed to discuss the main arguments of the proof we need to make some important observations. Specifically, let \mathbf{T} be a finite vector homogeneous tree and suppose that we are given a subset A of the level product $\otimes \mathbf{T}$ of \mathbf{T} . We need an effective way to measure the size of the set A where the word “effective” should be interpreted as “taking into account how the set A is distributed along the products of different levels of \mathbf{T} ”. Notice that the uniform probability measure on $\otimes \mathbf{T}$ is rather ineffective in this regard since it is highly concentrated on the products of very few of the last levels of \mathbf{T} . There is, however, a very natural way to overcome this problem, discovered by Furstenberg and Weiss in [10]. Specifically, let

$$(31) \quad \mu_{\mathbf{T}}(A) = \mathbb{E}_{n < h(\mathbf{T})} \frac{|A \cap \otimes \mathbf{T}(n)|}{|\otimes \mathbf{T}(n)|}.$$

Actually, Furstenberg and Weiss considered finite homogeneous trees instead of level products of finite vector homogeneous trees. It is completely straightforward, however, to extend their definition to the higher-dimensional case. The reader should have in mind that, in what follows, when we say that a certain property

holds for “many” or for “almost every” $\mathbf{t} \in \otimes \mathbf{T}$, then we will refer to the probability measure $\mu_{\mathbf{T}}$.

After this preliminary discussion we are ready to comment on the proof of the basic step of the inductive scheme described in (30). So assume that the parameters b_1, \dots, b_{d+1}, k and ε are fixed and that we are given: (i) a finite vector homogeneous tree \mathbf{T} with $b_{\mathbf{T}} = (b_1, \dots, b_d)$, (ii) a homogeneous tree W with $b_W = b_{d+1}$, and (iii) a level selection $D: \otimes \mathbf{T} \rightarrow \mathcal{P}(W)$ with $\delta(D) \geq \varepsilon$ and of sufficiently large height. What we need to find is a vector strong subtree \mathbf{S} of \mathbf{T} and a strong subtree R of W with $h(\mathbf{S}) = h(R) = k + 1$ and such that

$$(32) \quad R(n) \subseteq \bigcap_{\mathbf{s} \in \otimes \mathbf{S}(n)} D(\mathbf{s})$$

for every $n \in \{0, \dots, k\}$.

Let r be a small enough parameter depending on our data b_1, \dots, b_{d+1}, k and ε . The first observation we make is quite standard in proofs of this sort: we can assume that for every $w \in W$ we cannot increase the density of the level selection D to $\varepsilon + r$ by restricting its values to the subtree $\text{Succ}_W(w)$. Indeed, suppose that there exists a node $w \in W$ such that for “many” $\mathbf{t} \in \otimes \mathbf{T}$ the density of the set $D(\mathbf{t})$ relative to the node w is at least $\varepsilon + r$. Then we can find a new level selection D' whose graph is contained in D and such that $\delta(D') \geq \varepsilon + r$. The number of times this can happen is, of course, bounded by $\lceil 1/r \rceil$ until the density reaches 1 and we can finish the proof in a particularly simple way.

Thus, in what follows we can assume that we have “lack of density increment”, or equivalently, that the following concentration hypothesis holds true.

- (H) If \mathbf{Z} is a vector strong subtree of \mathbf{T} of sufficiently large height, then for “almost every” $w \in W$ and for “almost every” $\mathbf{z} \in \otimes \mathbf{Z}$ the density of the set $D(\mathbf{z})$ relative to the node w is roughly ε .

With this information at hand, we devise an algorithm in order to find the desired trees \mathbf{S} and R . The number of times we need to iterate this algorithm is at most $K_0 = \text{UDHL}(b_1, \dots, b_d | 2, \varepsilon / (4b_{d+1}))$, and so, it is *a priori* controlled. Each time we perform the following three basic steps.

Step 1. Suppose that we are at stage $n + 1$. From the previous iteration we will have as an input a vector strong subtree \mathbf{Z}_n of \mathbf{T} , a node $w_n \in W$ and $p_n \in \{0, \dots, b_{d+1} - 1\}$ satisfying certain properties. These properties are used to define a subset A_{n+1} of $\text{Succ}_W(w_n^{\wedge} p_n)$ with $\text{dens}(A_n | w_n^{\wedge} p_n) \geq \varepsilon/8$. We view the set A_{n+1} as an “admissible” subset of W . Next, we find a vector strong subtree \mathbf{V} of \mathbf{Z}_n with $\mathbf{V} \upharpoonright n + 1 = \mathbf{Z}_n \upharpoonright n + 1$ and of sufficiently large height, as well as, a node $w \in A_{n+1}$ such that for every integer $m > n + 1$ and every $\mathbf{v} \in \otimes \mathbf{V}(m)$ the density of the set $D(\mathbf{v})$ relative to *every* immediate successor w' of w is almost ε . This is achieved using hypothesis (H).

Step 2. We perform coloring arguments, using Milliken's theorem, in order to find the desired trees \mathbf{S} and R . If we do not succeed, then we will be able to select a vector strong subtree \mathbf{B} of \mathbf{V} with $\mathbf{B} \upharpoonright n+1 = \mathbf{V} \upharpoonright n+1$ and of sufficiently large height, a subset Γ_{n+1} of $\otimes \mathbf{V}(n+1)$ of cardinality at least $(\varepsilon/4b_{d+1})|\otimes \mathbf{V}(n+1)|$ and $p \in \{0, \dots, b_{d+1} - 1\}$ with the following property. For every $\mathbf{v} \in \Gamma_{n+1}$ and every $\mathbf{S} \in \text{Str}_{k+1}(\mathbf{B})$ with $\mathbf{S}(0) = \mathbf{v}$ the set

$$(33) \quad \bigcup_{n=1}^k \left(\bigcap_{\mathbf{s} \in \otimes \mathbf{S}(n)} D(\mathbf{s}) \cap \text{Succ}_W(w^{\wedge w} p) \right)$$

does not contain a strong subtree of W of height k .

Step 3. We use the aforementioned property to show that the level selection D is rather “thin” when restricted to $\text{Succ}_W(w^{\wedge w} p)$. Specifically, let $\Gamma_0, \dots, \Gamma_{n+1}$ be the sets obtained by Step 2 from all previous iterations. We find a vector strong subtree \mathbf{Z}' of \mathbf{B} with $\mathbf{Z}' \upharpoonright n+1 = \mathbf{B} \upharpoonright n+1$ and of sufficiently large height and satisfying the following. For every $\mathbf{F} \in \text{Str}_2(\mathbf{Z}')$ with $\mathbf{F}(0) \in \Gamma_0 \cup \dots \cup \Gamma_{n+1}$ and $\otimes \mathbf{F}(1) \subseteq \otimes \mathbf{Z}'(n+2)$ the density of the set

$$(34) \quad \bigcap_{\mathbf{t} \in \otimes \mathbf{F}(1)} D(\mathbf{t})$$

relative to the node $w^{\wedge w} p$ is essentially negligible. We set $\mathbf{Z}_{n+1} = \mathbf{Z}'$, $w_{n+1} = w$ and $p_{n+1} = p$ and we go back to Step 1.

If after K_0 iterations the desired trees \mathbf{S} and R have not been found, then using the sets $\{\Gamma_0, \dots, \Gamma_{K_0-1}\}$ obtained by Step 2 we can easily derive a contradiction. Having shown that the above algorithm does locate the trees \mathbf{S} and R , we can analyze each step separately and estimate the number $\text{LS}(b_1, \dots, b_{d+1} | k+1, \varepsilon)$. And as we have already pointed out, this is enough to complete the proof of Theorem 3.

5. STEP 1: OBTAINING STRONG DENSENESS

We start with the following definition. It is a crucial conceptual step towards the proof of Theorem 3.

Definition 14. Let \mathbf{T} be a finite vector homogeneous tree, W a homogeneous tree and $D: \otimes \mathbf{T} \rightarrow \mathcal{P}(W)$ a level selection. Also let \mathbf{S} be a vector strong subtree of \mathbf{T} , $w \in W$ and $0 < \varepsilon \leq 1$.

- (1) We say that D is $(w, \mathbf{S}, \varepsilon)$ -dense provided that $\ell_W(w) \leq \min L(D \upharpoonright \mathbf{S})$ and $\text{dens}(D(\mathbf{s}) | w) \geq \varepsilon$ for every $\mathbf{s} \in \otimes \mathbf{S}$.
- (2) We say that D is $(w, \mathbf{S}, \varepsilon)$ -strongly dense if D is $(w', \mathbf{S}, \varepsilon)$ -dense for every $w' \in \text{ImmSucc}_W(w)$.

5.1. Lack of density increment implies strong denseness. For every $0 < \alpha \leq \beta \leq 1$ and every $0 < \varrho \leq 1$ we set

$$(35) \quad \gamma_0 = \gamma_0(\alpha, \beta, \varrho) = (\beta + \varrho^2 - \alpha)^{1/2},$$

$$(36) \quad \gamma_1 = \gamma_1(\alpha, \beta, \varrho) = (\gamma_0 + \gamma_0^2)^{1/2}$$

and

$$(37) \quad \gamma_2 = \gamma_2(\alpha, \beta, \varrho) = (\gamma_1 + \gamma_1^2)^{1/2}.$$

The following lemma corresponds to the first step of the proof of Theorem 3. It is based on the phenomenon we described in §4, namely that “lack of density increment” implies a strong concentration hypothesis.

Lemma 15. *Let $d \in \mathbb{N}$ with $d \geq 1$ and $b_1, \dots, b_d \in \mathbb{N}$ with $b_i \geq 2$ for every $i \in \{1, \dots, d\}$ and assume that for every integer $n \geq 1$ and every $0 < \eta \leq 1$ the number $\text{UDHL}(b_1, \dots, b_d | n, \eta)$ has been defined.*

Also let $0 < \alpha \leq \beta \leq 1$, $0 < \varrho \leq 1$ and $b, q \in \mathbb{N}$ with $b \geq 2$ and $q \geq 1$ such that

$$(38) \quad \gamma_0 \leq \left(\frac{\alpha}{4qb} \right)^4.$$

Assume that we are given

- (a) *a finite vector homogeneous tree \mathbf{U} with $b_{\mathbf{U}} = (b_1, \dots, b_d)$,*
- (b) *a homogeneous tree W with $b_W = b$,*
- (c) *a node $w_0 \in W$ and $\ell \in \mathbb{N}$ with $\ell_W(w_0) \leq \ell$,*
- (d) *a subset A of $\text{Succ}_W(w_0) \cap W(\ell)$ with $\text{dens}(A | w_0) \geq \beta/8$,*
- (e) *a subset L of \mathbb{N} with $|L| = h(\mathbf{U})$ and $\ell < \min L$, and*
- (f) *for every $j \in \{1, \dots, q\}$ a level selection $D_j: \otimes \mathbf{U} \rightarrow \mathcal{P}(W)$ with $L(D_j) = L$ and which is $(w_0, \mathbf{U}, \alpha)$ -dense.*

Finally, let $N \in \mathbb{N}$ with $N \geq 1$ and suppose that

$$(39) \quad h(\mathbf{U}) \geq \frac{1}{\varrho^3} \text{UDHL}(b_1, \dots, b_d | N, \varrho^3).$$

Then, either

- (i) *there exist a vector strong subtree \mathbf{U}' of \mathbf{U} with $h(\mathbf{U}') = N$, $j_0 \in \{1, \dots, q\}$ and a node $w'_0 \in \text{Succ}_W(w_0) \cap W(\ell + 1)$ such that the level selection D_{j_0} is $(w'_0, \mathbf{U}', \beta + \varrho^2/2)$ -dense, or*
- (ii) *there exist a vector strong subtree \mathbf{U}'' of \mathbf{U} with $h(\mathbf{U}'') = N$ and a node $w''_0 \in A$ such that D_j is $(w''_0, \mathbf{U}'', \alpha - \gamma_0 - \gamma_1 - \gamma_2)$ -strongly dense for every $j \in \{1, \dots, q\}$.*

5.2. Proof of Lemma 15. We will consider four cases. The first three cases imply that alternative (i) holds true while the last one yields alternative (ii). Before we proceed to the details, we isolate for future use the following elementary facts.

$$(P1) \quad \alpha + \gamma_0^2 = \alpha - \gamma_0 + \gamma_1^2 = \alpha - \gamma_0 - \gamma_1 + \gamma_2^2 = \beta + \varrho^2.$$

$$(P2) \quad 0 < \varrho \leq \gamma_0 < \beta/(8bq) < 1/8.$$

$$(P3) \quad \gamma_1 \leq 2\gamma_0^{1/2} \text{ and } \gamma_2 \leq 2\gamma_0^{1/4}.$$

The above properties are straightforward consequences of (35), (36), (37) and (38). Also for every $j \in \{1, \dots, q\}$ and every $w \in \text{Succ}_W(w_0) \cap W(\ell+1)$ we set

$$(40) \quad \Delta_{j,w} = \{\mathbf{u} \in \otimes \mathbf{U} : \text{dens}(D_j(\mathbf{u}) \mid w) \geq \beta + \varrho^2/2\},$$

$$(41) \quad I_{j,w} = \{n < h(\mathbf{U}) : \mathbb{E}_{\mathbf{u} \in \otimes \mathbf{U}(n)} \text{dens}(D_j(\mathbf{u}) \mid w) \geq \beta + \varrho^2\}$$

and

$$(42) \quad K_{j,w} = \{n < h(\mathbf{U}) : |\Delta_{j,w} \cap \otimes \mathbf{U}(n)| \geq \varrho^3 |\otimes \mathbf{U}(n)|\}.$$

After this preliminary discussion, we are ready to distinguish cases.

CASE 1: *there exist $j_0 \in \{1, \dots, q\}$ and a node $w'_0 \in \text{Succ}_W(w_0) \cap W(\ell+1)$ such that $\mathbb{E}_{n < h(\mathbf{U})} \mathbb{E}_{\mathbf{u} \in \otimes \mathbf{U}(n)} \text{dens}(D_{j_0}(\mathbf{u}) \mid w'_0) \geq \beta + \varrho^2$.* Using our hypotheses and applying Fact 9 twice, we see that there exists $\Lambda \subseteq \{0, \dots, h(\mathbf{U}) - 1\}$ with $|\Lambda| \geq (\varrho^2/4)h(\mathbf{U})$ and such that $|\Delta_{j_0,w'_0} \cap \otimes \mathbf{U}(n)| \geq (\varrho^2/4)|\otimes \mathbf{U}(n)|$ for every $n \in \Lambda$. Notice that

$$(43) \quad |\Lambda| \geq \frac{\varrho^2}{4} h(\mathbf{U}) \stackrel{(39)}{\geq} \frac{1}{4\varrho} \text{UDHL}(b_1, \dots, b_d | N, \varrho^3) \stackrel{(P2)}{\geq} \text{UDHL}(b_1, \dots, b_d | N, \varrho^2/4).$$

Therefore, there exists a vector strong subtree \mathbf{U}' of \mathbf{U} with $h(\mathbf{U}') = N$ and such that $\otimes \mathbf{U}' \subseteq \Delta_{j_0,w'_0}$. Hence the level selection D_{j_0} is $(w'_0, \mathbf{U}', \beta + \varrho^2/2)$ -dense, and so this case implies part (i) of the lemma.

CASE 2: *there exist $j_0 \in \{1, \dots, q\}$ and a node $w'_0 \in \text{Succ}_W(w_0) \cap W(\ell+1)$ such that $|I_{j_0,w'_0}| \geq \varrho^3 h(\mathbf{U})$.* By Fact 9, we have $|\Delta_{j_0,w'_0} \cap \otimes \mathbf{U}(n)| \geq (\varrho^2/2)|\otimes \mathbf{U}(n)|$ for every $n \in I_{j_0,w'_0}$. Moreover,

$$(44) \quad |I_{j_0,w'_0}| \geq \varrho^3 h(\mathbf{U}) \stackrel{(39)}{\geq} \text{UDHL}(b_1, \dots, b_d | N, \varrho^3) \stackrel{(P2)}{\geq} \text{UDHL}(b_1, \dots, b_d | N, \varrho^2/2).$$

Arguing as above, we see that this case also implies part (i) of the lemma.

CASE 3: *there exist $j_0 \in \{1, \dots, q\}$ and a node $w'_0 \in \text{Succ}_W(w_0) \cap W(\ell+1)$ such that $|K_{j_0,w'_0}| \geq \varrho^3 h(\mathbf{U})$.* By (39), we have $|K_{j_0,w'_0}| \geq \text{UDHL}(b_1, \dots, b_d | N, \varrho^3)$. Moreover, by the definition of the set K_{j_0,w'_0} in (42), we see that $|\Delta_{j_0,w'_0} \cap \otimes \mathbf{U}(n)| \geq \varrho^3 |\otimes \mathbf{U}(n)|$ for every $n \in K_{j_0,w'_0}$. Thus, this case also implies part (i) of the lemma.

CASE 4: *none of the above cases holds true.* In this case we will show that the second alternative of the lemma is satisfied. It is useful at this point to isolate which hypotheses we have at our disposal. In particular, notice that for every $j \in \{1, \dots, q\}$ and every $w \in \text{Succ}_W(w_0) \cap W(\ell+1)$ we have

- (H1) $\mathbb{E}_{n < h(\mathbf{U})} \mathbb{E}_{\mathbf{u} \in \otimes \mathbf{U}(n)} \text{dens}(D_j(\mathbf{u}) \mid w) < \beta + \varrho^2$,
- (H2) $|I_{j,w}| < \varrho^3 h(\mathbf{U})$ and
- (H3) $|K_{j,w}| < \varrho^3 h(\mathbf{U})$.

We set

$$(45) \quad \alpha_0 = \alpha - \gamma_0$$

and we define $B \subseteq \text{Succ}_W(w_0) \cap W(\ell + 1)$ by the rule

$$(46) \quad w \in B \Leftrightarrow \mathbb{E}_{n < h(\mathbf{U})} \mathbb{E}_{\mathbf{u} \in \otimes \mathbf{U}(n)} \text{dens}(D_j(\mathbf{u}) \mid w) \geq \alpha_0 \text{ for every } j \in \{1, \dots, q\}.$$

Claim 16. *We have $|B| \geq (1 - q\gamma_0)|\text{Succ}_W(w_0) \cap W(\ell + 1)|$.*

Proof of Claim 16. For every $j \in \{1, \dots, q\}$ and every $w \in \text{Succ}_W(w_0) \cap W(\ell + 1)$ we set $\epsilon_{j,w} = \mathbb{E}_{n < h(\mathbf{U})} \mathbb{E}_{\mathbf{u} \in \otimes \mathbf{U}(n)} \text{dens}(D_j(\mathbf{u}) \mid w)$. Also let

$$(47) \quad B_j = \{w \in \text{Succ}_W(w_0) \cap W(\ell + 1) : \epsilon_{j,w} \geq \alpha_0\}.$$

Clearly, it is enough to show that $|B_j| \geq (1 - \gamma_0)|\text{Succ}_W(w_0) \cap W(\ell + 1)|$ for every $j \in \{1, \dots, q\}$. To this end let $j \in \{1, \dots, q\}$ be arbitrary. By (H1), for every node $w \in \text{Succ}_W(w_0) \cap W(\ell + 1)$ we have

$$(48) \quad \epsilon_{j,w} < \beta + \varrho^2 \stackrel{(\mathcal{P}1)}{=} \alpha + \gamma_0^2.$$

On the other hand, the level selection $D_j : \otimes \mathbf{U} \rightarrow \mathcal{P}(W)$ is $(w_0, \mathbf{U}, \alpha)$ -dense. Since the tree W is homogeneous, this yields that

$$(49) \quad \mathbb{E}_{w \in \text{Succ}_W(w_0) \cap W(\ell + 1)} \epsilon_{j,w} = \mathbb{E}_{n < h(\mathbf{U})} \mathbb{E}_{\mathbf{u} \in \otimes \mathbf{U}(n)} \text{dens}(D_j(\mathbf{u}) \mid w_0) \geq \alpha.$$

Combining (48) and (49) and using Fact 10, the result follows. \square

Claim 17. *There exists $w_0'' \in A$ such that $\text{ImmSucc}_W(w_0'') \subseteq B$.*

Proof of Claim 17. We set $F = \{w \in \text{Succ}_W(w_0) \cap W(\ell) : \text{ImmSucc}_W(w) \not\subseteq B\}$. By Claim 16 and using the fact that the branching number of the homogeneous tree W is b , we see that the cardinality of F is at most $bq\gamma_0|\text{Succ}_W(w_0) \cap W(\ell)|$. On the other hand, we have $|A| \geq (\beta/8)|\text{Succ}_W(w_0) \cap W(\ell)|$. Invoking property $(\mathcal{P}2)$, the result follows. \square

Now we set

$$(50) \quad \alpha_1 = \alpha_0 - \gamma_1 \stackrel{(45)}{=} \alpha - \gamma_0 - \gamma_1$$

and we define $\mathcal{N} \subseteq \{0, \dots, h(\mathbf{U}) - 1\}$ by the rule

$$(51) \quad n \in \mathcal{N} \Leftrightarrow \mathbb{E}_{\mathbf{u} \in \otimes \mathbf{U}(n)} \text{dens}(D_j(\mathbf{u}) \mid w) \geq \alpha_1 \text{ for every } j \in \{1, \dots, q\} \\ \text{and every } w \in \text{ImmSucc}_W(w_0'').$$

Claim 18. *We have $|\mathcal{N}| \geq (1 - bq\gamma_1)h(\mathbf{U})$.*

Proof of Claim 18. We will argue as in the proof of Claim 16. Specifically, for every $j \in \{1, \dots, q\}$, every $w \in \text{ImmSucc}_W(w_0'')$ and every integer $n < h(\mathbf{U})$ we set $\epsilon_{j,w,n} = \mathbb{E}_{\mathbf{u} \in \otimes \mathbf{U}(n)} \text{dens}(D_j(\mathbf{u}) | w)$ and $\mathcal{N}_{j,w} = \{n < h(\mathbf{U}) : \epsilon_{j,w,n} \geq \alpha_1\}$. Since the branching number of the homogeneous tree W is b , it is enough to show that $|\mathcal{N}_{j,w}| \geq (1 - \gamma_1)h(\mathbf{U})$ for every $j \in \{1, \dots, q\}$ and every $w \in \text{ImmSucc}_W(w_0'')$. So fix $j \in \{1, \dots, q\}$ and $w \in \text{ImmSucc}_W(w_0'')$. By (H2), we see that

$$(52) \quad |\{n < h(\mathbf{U}) : \epsilon_{j,w,n} \geq \beta + \varrho^2\}| < \varrho^3 h(\mathbf{U}) \stackrel{(\mathcal{P}2), (36)}{\leq} \gamma_1^3 h(\mathbf{U}).$$

By Claim 17, we have $w \in \text{ImmSucc}_W(w_0'') \subseteq B$. Therefore, invoking the definition of the set B given in (46), we obtain that

$$(53) \quad \mathbb{E}_{n < h(\mathbf{U})} \epsilon_{j,w,n} \geq \alpha_0.$$

Finally, by (P1) and (45), we have $\beta + \varrho^2 = \alpha_0 + \gamma_1^2$. Thus, combining the estimates in (52) and (53) and using Fact 10, the result follows. \square

Next we set

$$(54) \quad \alpha_2 = \alpha_1 - \gamma_2 \stackrel{(50)}{=} \alpha - \gamma_0 - \gamma_1 - \gamma_2$$

and

$$(55) \quad \mathcal{N}^* = \mathcal{N} \setminus \left(\bigcup_{j=1}^q \bigcup_{w \in \text{ImmSucc}_W(w_0'')} K_{j,w} \right).$$

Notice that if $n \in \mathcal{N}^*$, then for every $j \in \{1, \dots, q\}$ and every $w \in \text{ImmSucc}_W(w_0'')$ we have that $n \notin K_{j,w}$ and so

$$(56) \quad |\{\mathbf{u} \in \otimes \mathbf{U}(n) : \text{dens}(D_j(\mathbf{u}) | w) \geq \beta + \varrho^2\}| \stackrel{(40)}{\leq} |\Delta_{j,w} \cap \otimes \mathbf{U}(n)| \\ \stackrel{(42)}{\leq} \varrho^3 |\otimes \mathbf{U}(n)| \\ \stackrel{(\mathcal{P}2), (37)}{\leq} \gamma_2^3 |\otimes \mathbf{U}(n)|.$$

Claim 19. *The following hold.*

- (i) *We have $|\mathcal{N}^*| \geq (1 - bq\gamma_1 - bq\varrho^3)h(\mathbf{U})$.*
- (ii) *For every $n \in \mathcal{N}^*$, every $j \in \{1, \dots, q\}$ and every $w \in \text{ImmSucc}_W(w_0'')$ we have that $|\{\mathbf{u} \in \otimes \mathbf{U}(n) : \text{dens}(D_j(\mathbf{u}) | w) \geq \alpha_2\}| \geq (1 - \gamma_2)|\otimes \mathbf{U}(n)|$.*

Proof of Claim 19. By Claim 17 and our assumptions for the set A , we see that $\text{ImmSucc}_W(w_0'') \subseteq \text{Succ}_W(w_0) \cap W(\ell + 1)$. Thus, part (i) follows by Claim 18 and hypothesis (H3). On the other hand, by property (P1) and (50), we have that $\beta + \varrho^2 = \alpha_1 + \gamma_2^2$. Therefore, part (ii) follows by (54), (56) and Fact 10. \square

We are ready for the final step of the argument. Let Δ^* be the subset of $\otimes \mathbf{U}$ defined by the rule

$$(57) \quad \mathbf{u} \in \Delta^* \Leftrightarrow \text{dens}(D_j(\mathbf{u}) | w) \geq \alpha_2 \text{ for every } w \in \text{ImmSucc}_W(w_0'') \\ \text{and every } j \in \{1, \dots, q\}.$$

By part (ii) of Claim 19, we obtain that $|\Delta^* \cap \otimes \mathbf{U}(n)| \geq (1 - bq\gamma_2)|\otimes \mathbf{U}(n)|$ for every $n \in \mathcal{N}^*$. Moreover,

$$(58) \quad 1 - bq\gamma_2 \stackrel{(\mathcal{P}3)}{\geq} 1 - 2bq\gamma_0^{1/4} \stackrel{(38)}{\geq} 1 - 2bq\frac{\alpha}{4bq} \geq \frac{1}{2} \stackrel{(\mathcal{P}2)}{\geq} \varrho^3$$

and, by part (i) of Claim 19,

$$(59) \quad \begin{aligned} |\mathcal{N}^*| &\geq (1 - bq\gamma_1 - bq\varrho^3)h(\mathbf{U}) \stackrel{(\mathcal{P}2), (36)}{\geq} (1 - 2bq\gamma_1)h(\mathbf{U}) \\ &\stackrel{(\mathcal{P}3)}{\geq} (1 - 4bq\gamma_0^{1/2})h(\mathbf{U}) \stackrel{(38)}{\geq} \left(1 - 4bq\left(\frac{\alpha}{4bq}\right)^2\right)h(\mathbf{U}) \\ &\geq \frac{1}{2}h(\mathbf{U}) \stackrel{(\mathcal{P}2)}{\geq} \varrho^3h(\mathbf{U}) \stackrel{(39)}{\geq} \text{UDHL}(b_1, \dots, b_d|N, \varrho^3). \end{aligned}$$

Therefore, there exists a vector strong subtree \mathbf{U}'' of \mathbf{U} with $h(\mathbf{U}'') = N$ and such that $\otimes \mathbf{U}'' \subseteq \Delta^*$. Invoking the definition of Δ^* and (54), we conclude that for every $j \in \{1, \dots, q\}$ the level selection D_j is $(w''_0, \mathbf{U}'', \alpha - \gamma_0 - \gamma_1 - \gamma_2)$ -strongly dense. Thus, the proof of Lemma 15 is completed.

5.3. Consequences. Lemma 15 will be used, later on, in a rather special form. We isolate, below, the exact statement that we need.

Corollary 20. *Let $d \in \mathbb{N}$ with $d \geq 1$ and $b_1, \dots, b_d \in \mathbb{N}$ with $b_i \geq 2$ for every $i \in \{1, \dots, d\}$ and assume that for every integer $n \geq 1$ and every $0 < \eta \leq 1$ the number $\text{UDHL}(b_1, \dots, b_d|n, \eta)$ has been defined.*

Let $0 < \alpha \leq \beta \leq 1$ and $0 < \varrho \leq 1$ and define γ_0, γ_1 and γ_2 as in (35), (36) and (37) respectively. Also let $b, m \in \mathbb{N}$ with $b \geq 2$ and such that

$$(60) \quad \gamma_0 \leq \left(\frac{\alpha}{4(\prod_{i=1}^d b_i)^{m+1}b} \right)^4.$$

Assume that we are given

- (a) *a finite vector homogeneous tree \mathbf{Z} with $b_{\mathbf{Z}} = (b_1, \dots, b_d)$ and $h(\mathbf{Z}) \geq m+2$,*
- (b) *a homogeneous tree W with $b_W = b$,*
- (c) *a node $\tilde{w} \in W$ and $\ell \in \mathbb{N}$ with $\ell_W(\tilde{w}) \leq \ell$,*
- (d) *a subset A of $\text{Succ}_W(\tilde{w}) \cap W(\ell)$ with $\text{dens}(A|\tilde{w}) \geq \beta/8$,*
- (e) *a subset $L = \{l_0 < \dots < l_{h(\mathbf{Z})-1}\}$ of \mathbb{N} such that $l_m = \ell$, and*
- (f) *a level selection $D: \otimes \mathbf{Z} \rightarrow \mathcal{P}(W)$ which is $(\tilde{w}, \text{Succ}_{\mathbf{Z}}(\mathbf{z}), \alpha)$ -dense for every $\mathbf{z} \in \otimes \mathbf{Z}(m+1)$ and with $L(D) = L$.*

Finally, let $N \in \mathbb{N}$ with $N \geq 1$ and suppose that

$$(61) \quad h(\mathbf{Z}) \geq (m+1) + \frac{1}{\varrho^3} \text{UDHL}(b_1, \dots, b_d|N, \varrho^3).$$

Then, either

- (i) *there exist a vector strong subtree \mathbf{V} of \mathbf{Z} with $h(\mathbf{V}) = N$ and a node $w' \in \text{Succ}_W(\tilde{w}) \cap W(\ell+1)$ such that D is $(w', \mathbf{V}, \beta + \varrho^2/2)$ -dense, or*

- (ii) *there exist a vector strong subtree \mathbf{Z}' of \mathbf{Z} with $\mathbf{Z}' \upharpoonright m = \mathbf{Z} \upharpoonright m$ and $h(\mathbf{Z}') = (m+1) + N$ and a node $w'' \in A$ such that the level selection D is $(w'', \text{Succ}_{\mathbf{Z}'}(\mathbf{z}), \alpha - \gamma_0 - \gamma_1 - \gamma_2)$ -strongly dense for every $\mathbf{z} \in \otimes \mathbf{Z}'(m+1)$.*

Proof. Let $\lambda = h(\mathbf{Z}) - (m+1)$ and $q = (\prod_{i=1}^d b_i)^{m+1}$. Notice that $|\otimes \mathbf{Z}(m+1)| = q$. Also we write $\mathbf{Z} = (Z_1, \dots, Z_d)$ and we set $\mathbf{B} = (b_1^{<\lambda}, \dots, b_d^{<\lambda})$.

Let $\mathbf{z} \in \otimes \mathbf{Z}(m+1)$ be arbitrary and write $\text{Succ}_{\mathbf{Z}}(\mathbf{z}) = (Z_1^{\mathbf{z}}, \dots, Z_d^{\mathbf{z}})$. For every $i \in \{1, \dots, d\}$ the finite homogeneous trees $b_i^{<\lambda}$ and $Z_i^{\mathbf{z}}$ have the same branching number and the same height. Therefore, as we described in §2.5, we may consider the canonical isomorphism $\mathbf{I}_i^{\mathbf{z}}: b_i^{<\lambda} \rightarrow Z_i^{\mathbf{z}}$. The same remarks, of course, apply to the finite vector homogeneous trees \mathbf{B} and $\text{Succ}_{\mathbf{Z}}(\mathbf{z})$. Thus we may also consider the vector canonical isomorphism $\mathbf{I}_{\mathbf{z}}: \otimes \mathbf{B} \rightarrow \otimes \text{Succ}_{\mathbf{Z}}(\mathbf{z})$ given by the family of maps $\{\mathbf{I}_i^{\mathbf{z}}: i \in \{1, \dots, d\}\}$ via formula (13). Notice that for every $\mathbf{z}, \mathbf{t} \in \otimes \mathbf{Z}(m+1)$ and every $i \in \{1, \dots, d\}$ if $Z_i^{\mathbf{z}} = Z_i^{\mathbf{t}}$ (that is, if the finite sequences \mathbf{z} and \mathbf{t} agree on the i -th coordinate), then the maps $\mathbf{I}_i^{\mathbf{z}}$ and $\mathbf{I}_i^{\mathbf{t}}$ are identical.

For every $\mathbf{z} \in \otimes \mathbf{Z}(m+1)$ we define a level selection $D_{\mathbf{z}}: \otimes \mathbf{B} \rightarrow \mathcal{P}(W)$ by

$$(62) \quad D_{\mathbf{z}}(\mathbf{u}) = D(\mathbf{I}_{\mathbf{z}}(\mathbf{u})).$$

It is then clear that we may apply Lemma 15 to the family $\{D_{\mathbf{z}}: \mathbf{z} \in \otimes \mathbf{Z}(m+1)\}$.

If the first alternative of the lemma holds true, then we obtain a vector strong subtree \mathbf{U}' of \mathbf{B} of height N , $\mathbf{z}_0 \in \otimes \mathbf{Z}(m+1)$ and a node $w' \in \text{Succ}_W(\bar{w}) \cap W(\ell+1)$ such that the level selection $D_{\mathbf{z}_0}$ is $(w', \mathbf{U}', \beta + \varrho^2/2)$ -dense. We set $\mathbf{V} = \mathbf{I}_{\mathbf{z}_0}(\mathbf{U}')$ and we observe that with this choice part (i) of the corollary is satisfied.

Otherwise, we obtain a vector strong subtree $\mathbf{U}'' = (U_1, \dots, U_d)$ of \mathbf{B} of height N and a node $w'' \in A$ such that $D_{\mathbf{z}}$ is $(w'', \mathbf{U}'', \alpha - \gamma_0 - \gamma_1 - \gamma_2)$ -strongly dense for every $\mathbf{z} \in \otimes \mathbf{Z}(m+1)$. In this case, for every $i \in \{1, \dots, d\}$ let

$$(63) \quad Z'_i = (Z_i \upharpoonright m) \cup \{\mathbf{I}_i^{\mathbf{z}}(U_i): \mathbf{z} \in \otimes \mathbf{Z}(m+1)\}$$

and set $\mathbf{Z}' = (Z'_1, \dots, Z'_d)$. It is easy to check that with this choice part (ii) of the corollary is satisfied. The proof is completed. \square

6. STEP 2: OBTAINING THE SET Γ AND FIXING THE “DIRECTION”

Our goal in this section is to analyze the second step of the proof of Theorem 3. This is, essentially, the content of the following lemma.

Lemma 21. *Let $d \in \mathbb{N}$ with $d \geq 1$ and $b_1, \dots, b_d \in \mathbb{N}$ with $b_i \geq 2$ for every $i \in \{1, \dots, d\}$. Also let $b, q, k \in \mathbb{N}$ with $b \geq 2$ and $q, k \geq 1$.*

Assume that we are given

- (a) *a finite vector homogeneous tree \mathbf{U} with $b_{\mathbf{U}} = (b_1, \dots, b_d)$,*
- (b) *a homogeneous tree W with $b_W = b$,*
- (c) *a node $w_0 \in W$,*
- (d) *a subset L of \mathbb{N} with $|L| = h(\mathbf{U})$, and*

- (e) for every $j \in \{1, \dots, q\}$ a level selection $D_j: \otimes \mathbf{U} \rightarrow \mathcal{P}(W)$ with $L(D_j) = L$ and such that $w_0 \in D_j(\mathbf{U}(0))$.

Finally, let $N \in \mathbb{N}$ with $N \geq k$ and suppose that

$$(64) \quad h(\mathbf{U}) \geq \text{Mil}(\underbrace{b_1, \dots, b_1}_{b_1\text{-times}}, \dots, \underbrace{b_d, \dots, b_d}_{b_d\text{-times}} | N, k, b^q) + 1.$$

Then, either

- (i) there exist $j_0 \in \{1, \dots, q\}$, a vector strong subtree \mathbf{Q} of \mathbf{U} and a strong subtree P of W with $\mathbf{Q}(0) = \mathbf{U}(0)$, $P(0) = w_0$, $h(\mathbf{Q}) = h(P) = k + 1$ and such that

$$(65) \quad P(n) \subseteq \bigcap_{\mathbf{u} \in \otimes \mathbf{Q}(n)} D_{j_0}(\mathbf{u})$$

for every $n \in \{0, \dots, k\}$, or

- (ii) there exist a vector strong subtree \mathbf{U}' of \mathbf{U} with $\mathbf{U}'(0) = \mathbf{U}(0)$ and of height $N + 1$, $p_0 \in \{0, \dots, b - 1\}$ and $\mathcal{J} \subseteq \{1, \dots, q\}$ with $|\mathcal{J}| \geq q/b$ satisfying the following. For $j \in \mathcal{J}$ and every vector strong subtree \mathbf{R} of \mathbf{U}' of height $k + 1$ and with $\mathbf{R}(0) = \mathbf{U}'(0)$ the set

$$(66) \quad \bigcup_{n=1}^k \left(\bigcap_{\mathbf{r} \in \otimes \mathbf{R}(n)} D_j(\mathbf{r}) \right)$$

does not contain a strong subtree of $\text{Succ}_W(w_0^{\wedge^w} p_0)$ of height k .

6.1. Proof of Lemma 21. Assume that part (i) is not satisfied. This has, in particular, the following consequence.

(H): for every $\mathbf{R} \in \text{Str}_{k+1}^0(\mathbf{U})$ and every $j \in \{1, \dots, q\}$ there exists $p \in \{0, \dots, b-1\}$ (depending, possibly, on the choice of \mathbf{R} and j) such that the set

$$(67) \quad \bigcup_{n=1}^k \left(\bigcap_{\mathbf{r} \in \otimes \mathbf{R}(n)} D_j(\mathbf{r}) \right)$$

does not contain a strong subtree of $\text{Succ}_W(w_0^{\wedge^w} p)$ of height k .

This assumption permits us to define a coloring $\mathcal{C}: \text{Str}_{k+1}^0(\mathbf{U}) \rightarrow \{0, \dots, b-1\}^q$ by

$$(68) \quad \mathcal{C}(\mathbf{R}) = (p_j)_{j=1}^q \Leftrightarrow p_j = \min\{p : (\mathbf{H}) \text{ is satisfied for } \mathbf{R}, j \text{ and } p\} \\ \text{for every } j \in \{1, \dots, q\}.$$

By Corollary 7 and (64), there exist $\mathbf{U}' \in \text{Str}_{N+1}^0(\mathbf{U})$ and a finite sequence $(p_j)_{j=1}^q$ in $\{0, \dots, b-1\}$ such that $\mathcal{C}(\mathbf{R}) = (p_j)_{j=1}^q$ for every $\mathbf{R} \in \text{Str}_{k+1}^0(\mathbf{U}')$. By the classical pigeonhole principle, there exist $p_0 \in \{0, \dots, b-1\}$ and a subset \mathcal{J} of $\{1, \dots, q\}$ of cardinality at least q/b such that $p_j = p_0$ for every $j \in \mathcal{J}$. Thus, with these choices, the second alternative holds true. The proof of Lemma 21 is completed.

6.2. Consequences. We will need the following consequence of Lemma 21.

Corollary 22. *Let $d \in \mathbb{N}$ with $d \geq 1$ and $b_1, \dots, b_d \in \mathbb{N}$ with $b_i \geq 2$ for every $i \in \{1, \dots, d\}$. Also let $b, k, m \in \mathbb{N}$ with $b \geq 2$ and $k \geq 1$.*

Assume that we are given

- (a) *a finite vector homogeneous tree \mathbf{Z} with $b_{\mathbf{Z}} = (b_1, \dots, b_d)$ and $h(\mathbf{Z}) \geq m+1$,*
- (b) *a homogeneous tree W with $b_W = b$,*
- (c) *a node $w \in W$,*
- (d) *a nonempty subset Δ of $\otimes \mathbf{Z}(m)$, and*
- (e) *a level selection $D: \otimes \mathbf{Z} \rightarrow \mathcal{P}(W)$ such that $w \in D(\mathbf{z})$ for every $\mathbf{z} \in \Delta$.*

Finally, let $N \in \mathbb{N}$ with $N \geq k$ and suppose that

$$(69) \quad h(\mathbf{Z}) \geq m + \text{Mil}\left(\underbrace{b_1, \dots, b_1}_{b_1\text{-times}}, \dots, \underbrace{b_d, \dots, b_d}_{b_d\text{-times}} \mid N, k, b^{(\prod_{i=1}^d b_i)^m}\right) + 1.$$

Then, either

- (i) *there exist a vector strong subtree \mathbf{S} of \mathbf{Z} and a strong subtree R of W with $\mathbf{S}(0) \in \Delta$, $R(0) = w$, $h(\mathbf{S}) = h(R) = k+1$ and such that*

$$(70) \quad R(n) \subseteq \bigcap_{\mathbf{s} \in \otimes \mathbf{S}(n)} D(\mathbf{s})$$

for every $n \in \{0, \dots, k\}$, or

- (ii) *there exist a vector strong subtree \mathbf{Z}' of \mathbf{Z} with $\mathbf{Z}' \upharpoonright m = \mathbf{Z} \upharpoonright m$ and of height $(m+1) + N$, $p_0 \in \{0, \dots, b-1\}$ and $\Gamma \subseteq \Delta$ with $|\Gamma| \geq (1/b)|\Delta|$ satisfying the following. If \mathbf{R}' is a vector strong subtree of \mathbf{Z}' with $\mathbf{R}'(0) \in \Gamma$ and of height $k+1$, then the set*

$$(71) \quad \bigcup_{n=1}^k \left(\bigcap_{\mathbf{r} \in \otimes \mathbf{R}'(n)} D(\mathbf{r}) \right)$$

does not contain a strong subtree of $\text{Succ}_W(w^{\wedge^w} p_0)$ of height k .

Proof. We will argue as in the proof of Corollary 20. Specifically, let $\lambda = h(\mathbf{Z}) - m$. We write $\mathbf{Z} = (Z_1, \dots, Z_d)$ and we set $\mathbf{B} = (b_1^{<\lambda}, \dots, b_d^{<\lambda})$. For every $\mathbf{z} \in \otimes \mathbf{Z}(m)$ let $\text{Succ}_{\mathbf{Z}}(\mathbf{z}) = (Z_1^{\mathbf{z}}, \dots, Z_d^{\mathbf{z}})$ and notice that $h(\mathbf{B}) = h(\text{Succ}_{\mathbf{Z}}(\mathbf{z})) = \lambda$. Thus, as in the proof of Corollary 20, for every $i \in \{1, \dots, d\}$ we may consider the canonical isomorphism $\text{I}_i^{\mathbf{z}}$ between $b_i^{<\lambda}$ and $Z_i^{\mathbf{z}}$. The vector canonical isomorphism between $\otimes \mathbf{B}$ and $\otimes \text{Succ}_{\mathbf{Z}}(\mathbf{z})$ will be denoted by $\mathbf{I}_{\mathbf{z}}$.

For every $\mathbf{z} \in \Delta$ we define the level selection $D_{\mathbf{z}}: \otimes \mathbf{B} \rightarrow \mathcal{P}(W)$ exactly as we did in (62). By (69) and the estimate

$$(72) \quad |\Delta| \leq |\otimes \mathbf{Z}(m)| = \left(\prod_{i=1}^d b_i \right)^m$$

we obtain that

$$(73) \quad h(\mathbf{B}) \geq \text{Mil}\left(\underbrace{b_1, \dots, b_1}_{b_1\text{-times}}, \dots, \underbrace{b_d, \dots, b_d}_{b_d\text{-times}} | N, k, b^{|\Delta|}\right) + 1.$$

Hence, we may apply Lemma 21 to the family $\{D_{\mathbf{z}} : \mathbf{z} \in \Delta\}$. If the first alternative of the lemma holds true, then it is easily seen that part (i) is satisfied.

Otherwise, we obtain $\mathbf{U}' \in \text{Str}_{N+1}^0(\mathbf{B})$, $p_0 \in \{0, \dots, b-1\}$ and $\Gamma \subseteq \Delta$ of cardinality at least $(1/b)|\Delta|$ such that for every $\mathbf{z} \in \Gamma$ and every $\mathbf{R} \in \text{Str}_{k+1}^0(\mathbf{U}')$ the set

$$(74) \quad \bigcup_{n=1}^k \left(\bigcap_{\mathbf{r} \in \otimes \mathbf{R}(n)} D_{\mathbf{z}}(\mathbf{r}) \right)$$

does not contain a strong subtree of $\text{Succ}_W(w^{\wedge_W} p_0)$ of height k . We have already pointed out in the proof of Corollary 20 that for every $i \in \{1, \dots, d\}$ the family $\{\mathbf{I}_i^{\mathbf{z}} : \mathbf{z} \in \otimes \mathbf{Z}(m)\}$ has the following coherence property: for every $\mathbf{z}, \mathbf{t} \in \otimes \mathbf{Z}(m)$ if the finite sequences \mathbf{z} and \mathbf{t} agree on the i -th coordinate, then the maps $\mathbf{I}_i^{\mathbf{z}}$ and $\mathbf{I}_i^{\mathbf{t}}$ are identical. Therefore, it is possible to select a vector strong subtree \mathbf{Z}' of \mathbf{Z} of height $m + (N + 1)$ with $\mathbf{Z}' \upharpoonright m = \mathbf{Z} \upharpoonright m$ and such that for every $\mathbf{z} \in \Gamma$ we have $\mathbf{I}_{\mathbf{z}}(\mathbf{U}') = \text{Succ}_{\mathbf{Z}'}(\mathbf{z})$. It is then easy to check that part (ii) is satisfied for \mathbf{Z}' , p_0 and Γ . The proof is completed. \square

7. STEP 3: SMALL CORRELATION

This section is devoted to the proof of the following result and its consequences.

Lemma 23. *Let $d \in \mathbb{N}$ with $d \geq 1$ and $b_1, \dots, b_d, b_{d+1} \in \mathbb{N}$ with $b_i \geq 2$ for all $i \in \{1, \dots, d+1\}$. Also let $k \in \mathbb{N}$ with $k \geq 1$ and assume that for every $0 < \eta \leq 1$ the number $\text{LS}(b_1, \dots, b_{d+1} | k, \eta)$ has been defined.*

Let $q \in \mathbb{N}$ with $q \geq 1$ and $0 < \eta_0 \leq 1$. Assume that we are given

- (a) *a finite vector homogeneous tree \mathbf{U} with $b_{\mathbf{U}} = (b_1, \dots, b_d)$,*
- (b) *a homogeneous tree W with $b_W = b_{d+1}$,*
- (c) *a node $w_0 \in W$,*
- (d) *a subset L of \mathbb{N} with $|L| = h(\mathbf{U})$ and $\ell_W(w_0) \leq \min L$, and*
- (e) *for every $j \in \{1, \dots, q\}$ a level selection $D_j : \otimes \mathbf{U} \rightarrow \mathcal{P}(W)$ with $L(D_j) = L$.*

Finally, let $N \in \mathbb{N}$ with $N \geq 1$ and suppose that

$$(75) \quad h(\mathbf{U}) \geq N + \text{Mil}(b_1, \dots, b_d | \text{LS}(b_1, \dots, b_{d+1} | k, \eta_0), 1, q) - 1.$$

Then, either

- (i) *there exist $j_0 \in \{1, \dots, q\}$, a vector strong subtree \mathbf{Q} of \mathbf{U} and a strong subtree P of $\text{Succ}_W(w_0)$ with $h(\mathbf{Q}) = h(P) = k$ and such that*

$$(76) \quad P(n) \subseteq \bigcap_{\mathbf{u} \in \otimes \mathbf{Q}(n)} D_{j_0}(\mathbf{u})$$

for every $n \in \{0, \dots, k-1\}$, or

- (ii) *there exists a vector strong subtree \mathbf{U}' of \mathbf{U} with $h(\mathbf{U}') = N$ and such that $\text{dens}(D_j(\mathbf{U}'(0)) \mid w_0) < \eta_0$ for every $j \in \{1, \dots, q\}$.*

Lemma 23 corresponds to the third step of the proof of Theorem 3. It will be used, however, in a more convenient form which is stated and proved in §7.2.

7.1. Proof of Lemma 23. We set

$$(77) \quad N_0 = \text{LS}(b_1, \dots, b_{d+1} \mid k, \eta_0)$$

and

$$(78) \quad M_0 = \text{Mil}(b_1, \dots, b_d \mid N_0, 1, q).$$

Also let

$$(79) \quad \mathbf{V} = \mathbf{U} \upharpoonright (M_0 - 1).$$

We consider the following cases.

CASE 1: *for every $\mathbf{v} \in \otimes \mathbf{V}$ there exists $j \in \{1, \dots, q\}$ with $\text{dens}(D_j(\mathbf{v}) \mid w_0) \geq \eta_0$.* In this case we will show that the first alternative of the lemma holds true. Specifically, we define a coloring $\mathcal{C}: \otimes \mathbf{V} \rightarrow \{1, \dots, q\}$ by the rule

$$(80) \quad \mathcal{C}(\mathbf{v}) = \min \{j \in \{1, \dots, q\} : \text{dens}(D_j(\mathbf{v}) \mid w_0) \geq \eta_0\}.$$

Since $h(\mathbf{V}) = M_0$, by the choice of M_0 in (78), there exist $j_0 \in \{1, \dots, q\}$ and a vector strong subtree \mathbf{V}' of \mathbf{V} with $h(\mathbf{V}') = N_0$ such that $\mathcal{C}(\mathbf{v}) = j_0$ for every $\mathbf{v} \in \otimes \mathbf{V}'$. We define $D_{j_0}^{w_0}: \otimes \mathbf{V}' \rightarrow \mathcal{P}(\text{Succ}_W(w_0))$ by $D_{j_0}^{w_0}(\mathbf{v}) = D_{j_0}(\mathbf{v}) \cap \text{Succ}_W(w_0)$. It follows that $D_{j_0}^{w_0}$ is a level selection with density at least η_0 and of height N_0 . Therefore, by the choice of N_0 in (77), we conclude that part (i) of the lemma is satisfied.

CASE 2: *there is $\mathbf{v}_0 \in \otimes \mathbf{V}$ such that $\text{dens}(D_j(\mathbf{v}_0) \mid w_0) < \eta_0$ for every $j \in \{1, \dots, q\}$.* By (75) and (79), we have $h(\text{Succ}_{\mathbf{U}}(\mathbf{v}_0)) \geq N$. Thus, part (ii) is satisfied for “ $\mathbf{U}' = \text{Succ}_{\mathbf{U}}(\mathbf{v}_0) \upharpoonright (N - 1)$ ”. The proof of Lemma 23 is completed.

7.2. Consequences. We have the following corollary.

Corollary 24. *Let $d \in \mathbb{N}$ with $d \geq 1$ and $b_1, \dots, b_d, b_{d+1} \in \mathbb{N}$ with $b_i \geq 2$ for all $i \in \{1, \dots, d+1\}$. Also let $k \in \mathbb{N}$ with $k \geq 1$ and assume that for every $0 < \eta \leq 1$ the number $\text{LS}(b_1, \dots, b_{d+1} \mid k, \eta)$ has been defined.*

Let $0 < \eta_0 \leq 1$ and $m \in \mathbb{N}$ and define $q = q(b_1, \dots, b_d, m)$ as in (14). Assume that we are given

- (a) *a finite vector homogeneous tree \mathbf{Z} with $b_{\mathbf{Z}} = (b_1, \dots, b_d)$ and $h(\mathbf{Z}) \geq m+2$,*
- (b) *a homogeneous tree W with $b_W = b_{d+1}$,*
- (c) *a node $\tilde{w} \in W$,*
- (d) *a subset $L = \{l_0 < \dots < l_{h(\mathbf{Z})-1}\}$ of \mathbb{N} with $\ell_W(\tilde{w}) \leq l_{m+1}$,*
- (e) *for every $n \in \{0, \dots, m\}$ a nonempty subset Γ_n of $\otimes \mathbf{Z}(n)$, and*
- (f) *a level selection $D: \otimes \mathbf{Z} \rightarrow \mathcal{P}(W)$ with $L(D) = L$.*

Finally, let $N \in \mathbb{N}$ with $N \geq 1$ and suppose that

$$(81) \quad h(\mathbf{Z}) \geq (m+1) + N + \text{Mil}(b_1, \dots, b_d | \text{LS}(b_1, \dots, b_{d+1} | k, \eta_0), 1, q) - 1.$$

Then, either

- (i) there exist a vector strong subtree \mathbf{S} of \mathbf{Z} with $\mathbf{S}(0) \in \Gamma_0 \cup \dots \cup \Gamma_m$ and $h(\mathbf{S}) = k+1$ and a strong subtree R of $\text{Succ}_W(\tilde{w})$ with $h(R) = k$ such that

$$(82) \quad R(n) \subseteq \bigcap_{\mathbf{s} \in \otimes \mathbf{S}(n+1)} D(\mathbf{s})$$

for every $n \in \{0, \dots, k-1\}$, or

- (ii) there exists a vector strong subtree \mathbf{Z}' of \mathbf{Z} of height $(m+1) + N$ and with $\mathbf{Z}' \upharpoonright m = \mathbf{Z} \upharpoonright m$ and satisfying the following. For every $\mathbf{F}' \in \text{Str}_2(\mathbf{Z}')$ with $\mathbf{F}'(0) \in \Gamma_0 \cup \dots \cup \Gamma_m$ and $\otimes \mathbf{F}'(1) \subseteq \otimes \mathbf{Z}'(m+1)$ we have

$$(83) \quad \text{dens} \left(\bigcap_{\mathbf{z} \in \otimes \mathbf{F}'(1)} D(\mathbf{z}) \mid \tilde{w} \right) < \eta_0.$$

Proof. We will reduce the proof to Lemma 23 using the notion of vector canonical isomorphism exactly as we did in the proofs of Corollary 20 and Corollary 22. The reduction in this case is slightly more involved but the overall strategy is identical.

Specifically, let $\lambda = h(\mathbf{Z}) - (m+1)$. We write $\mathbf{Z} = (Z_1, \dots, Z_d)$ and we set $\mathbf{B} = (b_1^{<\lambda}, \dots, b_d^{<\lambda})$. Also, for every $\mathbf{z} \in \otimes \mathbf{Z}(m+1)$ we set $\text{Succ}_{\mathbf{Z}}(\mathbf{z}) = (Z_1^{\mathbf{z}}, \dots, Z_d^{\mathbf{z}})$. As in the proof of Corollary 20, for every $i \in \{1, \dots, d\}$ by $\mathbf{I}_i^{\mathbf{z}}$ we shall denote the canonical isomorphism between $b_i^{<\lambda}$ and $Z_i^{\mathbf{z}}$. The vector canonical isomorphism between $\otimes \mathbf{B}$ and $\otimes \text{Succ}_{\mathbf{Z}}(\mathbf{z})$ will be denoted by $\mathbf{I}_{\mathbf{z}}$. We define

$$(84) \quad \mathcal{F} = \{ \mathbf{F} \in \text{Str}_2(\mathbf{Z}) : \mathbf{F}(0) \in \Gamma_0 \cup \dots \cup \Gamma_m \text{ and } \otimes \mathbf{F}(1) \subseteq \otimes \mathbf{Z}(m+1) \}.$$

By Fact 4 and the choice of q , we have the estimate

$$(85) \quad |\mathcal{F}| \leq q.$$

For every $\mathbf{F} \in \mathcal{F}$ we define $D_{\mathbf{F}}: \otimes \mathbf{B} \rightarrow \mathcal{P}(W)$ by the rule

$$(86) \quad D_{\mathbf{F}}(\mathbf{u}) = \bigcap_{\mathbf{z} \in \otimes \mathbf{F}(1)} D(\mathbf{I}_{\mathbf{z}}(\mathbf{u})).$$

Notice that $D_{\mathbf{F}}$ is a level selection with $L(D_{\mathbf{F}}) = \{l_{m+1} < \dots < l_{h(\mathbf{Z})-1}\}$. Moreover,

$$(87) \quad \begin{aligned} h(\mathbf{B}) &\stackrel{(81)}{\geq} N + \text{Mil}(b_1, \dots, b_d | \text{LS}(b_1, \dots, b_{d+1} | k, \eta_0), 1, q) - 1 \\ &\stackrel{(85)}{\geq} N + \text{Mil}(b_1, \dots, b_d | \text{LS}(b_1, \dots, b_{d+1} | k, \eta_0), 1, |\mathcal{F}|) - 1. \end{aligned}$$

Therefore, by Lemma 23 applied to the family $\{D_{\mathbf{F}} : \mathbf{F} \in \mathcal{F}\}$, we obtain that one of the following cases is satisfied.

CASE 1: *there exist $\mathbf{G} \in \mathcal{F}$, a vector strong subtree \mathbf{Q} of \mathbf{B} and a strong subtree R of $\text{Succ}_W(\tilde{w})$ with $h(\mathbf{Q}) = h(R) = k$ and such that*

$$(88) \quad R(n) \subseteq \bigcap_{\mathbf{u} \in \otimes \mathbf{Q}(n)} D_{\mathbf{G}}(\mathbf{u})$$

for every $n \in \{0, \dots, k-1\}$. In this case, we will show that the first alternative of the corollary holds true. Specifically, write $\mathbf{G} = (G_1, \dots, G_d)$ and $\mathbf{Q} = (Q_1, \dots, Q_d)$. Since $\mathbf{G} \in \mathcal{F}$ there exists $m_0 \in \{0, \dots, m\}$ such that $\mathbf{G}(0) \in \Gamma_{m_0}$. For every $i \in \{1, \dots, d\}$ we set

$$(89) \quad S_i = G_i(0) \cup \bigcup_{\mathbf{z} \in \otimes \mathbf{G}(1)} \mathbf{I}_i^{\mathbf{z}}(Q_i)$$

and we define

$$(90) \quad \mathbf{S} = (S_1, \dots, S_d).$$

It is easy to see that \mathbf{S} is a vector strong subtree of \mathbf{Z} with $h(\mathbf{S}) = k+1$ and such that $\mathbf{S}(0) = \mathbf{G}(0) \in \Gamma_{m_0}$. Moreover, we have the following fact.

Fact 25. *We have $\otimes \mathbf{S}(n+1) = \{\mathbf{I}_{\mathbf{z}}(\mathbf{u}) : \mathbf{u} \in \otimes \mathbf{Q}(n) \text{ and } \mathbf{z} \in \otimes \mathbf{G}(1)\}$ for every $n \in \{0, \dots, k-1\}$.*

Fact 25 is a rather straightforward consequence of the relevant definitions. Now let $n \in \{0, \dots, k-1\}$ be arbitrary. By (88), (86) and Fact 25, we have

$$(91) \quad R(n) \subseteq \bigcap_{\mathbf{u} \in \otimes \mathbf{Q}(n)} \bigcap_{\mathbf{z} \in \otimes \mathbf{G}(1)} D(\mathbf{I}_{\mathbf{z}}(\mathbf{u})) = \bigcap_{\mathbf{s} \in \otimes \mathbf{S}(n+1)} D(\mathbf{s}).$$

Thus, in this case part (i) of the corollary is satisfied.

CASE 2: *there exists a vector strong subtree \mathbf{U}' of \mathbf{B} with $h(\mathbf{U}') = N$ and such that*

$$(92) \quad \text{dens}(D_{\mathbf{F}}(\mathbf{U}'(0)) \mid \tilde{w}) < \eta_0$$

for every $\mathbf{F} \in \mathcal{F}$. As the reader might have already guessed, we will show that the second alternative of the corollary holds true. To this end write $\mathbf{U}' = (U'_1, \dots, U'_d)$. For every $i \in \{1, \dots, d\}$ we set

$$(93) \quad Z'_i = (Z_i \upharpoonright m) \cup \bigcup_{\mathbf{z} \in \otimes \mathbf{Z}(m+1)} \mathbf{I}_i^{\mathbf{z}}(U'_i)$$

and we define

$$(94) \quad \mathbf{Z}' = (Z'_1, \dots, Z'_d).$$

It is easy to check that \mathbf{Z}' is a vector strong subtree of \mathbf{Z} of height $(m+1) + N$ and such that $\mathbf{Z}' \upharpoonright m = \mathbf{Z} \upharpoonright m$. Also notice that there exists a natural bijection ϕ between $\otimes \mathbf{Z}(m+1)$ and $\otimes \mathbf{Z}'(m+1)$. It is defined by the rule

$$(95) \quad \phi(\mathbf{z}) = \mathbf{I}_{\mathbf{z}}(\mathbf{U}'(0)).$$

Observe that the map ϕ has the following coherence property: if $\mathbf{F}' \in \text{Str}_2(\mathbf{Z}')$ with $\otimes \mathbf{F}'(1) \subseteq \otimes \mathbf{Z}'(m+1)$, then there exists $\mathbf{F} \in \text{Str}_2(\mathbf{Z})$ with $\mathbf{F}'(0) = \mathbf{F}(0)$,

$\otimes \mathbf{F}(1) \subseteq \otimes \mathbf{Z}(m+1)$ and such that $\phi(\otimes \mathbf{F}(1)) = \otimes \mathbf{F}'(1)$. These remarks yield the following fact.

Fact 26. *Let $\mathbf{F}' \in \text{Str}_2(\mathbf{Z}')$ with $\mathbf{F}'(0) \in \Gamma_0 \cup \dots \cup \Gamma_m$ and $\otimes \mathbf{F}'(1) \subseteq \otimes \mathbf{Z}'(m+1)$. Then there exists $\mathbf{F} \in \mathcal{F}$ such that $\otimes \mathbf{F}'(1) = \{\mathbf{I}_{\mathbf{z}}(\mathbf{U}'(0)) : \mathbf{z} \in \otimes \mathbf{F}(1)\}$.*

The vector strong subtree \mathbf{Z}' of \mathbf{Z} satisfies all requirements of part (ii). Indeed, we have already pointed out that $h(\mathbf{Z}') = (m+1) + N$ and $\mathbf{Z}' \upharpoonright m = \mathbf{Z} \upharpoonright m$. Now let $\mathbf{F}' \in \text{Str}_2(\mathbf{Z}')$ with $\mathbf{F}'(0) \in \Gamma_0 \cup \dots \cup \Gamma_m$ and $\otimes \mathbf{F}'(1) \subseteq \otimes \mathbf{Z}'(m+1)$ be arbitrary. By Fact 25, there exists $\mathbf{F} \in \mathcal{F}$ such that $\otimes \mathbf{F}'(1) = \{\mathbf{I}_{\mathbf{z}}(\mathbf{U}'(0)) : \mathbf{z} \in \otimes \mathbf{F}(1)\}$. It follows that

$$(96) \quad \text{dens} \left(\bigcap_{\mathbf{z} \in \otimes \mathbf{F}'(1)} D(\mathbf{z}) \mid \tilde{w} \right) = \text{dens} \left(\bigcap_{\mathbf{z} \in \otimes \mathbf{F}(1)} D(\mathbf{I}_{\mathbf{z}}(\mathbf{U}'(0))) \mid \tilde{w} \right) \\ \stackrel{(86)}{=} \text{dens}(D_{\mathbf{F}}(\mathbf{U}'(0)) \mid \tilde{w}) \stackrel{(92)}{<} \eta_0.$$

The proof of Corollary 24 is completed. \square

8. PERFORMING THE ALGORITHM

Our goal in this section is to perform the algorithm outlined in §4 using the analysis of the three basic steps given in §5, §6 and §7. We recall that this algorithm is part of the inductive scheme described in (30). In particular, we make the following assumptions which will be repeatedly used throughout this section.

Assumptions. *We fix $d \in \mathbb{N}$ with $d \geq 1$, $b_1, \dots, b_d, b_{d+1} \in \mathbb{N}$ with $b_i \geq 2$ for all $i \in \{1, \dots, d+1\}$, $k \in \mathbb{N}$ with $k \geq 1$ and $0 < \varepsilon \leq 1$. We will assume that for every integer $\ell \geq 1$ and every real $0 < \eta \leq 1$ the numbers $\text{UDHL}(b_1, \dots, b_d | \ell, \eta)$ and $\text{LS}(b_1, \dots, b_{d+1} | k, \eta)$ have been defined.*

8.1. Initializing various numerical parameters. First we set

$$(97) \quad K_0 = \text{UDHL}(b_1, \dots, b_d | 2, \varepsilon / (4b_{d+1})).$$

The number K_0 is the number of iterations of the algorithm. Next we set

$$(98) \quad r = \left(\frac{\varepsilon}{16(\prod_{i=1}^d b_i)^{K_0} b_{d+1}} \right)^{2^{3K_0-1}}.$$

The quantity r will be used to control the density increment. Finally let

$$(99) \quad Q_0 = \frac{(\prod_{i=1}^d b_i^{b_i})^{K_0} - (\prod_{i=1}^d b_i)^{K_0}}{\prod_{i=1}^d b_i^{b_i} - \prod_{i=1}^d b_i}$$

and set

$$(100) \quad \theta_0 = \frac{\varepsilon}{8Q_0}.$$

The quantity θ_0 will be used to quantify what “negligible” means in the third step of each iteration.

8.2. Functions that control the height. Each time we perform a basic step of the algorithm we refine the finite vector homogeneous tree that we have as an input to achieve further properties. The height of the resulting vector strong subtree will be controlled by three functions f_1, f_2 and f_3 corresponding to the first, second and third step respectively.

Specifically, we define $f_1: \mathbb{N} \rightarrow \mathbb{N}$ by $f_1(0) = 0$ and

$$(101) \quad f_1(n) = \lceil 1/r^3 \rceil \text{UDHL}(b_1, \dots, b_d | n, r^3)$$

for every integer $n \geq 1$. Next let $f_2: \mathbb{N} \rightarrow \mathbb{N}$ be defined by $f_2(n) = 0$ if $n < k$ and

$$(102) \quad f_2(n) = \text{Mil}\left(\underbrace{b_1, \dots, b_1}_{b_1\text{-times}}, \dots, \underbrace{b_d, \dots, b_d}_{b_d\text{-times}} | n, k, b_{d+1}^{(\prod_{i=1}^d b_i)^{K_0-1}}\right)$$

for every integer $n \geq k$. Also we define $f_3: \mathbb{N} \rightarrow \mathbb{N}$ by

$$(103) \quad f_3(n) = \text{Mil}(b_1, \dots, b_d | \text{LS}(b_1, \dots, b_{d+1} | k, \theta_0), 1, Q_0) + n - 1.$$

Finally, let $g: \mathbb{N} \rightarrow \mathbb{N}$ be defined by the rule

$$(104) \quad g(n) = (f_1 \circ f_2 \circ f_3)(n) + 1.$$

It is, of course, clear that the function g will be used to control the height of the resulting vector strong subtree after all steps have been performed.

8.3. Control of loss of density. Recall that, by Lemma 15, if we have “lack of density increment” then it is always possible to have strong denseness losing just a small amount of density. This basic fact will be repeatedly used while performing the algorithm and is appropriately quantified as follows.

We define $(\delta_n)_{n=0}^{3K_0-1}$ and $(\varepsilon_n)_{n=0}^{K_0}$ recursively by the rule

$$(105) \quad \begin{cases} \delta_0 = r, \\ \delta_{n+1} = (\delta_n + \delta_n^2)^{1/2} \end{cases} \quad \text{and} \quad \begin{cases} \varepsilon_0 = \varepsilon, \\ \varepsilon_{n+1} = \varepsilon_n - (\delta_{3n} + \delta_{3n+1} + \delta_{3n+2}). \end{cases}$$

The sequence $(\delta_n)_{n=0}^{3K_0-1}$ will be used to control the loss of density at each step of the iteration while the sequence $(\varepsilon_n)_{n=0}^{K_0}$ stands for the density left at our disposal.

We will need the following properties satisfied by these sequences.

- (P1) For every $n \in \{0, \dots, 3K_0 - 1\}$ we have $\delta_n \leq 2r^{2^{-n}}$.
- (P2) For every $n \in \{0, \dots, 3K_0 - 2\}$ we have $\sum_{i=0}^n \delta_i = \delta_{n+1}^2 - r^2$.
- (P3) We have $\sum_{n=0}^{3K_0-1} \delta_n \leq \varepsilon/2$.
- (P4) For every $n \in \{0, \dots, K_0 - 1\}$ we have $\varepsilon_{n+1} = \varepsilon - \sum_{i=0}^{3n+2} \delta_i$.
- (P5) For every $n \in \{0, \dots, K_0\}$ we have $\varepsilon/2 \leq \varepsilon_n \leq \varepsilon$.
- (P6) For every $n \in \{0, \dots, K_0 - 1\}$ we have

$$(106) \quad \delta_{3n} \leq \delta_{3K_0-3} \leq \left(\frac{\varepsilon/2}{4(\prod_{i=1}^d b_i)^{K_0} b_{d+1}} \right)^4 \leq \left(\frac{\varepsilon/2}{4(\prod_{i=1}^d b_i)^{n+1} b_{d+1}} \right)^4.$$

The verification of these properties is fairly elementary and is left to the reader.

We simply notice that properties (P3) and (P6) follow by the choice of r in (98).

8.4. The main dichotomy. We are now ready to state the main result in this section which is the last step towards the proof of Theorem 3.

Lemma 27. *Let \mathbf{T} be a finite vector homogeneous tree with $b_{\mathbf{T}} = (b_1, \dots, b_d)$, W a homogeneous tree with $b_W = b_{d+1}$ and $D: \otimes \mathbf{T} \rightarrow \mathcal{P}(W)$ a level selection with $\delta(D) \geq \varepsilon$. Also let $M \in \mathbb{N}$ with $M \geq k$ and assume that*

$$(107) \quad h(\mathbf{T}) \geq g^{(K_0)}(M).$$

Then, either

- (i) *there exist a vector strong subtree \mathbf{T}' of \mathbf{T} with $h(\mathbf{T}') = M$ and $w' \in W$ such that D is $(w', \mathbf{T}', \varepsilon + r^2/2)$ -dense, or*
- (ii) *there exist a vector strong subtree \mathbf{S} of \mathbf{T} and a strong subtree R of W with $h(\mathbf{S}) = h(R) = k + 1$ and such that for every $n \in \{0, \dots, k\}$ we have*

$$(108) \quad R(n) \subseteq \bigcap_{\mathbf{s} \in \otimes \mathbf{S}(n)} D(\mathbf{s}).$$

8.5. Proof of Lemma 27. Assuming that neither (i) nor (ii) are satisfied we will derive a contradiction. In particular, recursively we will construct

- (a) a finite sequence $(\mathbf{Z}_n)_{n=0}^{K_0-1}$ of vector strong subtrees of \mathbf{T} ,
- (b) a finite sequence $(A_n)_{n=0}^{K_0-1}$ of subsets of W ,
- (c) two finite sequences $(w_n)_{n=0}^{K_0-1}$ and $(\tilde{w}_n)_{n=0}^{K_0-1}$ of nodes of W ,
- (d) two finite sequences $(\Delta_n)_{n=0}^{K_0-1}$ and $(\Gamma_n)_{n=0}^{K_0-1}$ of subsets of $\otimes \mathbf{T}$, and
- (e) a strictly increasing finite sequence $(\ell_n)_{n=0}^{K_0-1}$ in \mathbb{N}

such that, setting

$$(109) \quad \mathcal{F}_n = \{\mathbf{F} \in \text{Str}_2(\mathbf{Z}_n) : \mathbf{F}(0) \in \Gamma_0 \cup \dots \cup \Gamma_n \text{ and } \otimes \mathbf{F}(1) \subseteq \otimes \mathbf{Z}_n(n+1)\},$$

for every $n \in \{0, \dots, K_0 - 1\}$ the following conditions are satisfied.

- (C1) We have $\ell_0 = \min L(D)$, $A_0 = D(\mathbf{T}(0))$ and $\Delta_0 = \Gamma_0 = \mathbf{T}(0)$.
- (C2) We have $h(\mathbf{Z}_n) = n + 1 + g^{(K_0-n-1)}(M)$ and $\mathbf{Z}_n(0) = \mathbf{T}(0)$. Moreover, if $n \geq 1$, then $\mathbf{Z}_n \upharpoonright n = \mathbf{Z}_{n-1} \upharpoonright n$.
- (C3) For every $\mathbf{z} \in \otimes \mathbf{Z}_n(n)$ we have $D(\mathbf{z}) \subseteq W(\ell_n)$.
- (C4) The level selection D is $(\tilde{w}_n, \text{Succ}_{\mathbf{Z}_n}(\mathbf{z}), \varepsilon_{n+1})$ -dense for all $\mathbf{z} \in \otimes \mathbf{Z}_n(n+1)$.
- (C5) If $n \geq 1$, then $A_n \subseteq \text{Succ}_W(\tilde{w}_{n-1}) \cap W(\ell_n)$ and $\text{dens}(A_n \upharpoonright \tilde{w}_{n-1}) \geq \varepsilon/8$.
- (C6) If $n \geq 1$, then Δ_n is a subset of $\otimes \mathbf{Z}_{n-1}(n)$ of cardinality $(\varepsilon/4)|\otimes \mathbf{Z}_{n-1}(n)|$.
- (C7) We have $\Gamma_n \subseteq \Delta_n$ with $|\Gamma_n| \geq (1/b_{d+1})|\Delta_n|$.
- (C8) We have

$$(110) \quad \text{dens}\left(\bigcup_{\mathbf{F} \in \mathcal{F}_n} \bigcap_{\mathbf{z} \in \otimes \mathbf{F}(1)} D(\mathbf{z}) \mid \tilde{w}_n\right) \leq \varepsilon/8.$$

- (C9) If $n \geq 1$, then

$$(111) \quad A_n \cap \left(\bigcup_{\mathbf{F} \in \mathcal{F}_{n-1}} \bigcap_{\mathbf{z} \in \otimes \mathbf{F}(1)} D(\mathbf{z})\right) = \emptyset.$$

(C10) We have

$$(112) \quad w_n \in A_n \cap \bigcap_{\mathbf{z} \in \Delta_n} D(\mathbf{z}).$$

(C11) We have $\tilde{w}_n \in \text{ImmSucc}_W(w_n)$.

(C12) For every $\mathbf{R} \in \text{Str}_{k+1}(\mathbf{Z}_n)$ with $\mathbf{R}(0) \in \Gamma_0 \cup \dots \cup \Gamma_n$ the set

$$(113) \quad \bigcup_{j=1}^k \bigcap_{\mathbf{z} \in \otimes \mathbf{R}(j)} D(\mathbf{z})$$

does not contain a strong subtree of $\text{Succ}_W(\tilde{w}_n)$ of height k .

As the first step is identical to the general one, let $n \in \{0, \dots, K_0 - 2\}$ and assume that the construction has been carried out up to n so that the above conditions are satisfied.

Step 1: selection of $\ell_{n+1}, A_{n+1}, w_{n+1}$ and Δ_{n+1} . Let ℓ_{n+1} be the unique element of the level set $L(D)$ of D such that

$$(114) \quad D(\mathbf{z}) \subseteq W(\ell_{n+1})$$

for every $\mathbf{z} \in \otimes \mathbf{Z}_n(n+1)$. For every $w \in W(\ell_{n+1})$ we set

$$(115) \quad \Delta_w = \{\mathbf{z} \in \otimes \mathbf{Z}_n(n+1) : w \in D(\mathbf{z})\}$$

and we define

$$(116) \quad B_{n+1} = \{w \in \text{Succ}_W(\tilde{w}_n) \cap W(\ell_{n+1}) : |\Delta_w| \geq (\varepsilon_{n+1}/2) |\otimes \mathbf{Z}_n(n+1)|\}.$$

By condition (C4), the level selection D is $(\tilde{w}_n, \text{Succ}_{\mathbf{Z}_n}(\mathbf{z}), \varepsilon_{n+1})$ -dense for every $\mathbf{z} \in \otimes \mathbf{Z}_n(n+1)$. Therefore,

$$(117) \quad \text{dens}(B_{n+1} \mid \tilde{w}_n) \geq \varepsilon_{n+1}/2 \stackrel{(\mathcal{P}5)}{\geq} \varepsilon/4.$$

Next we set

$$(118) \quad A_{n+1} = B_{n+1} \setminus \left(\bigcup_{\mathbf{F} \in \mathcal{F}_n} \bigcap_{\mathbf{z} \in \otimes \mathbf{F}(1)} D(\mathbf{z}) \right).$$

Using estimates (110) and (117), we see that with these choices conditions (C5) and (C9) are satisfied.

We proceed to select the node w_{n+1} and the set Δ_{n+1} . This will be done with an appropriate application of Corollary 20. Specifically, we set

$$(119) \quad M_2 = (f_2 \circ f_3)(g^{(K_0-n-2)}(M)).$$

Notice that $M_2 \geq M \geq k \geq 1$. Moreover,

$$\begin{aligned}
 (120) \quad h(\mathbf{Z}_n) &\stackrel{(C2)}{=} (n+1) + g^{(K_0-n-1)}(M) \\
 &\stackrel{(104)}{=} (n+2) + (f_1 \circ f_2 \circ f_3)(g^{(K_0-n-2)}(M)) \\
 &\stackrel{(119)}{=} (n+2) + f_1(M_2) \\
 &\stackrel{(101)}{\geq} (n+2) + \frac{1}{r^3} \text{UDHL}(b_1, \dots, b_d | M_2, r^3).
 \end{aligned}$$

Also,

$$\begin{aligned}
 (121) \quad \gamma_0(\varepsilon_{n+1}, \varepsilon, r) &\stackrel{(36)}{=} (\varepsilon + r^2 - \varepsilon_{n+1})^{1/2} \\
 &\stackrel{(P4)}{=} \left(\sum_{i=0}^{3n+2} \delta_i + r^2 \right)^{1/2} \stackrel{(P2)}{=} \delta_{3(n+1)}.
 \end{aligned}$$

By (37), (38), (105) and the above identity, we see that

$$(122) \quad \gamma_1(\varepsilon_{n+1}, \varepsilon, r) = \delta_{3(n+1)+1} \quad \text{and} \quad \gamma_2(\varepsilon_{n+1}, \varepsilon, r) = \delta_{3(n+1)+2}.$$

Finally, by properties (P5) and (P6), we obtain that

$$(123) \quad \delta_{3(n+1)} \leq \left(\frac{\varepsilon_{n+1}}{4(\prod_{i=1}^d b_i)^{n+2} b_{d+1}} \right)^4.$$

It follows by condition (C4) and the above discussion that we may apply Corollary 20 for “ $\alpha = \varepsilon_{n+1}$ ”, “ $\beta = \varepsilon$ ”, “ $\varrho = r$ ”, “ $\gamma_i = \delta_{3(n+1)+i}$ ” for $i \in \{0, 1, 2\}$, “ $b = b_{d+1}$ ”, “ $m = n+1$ ”, “ $\mathbf{Z} = \mathbf{Z}_n$ ”, “ $\tilde{w} = \tilde{w}_n$ ”, “ $\ell = \ell_{n+1}$ ”, “ $A = A_{n+1}$ ”, “ $D = D \upharpoonright \mathbf{Z}_n$ ” and “ $N = M_2$ ”. (For the first step of the recursive selection we set “ $\mathbf{Z} = \mathbf{T}$ ”, “ $\tilde{w} = W(0)$ ” and “ $m = 1$ ”; the rest of the parameters are chosen *mutatis mutandis* taking into account the choices we made in condition (C1) of the recursive construction.) We have already pointed out that $M_2 \geq M$. Since we have assumed that part (i) of the lemma is not satisfied, we see that the second alternative of Corollary 20 holds true. Therefore, there exist a vector strong subtree \mathbf{V}_1 of \mathbf{Z}_n and a node $w \in A_{n+1}$ such that

- (1a) $h(\mathbf{V}_1) = (n+2) + M_2$,
- (1b) $\mathbf{V}_1 \upharpoonright (n+1) = \mathbf{Z}_n \upharpoonright (n+1)$ and
- (1c) D is $(w, \text{Succ}_{\mathbf{V}_1}(\mathbf{z}), \varepsilon_{n+2})$ -strongly dense for every $\mathbf{z} \in \otimes \mathbf{V}_1(n+2)$.

We set

$$(124) \quad w_{n+1} = w \quad \text{and} \quad \Delta_{n+1} = \Delta_w$$

and we observe that with these choices condition (C6) and (C10) are satisfied. The first step of the recursive selection is completed and, so far, conditions (C5), (C6), (C9) and (C10) are satisfied.

Step 2: selection of Γ_{n+1} and \tilde{w}_{n+1} . In this step we will rely on Corollary 22. Precisely, we set

$$(125) \quad M_1 = f_3(g^{(K_0-n-2)}(M))$$

and we observe that $M_1 \geq M \geq k$. Moreover,

$$(126) \quad \begin{aligned} h(\mathbf{V}_1) &\stackrel{(1a)}{=} (n+2) + M_2 \\ &\stackrel{(119)}{=} (n+2) + (f_2 \circ f_3)(g^{(K_0-n-2)}(M)) \stackrel{(125)}{=} (n+2) + f_2(M_1) \\ &\stackrel{(102)}{\geq} (n+1) + \text{Mil}\left(\underbrace{b_1, \dots, b_1}_{b_1\text{-times}}, \dots, \underbrace{b_d, \dots, b_d}_{b_d\text{-times}} \mid M_1, k, b_{d+1}^{(\prod_{i=1}^d b_i)^{n+1}}\right) + 1. \end{aligned}$$

Using this estimate and the fact that condition (C10) has already been verified for $n+1$, we may apply Corollary 22 for “ $b = b_{d+1}$ ”, “ $m = n+1$ ”, “ $\mathbf{Z} = \mathbf{V}_1$ ”, “ $w = w_{n+1}$ ”, “ $\Delta = \Delta_{n+1}$ ”, “ $D = D \upharpoonright \mathbf{V}_1$ ” and “ $N = M_1$ ”. Recall that, by our assumptions, part (ii) of the lemma is not satisfied. It follows that the second alternative of Corollary 22 holds true. Therefore, there exist a vector strong subtree \mathbf{V}_2 of \mathbf{V}_1 , $p_0 \in \{0, \dots, b_{d+1} - 1\}$ and a subset Γ of Δ_{n+1} such that

- (2a) $h(\mathbf{V}_2) = (n+2) + M_1$,
- (2b) $\mathbf{V}_2 \upharpoonright (n+1) = \mathbf{V}_1 \upharpoonright (n+1)$,
- (2c) $|\Gamma| \geq (1/b_{d+1})|\Delta_{n+1}|$ and
- (2d) for every $\mathbf{R} \in \text{Str}_{k+1}(\mathbf{V}_2)$ with $\mathbf{R}(0) \in \Gamma$ the set

$$(127) \quad \bigcup_{j=1}^k \bigcap_{\mathbf{z} \in \otimes \mathbf{R}(j)} D(\mathbf{z})$$

does not contain a strong subtree of $\text{Succ}_W(w_{n+1}^w p_0)$ of height k .

We set

$$(128) \quad \tilde{w}_{n+1} = w_{n+1}^w p_0 \quad \text{and} \quad \Gamma_{n+1} = \Gamma.$$

and we observe that with these choices conditions (C7) and (C11) are satisfied. The second step of the recursive selection is completed.

Step 3: selection of \mathbf{Z}_{n+1} . As the reader might have already guessed, the selection of \mathbf{Z}_{n+1} will be achieved with the help of Corollary 24. To apply Corollary 24, however, we need to do some preparatory work.

Firstly, we will use our inductive hypotheses to strengthen property (2d) above. Specifically, by (1b) and (2b), we have that \mathbf{V}_2 is a vector strong subtree of \mathbf{Z}_n with $\mathbf{V}_2 \upharpoonright (n+1) = \mathbf{Z}_n \upharpoonright (n+1)$. Moreover, $\tilde{w}_{n+1} \in \text{ImmSucc}_W(w_{n+1})$ and

$$(129) \quad w_{n+1} \in A_{n+1} \stackrel{(118)}{\subseteq} B_{n+1} \stackrel{(116)}{\subseteq} \text{Succ}_W(\tilde{w}_n).$$

Taking into account these remarks and using condition (C12) for \mathbf{Z}_n , we arrive at the following property.

(2e) For every $\mathbf{R} \in \text{Str}_{k+1}(\mathbf{V}_2)$ with $\mathbf{R}(0) \in \Gamma_0 \cup \dots \cup \Gamma_{n+1}$ the set

$$(130) \quad \bigcup_{j=1}^k \bigcap_{\mathbf{z} \in \otimes \mathbf{R}(j)} D(\mathbf{z})$$

does not contain a strong subtree of $\text{Succ}_W(\tilde{w}_{n+1})$ of height k .

Next we set

$$(131) \quad q_n = q(b_1, \dots, b_d, n+1) \stackrel{(14)}{=} \frac{(\prod_{i=1}^d b_i^{b_i})^{n+2} - (\prod_{i=1}^d b_i)^{n+2}}{\prod_{i=1}^d b_i^{b_i} - \prod_{i=1}^d b_i}$$

and we notice that

$$(132) \quad q_n \stackrel{(99)}{\leq} Q_0.$$

Finally, let

$$(133) \quad M_0 = g^{(K_0-n-2)}(M)$$

and observe that

$$(134) \quad \begin{aligned} h(\mathbf{V}_2) &\stackrel{(2a)}{=} (n+2) + M_1 \\ &\stackrel{(125)}{=} (n+2) + f_3(g^{(K_0-n-2)}(M)) \stackrel{(133)}{=} (n+2) + f_3(M_0) \\ &\stackrel{(103)}{=} (n+2) + M_0 + \text{Mil}(b_1, \dots, b_d | \text{LS}(b_1, \dots, b_{d+1} | k, \theta_0), 1, Q_0) - 1 \\ &\stackrel{(132)}{\geq} (n+2) + M_0 + \text{Mil}(b_1, \dots, b_d | \text{LS}(b_1, \dots, b_{d+1} | k, \theta_0), 1, q_n) - 1. \end{aligned}$$

Therefore, we may apply Corollary 24 for “ $\eta_0 = \theta_0$ ”, “ $m = n+1$ ”, “ $\mathbf{Z} = \mathbf{V}_2$ ”, “ $\tilde{w} = \tilde{w}_{n+1}$ ”, the family “ $\{\Gamma_0, \dots, \Gamma_{n+1}\}$ ”, “ $D = D \upharpoonright \mathbf{V}_2$ ” and “ $N = M_0$ ”. The first alternative of Corollary 24 contradicts (2e) isolated above. Thus, there exists a vector strong subtree \mathbf{V}_3 of \mathbf{V}_2 such that

$$(3a) \quad h(\mathbf{V}_3) = (n+2) + M_0 \stackrel{(133)}{=} (n+2) + g^{(K_0-n-2)}(M),$$

$$(3b) \quad \mathbf{V}_3 \upharpoonright (n+1) = \mathbf{V}_2 \upharpoonright (n+1) \text{ and}$$

$$(3c) \quad \text{for every } \mathbf{F} \in \text{Str}_2(\mathbf{V}_3) \text{ with } \mathbf{F}(0) \in \Gamma_0 \cup \dots \cup \Gamma_{n+1} \text{ and } \otimes \mathbf{F}(1) \subseteq \otimes \mathbf{V}_3(n+2) \text{ we have}$$

$$(135) \quad \text{dens}\left(\bigcap_{\mathbf{z} \in \otimes \mathbf{F}(1)} D(\mathbf{z}) \mid \tilde{w}_{n+1}\right) < \theta_0.$$

We set

$$(136) \quad \mathbf{Z}_{n+1} = \mathbf{V}_3$$

and we claim that with this choice all conditions are satisfied. Recall that conditions (C5), (C6), (C7), (C9), (C10) and (C11) have already been verified and so we only have to argue for the rest. Condition (C1) is just an initial assumption. Conditions (C2) and (C3) follow by (114), (1b), (2b), (3a) and (3b). Condition (C4) follows by

(1c) and the fact that \mathbf{Z}_{n+1} is a vector strong subtree of \mathbf{V}_1 . To see that condition (C8) is satisfied notice that, by Fact 4, we have

$$(137) \quad |\mathcal{F}_{n+1}| \stackrel{(14)}{\leq} q(b_1, \dots, b_d, n+1) \stackrel{(99)}{\leq} Q_0.$$

Hence,

$$(138) \quad \text{dens} \left(\bigcup_{\mathbf{F} \in \mathcal{F}_{n+1}} \bigcap_{\mathbf{z} \in \otimes \mathbf{F}(1)} D(\mathbf{z}) \mid \tilde{w}_{n+1} \right) \stackrel{(135), (137)}{\leq} \theta_0 Q_0 \stackrel{(100)}{\leq} \varepsilon/8.$$

Finally, condition (C12) follows by (2e) and the fact that \mathbf{Z}_{n+1} is a vector strong subtree of \mathbf{V}_2 . The recursive selection is completed.

Obtaining the contradiction. We are now in a position to derive the contradiction. By condition (C2), we have $h(\mathbf{Z}_{K_0-1}) \geq K_0$. We set $\mathbf{B} = \mathbf{Z}_{K_0-1} \upharpoonright (K_0 - 1)$. By conditions (C1), (C2), (C6) and (C7), we see that Γ_n is a subset of $\otimes \mathbf{B}(n)$ of cardinality at least $(\varepsilon/4b_{d+1})|\otimes \mathbf{B}(n)|$ for every $n \in \{0, \dots, K_0 - 1\}$. Next observe that $b_{\mathbf{B}} = (b_1, \dots, b_d)$. Hence, by the choice of K_0 in (97), there exist $\mathbf{G} \in \text{Str}_2(\mathbf{B})$ and $0 \leq n_0 < n_1 < K_0$ such that $\mathbf{G}(0) \in \Gamma_{n_0}$ and $\otimes \mathbf{G}(1) \subseteq \Gamma_{n_1}$.

By condition (C2) and the choice of \mathbf{B} , we have that $\mathbf{B} \upharpoonright n_1 = \mathbf{Z}_{n_1-1} \upharpoonright n_1$. Therefore, by (109) and the properties of \mathbf{G} , we see that $\mathbf{G} \in \mathcal{F}_{n_1-1}$. Moreover, by condition (C10), we have $w_{n_1} \in A_{n_1}$. Thus, invoking condition (C9), we conclude

$$(139) \quad w_{n_1} \notin \bigcap_{\mathbf{z} \in \otimes \mathbf{G}(1)} D(\mathbf{z}).$$

On the other hand, however, invoking condition (C10) we obtain that

$$(140) \quad w_{n_1} \in \bigcap_{\mathbf{z} \in \Delta_{n_1}} D(\mathbf{z}) \stackrel{(C7)}{\subseteq} \bigcap_{\mathbf{z} \in \Gamma_{n_1}} D(\mathbf{z}) \subseteq \bigcap_{\mathbf{z} \in \otimes \mathbf{G}(1)} D(\mathbf{z}).$$

This is clearly a contradiction. The proof of Lemma 27 is thus completed.

9. PROOF OF THE MAIN RESULTS

This section is devoted to the proofs of Theorem 3 and Theorem 12. We will give the proof simultaneously for both results following the inductive scheme outlined in §4. As we have already mentioned in §2.7, the numbers $\text{UDHL}(b|k, \varepsilon)$ are defined by Lemma 8. In fact, we have $\text{UDHL}(b|k, \varepsilon) = O_{b, \varepsilon}(k)$.

So let $d \in \mathbb{N}$ with $d \geq 1$ and $b_1, \dots, b_d \in \mathbb{N}$ with $b_i \geq 2$ for all $i \in \{1, \dots, d\}$ and assume that the numbers $\text{UDHL}(b_1, \dots, b_d|\ell, \eta)$ have been defined for every integer $\ell \geq 1$ and every real $0 < \eta \leq 1$. Let $b_{d+1} \in \mathbb{N}$ with $b_{d+1} \geq 2$ be arbitrary. We will define, recursively, the numbers $\text{LS}(b_1, \dots, b_d, b_{d+1}|k, \varepsilon)$ for every integer $k \geq 1$ and every $0 < \varepsilon \leq 1$.

To this end notice that $\text{LS}(b_1, \dots, b_d, b_{d+1}|1, \varepsilon) = 1$. Let $k \in \mathbb{N}$ with $k \geq 1$ and assume that the numbers $\text{LS}(b_1, \dots, b_d, b_{d+1}|k, \eta)$ have been defined for every $0 < \eta \leq 1$. Also let $0 < \varepsilon \leq 1$ be arbitrary. From our data $b_1, \dots, b_d, b_{d+1}, k$ and ε and the inductive assumptions, we may define the integer K_0 , the positive constant

r and the function $g: \mathbb{N} \rightarrow \mathbb{N}$ as in (97), (98) and (104) respectively. By Lemma 27, setting

$$(141) \quad K_1 = K_0 \lceil 2/r^2 \rceil,$$

we see that

$$(142) \quad \text{LS}(b_1, \dots, b_d, b_{d+1} | k+1, \varepsilon) \leq g^{(K_1)}(k+1).$$

Having define the numbers $\text{LS}(b_1, \dots, b_d, b_{d+1} | k+1, \varepsilon)$ for every $0 < \varepsilon \leq 1$, the corresponding numbers $\text{UDHL}(b_1, \dots, b_d, b_{d+1} | k+1, \varepsilon)$ can then be estimated easily using Fact 13. The proofs of Theorem 3 and Theorem 12 are completed.

10. APPENDIX: PROOF OF THEOREM 6

The family of functions $\{\phi_k : k \geq 1\}$ will be defined recursively. Notice that the case “ $k = 1$ ” is just the finite version of the Halpern–Läuchli theorem for vector homogeneous trees. The existence of the corresponding function $\phi_1: \mathbb{N}^3 \rightarrow \mathbb{N}$ follows by [29, Theorem 5]. An essential ingredient for obtaining this bound is the work of Shelah [28] on the “Hales–Jewett” numbers [14]. The relation between the Halpern–Läuchli theorem for vector homogeneous trees and the Hales–Jewett theorem is well understood (see, e.g., [24, 33]) and can be traced in the works of Carlson and Simpson [5], and Carlson [4].

Now let $k \in \mathbb{N}$ with $k \geq 1$ and assume that the function ϕ_k has been defined. To define the function ϕ_{k+1} we need, first, to briefly outline the general inductive step of the proof of Milliken’s theorem emphasizing, in particular, the bounds we get from the argument. The details are fairly standard (see, e.g., [33]) and are left to the reader.

So assume that the numbers $\text{Mil}(b'_1, \dots, b'_{d'} | \ell, k, r')$ have been defined for every integer $d' \geq 1$, every $b'_1, \dots, b'_{d'} \in \mathbb{N}$ with $b'_i \geq 2$ for all $i \in \{1, \dots, d'\}$, every integer $\ell \geq k$ and every integer $r' \geq 1$.

We fix $d \in \mathbb{N}$ with $d \geq 1$ and $b_1, \dots, b_d \in \mathbb{N}$ with $b_i \geq 2$ for all $i \in \{1, \dots, d\}$. Also let $r \in \mathbb{N}$ with $r \geq 1$ be arbitrary. It is convenient to introduce some notation. Specifically, for every finite vector homogeneous tree \mathbf{T} with $h(\mathbf{T}) \geq k+1$ and every integer $n \leq h(\mathbf{T}) - (k+1)$ we set

$$(143) \quad \text{Str}_{k+1}^n(\mathbf{T}) = \{\mathbf{S} \in \text{Str}_{k+1}(\mathbf{T}) : \mathbf{S}(0) \in \otimes \mathbf{T}(n)\}.$$

We have the following.

Lemma 28. *Let $n, N \in \mathbb{N}$ with $N \geq k$. Also let \mathbf{T} be a finite vector homogeneous tree with $b_{\mathbf{T}} = (b_1, \dots, b_d)$ and such that*

$$(144) \quad h(\mathbf{T}) \geq (n+1) + \text{Mil}\left(\underbrace{b_1, \dots, b_1}_{b_1\text{-times}}, \dots, \underbrace{b_d, \dots, b_d}_{b_d\text{-times}} | N, k, r^{(\prod_{i=1}^d b_i)^n}\right).$$

Then for every r -coloring of $\text{Str}_{k+1}^n(\mathbf{T})$ there exists a vector strong subtree \mathbf{S} of \mathbf{T} with $\mathbf{S} \upharpoonright n = \mathbf{T} \upharpoonright n$, $h(\mathbf{S}) = (n+1) + N$ and such that for every $\mathbf{R}, \mathbf{R}' \in \text{Str}_{k+1}^n(\mathbf{S})$ if $\mathbf{R}(0) = \mathbf{R}'(0)$, then \mathbf{R} and \mathbf{R}' have the same color.

Now let $m \in \mathbb{N}$ with $m \geq k+1$ be arbitrary and set

$$(145) \quad M_1(m) = \text{Mil}(b_1, \dots, b_b | m, 1, r).$$

Also let $g: \mathbb{N} \rightarrow \mathbb{N}$ be defined by $g(n) = 0$ if $n < k$ and

$$(146) \quad g(n) = \text{Mil}\left(\underbrace{b_1, \dots, b_1}_{b_1\text{-times}}, \dots, \underbrace{b_d, \dots, b_d}_{b_d\text{-times}} | n, k, r^{(\prod_{i=1}^d b_i)^{M_1(m)-2}}\right) + 1$$

for every $n \geq k$. The following result is based on Lemma 28 and completes the outline of the general inductive step of the proof of Milliken's theorem.

Lemma 29. *We have*

$$(147) \quad \text{Mil}(b_1, \dots, b_d | m, k+1, r) \leq g^{(M_1(m))}(k).$$

We are now ready to define the function ϕ_{k+1} . First we define $\zeta: \mathbb{N}^3 \rightarrow \mathbb{N}$ by

$$(148) \quad \zeta(b, m, r) = \phi_1(b, m, r).$$

Also let $\omega: \mathbb{N}^3 \rightarrow \mathbb{N}$ be defined by

$$(149) \quad \omega(b, m, r) = r^{b^{\zeta(b, m, r)}}.$$

Using the fact that the function ϕ_1 belongs to the class \mathcal{E}^6 and elementary properties of primitive recursive functions (see, e.g., [26]), it is easy to check that ζ and ω belong to the class \mathcal{E}^6 . Next we define $f: \mathbb{N}^3 \rightarrow \mathbb{N}$ by

$$(150) \quad f(b, m, r) = \phi_k(b, m, \omega(b, m, r)) + 1.$$

By our inductive assumptions, the function ϕ_k belongs to the class \mathcal{E}^{5+k} . Hence, so does f . Finally, define $\psi: \mathbb{N}^4 \rightarrow \mathbb{N}$ recursively by the rule

$$(151) \quad \begin{cases} \psi(0, b, m, r) = m, \\ \psi(i+1, b, m, r) = f(b, \psi(i, b, m, r), r). \end{cases}$$

and set

$$(152) \quad \phi_{k+1}(b, m, r) = \psi(\zeta(b, m, r), b, k, r).$$

The function ϕ_{k+1} is the desired one. Indeed, notice that ϕ_{k+1} belongs to the class \mathcal{E}^{5+k+1} . Moreover, using Lemma 29, it is easily checked that

$$(153) \quad \text{Mil}(b_1, \dots, b_d | m, k+1, r) \leq \phi_{k+1}\left(\prod_{i=1}^d b_i^{b_i}, m, r\right).$$

The proof of Theorem 6 is completed.

REFERENCES

- [1] S. A. Argyros, P. Dodos and V. Kanellopoulos, *Unconditional families in Banach spaces*, Math. Annalen 341 (2008), 15–38.
- [2] R. Bicker and B. Voigt, *Density theorems for finitistic trees*, Combinatorica 3 (1983), 305–313.
- [3] A. Blass, *A partition theorem for perfect sets*, Proc. Amer. Math. Soc. 82 (1981), 271–277.
- [4] T. J. Carlson, *Some unifying principles in Ramsey Theory*, Discrete Math. 68 (1988), 117–169.
- [5] T. J. Carlson and S. G. Simpson, *A dual form of Ramsey’s theorem*, Adv. Math. 53 (1984), 265–290.
- [6] P. Dodos, *Operators whose dual has non-separable range*, J. Funct. Anal. 260 (2011), 1285–1303.
- [7] P. Dodos, V. Kanellopoulos and N. Karagiannis, *A density version of the Halpern–Läuchli theorem*, Adv. Math. 244 (2013), 955–978.
- [8] P. Dodos, V. Kanellopoulos and K. Tyros, *Measurable events indexed by products of trees*, Combinatorica 34 (2014), 427–470.
- [9] H. Furstenberg and Y. Katznelson, *A density version of the Hales–Jewett theorem*, Journal d’Anal. Math. 57 (1991), 64–119.
- [10] H. Furstenberg and B. Weiss, *Markov processes and Ramsey theory for trees*, Comb. Probab. Comput. 12 (2003), 547–563.
- [11] F. Galvin, *Partition theorems for the real line*, Notices Amer. Math. Soc. 15 (1968), 660.
- [12] W. T. Gowers, *Hypergraph regularity and the multidimensional Szemerédi theorem*, Ann. Math. 166 (2007), 897–946.
- [13] R. L. Graham, B. L. Rothschild and J. H. Spencer, *Ramsey Theory*, 2nd edition, John Wiley & Sons, 1980.
- [14] A. H. Hales and R. I. Jewett, *Regularity and positional games*, Trans. Amer. Math. Soc. 106 (1963), 222–229.
- [15] J. D. Halpern and H. Läuchli, *A partition theorem*, Trans. Amer. Math. Soc. 124 (1966), 360–367.
- [16] V. Kanellopoulos, *Ramsey families of subtrees of the dyadic tree*, Trans. Amer. Math. Soc. 357 (2005), 3865–3886.
- [17] R. Laver, *Products of infinitely many perfect trees*, J. London Math. Soc. 29 (1984), 385–396.
- [18] A. Louveau, S. Shelah and B. Veličković, *Borel partitions of infinite trees of a perfect tree*, Annals Pure and Applied Logic 63 (1993), 271–281.
- [19] K. Milliken, *A Ramsey theorem for trees*, J. Comb. Theory, Ser. A 26 (1979), 215–237.
- [20] K. Milliken, *A partition theorem for the infinite subtrees of a tree*, Trans. Amer. Math. Soc. 263 (1981), 137–148.
- [21] J. Pach, J. Solymosi and G. Tardos, *Remarks on a Ramsey theory for trees*, Combinatorica 32 (2012), 473–482.
- [22] J. B. Paris and L. A. Harrington, *A mathematical incompleteness in Peano arithmetic*, in “Handbook for Mathematical Logic”, North-Holland, Amsterdam (1977), 1133–1142.
- [23] D. H. J. Polymath, *A new proof of the density Hales–Jewett theorem*, Ann. Math. 175 (2012), 1283–1327.
- [24] H. J. Prömel and B. Voigt, *Graham–Rothschild parameter sets*, in “Mathematics of Ramsey Theory”, Springer-Verlag, Berlin (1990), 113–149.
- [25] V. Rödl, B. Nagle, J. Skokan, M. Schacht and Y. Kohayakawa, *The hypergraph regularity method and its applications*, Proc. Nat. Acad. Sci. USA 102 (2005), 8109–8113.
- [26] H. E. Rose, *Subrecursion: functions and hierarchies*, Oxford Logic Guide 9, Oxford Univ. Press, Oxford, 1984.
- [27] K. F. Roth, *On certain sets of integers*, J. London Math. Soc. 28 (1953), 104–109.

- [28] S. Shelah, *Primitive recursive bounds for van der Waerden numbers*, J. Amer. Math. Soc. 1 (1988), 683–697.
- [29] M. Sokić, *Bounds on trees*, Discrete Math. 311 (2011), 398–407.
- [30] J. Stern, *A Ramsey theorem for trees with an application to Banach spaces*, Israel J. Math. 29 (1978), 179–188.
- [31] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. 27 (1975), 199–245.
- [32] S. Todorcevic, *Compact subsets of the first Baire class*, J. Amer. Math. Soc. 12 (1999), 1179–1212.
- [33] S. Todorcevic, *Introduction to Ramsey Spaces*, Annals Math. Studies, No. 174, Princeton Univ. Press, 2010.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ATHENS, PANEPISTIMIOPOLIS 157 84, ATHENS, GREECE

E-mail address: `pdodos@math.uoa.gr`

NATIONAL TECHNICAL UNIVERSITY OF ATHENS, FACULTY OF APPLIED SCIENCES, DEPARTMENT OF MATHEMATICS, ZOGRAFOU CAMPUS, 157 80, ATHENS, GREECE

E-mail address: `bkanel@math.ntua.gr`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, CANADA M5S 2E4

E-mail address: `k.tyros@utoronto.ca`