PARTITION IDEALS BELOW $\aleph_\omega$

PANDELIS DODOS, JORDI LOPEZ-ABAD AND STEVO TODORCEVIC

Abstract. Motivated by an application to the unconditional basic sequence problem appearing in our previous paper [DLT], we introduce analogues of the Laver ideal on $\aleph_2$ ([GJM]) living on index sets of the form $[\aleph_k]^\omega$ and use this to refine the well-known high-dimensional polarized partition relation for $\aleph_\omega$ of Shelah [Sh2].

1. Introduction

The results of this paper were originally motivated by the general unconditional basic sequence problem asking under which conditions a given Banach space must contain an infinite unconditional basic sequence (see [LT, page 27]). More precisely, in our previous paper [DLT], we were interested in properties of cardinals κ that guarantee that Banach spaces of densities at least κ must contain an infinite unconditional basic sequence. The first advance on this problem is given in a paper of Ketonen [Ke] which shows that if a density of a given Banach space E is greater than or equal to the ω-Erdős cardinal $\kappa(\omega)$ (see Section 2.2), then E contains an infinite unconditional basic sequence. Let $\mathfrak{nc}$ be the minimal cardinal λ such that every Banach space of density at least λ contains an infinite unconditional basic sequence. Then Ketonen's result can be restated as saying that $\kappa(\omega) \geqslant \mathfrak{nc}$. Since $\kappa(\omega)$ is a considerably large cardinal (strongly inaccessible and more) we wanted to determine if $\mathfrak{nc}$ is really a large cardinal or not. In particular, we wanted to know if $\exp_\omega(\aleph_0)$ could be an upper bound of $\mathfrak{nc}$ or not. In fact, in our previous paper [DLT], we managed to establish that indeed the inequality $\aleph_\omega \geqslant \mathfrak{nc}$ and the GCH are jointly consistent with the usual axioms of set theory. The consistency proof relies heavily on a Ramsey-theoretic property of $\exp_\omega(\aleph_0)$ established in a previous work of Shelah [Sh2] (see also [Mi]). Having in mind similar further applications, in this paper we refine Shelah's work by adding to it a feature present in Laver's well-known forcing construction of a normal ideal $\mathcal{I}$ on ω_2 for which the quotient algebra $\mathcal{P}(\omega_2)/\mathcal{I}$ contains a σ-closed dense subset (see [GJM, Theorem 4 and Remark (4)]). We find ideals $\mathcal{I}$ on index sets of the form $[\omega_k]^\omega$ such that the corresponding co-ideal $\mathcal{I}^+ = \mathcal{P}([\omega_k]^\omega) \setminus \mathcal{I}$ is at the same time rich enough to contain collections of homogeneous sets for any given partition of the form $f\colon [\omega_k]^d \to \omega_l$ for some fixed dimension d, and is sufficiently complete, in the sense that it contains a dense

2000 *Mathematics Subject Classification.* Primary 03E35; Secondary 03E02, 03E55.
Key words: forcing, polarized Ramsey property, strongly compact cardinal.

subset that is ω_m-closed for large enough $m < k$. We state the precise version of our result in the case $d = 2$.

Theorem 1. *Assuming the existence of infinitely many strongly compact cardinals there is a forcing extension of the universe satisfying the* GCH[1] *and for every positive integer k an ideal $\mathcal{I}_k$ on $[(2^{\aleph_{5k}})^+]^\omega$ and a subset $\mathcal{D}_k$ of $\mathcal{I}_k^+$ such that the following are satisfied.*

(P1) *$\mathcal{I}_k$ is $\aleph_{5k}$-complete.*
(P2) *$\mathcal{D}_k$ is dense and $\aleph_{5k}$-closed in $\mathcal{I}_k^+$; that is, every element of $\mathcal{I}_k^+$ includes an element of $\mathcal{D}_k$ and every decreasing sequence of elements of $\mathcal{D}_k$ of length less than $\aleph_{5k}$ has a lower bound in $\mathcal{I}_k^+$.*
(P3) *For every $\mu < \aleph_{5k}$, every coloring $c\colon [(2^{\aleph_{5k}})^+]^2 \to \mu$ and every $A \in \mathcal{I}_k^+$ there exist a color $\xi < \mu$ and an element $D \in \mathcal{D}_k$ with $D \subseteq A$ and such that for every $\mathbf{x} \in D$ the restriction $c \restriction [\mathbf{x}]^2$ is constantly equal to ξ.*

We have a similar conclusion for all other finite dimensions d true in the same forcing extension of the universe of sets containing infinitely many strongly compact cardinals. The point of involving co-ideals with highly closed dense subsets is that one has the opportunity to combine results of various Ramsey applications at different levels k into a single sequence of sets simultaneously homogeneous for larger and larger families of partitions. Our proof of Proposition 5 below is a good example that shows this advantage.

2. Preliminaries

Our set theoretic terminology and notation is standard and follows texts like [Ku]. For example, we assume the reader is familiar with the following standard set theoretic notions that we freely use in the rest of the paper.

2.1. **Ideals on Fields of Sets.** Let X be a nonempty set. An *ideal* $\mathcal{I}$ on X is a collection of subsets of X satisfying the following conditions.

(i) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
(ii) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

If $\mathcal{I}$ is an ideal on X and κ is a cardinal, then we say that $\mathcal{I}$ is *κ-complete* if for every $\lambda < \kappa$ and every sequence $(A_\xi : \xi < \lambda)$ in $\mathcal{I}$ we have $\bigcup_{\xi<\lambda} A_\xi \in \mathcal{I}$.

A subset A of X is said to be *positive* with respect to an ideal $\mathcal{I}$ if $A \notin \mathcal{I}$. The set of all positive sets with respect to $\mathcal{I}$ is denoted by $\mathcal{I}^+$. If $\mathcal{D}$ is a subset of $\mathcal{I}^+$ and κ is a cardinal, then we say that $\mathcal{D}$ is *κ-closed in* $\mathcal{I}^+$ if for every $\lambda < \kappa$ and every decreasing sequence $(D_\xi : \xi < \lambda)$ in $\mathcal{D}$ we have $\bigcap_{\xi<\kappa} D_\xi \in \mathcal{I}^+$. We also say that such a set $\mathcal{D}$ is *dense* in $\mathcal{I}^+$ if for every $A \in \mathcal{I}^+$ there exists $D \in \mathcal{D}$ with $D \subseteq A$.

[1]This makes $(2^{\aleph_{5k}})^+ = \aleph_{5k+2}$ but in the conclusion (P3) we use the exponential notation for a reason that will be clear later on when we define the partition ideals.

If $\mathcal{F}$ is a filter on X, then the family $\{X \setminus A : A \in \mathcal{F}\}$ is an ideal. Having in mind this correspondence, we will continue to use the above terminology for the filter $\mathcal{F}$. Notice that if the given filter is actually an ultrafilter $\mathcal{U}$, then, setting $\mathcal{I} = \mathcal{P}(X) \setminus \mathcal{U}$, we have that $\mathcal{I}^+ = \mathcal{U}$.

2.2. **Large Cardinals.** Let θ be a cardinal.

(a) θ is said to be *inaccessible* if it is regular and strong limit; that is, $2^\lambda < \theta$ for every $\lambda < \theta$.
(b) An α-*Erdős* cardinal, usually denoted by $\kappa(\alpha)$ if exists, is the minimal cardinal λ such that $\lambda \to (\alpha)^{<\omega}_2$; that is, λ is the least cardinal with the property that for every coloring $c\colon [\lambda]^{<\omega} \to 2$ there is $H \subseteq \lambda$ of order-type α such that c is constant on $[H]^n$ for every $n < \omega$. A cardinal λ that is λ-Erdős (in other words, a cardinal λ which has the partition property $\lambda \to (\lambda)^{<\omega}_2$) is called a *Ramsey* cardinal.
(c) θ is said to be *measurable* if there exists a θ-complete normal ultrafilter $\mathcal{U}$ on θ. Looking at the ultrapower of the universe using $\mathcal{U}$ one can observe that the set $\{\lambda < \theta : \lambda \text{ is inaccessible}\}$ belongs to $\mathcal{U}$. Similarly, one shows that set $\{\lambda < \theta : \lambda \text{ is Ramsey}\}$ belongs to $\mathcal{U}$.
(d) θ is said to be *strongly compact* if every θ-complete filter can be extended to a θ-complete ultrafilter.

Finally, for every cardinal κ and every $n \in \omega$ we define recursively the cardinal $\exp_n(\kappa)$ by the rule $\exp_0(\kappa) = \kappa$ and $\exp_{n+1}(\kappa) = 2^{\exp_n(\kappa)}$.

2.3. **The Lévy Collapse.** Let λ be a regular infinite cardinal and let $\kappa > \lambda$ be an inaccessible cardinal. By $\mathrm{Col}(\lambda, <\kappa)$ we shall denote the set of all partial mappings p satisfying the following.

(i) $\mathrm{dom}(p) \subseteq \lambda \times \kappa$ and $\mathrm{range}(p) \subseteq \kappa$.
(ii) $|p| < \lambda$.
(iii) For every $(\alpha, \beta) \in \mathrm{dom}(p)$ with $\beta > 0$ we have $p(\alpha, \beta) < \beta$.

We equip the set $\mathrm{Col}(\lambda, <\kappa)$ with the partial order $\leqslant$ defined by

$$p \leqslant q \Leftrightarrow \mathrm{dom}(q) \subseteq \mathrm{dom}(p) \text{ and } p \upharpoonright \mathrm{dom}(q) = q.$$

If p and q is a pair in $\mathrm{Col}(\lambda, <\kappa)$, then by $p \parallel q$ we denote the fact that p and q are *compatible* (that is, there exists r in $\mathrm{Col}(\lambda, <\kappa)$ with $r \leqslant p$ and $r \leqslant q$), while by $p \perp q$ we denote the fact that p and q are *incompatible*.

We will need the following well-known properties of the Lévy collapse (see, for instance, [Ka]). In what follows, G will be a $\mathrm{Col}(\lambda, <\kappa)$-generic filter.

(a) The generic filter G is λ-complete (this is a consequence of the fact that the forcing $\mathrm{Col}(\lambda, <\kappa)$ is λ-closed).
(b) $\mathrm{Col}(\lambda, <\kappa)$ has the κ-cc (this follows from the fact that the cardinal κ is inaccessible).
(c) In $V[G]$ we have $\kappa = \lambda^+$.

(d) In $V[G]$ the sets ${}^{\kappa}2$ and ${}^{\kappa}2 \cap V$ are equipotent.

Finally, let us introduce some pieces of notation (actually, these pieces of notation will be used only in Section 4). For every $p \in \mathrm{Col}(\lambda, <\kappa)$ and every $\alpha < \kappa$ by $p \upharpoonright \alpha$ we shall denote the restriction of the partial map p to $\mathrm{dom}(p) \cap (\lambda \times \alpha)$. Moreover, for every $p \in \mathrm{Col}(\lambda, <\kappa)$ we set $(\mathrm{dom}(p))_1 := \{\alpha < \kappa : \exists \xi < \lambda \text{ with } (\xi, \alpha) \in \mathrm{dom}(p)\}$.

3. Co-ideals and polarized partition relations

We start by recalling the following polarized partition property, originally appearing in the problem lists of Erdős and Hajnal [EH1], [EH2, Problem 29] (see also [Sh2]).

Definition 2. *Let κ be a cardinal and $d \in \omega$ with $d \geqslant 1$. By $\mathrm{Pl}_d(\kappa)$ we shall denote the combinatorial principle asserting that for every coloring $c\colon \big[[\kappa]^d\big]^{<\omega} \to \omega$ there exists a sequence $(\mathbf{x}_n)$ of infinite disjoint subsets of κ such that for every $m \in \omega$ the restriction $c \upharpoonright \prod_{n=0}^{m} [\mathbf{x}_n]^d$ is constant.*

Clearly property $\mathrm{Pl}_d(\kappa)$ implies property $\mathrm{Pl}_{d'}(\kappa)$ for any cardinal κ and any pair $d, d' \in \omega$ with $d \geqslant d' \geqslant 1$. From known results one can easily deduce that the principle $\mathrm{Pl}_d(\exp_{d-1}(\aleph_0)^{+n})$ is false for every $n \in \omega$ and every integer $d \geqslant 1$ (see, for instance, [EHMR, CDPM, DT]). Thus, the minimal cardinal κ for which $\mathrm{Pl}_d(\kappa)$ could possibly be true is $\exp_{d-1}(\aleph_0)^{+\omega}$. Indeed, Di Prisco and Todorcevic [DT] have established the consistency of $\mathrm{Pl}_1(\aleph_\omega)$ relative the consistency of a single measurable cardinal, an assumption that also happens to be optimal. On the other hand, Shelah [Sh2] was able to establish that GCH and principles $\mathrm{Pl}_d(\aleph_\omega)$ $(d \geqslant 1)$ are jointly consistent, relative to the consistency of GCH and the existence of an infinite sequence of strongly compact cardinals.

It turns out that Shelah's forcing construction can be refined by adding to it a feature present in Laver's well-known construction of a normal ideal $\mathcal{I}$ on ω_2 for which the quotient algebra $\mathcal{P}(\omega_2)/\mathcal{I}$ contains a σ-closed dense subset (see, [GJM, Theorem 4 and Remark(4)]). The key to this refinement is contained in the following lemma whose proof (given in the final section of the paper) combines the ideas of Laver and Shelah.

Lemma 3. *Suppose that κ is a strongly compact cardinal and that $\lambda < \kappa$ is an infinite regular cardinal. Let G be a $\mathrm{Col}(\lambda, <\kappa)$-generic filter over V. Then, in $V[G]$, for every integer $d \geqslant 1$ there exists an ideal $\mathcal{I}_d$ on $[(\exp_d(\kappa))^+]^\omega$ and a subset $\mathcal{D}_d$ of $\mathcal{I}_d^+$ such that the following are satisfied.*

(1) *$\mathcal{I}_d$ is κ-complete.*
(2) *$\mathcal{D}_d$ is dense in $\mathcal{I}_d^+$ and is λ-closed in $\mathcal{I}_d^+$.*
(3) *For every $\mu < \kappa$, every coloring $c\colon [(\exp_d(\kappa))^+]^{d+1} \to \mu$ and every set $A \in \mathcal{I}_d^+$ there exist a color $\xi < \mu$ and an element $D \in \mathcal{D}_d$ with $D \subseteq A$ and such that for every $\mathbf{x} \in D$ the restriction $c \upharpoonright [\mathbf{x}]^{d+1}$ is constantly equal to ξ.*

To state an application of this lemma, it is convenient to introduce a sequence (Θ_n) of cardinals, defined recursively by the rule

$$\Theta_0 = \aleph_0 \ \text{ and } \ \Theta_{n+1} = \big(2^{(2^{\Theta_n})^+}\big)^{++}. \tag{1}$$

Observe that the sequence (Θ_n) is strictly increasing and that $\exp_\omega(\aleph_0) = \sup\{\Theta_n : n \in \omega\}$. Also note that $\exp_n(\aleph_0) < \Theta_n \leqslant \exp_{5n}(\aleph_0)$ for every $n \in \omega$ with $n \geqslant 1$, and so if GCH holds, then $\Theta_n = \aleph_{5n}$ for every $n \in \omega$.

Theorem 4. *Suppose that (κ_n) is a strictly increasing sequence of strongly compact cardinals with $\kappa_0 = \aleph_0$. For every $n \in \omega$ set $\lambda_n = \big(2^{(2^{\kappa_n})^+}\big)^+$. Let*

$$\mathbb{P} = \bigotimes_{n\in\omega} \mathrm{Col}(\lambda_n, < \kappa_{n+1})$$

be the iteration of the sequence of Lévy collapses. Let G be a $\mathbb{P}$-generic filter over V. Then, in $V[G]$, for every $n \in \omega$ we have $\kappa_n = \Theta_n$ and there exist an ideal $\mathcal{I}_n$ on $[(2^{\Theta_{n+1}})^+]^\omega$ and a subset $\mathcal{D}_n$ of $\mathcal{I}_n^+$ such that the following are satisfied.

- (P1) *$\mathcal{I}_n$ is Θ_{n+1}-complete.*
- (P2) *$\mathcal{D}_n$ is $(< \Theta_{n+1})$-closed in $\mathcal{I}_n^+$; that is, $\mathcal{D}_n$ is μ-closed in $\mathcal{I}_n^+$ for every $\mu < \Theta_{n+1}$.*
- (P3) *For every $\mu < \Theta_{n+1}$, every coloring $c\colon [(2^{\Theta_{n+1}})^+]^2 \to \mu$ and every $A \in \mathcal{I}_n^+$ there exist a color $\xi < \mu$ and an element $D \in \mathcal{D}_n$ with $D \subseteq A$ and such that for every $\mathbf{x} \in D$ the restriction $c \restriction [\mathbf{x}]^2$ is constantly equal to ξ.*

Moreover, if GCH *holds in V, then* GCH *also holds in $V[G]$.*

Proof. Let $n \in \omega$ be arbitrary. Let G_n be the restriction of G to the finite iteration

$$\mathbb{P}_n = \bigotimes_{m<n} \mathrm{Col}(\lambda_m, < \kappa_{m+1}).$$

Notice first that the small forcing extension $V[G_n]$ preserves the strong compactness of κ_{n+1}. This fact follows immediately from the elementary-embedding characterization of strong compactness (see [Ka, Theorem 22.17]). Working in $V[G_n]$ and applying Lemma 3 for $d = 1$, we see that the intermediate forcing extension $V[G_{n+1}]$ has the ideal $\mathcal{I}_n$ whose quotient has properties (P1), (P2) and (P3) described in Lemma 3. Working still in the intermediate forcing extension $V[G_{n+1}]$, we see that the rest of the forcing

$$\mathbb{P}^{n+1} = \bigotimes_{n<m<\omega} \mathrm{Col}(\lambda_m, < \kappa_{m+1})$$

is λ_{n+1}-closed, and so, in particular, it adds no new subsets to the index set on which the ideal $\mathcal{I}_n$ lives. Therefore, properties (P1), (P2) and (P3) of the quotient of $\mathcal{I}_n$ are preserved in $V[G]$. Since n was arbitrary, the proof is completed. □

Let us see how this could easily be used in deducing the 2-dimensional polarized partition property $\mathrm{Pl}_2\big(\exp_\omega(\aleph_0)\big)$.

Proposition 5. *Let* (Θ_n) *be the sequence of cardinals defined in* (1) *above. Suppose that for every* $n \in \omega$ *there exist an ideal* $\mathcal{I}_n$ *on* $[(2^{\Theta_{n+1}})^+]^\omega$ *and a subset* $\mathcal{D}_n$ *of* $\mathcal{I}_n^+$ *which satisfy properties* (P1), (P2) *and* (P3) *described in Theorem* 4*. Then the principle* $\mathrm{Pl}_2\big(\exp_\omega(\aleph_0)\big)$ *holds.*

Proof. The proof is based on the following claim.

Claim 6. *Let* $n \in \omega$*. Also let* $c\colon \prod_{i=0}^n [(2^{\Theta_{i+1}})^+]^2 \to \omega$ *be a coloring and* $(D_i)_{i=0}^n \in \prod_{i=0}^n \mathcal{D}_i$*. Then there exist* $(E_i)_{i=0}^n \in \prod_{i=0}^n \mathcal{D}_i \upharpoonright D_i$ *and a color* $m_0 \in \omega$ *such that for every* $(\mathbf{x}_i)_{i=0}^n \in \prod_{i=0}^n E_i$ *the restriction* $c \upharpoonright \prod_{i=0}^n [\mathbf{x}_i]^2$ *is constantly equal to* m_0*.*

Proof of Claim 6*.* By induction on n. The case $n=0$ is an immediate consequence of property (P3) in Theorem 4. So, let $n \in \omega$ with $n \geqslant 1$ and assume that the result has been proved for all $k \in \omega$ with $k < n$. Fix a coloring $c\colon \prod_{i=0}^n [(2^{\Theta_{i+1}})^+]^2 \to \omega$. Also fix $(D_i)_{i=0}^n \in \prod_{i=0}^n \mathcal{D}_i$ and set

$$\mathcal{F} := \Big\{ f\colon \prod_{i=0}^{n-1} [(2^{\Theta_{i+1}})^+]^2 \to \omega : f \text{ is a coloring} \Big\}.$$

Notice that $|\mathcal{F}| = 2^{(2^{\Theta_n})^+}$, and so, $|\mathcal{F}| < \Theta_{n+1}$. We define a coloring

$$d\colon [(2^{\Theta_{n+1}})^+]^2 \to \mathcal{F}$$

by the rule $d\big(\{\alpha,\beta\}\big)(\bar{s}) = c\big(\bar{s}^\frown\{\alpha,\beta\}\big)$ for every $\bar{s} \in \prod_{i=0}^{n-1} [(2^{\Theta_{i+1}})^+]^2$. By (P3) in Theorem 4, there exist $E_n \in \mathcal{D}_n \upharpoonright D_n$ and $f_0 \in \mathcal{F}$ such that for every $\mathbf{x} \in E_n$ the restriction $d \upharpoonright [\mathbf{x}]^2$ is constantly equal to f_0. The result now follows by applying our inductive hypothesis to the coloring f_0. □

By Claim 6 and the fact that every $\mathcal{D}_n$ is σ-closed (property (P2) in Theorem 4), the proof of Proposition 5 is completed. □

As a consequence of the previous analysis we obtain the following corollary.

Corollary 7 ([Sh2])**.** *Suppose that in our universe* V *there exists a strictly increasing sequence* (κ_n) *of strongly compact cardinals with* $\kappa_0 = \aleph_0$*. Then there is a forcing extension of* V *in which the principle* $\mathrm{Pl}_2\big(\exp_\omega(\aleph_0)\big)$ *holds. Moreover, if* GCH *holds in* V*, then* GCH *also holds in the extension.*

Proof. Follows by Theorem 4 and Proposition 5. □

Clearly, in the forcing extension obtained above the combinatorial principle $\mathrm{Pl}_1\big(\exp_\omega(\aleph_0)\big)$ holds as well. However, as we have already indicated, one can obtain the consistency of $\mathrm{Pl}_1\big(\exp_\omega(\aleph_0)\big)$ using a considerably weaker (and, in fact, optimal) large-cardinal assumption from the one used for $\mathrm{Pl}_2\big(\exp_\omega(\aleph_0)\big)$. More precisely, it is shown in [DT] that, assuming the existence of a measurable cardinal, there is a forcing extension in which GCH and $\mathrm{Pl}_1(\aleph_\omega)$ hold. In our applications to Banach space theory (see [DLT]), we were able to use so far only the combinatorial principles $\mathrm{Pl}_2\big(\exp_\omega(\aleph_0)\big)$ and $\mathrm{Pl}_1(\aleph_\omega)$ but it is likely that the higher-dimensional versions will find applications as well. So we would like to record here also the higher-dimensional analogues of Theorem 4, Proposition 5 and Corollary 7 respectively. Their proofs are straightforward adaptations of our previous arguments, so we leave the details to the interested reader.

Theorem 8. *Suppose that (κ_n) is a strictly increasing sequence of strongly compact cardinals with $\kappa_0 = \aleph_0$. For every $n \in \omega$ we can select cardinals $\lambda_n, \theta_n \in \big[\kappa_n, \exp_\omega(\kappa_n)\big)$ in such a way that, if we let*

$$\mathbb{P} = \bigotimes_{n\in\omega} \mathrm{Col}(\lambda_n, <\kappa_{n+1})$$

be the iteration of the sequence of Lévy collapses and if we select a $\mathbb{P}$-generic filter G over V, then, in $V[G]$, we have

$$\sup_{n\in\omega} \kappa_n = \sup_{n\in\omega} \lambda_n = \sup_{n\in\omega} \theta_n = \exp_\omega(\aleph_0)$$

and for every $n \in \omega$ there exist an ideal $\mathcal{I}_n$ on $[\theta_{n+1}]^\omega$ and a subset $\mathcal{D}_n$ of $\mathcal{I}_n^+$ such that the following are satisfied.

- (P1) *$\mathcal{I}_n$ is κ_{n+1}-complete.*
- (P2) *$\mathcal{D}_n$ is $(<\lambda_n)$-closed in $\mathcal{I}_n^+$; that is, $\mathcal{D}_n$ is μ-closed in $\mathcal{I}_n^+$ for every $\mu < \lambda_n$.*
- (P3) *For every $\mu < \kappa_{n+1}$, every coloring $c\colon [\theta_{n+1}]^{n+1} \to \mu$ and every $A \in \mathcal{I}_n^+$ there exist a color $\xi < \mu$ and an element $D \in \mathcal{D}_n$ with $D \subseteq A$ and such that for every $\mathbf{x} \in D$ the restriction $c \restriction [\mathbf{x}]^{n+1}$ is constantly equal toξ.*

Moreover, if GCH *holds in V, then* GCH *also holds in $V[G]$.*

Proposition 9. *Suppose (κ_n), (λ_n) and (θ_n) are strictly increasing sequences of regular cardinal that all converge to $\exp_\omega(\aleph_0)$. Suppose further that for every $n \in \omega$ there exist an ideal $\mathcal{I}_n$ on $[\theta_{n+1}]^\omega$ and a subset $\mathcal{D}_n$ of $\mathcal{I}_n^+$ which satisfy properties* (P1)*,* (P2) *and* (P3) *of Theorem* 8*. Then the principle $\mathrm{Pl}_d\big(\exp_\omega(\aleph_0)\big)$ holds for every integer $d \geqslant 1$.*

Corollary 10 ([Sh2])**.** *Suppose that in our universe V there exists a strictly increasing sequence (κ_n) of strongly compact cardinals with $\kappa_0 = \aleph_0$. Then, there is a forcing extension of V in which the principle $\mathrm{Pl}_d\big(\exp_\omega(\aleph_0)\big)$ holds for every integer $d \geqslant 1$. Moreover, if* GCH *holds in V, then* GCH *also holds in the forcing extension.*

4. Proof of Lemma 3

Assume that $\lambda < \kappa$ is a pair of two infinite cardinals with λ regular uncountable and κ strongly compact. Let G be a $\mathrm{Col}(\lambda, <\kappa)$-generic filter. *The generic filter G will be fixed until the end of the proof. We also fix a κ-complete normal ultrafilter $\mathcal{U}$ on κ.*

Fix an arbitrary integer $d \geqslant 1$. Let $\{V_\alpha : \alpha \in \mathrm{Ord}\}$ be the von-Neumann hierarchy of V. As κ is inaccessible (being strongly compact), we see that $|V_\kappa| = \kappa$. For every coloring $c\colon [(\exp_d(\kappa))^+]^{d+1} \to V_\kappa$ we set

$$\mathrm{Sol}^\omega_{d,\kappa}(c) \coloneqq \{\mathbf{x} \in [(\exp_d(\kappa))^+]^\omega : c \upharpoonright [\mathbf{x}]^{d+1} \text{ is constant}\} \tag{2}$$

and we define

$$\mathrm{Sol}^\omega_{d,\kappa} \coloneqq \{\mathrm{Sol}^\omega_{d,\kappa}(c) : c\colon [(\exp_d(\kappa))^+]^{d+1} \to V_\kappa \text{ is a coloring}\}. \tag{3}$$

The idea of considering the family of sets which are monochromatic with respect to a coloring is taken from Shelah's paper [Sh2] and has been also used by other authors (see, for instance, [Mi]).

Fact 11. *The following hold.*

(a) *For every coloring* $c\colon [(\exp_d(\kappa))^+]^{d+1} \to V_\kappa$ *we have* $\mathrm{Sol}^\omega_{d,\kappa}(c) \neq \emptyset$.
(b) *The family* $\mathrm{Sol}^\omega_{d,\kappa}$ *is* κ*-complete. That is, for every* $\delta < \kappa$ *and every sequence* $(A_\xi : \xi < \delta)$ *in* $\mathrm{Sol}^\omega_{d,\kappa}$ *we have that* $\bigcap_{\xi<\delta} A_\xi \in \mathrm{Sol}^\omega_{d,\kappa}$.

Proof. (a) By our assumptions we see that $|V_\kappa| = \kappa$. Moreover, by the classical Erdős–Rado partition theorem (see [Ku]), we have

$$(\exp_d(\kappa))^+ \to (\kappa^+)^{d+1}_\kappa$$

and the result follows.

(b) For every $\xi < \delta$ let $c_\xi\colon [\exp_d(\kappa))^+]^{d+1} \to V_\kappa$ be a coloring such that $A_\xi = \mathrm{Sol}^\omega_{d,\kappa}(c_\xi)$. Observe that $(V_\kappa)^\delta \subseteq V_\kappa$. Define a coloring $c\colon [\exp_d(\kappa))^+]^{d+1} \to (V_\kappa)^\delta$ by $c(s) = (c_\xi(s) : \xi < \delta)$. Noticing that

$$\bigcap_{\xi<\delta} \mathrm{Sol}^\omega_{d,\kappa}(c_\xi) = \mathrm{Sol}^\omega_{d,\kappa}(c)$$

the proof is completed. □

By part (b) of Fact 11 and our hypothesis that κ is a strongly compact cardinal, we see that there exists a κ-complete ultrafilter $\mathcal{V}$ on $[\exp_d(\kappa))^+]^\omega$ extending the family $\mathrm{Sol}^\omega_{d,\kappa}$. *We fix such an ultrafilter* $\mathcal{V}$.

Definition 12. *A* $\mathcal{V}$–sequence of conditions *is a sequence* $\overline{p} = (p_{\mathbf{x}} : \mathbf{x} \in A)$ *in* $\mathrm{Col}(\lambda, <\kappa)$*, belonging to the ground model* V *and indexed by a member* A *of the ultrafilter* $\mathcal{V}$*. We will refer to the set* A *as the* index set *of* $\overline{p}$ *and we shall denote it by* $I(\overline{p})$.

Definition 13. *Let* $\overline{p} = (p_{\mathbf{x}} : \mathbf{x} \in I(\overline{p}))$ *be a* $\mathcal{V}$*–sequence of conditions. We say that a condition* r *in* $\mathrm{Col}(\lambda, <\kappa)$ *is a* root *of* $\overline{p}$ *if* [2]

$$(\mathcal{U}\alpha)\ (\mathcal{V}\mathbf{x})\ \ p_{\mathbf{x}} \upharpoonright \alpha = r. \tag{4}$$

Related to the above definitions, we have the following fact.

[2]The formula (4) is an abbreviation of the fact that $\{\alpha : \{\mathbf{x} : p_{\mathbf{x}} \upharpoonright \alpha = r\} \in \mathcal{V}\} \in \mathcal{U}$.

Fact 14. *Every $\mathcal{V}$–sequence of conditions $\overline{p}$ has a unique root $r(\overline{p})$.*

Proof. For every $\alpha < \kappa$ the map $I(\overline{p}) \ni \mathbf{x} \mapsto p_{\mathbf{x}} \restriction \alpha$ has fewer than κ values. So, by the κ-completeness of $\mathcal{V}$, there exist $p_\alpha \in \mathrm{Col}(\lambda, <\kappa)$ and $I_\alpha \in \mathcal{V} \restriction I(\overline{p})$ so that $p_{\mathbf{x}} \restriction \alpha = p_\alpha$ for all $\mathbf{x} \in I_\alpha$. Hence, we can select a sequence $(p_\alpha : \alpha < \kappa)$ in $\mathrm{Col}(\lambda, <\kappa)$ and a decreasing sequence $(I_\alpha : \alpha < \kappa)$ of elements of $\mathcal{V} \restriction I(\overline{p})$ such that for every $\alpha < \kappa$ and every $\mathbf{x} \in I_\alpha$ we have that $p_{\mathbf{x}} \restriction \alpha = p_\alpha$.

Let $A \subseteq \kappa$ be the set of all limit ordinals $\alpha < \kappa$ with $\mathrm{cf}(\alpha) > \lambda$. Since $\mathcal{U}$ is normal, the set A is in $\mathcal{U}$. Consider the mapping $c\colon A \to \kappa$ defined by

$$c(\alpha) = \sup\{\xi : \xi \in (\mathrm{dom}(p_\alpha \restriction \alpha))_1\}$$

for every $\alpha \in A$. As $\mathrm{cf}(\alpha) > \lambda$, we see that c is a regressive mapping. The ultrafilter $\mathcal{U}$ is normal, and so, there exist $A' \in \mathcal{U} \restriction A$ and $\gamma_0 < \kappa$ such that $c(\alpha) = \gamma_0$ for every $\alpha \in A'$. Now consider the map

$$A' \ni \alpha \mapsto p_\alpha \restriction \alpha = p_\alpha \restriction \gamma_0 \subseteq (\lambda \times \gamma_0) \times \gamma_0.$$

Noticing that $|\mathcal{P}\big((\lambda \times \gamma_0) \times \gamma_0\big)| < \kappa$ and recalling that $\mathcal{U}$ is κ-complete, we see that there exist $A'' \in \mathcal{U} \restriction A'$ and $r(\overline{p})$ in $\mathrm{Col}(\lambda, <\kappa)$ such that $p_\alpha \restriction \alpha = r(\overline{p})$ for every $\alpha \in A''$. It follows that for every $\alpha \in A''$ the set $\{\mathbf{x} \in [\exp_d(\kappa))^+]^\omega : p_{\mathbf{x}} \restriction \alpha = r(\overline{p})\}$ contains the set I_α, and so

$$(\mathcal{U}\alpha)\ (\mathcal{V}\mathbf{x})\ \ p_{\mathbf{x}} \restriction \alpha = r(\overline{p}).$$

The uniqueness of $r(\overline{p})$ is an immediate consequence of property (4) in Definition 13. The proof is completed. □

We are ready to introduce the ideal $\mathcal{I}_d$.

Definition 15. *In $V[G]$ we define*

$$\mathcal{I}_d := \{I \subseteq [\exp_d(\kappa))^+]^\omega : \textit{there is some } A \in \mathcal{V} \textit{ such that } I \cap A = \emptyset\}.$$

We isolate, for future use, the following (easily verified) properties of $\mathcal{I}_d$.

(P1) $\mathcal{I}_d$ is an ideal; in fact, $\mathcal{I}_d$ is a κ-complete ideal.
(P2) $\mathcal{V} \subseteq \mathcal{I}_d^+$.
(P3) If $A \in \mathcal{V}$ and $B \in \mathcal{I}_d^+$, then $A \cap B \in \mathcal{I}_d^+$.

For every $\mathcal{V}$–sequence of conditions $\overline{p}$ we set

$$D_{\overline{p}} := \{\mathbf{x} \in I(\overline{p}) : p_{\mathbf{x}} \in G\}. \tag{5}$$

Now we are ready to introduce the set $\mathcal{D}_d$.

Definition 16. *In $V[G]$ we define*

$$\mathcal{D}_d := \{D_{\overline{p}} : \overline{p} \textit{ is a } \mathcal{V}\textit{–sequence of conditions in the ground model } V\} \cap \mathcal{I}_d^+.$$

By definition, we have $\mathcal{D}_d \subseteq \mathcal{I}_d^+$. The rest of the proof will be devoted to the verification that the ideal $\mathcal{I}_d$ and the set $\mathcal{D}_d$ satisfy the requirements of Lemma 3. To this end, we need the following lemma.

Lemma 17. *Let $\overline{p} = (p_{\mathbf{x}} : \mathbf{x} \in I(\overline{p}))$ be a $\mathcal{V}$–sequence of conditions. Then the following are equivalent.*

(1) $D_{\overline{p}} \in \mathcal{D}_d$.
(2) $r(\overline{p}) \in G$.

Proof. (1)⇒(2) Assume that $D_{\overline{p}} \in \mathcal{D}_d$. We use the fact that $D_{\overline{p}} \in \mathcal{I}_d^+$ and that

$$(\mathcal{U}\alpha)\ (\mathcal{V}\mathbf{x})\ \ p_{\mathbf{x}} \upharpoonright \alpha = r(\overline{p})$$

to find $\mathbf{x} \in D_{\overline{p}}$ such that $p_{\mathbf{x}} \leqslant r(\overline{p})$. By the definition of $D_{\overline{p}}$, we see that $p_{\mathbf{x}} \in G$, and so, $r(\overline{p}) \in G$ as well.

(2)⇒(1) Suppose that $r(\overline{p}) \in G$. Fix a ground model set A which is in $\mathcal{V}$. It is enough to show that $D_{\overline{p}} \cap A \neq \emptyset$. To this end, set

$$E \coloneqq \{q \in \mathrm{Col}(\lambda, <\kappa) : q \perp r(\overline{p}) \text{ or there is } \mathbf{x} \in I(\overline{p}) \cap A \text{ with } q \leqslant p_{\mathbf{x}}\}.$$

We claim that E is a dense subset of $\mathrm{Col}(\lambda, <\kappa)$. To see this, let $r \in \mathrm{Col}(\lambda, <\kappa)$ be arbitrary. If $r \perp r(\overline{p})$, then $r \in E$. So, suppose that $r \parallel r(\overline{p})$. Using this and the fact that

$$(\mathcal{U}\alpha)\ (\mathcal{V}\mathbf{x})\ \ p_{\mathbf{x}} \upharpoonright \alpha = r(\overline{p}),$$

we may select $\mathbf{x} \in I(\overline{p}) \cap A$ such that $p_{\mathbf{x}} \parallel r$. So, there exist $q \in \mathrm{Col}(\lambda, <\kappa)$ and $\mathbf{x} \in I(\overline{p}) \cap A$ such that $q \leqslant p_{\mathbf{x}}$ and $q \leqslant r$. In other words, there exists $q \in E$ with $q \leqslant r$. This establishes our claim that E is a dense subset of $\mathrm{Col}(\lambda, <\kappa)$.

It follows from the above discussion that there exists $q \in G$ with $q \in E$. Since $r(\overline{p}) \in G$, we have $r(\overline{p}) \parallel q$. Therefore, by the definition of the set E, there exists $\mathbf{x} \in I(\overline{p}) \cap A$ with $q \leqslant p_{\mathbf{x}}$. It follows that $p_{\mathbf{x}} \in G$, and so, $\mathbf{x} \in D_{\overline{p}} \cap A$. The proof is completed. □

Lemma 18. *$\mathcal{D}_d$ is dense in $\mathcal{I}_d^+$.*

Proof. Fix $J \in \mathcal{I}_d^+$. We will prove that there exists a $\mathcal{V}$–sequence of conditions $\overline{q}$ in the ground model V satisfying $D_{\overline{q}} \in \mathcal{D}_d$ and $D_{\overline{q}} \subseteq J$. This will finish the proof.

To this end, we fix a $\mathrm{Col}(\lambda, <\kappa)$-name $\dot{J}$ for J. Let $p \in \mathrm{Col}(\lambda, <\kappa)$ be an arbitrary condition such that $p \Vdash \dot{J} \notin \mathcal{I}_d$. Define, in the ground model V, the set

$$A_p \coloneqq \{\mathbf{x} \in [\exp_d(\kappa))^+]^\omega : \text{there is } q \leq p \text{ such that } q \Vdash \check{\mathbf{x}} \in \dot{J}\}.$$

First we claim that $A_p \in \mathcal{V}$. Suppose, towards a contradiction, that the set $C \coloneqq [\exp_d(\kappa))^+]^\omega \setminus A_p$ is in $\mathcal{V}$. Since $J \in \mathcal{I}_d^+$ we see that $J \cap C \neq \emptyset$ in $V[G]$. Using the fact that $p \Vdash \dot{J} \notin \mathcal{I}_d$ and that the forcing $\mathrm{Col}(\lambda, <\kappa)$ is σ-closed, we may find $\mathbf{x} \in C$ and a condition $q \leqslant p$ such that $q \Vdash \check{\mathbf{x}} \in \dot{J}$. But this implies that $\mathbf{x} \in A_p$, a contradiction.

It follows that we may select a $\mathcal{V}$–sequence of conditions $\overline{q} = (q_{\mathbf{x}} : \mathbf{x} \in A_p)$ such that $q_{\mathbf{x}} \leqslant p$ and $q_{\mathbf{x}} \Vdash \check{\mathbf{x}} \in \dot{J}$ for every $\mathbf{x} \in A_p$. By Fact 14, let $r(\overline{q})$ be the root of $\overline{q}$. Clearly we have $r(\overline{q}) \leqslant p$.

Now, fix a condition r such that $r \Vdash \dot{J} \notin \mathcal{I}_d$. What we have just proved is that the set of conditions $r(\overline{q})$ such that

($*$) $r(\overline{q})$ is the root of a $\mathcal{V}$–sequence of conditions $\overline{q} = (q_{\mathbf{x}} : \mathbf{x} \in I(\overline{q}))$ with the property that $q_{\mathbf{x}} \Vdash \check{\mathbf{x}} \in \dot{J}$ for every $\mathbf{x} \in I(\overline{q})$

is dense below r. As G is generic, we see that there exists a $\mathcal{V}$–sequence of conditions $\overline{q}$ as in ($*$) above such that $r(\overline{q}) \in G$. On the one hand, by Lemma 17, we see that $D_{\overline{q}} \in \mathcal{D}_d$. On the other hand, property ($*$) above implies that $D_{\overline{q}} \subseteq J$; indeed, if $\mathbf{x} \in D_{\overline{q}}$, then $q_{\mathbf{x}} \in G$ and, by ($*$), $q_{\mathbf{x}} \Vdash \check{\mathbf{x}} \in \dot{J}$. The proof is completed. □

Lemma 19. *$\mathcal{D}_d$ is λ-closed in $\mathcal{I}_d^+$.*

Proof. Fix $\mu < \lambda$ and a decreasing sequence $(D_\xi : \xi < \mu)$ in $\mathcal{D}_d$. For every $\xi < \mu$ let $\overline{p}_\xi = (p^\xi_{\mathbf{x}} : \mathbf{x} \in I(\overline{p}_\xi))$ be a $\mathcal{V}$–sequence of conditions in V such that $D_\xi = D_{\overline{p}_\xi}$. Our forcing $\mathrm{Col}(\lambda, <\kappa)$ is λ-closed, and so, the sequence $(\overline{p}_\xi : \xi < \mu)$ is in the ground model V as well. Applying Fact 14 to every $\overline{p}_\xi$, we find a sequence $(r_\xi : \xi < \mu)$ in $\mathrm{Col}(\lambda, <\kappa)$ such that r_ξ is the root of $\overline{p}_\xi$ for every $\xi < \mu$. By Lemma 17, we obtain that $r_\xi \in G$ for every $\xi < \mu$.

We claim, first, that for every $\xi < \zeta < \mu$ we have

$$(\mathcal{V}\mathbf{x}) \ \ p^\xi_{\mathbf{x}} \parallel p^\zeta_{\mathbf{x}}. \tag{6}$$

Suppose, towards a contradiction, that there exist $\xi < \zeta < \mu$ such that the set $L := \{\mathbf{x} \in A : p^\xi_{\mathbf{x}} \perp p^\zeta_{\mathbf{x}}\}$ is in $\mathcal{V}$. As $D_{\overline{p}_\zeta} \in \mathcal{D}_d \subseteq \mathcal{I}_d^+$ and $L \in \mathcal{V}$, there exists $\mathbf{x} \in D_{\overline{p}_\zeta} \cap L$. And since $D_{\overline{p}_\zeta} = D_\zeta \subseteq D_\xi = D_{\overline{p}_\xi}$ we have $\mathbf{x} \in D_{\overline{p}_\xi}$ as well. But this implies that both $p^\xi_{\mathbf{x}}$ and $p^\zeta_{\mathbf{x}}$ are in G and at the same time $p^\xi_{\mathbf{x}} \perp p^\zeta_{\mathbf{x}}$, a contradiction.

Invoking (6) above, we may find $A \in \mathcal{V}$ such that for every $\xi < \zeta < \mu$ and every $\mathbf{x} \in A$ we have that $p^\xi_{\mathbf{x}} \parallel p^\zeta_{\mathbf{x}}$. We set

$$p_{\mathbf{x}} = \bigcup_{\xi<\mu} p^\xi_{\mathbf{x}} \quad \text{for every } \mathbf{x} \in A$$

and we define $\overline{p} = (p_{\mathbf{x}} : \mathbf{x} \in A)$. It is clear that $\overline{p}$ is a well-defined $\mathcal{V}$–sequence of conditions. Also observe that $D_{\overline{p}} \subseteq D_\xi$ for every $\xi < \mu$. We are going to show that $D_{\overline{p}} \in \mathcal{D}_d$. This will finish the proof.

To this end, let r be the root of $\overline{p}$. By Lemma 17, it is enough to show that $r \in G$. Notice, first, that

$$(\mathcal{U}\alpha) \ (\mathcal{V}\mathbf{x}) \ \ \bigcup_{\xi<\mu} p^\xi_{\mathbf{x}} \restriction \alpha = p_{\mathbf{x}} \restriction \alpha = r. \tag{7}$$

On the other hand, since r_ξ is the root of $\overline{p}_\xi$, we have

$$(\forall \xi < \mu) \ (\mathcal{U}\alpha) \ (\mathcal{V}\mathbf{x}) \ \ p^\xi_{\mathbf{x}} \restriction \alpha = r_\xi. \tag{8}$$

Both $\mathcal{U}$ and $\mathcal{V}$ are κ-complete, and so, (8) is equivalent to

$$(\mathcal{U}\alpha) \ (\mathcal{V}\mathbf{x}) \ (\forall \xi < \mu) \ \ p^\xi_{\mathbf{x}} \restriction \alpha = r_\xi. \tag{9}$$

Combining (7) and (9), we obtain that

$$(\mathcal{U}\alpha)\ (\mathcal{V}\mathbf{x})\ \ r = \bigcup_{\xi<\mu} p^{\xi}_{\mathbf{x}} \upharpoonright \alpha = \bigcup_{\xi<\mu} r_{\xi}. \tag{10}$$

Summing up, we see that the root r of $\overline{p}$ is the union $\bigcup_{\xi<\mu} r_\xi$ of the roots of the $\overline{p}_\xi$'s. Since the generic filter G is λ-complete, we conclude that $r \in G$. The proof is completed. □

Lemma 20. *Work in $V[G]$. Let $\mu < \kappa$ and let $c\colon [\exp_d(\kappa))^+]^{d+1} \to \mu$ be a coloring. Also let $A \in \mathcal{I}^+_d$ be arbitrary. Then there exist a color $\xi < \mu$ and an element $D \in \mathcal{D}_d$ with $D \subseteq A$ and such that for every $\mathbf{x} \in D$ and every $\{\alpha_0, \dots, \alpha_d\} \in [\mathbf{x}]^{d+1}$ we have $c(\{\alpha_0, \dots, \alpha_d\}) = \xi$.*

Proof. Fix a coloring $c\colon [\exp_d(\kappa))^+]^{d+1} \to \mu$ and let $A \in \mathcal{I}^+_d$. Also let $\dot{c}$ be a $\mathrm{Col}(\lambda, <\kappa)$-name for the coloring c. In V, let $\mathrm{RO}(\mathrm{Col}(\lambda, <\kappa))$ be the collection of all regular open subsets of $\mathrm{Col}(\lambda, <\kappa)$. Working in V, we define another coloring $C\colon [\exp_d(\kappa))^+]^{d+1} \to (\mathrm{RO}(\mathrm{Col}(\lambda, <\kappa)))^\mu$ by the rule

$$C(s) = (\llbracket \dot{c}(\check{s}) = \check{\xi} \rrbracket : \xi < \mu)$$

where $\llbracket \dot{c}(\check{s}) = \check{\xi} \rrbracket = \{p \in \mathrm{Col}(\lambda, <\kappa) \ : \ p \Vdash \dot{c}(\check{s}) = \check{\xi}\}$ is the boolean value of the formula "$c(s) = \xi$".

The forcing $\mathrm{Col}(\lambda, <\kappa)$ is κ-cc, and so, $(\mathrm{RO}(\mathrm{Col}(\lambda, <\kappa)))^\mu \subseteq V_\kappa$. Hence, $\mathrm{Sol}^\omega_{d,\kappa}(C) \in \mathcal{V}$. We set $J = A \cap \mathrm{Sol}^\omega_{d,\kappa}(C)$. Then J is in $\mathcal{I}^+_d$. Notice that for every $\mathbf{x} \in J$ and every $s, s' \in [\mathbf{x}]^{d+1}$ we have $C(s) = C(s')$. It follows that for every $\mathbf{x} \in J$ we may select a sequence $\overline{U}_{\mathbf{x}} = (U^\xi_{\mathbf{x}} : \xi < \mu)$ in $(\mathrm{RO}(\mathrm{Col}(\lambda, <\kappa)))^\mu$ such that for every $s \in [\mathbf{x}]^{d+1}$ and every $\xi < \mu$ we have $\llbracket \dot{c}(\check{s}) = \check{\xi} \rrbracket = U^\xi_{\mathbf{x}}$.

Now observe that for every $s \in [\exp_d(\kappa))^+]^{d+1}$ the set

$$\{\llbracket \dot{c}(\check{s}) = \check{\xi} \rrbracket \ : \ \xi < \mu\}$$

is a maximal antichain. So, we can naturally define in $V[G]$ a coloring $e\colon J \to \mu$ by the rule

$$e(\mathbf{x}) = \xi \ \text{ if and only if } \ U^\xi_{\mathbf{x}} \in G.$$

Equivalently, for every $\mathbf{x} \in J$ we have that $e(\mathbf{x}) = \xi$ if and only if $c \upharpoonright [\mathbf{x}]^{d+1}$ is constant with value ξ. The ideal $\mathcal{I}_d$ is κ-complete and $J \in \mathcal{I}^+_d$. Hence, there exists $\xi_0 < \mu$ such that $e^{-1}\{\xi_0\} \in \mathcal{I}^+_d$. By Lemma 18, we may select $D \in \mathcal{D}_d$ with $D \subseteq e^{-1}\{\xi_0\} \subseteq J \subseteq A$. Finally, notice that for every $\mathbf{x} \in D$ the restriction $c \upharpoonright [\mathbf{x}]^{d+1}$ is constant with value ξ_0. The proof is completed. □

We are ready to finish the proof of Lemma 3. As we have already mentioned, the ideal $\mathcal{I}_d$ will be the one defined in Definition 15, while the dense subset $\mathcal{D}_d$ of $\mathcal{I}^+_d$ will be the one defined in Definition 16. First, we notice that property (1) in Lemma 3 (that is, the fact that $\mathcal{I}_d$ is κ-complete) follows easily by the definition $\mathcal{I}_d$ and the fact that $\mathcal{V}$ is κ-complete (in fact, we have already isolated this property of $\mathcal{I}_d$ in (P1) above). Property (2) in Lemma 3 (that is, the fact that $\mathcal{D}_d$ is λ-closed in $\mathcal{I}_d^+$) has been established in Lemma 19. Finally, property (3) was proved in Lemma 20. Since $d \geqslant 1$ was arbitrary, the proof of Lemma 3 is completed.

References

[CDPM] M. Carrasco, C. A. Di Prisco and A. Millán, *Partitions of the set of finite sequences*, J. Comb. Theory, Ser. A 71 (1995), 255–274.

[DT] C. A. Di Prisco and S. Todorcevic, *A cardinal defined by a polarized partition relation*, Israel J. Math. 109 (1999), 41–52.

[DLT] P. Dodos, J. Lopez Abad and S. Todorcevic, *Unconditional basic sequences in spaces of large density*, Adv. Math. 226 (2011), 3297–3308.

[EH1] P. Erdős and A. Hajnal, *Unsolved problems in set theory*, 1971 Axiomatic Set Theory (Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967), pp. 17–48, American Mathematical Society, Providence, R.I.

[EH2] P. Erdős and A. Hajnal, *Unsolved and solved problems in set theory*, Proceedings of the Tarski Symposium (Proc. Sympos. Pure Math., Vol. XXV, Univ. California, Berkeley, Calif., 1971), pp. 269–287, American Mathematical Society, Providence, R.I., 1974.

[EHMR] P. Erdős, A. Hajnal, A. Máté and R. Rado, *Combinatorial set theory: partition relations for cardinals*, Studies in Logic and the Foundations of Mathematics, 106, North-Holland Publishing Co., Amsterdam, 1984.

[GJM] F. Galvin, T. Jech and M. Magidor, *An ideal game*, J. Symb. Logic 43 (1978), 284–292.

[Ka] A. Kanamori, *The Higher Infinite: Large Cardinals in Set Theory*, Springer Monogr. Math., Springer, 2003.

[Ke] J. Ketonen, *Banach spaces and large cardinals*, Fund. Math. 81 (1974), 291–303.

[Ko] P. Koepke, *The consistency strength of the free subset property at* ω_ω, J. Symb. Logic 49 (1984), 1198–1204.

[Ku] K. Kunen, *Set Theory. An introduction to independence proofs*, Studies in Logic and the Foundations of Mathematics, 102, North-Holland, Amsterdam, 1983.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I, sequence spaces*, Springer Verlag, 92, 1977.

[Mi] H. Mildenberger, *More canonical forms and dense free subsets*, Ann. Pure Appl. Logic 125 (2004), 75–99.

[Sh1] S. Shelah, *Independence of strong partition relation for small cardinals, and the free-subset problem*, J. Symb. Logic, 45 (1980), 505–509.

[Sh2] S. Shelah, $\aleph_\omega$ *may have a strong partition relation*, Israel J. Math. 38 (1981), 283–288.

Department of Mathematics, University of Athens, Panepistimiopolis 157 84, Athens, Greece

E-mail address: pdodos@math.ntua.gr

Instituto de Ciencias Matematicas, CSIC-UAM-UC3M-UCM, C/ Nicolá Cabrera, n 13-15, Campus Cantoblanco UAM, 28049 Madrid, Spain

E-mail address: abad@icmat.es

Université Paris 7, CNRS-FRE 3233, 2 place Jussieu - Case 7012, 72521 Paris Cedex 05, France

and

Department of Mathematics, University of Toronto, Toronto, Canada, M5S 2E4.

E-mail address: stevo@math.toronto.edu