# A DENSITY VERSION OF THE HALPERN–LÄUCHLI THEOREM

PANDELIS DODOS, VASSILIS KANELLOPOULOS AND NIKOLAOS KARAGIANNIS

Abstract. We prove a density version of the Halpern–Läuchli theorem. This settles in the affirmative a conjecture of Laver.

Specifically, we say that a tree $T$ is homogeneous if $T$ has a unique root and there exists an integer $b \geqslant 2$ such that every $t \in T$ has exactly $b$ immediate successors. We show that for every $d \geqslant 1$ and every tuple $(T_1, \dots, T_d)$ of homogeneous trees, if $D$ is a subset of the level product of $(T_1, \dots, T_d)$ satisfying

$$\limsup_{n\to\infty} \frac{|D \cap \big(T_1(n) \times \cdots \times T_d(n)\big)|}{|T_1(n) \times \cdots \times T_d(n)|} > 0$$

then there exist strong subtrees $(S_1, \dots, S_d)$ of $(T_1, \dots, T_d)$ having a common level set such that the level product of $(S_1, \dots, S_d)$ is a subset of $D$.

## 1. Introduction

1.1. **Statement of the problem and the main result.** Ramsey theory is the collection of a number of partition results asserting that for every finite coloring of a "structure" one can find a "substructure" which is monochromatic. In several cases, however, one can actually prove a significantly stronger *density* result asserting that every large subset of a "structure" must contain a "substructure". This phenomenon, investigated from the early beginnings of Ramsey theory, has seen some dramatic developments in recent years and, by now, there are several results in this direction. A well-known and illuminative example is the density version of the Hales–Jewett theorem [13] obtained by Furstenberg and Katznelson [11] (see also [10]).

The main goal of the present paper is to prove a density version of the Halpern–Läuchli theorem [15]. The Halpern–Läuchli theorem is a rather deep pigeonhole principle for trees. It was discovered in 1966, three years after the discovery of the Hales–Jewett theorem, as a result needed for the construction of a model of set theory in which the boolean prime ideal theorem is true but not the full axiom of choice (see [16]). The original proof was based on tools from logic; since then, other proofs have been found some of which are purely combinatorial (see [24, §3] for a detailed exposition). It has been the main tool for the development of Ramsey theory for trees, a rich area of combinatorics with important applications in functional

2000 *Mathematics Subject Classification*: 05D10, 05C05.

*Key words*: trees, strong subtrees, level product, density.

analysis and topology (see, for example, [4, 5, 6, 12, 14, 17, 18, 19, 20, 21, 22, 24] and [1, 2, 7, 8, 23] for applications).

The Halpern–Läuchli theorem has several equivalent forms (see [24, §3.1]). To proceed with our discussion it is useful at this point to recall one of these forms, known as the "strong subtree version of the Halpern–Läuchli theorem".

**Theorem 1.** *For every $d \geqslant 1$ we have that $\mathrm{HL}(d)$ holds, that is, for every tuple $(T_1, \dots, T_d)$ of uniquely rooted and finitely branching trees without maximal nodes and for every finite coloring of the level product*

$$\bigcup_{n \in \mathbb{N}} T_1(n) \times \cdots \times T_d(n)$$

*of $(T_1, \dots, T_d)$ there exist strong subtrees $(S_1, \dots, S_d)$ of $(T_1, \dots, T_d)$ having a common level set such that the level product of $(S_1, \dots, S_d)$ is monochromatic.*

We recall that a subtree $S$ of a tree $(T, <)$ is said to be *strong* if: (a) $S$ is uniquely rooted, (b) there exists an infinite subset $L_T(S) = \{l_0 < l_1 < \cdots\}$ of $\mathbb{N}$, called the *level set* of $S$, such that for every $n \in \mathbb{N}$ the $n$-level $S(n)$ of $S$ is a subset of $T(l_n)$, and (c) for every $s \in S$ and every immediate successor $t$ of $s$ in $T$ there exists a unique immediate successor $s'$ of $s$ in $S$ with $t \leqslant s'$. The last condition is the most important one and expresses a basic combinatorial requirement, namely that a strong subtree of $T$ must respect the "tree structure" of $T$. The notion of a strong subtree was highlighted with the work of Milliken [20, 21] who used Theorem 1 to show that the family of strong subtrees of a uniquely rooted and finitely-branching tree is partition regular.

The natural problem whether there exists a density version of Theorem 1 was first asked by Laver in the late 1960s who actually conjectured that there is such a version. The conjecture was circulated among experts in the area and it was explicitly stated in the paper [3] by Bicker and Voigt who made two important observations. Firstly, by providing several examples—see, in particular, [3, Theorems 2.4 and 2.5]—they isolated the largest class of trees for which a density version of Theorem 1 could be true. This is the class of *homogeneous* trees: a tree $T$ is said to be homogeneous if it has a unique root and there exists $b \geqslant 2$, called the *branching number* of $T$, such that every $t$ in $T$ has exactly $b$ immediate successors. Secondly, they showed that for a single homogeneous tree Theorem 1 does have a density version. Specifically, it was shown in [3, Theorem 2.3] that for every homogeneous tree $T$ and every subset $D$ of $T$ satisfying

$$\limsup_{n \to \infty} \frac{|D \cap T(n)|}{|T(n)|} > 0$$

there exists a strong subtree $S$ of $T$ with $S \subseteq D$.

Our main result shows that a density version of Theorem 1 is valid for an arbitrary finite number of homogeneous trees and thereby settles in the affirmative Laver's conjecture.

**Theorem 2.** *For every $d \geqslant 1$ we have that* $\mathrm{DHL}(d)$ *holds, that is, for every tuple $(T_1,\dots,T_d)$ of homogeneous trees and every subset $D$ of the level product of $(T_1,\dots,T_d)$ satisfying*

$$\limsup_{n\to\infty} \frac{|D\cap \big(T_1(n)\times\dots\times T_d(n)\big)|}{|T_1(n)\times\dots\times T_d(n)|} > 0$$

*there exist strong subtrees $(S_1,\dots,S_d)$ of $(T_1,\dots,T_d)$ having a common level set such that the level product of $(S_1,\dots,S_d)$ is a subset of $D$.*

Notice that the strong subtrees $S_1,\dots,S_d$ obtained by Theorem 2 are *infinite*. This is the first result in Ramsey theory where a density condition yields the existence of an infinite object instead of a sequence of finite objects of arbitrarily large cardinality.

1.2. **Outline of the argument.** The proof of Theorem 2 proceeds by induction on the number of trees and is based on combinatorial tools. In particular, at the process of establishing $\mathrm{DHL}(d+1)$ we use, as pigeonhole principles, $\mathrm{DHL}(d)$ as well as Theorem 1. One can actually determine which instance of Theorem 1 is needed in order to prove Theorem 2 for a fixed tuple $(T_1,\dots,T_{d+1})$ of homogeneous trees: if $b_i$ is the branching number of $T_i$ for every $i\in\{1,\dots,d+1\}$, then one needs to use $\mathrm{HL}\big(\sum_{i=1}^d b_i\big)$.

Let us briefly discuss the main steps (for unexplained terminology and notation we refer to §2). Assume that we have proven $\mathrm{DHL}(d)$ for some $d\geqslant 1$ and that we are given a tuple $(T_1,\dots,T_d,W)$ of homogeneous trees, a constant $0<\varepsilon\leqslant 1$ and a subset $D$ of the level product of $(T_1,\dots,T_d,W)$ satisfying

$$|D\cap\big(T_1(n)\times\dots\times T_d(n)\times W(n)\big)| \geqslant \varepsilon |T_1(n)\times\dots\times T_d(n)\times W(n)|$$

for infinitely many $n\in\mathbb{N}$. Using a Fubini-type argument and $\mathrm{DHL}(d)$, we can find a vector strong subtree $\mathbf{S}$ of $(T_1,\dots,T_d)$, with the following property: there exists a strictly increasing sequence $(l_n)$ in $\mathbb{N}$ such that for every $n\in\mathbb{N}$ and every $\mathbf{s}\in\otimes\mathbf{S}(n)$ the section $D(\mathbf{s})=\{w\in W:(\mathbf{s},w)\in D\}$ of $D$ at $\mathbf{s}$ is a subset of $W(l_n)$ of cardinality at least $\varepsilon/2|W(l_n)|$. This property of the section map $D\colon \otimes\mathbf{S}\to 2^W$ is abstracted in Definition 7 in the main text. We call such maps *dense level selections.*

The next step (which is the most demanding part of the proof) is to show that for every dense level selection $D\colon \otimes\mathbf{S}\to 2^W$ there exists a vector strong subtree $\mathbf{R}$ of $\mathbf{S}$ such that the sets $\{D(\mathbf{r}):\mathbf{r}\in\otimes\mathbf{R}\}$ are mutually "correlated"; this is the content of Theorem 9 in the main text. Precisely, we show that there exist a constant $0<\theta\leqslant 1$, a vector strong subtree $\mathbf{R}$ of $\mathbf{S}$ and a node $w\in D(\mathbf{r})$, where $\mathbf{r}$ is the root of $\mathbf{R}$, such that for every vector strong subtree $\mathbf{Z}$ of $\mathbf{R}$ with the same root as $\mathbf{R}$ the density of the set

$$\bigcap_{\mathbf{z}\in\otimes\mathbf{Z}(1)} D(\mathbf{z})$$

relative to *every* immediate successor $w'$ of $w$, is at least $\theta$. The main difficulty in the proof of this result lies in the fact that the number of sets in the above intersection increases *exponentially* with respect to the dimension[1]. It is worth pointing out that in this step we use again DHL($d$) as pigeonhole principle, but in a slightly different form (Proposition 11 in the main text).

With Theorem 9 at hand, one can perform a recursive construction in order to find a vector strong subtree $(Z_1,\dots,Z_d,V)$ of $(T_1,\dots,T_d,W)$ whose level product is a subset of $D$. This recursive selection, however, is rather unusual since we actually construct an infinite chain of $(T_1,\dots,T_d)$ and a strong subtree $V$ of $W$ with special properties. The desired vector strong subtree is then obtained using an "unfolding" argument.

1.3. **Organization of the paper.** The paper is organized as follows. In §2 we set up our notation and terminology. The next section is devoted to the study of a natural class of finite vector trees, which we call *vector fans*. In §4 we introduce the notion of a dense level selection we mentioned above; the main result in this section is Theorem 9. The proof of Theorem 2 is completed in §5. Finally, in §6 we make some comments.

1.4. **Acknowledgments.** We would like to thank S. Todorcevic for his comments and remarks as well as for historical information concerning the problem solved in the paper. The first author was supported by NSF grant DMS-0903558.

## 2. Background material

By $\mathbb{N}=\{0,1,2,\dots\}$ we denote the natural numbers. The cardinality of a set $X$ will be denoted by $|X|$.

2.1. **Trees and subtrees.** By the term *tree* we mean a partially ordered set $(T,<)$ such that the set $\{s\in T: s<t\}$ is finite and linearly ordered under $<$ for every $t\in T$. The cardinality of this set is defined to be the *length of $t$ in $T$* and is denoted by $\ell_T(t)$. For every $n\in\mathbb{N}$ the *$n$-level of $T$*, denoted by $T(n)$, is defined to be the set $\{t\in T:\ell_T(t)=n\}$. The *height* of $T$, denoted by $h(T)$, is defined as follows. If there exists $k\in\mathbb{N}$ with $T(k)=\emptyset$, then we set $h(T)=\max\{n\in\mathbb{N}:T(n)\neq\emptyset\}+1$; otherwise, we set $h(T)=\infty$.

For every $t\in T$ by $\mathrm{Succ}_T(t)$ we denote the set of *successors of $t$ in $T$*, that is,

$$\mathrm{Succ}_T(t)=\{s\in T: t\leqslant s\}. \tag{1}$$

[1] A typical phenomenon in the proof of several combinatorial results is that the "low dimensional" cases are relatively easy to prove and the full complexity appears after a critical threshold. In the case of the density Halpern–Läuchli theorem this critical threshold is dimension 3. In particular, the authors are aware of a different, and in a sense more effective, proof of DHL(2). This proof, however, cannot be generalized to higher dimensions.

The *set of immediate successors of $t$ in $T$* is the subset of $\mathrm{Succ}_T(t)$ defined by $\mathrm{ImmSucc}_T(t) = \{s \in T : t \leqslant s \text{ and } \ell_T(s) = \ell_T(t)+1\}$. More generally, for every subset $F$ of $T$ we set $\mathrm{Succ}_T(F) = \{s \in T : \text{exists } t \in F \text{ with } t \leqslant s\}$.

Let $n \in \mathbb{N}$ with $n < h(T)$ and $F \subseteq T(n)$. The *density* of $F$ is defined by

$$\mathrm{dens}(F) \coloneqq \frac{|F|}{|T(n)|}. \tag{2}$$

More generally, for every $m \in \mathbb{N}$ with $m \leqslant n$ and every $t \in T(m)$ the *density of $F$ relative to the node $t$* is defined by

$$\mathrm{dens}(F \,|\, t) \coloneqq \frac{|F \cap \mathrm{Succ}_T(t)|}{|T(n) \cap \mathrm{Succ}_T(t)|}. \tag{3}$$

A *subtree* $S$ of a tree $(T, <)$ is a subset of $T$ viewed as a tree equipped with the induced partial ordering. For every $n \in \mathbb{N}$ with $n < h(T)$ we set

$$T \upharpoonright n = T(0) \cup \cdots \cup T(n). \tag{4}$$

Notice that $h(T \upharpoonright n) = n+1$. An *initial subtree* of $T$ is a subtree of $T$ of the form $T \upharpoonright n$ for some $n \in \mathbb{N}$.

Finally, we recall that a tree $T$ is said to be *pruned* (respectively, *finitely branching*) if for every $t \in T$ the set of immediate successors of $t$ in $T$ is nonempty (respectively, finite). It is said to be *uniquely rooted* if $|T(0)| = 1$. The *root* of a uniquely rooted tree $T$ is defined to be the node $T(0)$.

2.2. **Vector trees and level products.** A *vector tree* $\mathbf{T}$ is a nonempty finite sequence of trees having a common height; this common height is defined to be the *height* of $\mathbf{T}$ and will be denoted by $h(\mathbf{T})$. We notice that, throughout the paper, we will start the enumeration of vector trees with 1 instead of 0.

For every vector tree $\mathbf{T} = (T_1, \dots, T_d)$ and every $n \in \mathbb{N}$ with $n < h(\mathbf{T})$ we set

$$\mathbf{T} \upharpoonright n = (T_1 \upharpoonright n, \dots, T_d \upharpoonright n). \tag{5}$$

A vector tree of this form is called a *vector initial subtree* of $\mathbf{T}$. Also let

$$\mathbf{T}(n) = \big(T_1(n), \dots, T_d(n)\big) \tag{6}$$

and

$$\otimes\mathbf{T}(n) = T_1(n) \times \cdots \times T_d(n). \tag{7}$$

The *level product of* $\mathbf{T}$, denoted by $\otimes\mathbf{T}$, is defined to be the set

$$\bigcup_{n<h(\mathbf{T})} \otimes\mathbf{T}(n). \tag{8}$$

If $\mathbf{t} = (t_1, \dots, t_d) \in \otimes\mathbf{T}$, then we define $\ell_{\mathbf{T}}(\mathbf{t})$ to be the unique $n \in \mathbb{N}$ such that $\mathbf{t} \in \otimes\mathbf{T}(n)$. Also we set

$$\mathrm{Succ}_{\mathbf{T}}(\mathbf{t}) = \big(\mathrm{Succ}_{T_1}(t_1), \dots, \mathrm{Succ}_{T_d}(t_d)\big). \tag{9}$$

Finally, we say that a vector tree $\mathbf{T} = (T_1, \dots, T_d)$ is *pruned* (respectively, *finitely branching*, *uniquely rooted*) if $T_i$ is pruned (respectively, finitely branching, uniquely rooted) for every $i \in \{1, \dots, d\}$. Notice that if $\mathbf{T}$ is uniquely rooted, then $\mathbf{T}(0) = \otimes\mathbf{T}(0)$; the element $\mathbf{T}(0)$ will be called the *root* of $\mathbf{T}$.

2.3. **Strong subtrees and vector strong subtrees.** Let $T$ be a pruned, finitely branching and uniquely rooted tree. A subtree $S$ of $T$ is said to be *strong* provided that: (a) $S$ is uniquely rooted, (b) for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ with $S(n) \subseteq T(m)$, and (c) for every $s \in S$ and every $t \in \mathrm{ImmSucc}_T(s)$ there exists a unique node $s' \in \mathrm{ImmSucc}_S(s)$ such that $t \leqslant s'$. The *level set* of a strong subtree $S$ of $T$ is defined to be the set

$$L_T(S) \coloneqq \{m \in \mathbb{N} : \text{exists } n \in \mathbb{N} \text{ with } S(n) \subseteq T(m)\}. \tag{10}$$

A *finite strong subtree* of $T$ is an initial subtree of a strong subtree of $T$.

The above concepts are naturally extended to vector trees. Specifically, let $\mathbf{T} = (T_1, \dots, T_d)$ be a pruned, finitely branching and uniquely rooted vector tree. A *vector strong subtree of* $\mathbf{T}$ is a vector tree $\mathbf{S} = (S_1, \dots, S_d)$ such that $S_i$ is a strong subtree of $T_i$ for every $i \in \{1, \dots, d\}$ and $L_{T_1}(S_1) = \dots = L_{T_d}(S_d)$. A *finite vector strong subtree of* $\mathbf{T}$ is a vector initial subtree of a vector strong subtree of $\mathbf{T}$.

2.4. **Homogeneous trees and vector homogeneous trees.** Let $b \in \mathbb{N}$ with $b \geqslant 2$. By $b^{<\mathbb{N}}$ we shall denote the set of all finite sequences having values in $\{0, \dots, b-1\}$. The empty sequence is denoted by $\emptyset$ and is included in $b^{<\mathbb{N}}$. We view $b^{<\mathbb{N}}$ as a tree equipped with the (strict) partial order $\sqsubset$ of end-extension. For every $n \in \mathbb{N}$ by $b^n$ we denote the $n$-level of $b^{<\mathbb{N}}$. If $n \geqslant 1$, then $b^{<n}$ stands for the initial subtree of $b^{<\mathbb{N}}$ of height $n$. By $<_{\mathrm{lex}}$ we denote the usual lexicographical order on $b^n$. For every $t, s \in b^{<\mathbb{N}}$ by $t^\frown s$ we denote their concatenation.

As we have already mentioned in the introduction, a homogeneous tree $T$ is a uniquely rooted tree such that every node in $T$ has exactly $b$ immediate successors, where $b \geqslant 2$ is the branching number of $T$. In several cases, we need to enumerate the set of nodes of a level of $T$. There is, of course, no problem for selecting an enumeration. But an arbitrary enumeration might lack compatibility when passing to subtrees. This problem can be resolved by restricting our attention to the class of strong subtrees of a *fixed* homogeneous tree. It is, of course, clear that all homogeneous trees with the same branching number are pairwise isomorphic, and so, such a restriction will have no effect in the generality of our results.

**Convention.** *In the rest of this paper by the term "homogeneous tree" we mean a strong subtree of $b^{<\mathbb{N}}$ for some $b \in \mathbb{N}$ with $b \geqslant 2$. For every homogeneous tree $T$ by $b_T$ we shall denote the branching number of $T$ and we set $B_T = b_T^{<\mathbb{N}}$. We follow the same conventions for vector trees. Precisely, by the term "vector homogeneous tree" we mean a vector strong subtree of $(b_1^{<\mathbb{N}}, \dots, b_d^{<\mathbb{N}})$ for some $b_1, \dots, b_d \in \mathbb{N}$ with*

*$b_i \geqslant 2$ for every $i \in \{1,\dots,d\}$. For every vector homogeneous tree $\mathbf{T} = (T_1,\dots,T_d)$ we set $b_{\mathbf{T}} \coloneqq (b_{T_1},\dots,b_{T_d})$ and $\mathbf{B_T} \coloneqq (b_{T_1}^{<\mathbb{N}},\dots,b_{T_d}^{<\mathbb{N}})$.*

The above convention enables us to effectively enumerate the set of immediate successors of a given node of a homogeneous tree $T$. Specifically, for every $t \in T$ and every $p \in \{0,\dots,b_T - 1\}$ we set

$$t^{\frown_T} p \coloneqq \mathrm{ImmSucc}_T(t) \cap \mathrm{Succ}_{B_T}(t^\frown p). \tag{11}$$

Notice that

$$\mathrm{ImmSucc}_T(t) = \big\{ t^{\frown_T} p : p \in \{0,\dots,b_T - 1\}\big\}.$$

Also observe that for every $p,q \in \{0,\dots,b_T - 1\}$ we have $t^{\frown_T} p <_{\mathrm{lex}} t^{\frown_T} q$ if and only if $p < q$.

2.5. **Canonical embeddings and vector canonical embeddings.** Let $T$ and $S$ be two homogeneous trees with the same branching number. We say that a map $f\colon T \to S$ is a *canonical embedding* if the following conditions are satisfied.

(a) For every $t, t' \in T$ we have $\ell_T(t) = \ell_T(t')$ if and only if $\ell_S\big(f(t)\big) = \ell_S\big(f(t')\big)$.
(b) For every $t \in T$ and $p \in \{0,\dots,b_T - 1\}$ we have $f(t^{\frown_T} p) \in \mathrm{Succ}_S\big(f(t)^{\frown_S} p\big)$.

It is easy to verify that for every canonical embedding $f\colon T \to S$ the following hold: (i) for every $t,t' \in T$ we have $t \sqsubset t'$ if and only if $f(t) \sqsubset f(t')$, (ii) $f$ is an injection, and (iii) the image $f(T)$ of $T$ under $f$ is a strong subtree of $S$.

Also notice that there exists a *unique* bijection between $T$ and $S$ satisfying the above conditions. This unique bijection will be called the *canonical isomorphism* between $T$ and $S$ and will be denoted by $\mathrm{I}(T,S)$.

We proceed to define the notion of a "vector canonical embedding". It is a kind of "tensorization" of a finite sequence of canonical embeddings with special properties. Specifically, let $\mathbf{T} = (T_1,\dots,T_d)$ and $\mathbf{S} = (S_1,\dots,S_d)$ be two vector homogeneous trees with $b_{\mathbf{T}} = b_{\mathbf{S}}$. For every $i \in \{1,\dots,d\}$ let $f_i\colon T_i \to S_i$ be a canonical embedding and assume that for every $n \in \mathbb{N}$ and every $\mathbf{t} = (t_1,\dots,t_d) \in \otimes\mathbf{T}(n)$ we have $\ell_{S_1}\big(f_1(t_1)\big) = \dots = \ell_{S_d}\big(f_d(t_d)\big)$. This assumption permits us to define a map $(\otimes_{i=1}^d f_i)\colon \otimes\mathbf{T} \to \otimes\mathbf{S}$ by the rule

$$(\otimes_{i=1}^d f_i)\big((t_1,\dots,t_d)\big) = \big(f_1(t_1),\dots,f_d(t_d)\big). \tag{12}$$

A map of this form will be called a *vector canonical embedding* of $\otimes\mathbf{T}$ into $\otimes\mathbf{S}$. The *vector canonical isomorphism* between $\otimes\mathbf{T}$ and $\otimes\mathbf{S}$ is defined to be the map $(\otimes_{i=1}^d \mathrm{I}(T_i,S_i))$ and will be denoted by $\mathbf{I}(\mathbf{T},\mathbf{S})$.

## 3. Fans and vector fans

We start with the following definition.

**Definition 3.** *Let $T$ be a homogeneous tree. We say that a tree $F$ is a* fan of $T$ *if $F$ is of the form $R \upharpoonright 1$ for some strong subtree $R$ of $T$. The set of all fans of $T$ will be denoted by* $\mathrm{Fan}(T)$.

Next we introduce the higher-dimensional analogues of fans.

**Definition 4.** *Let* $\mathbf{T}$ *be a vector homogeneous tree. We say that vector tree* $\mathbf{F}$ *is a* vector fan of $\mathbf{T}$ *if* $\mathbf{F}$ *is of the form* $\mathbf{R} \upharpoonright 1$ *for some vector strong subtree* $\mathbf{R}$ *of* $\mathbf{T}$*. The set of all vector fans of* $\mathbf{T}$ *will be denoted by* $\mathrm{Fan}(\mathbf{T})$*.*

We view vector fans as the fundamental building blocks of vector homogeneous trees. This point of view is crucial for the proof of Theorem 2. Also we make two simple observations. Firstly, we notice that if $\mathbf{R}$ is a vector strong subtree of a vector homogeneous tree $\mathbf{T}$, then $\mathrm{Fan}(\mathbf{R}) \subseteq \mathrm{Fan}(\mathbf{T})$. Secondly, we observe that if $\mathbf{F}$ is a vector fan of $\mathbf{T}$, then $\mathbf{F}(0) \in \otimes\mathbf{T}$ and $\otimes\mathbf{F}(1) \subseteq \otimes\mathbf{T}$.

We will need two combinatorial results concerning vector fans. The first result follows from [20, Theorem 1.3].

**Proposition 5.** *Let* $\mathbf{T}$ *be a vector homogeneous tree and set* $\mathbf{t}_0 = \mathbf{T}(0)$*. Then for every finite coloring*

$$\mathrm{Fan}(\mathbf{T}) = \mathcal{C}_0 \cup \cdots \cup \mathcal{C}_r$$

*there exist* $m \in \{0, \ldots, r\}$ *and a vector strong subtree* $\mathbf{Z}$ *of* $\mathbf{T}$ *with* $\mathbf{Z}(0) = \mathbf{t}_0$ *such that* $\mathbf{F} \in \mathcal{C}_m$ *for every* $\mathbf{F} \in \mathrm{Fan}(\mathbf{Z})$ *with* $\mathbf{F}(0) = \mathbf{t}_0$*.*

To state the second result we need to introduce some notation. For every vector homogeneous tree $\mathbf{T}$ and every $n \in \mathbb{N}$ with $n \geqslant 1$ we set

$$\mathrm{Fan}(\mathbf{T}, n) \coloneqq \big\{\mathbf{F} \in \mathrm{Fan}(\mathbf{T}) : \otimes\mathbf{F}(1) \subseteq \otimes\mathbf{T}(n)\big\}. \tag{13}$$

**Proposition 6.** *Let* $\mathbf{T} = (T_1, \ldots, T_d)$ *be a vector homogeneous tree. For every* $n \in \mathbb{N}$ *with* $n \geqslant 1$ *let* $\mathcal{F}_n$ *be a subset of* $\mathrm{Fan}(\mathbf{T})$ *with the following property.*

(P) *For every vector strong subtree* $\mathbf{S}$ *of* $\mathbf{T}$ *there exists* $\mathbf{F} \in \mathrm{Fan}(\mathbf{S}) \cap \mathcal{F}_n$ *with* $\mathbf{F}(0) = \mathbf{S}(0)$*.*

*Also let* $\mathbf{R}$ *be a vector strong subtree* $\mathbf{T}$ *and set* $\mathbf{r}_0 = \mathbf{R}(0)$*. Then there exists a vector strong subtree* $\mathbf{Z}$ *of* $\mathbf{R}$ *with* $\mathbf{Z}(0) = \mathbf{r}_0$ *such that for every* $n \in \mathbb{N}$ *with* $n \geqslant 1$ *and every* $\mathbf{F} \in \mathrm{Fan}(\mathbf{Z}, n)$ *with* $\mathbf{F}(0) = \mathbf{r}_0$ *we have* $\mathbf{F} \in \mathcal{F}_n$*.*

Proposition 6 can be hardly characterized as new since it follows using fairly standard arguments (see, e.g., [5, 20, 21]). Nevertheless, we have decided to include a proof for two reasons. The first one is for self-containedness. Secondly, because we want to emphasize which instance of Theorem 1 is needed for the proof.

*Proof of Proposition* 6. We write $\mathbf{R} = (R_1, \ldots, R_d)$ and $\mathbf{r}_0 = (r_1, \ldots, r_d)$. Recursively, we shall construct a sequence $(\mathbf{R}_n)$ of vector strong subtrees of $\mathbf{R}$ such that for every $n \in \mathbb{N}$ the following are satisfied.

(a) $\mathbf{R}_n(0) = \mathbf{r}_0$.
(b) $\mathbf{R}_{n+1}$ is a vector strong subtree of $\mathbf{R}_n$ and $\mathbf{R}_{n+1} \upharpoonright n = \mathbf{R}_n \upharpoonright n$.
(c) If $n \geqslant 1$, then for every $\mathbf{F} \in \mathrm{Fan}(\mathbf{R}_n, n)$ with $\mathbf{F}(0) = \mathbf{r}_0$ we have $\mathbf{F} \in \mathcal{F}_n$.

Assuming that the construction has been carried out, we define $\mathbf{Z}$ to be the unique vector strong subtree of $\mathbf{R}$ satisfying $\mathbf{Z}(n) = \mathbf{R}_n(n)$ for every $n \in \mathbb{N}$. It is easily seen that $\mathbf{Z}$ is the desired vector tree.

We proceed to the construction. For $n = 0$ we set $\mathbf{R}_0 = \mathbf{R}$ and we notice that with this choice property (a) is satisfied (the other properties are meaningless for $n = 0$). Assume that for some $n \in \mathbb{N}$ we have constructed the vector trees $\mathbf{R}_0, \dots, \mathbf{R}_n$ so that (a), (b) and (c) are satisfied and write $\mathbf{R}_n = (R_1^n, \dots, R_d^n)$. For the construction of the vector tree $\mathbf{R}_{n+1}$ we need to introduce some notation and terminology.

**(A)** Let $i \in \{1, \dots, d\}$ be arbitrary. Set $M_i = b_{T_i}^{n+1}$ and notice that the cardinality of the $(n+1)$-level $R_i^n(n+1)$ of $R_i^n$ is $M_i$. We write the set $R_i^n(n+1)$ in lexicographical increasing order as $\{t_1^i <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} t_{M_i}^i\}$ and we set $V_j^i = \mathrm{Succ}_{R_i^n}(t_j^i)$ for every $j \in \{1, \dots, M_i\}$.

**(B)** We define $\mathbf{V}$ to be the vector tree $(V_j^i)_{i=1,j=1}^{d \;\; M_i}$. For every vector strong subtree $\mathbf{U}$ of $\mathbf{V}$ we can naturally associate a vector strong subtree $\mathbf{R}^{\mathbf{U}} = (R_1^{\mathbf{U}}, \dots, R_d^{\mathbf{U}})$ of $\mathbf{R}_n$. Precisely, write $\mathbf{U}$ as $(U_j^i)_{i=1,j=1}^{d \;\; M_i}$ and for every $i \in \{1, \dots, d\}$ set

$$R_i^{\mathbf{U}} \coloneqq (R_i^n \upharpoonright n) \cup U_1^i \cup \dots \cup U_{M_i}^i.$$

Observe that $\mathbf{R}^{\mathbf{U}} \upharpoonright n = \mathbf{R}_n \upharpoonright n$. Also notice that $\mathbf{R}^{\mathbf{V}} = \mathbf{R}_n$.

**(C)** Next we introduce the notion of a *strong position.* It is a technical tool for the construction of the vector tree $\mathbf{R}_{n+1}$. Specifically, a strong position $\mathcal{P}$ is defined to be a finite sequence $(P_1, \dots, P_d)$ such that

- (I) $P_i \subseteq \{1, \dots, M_i\}$ for every $i \in \{1, \dots, d\}$, and
- (II) if $F_i = \{r_i\} \cup \{t_j^i : j \in P_i\}$ for every $i = \{1, \dots, d\}$, then $\mathbf{F} = (F_1, \dots, F_d)$ is a vector fan of $\mathbf{R}_n$.

By (II), if $\mathcal{P} = (P_1, \dots, P_d)$ is a strong position, then $|P_i| = b_{T_i}$ for every $i \in \{1, \dots, d\}$. For every $\mathbf{v} = (v_j^i)_{i=1,j=1}^{d \;\; M_i} \in \otimes\mathbf{V}$ and every strong position $\mathcal{P} = (P_1, \dots, P_d)$ we set

$$\mathbf{F}_{\mathbf{v},\mathcal{P}} \coloneqq \big(\{r_1\} \cup \{v_j^1 : j \in P_1\}, \dots, \{r_d\} \cup \{v_j^d : j \in P_d\}\big).$$

By (II), it is clear that $\mathbf{F}_{\mathbf{v},\mathcal{P}} \in \mathrm{Fan}(\mathbf{R}^{\mathbf{V}})$ and $\mathbf{F}_{\mathbf{v},\mathcal{P}}(0) = \mathbf{r}_0$. We isolate the following fact: *if* $\mathbf{U}$ *is a vector strong subtree of* $\mathbf{V}$*,* $k \in \mathbb{N}$ *with* $k \geqslant n+1$ *and* $\mathbf{F} \in \mathrm{Fan}(\mathbf{R}^{\mathbf{U}}, k)$ *with* $\mathbf{F}(0) = \mathbf{r}_0$*, then there exist a unique strong position* $\mathcal{P}$ *and an element* $\mathbf{u}$ *of* $\otimes\mathbf{U}$ (*not necessarily unique*) *such that* $\mathbf{F} = \mathbf{F}_{\mathbf{u},\mathcal{P}}$. The existence of $\mathcal{P}$ and $\mathbf{u}$ is a rather direct consequence of the relevant definitions.

After this preliminary discussion we are ready to proceed to the construction of the vector tree $\mathbf{R}_{n+1}$. For every strong position $\mathcal{P}$ let

$$\mathcal{G}_{\mathcal{P}} = \big\{\mathbf{v} \in \otimes\mathbf{V} : \mathbf{F}_{\mathbf{v},\mathcal{P}} \in \mathcal{F}_{n+1}\big\}.$$

Applying successively Theorem 1, we find a vector strong subtree $\mathbf{U}_0$ of $\mathbf{V}$ such that for every strong position $\mathcal{P}$ we have that either $\otimes\mathbf{U}_0 \subseteq \mathcal{G}_{\mathcal{P}}$ or $\otimes\mathbf{U}_0 \cap \mathcal{G}_{\mathcal{P}} = \emptyset$. Notice that the set $\mathcal{G}_{\mathcal{P}}$ depends only on the coordinates determined by $\mathcal{P}$, and so, each time we need to apply $\mathrm{HL}\big(\sum_{i=1}^{d} b_{T_i}\big)$.

We set $\mathbf{R}_{n+1} = \mathbf{R}^{\mathbf{U}_0}$. The vector tree $\mathbf{R}_{n+1}$ is the desired one. It is clear that we only need to check that property (c) is satisfied. So, let $\mathbf{F} \in \mathrm{Fan}(\mathbf{R}_{n+1}, n+1)$ with $\mathbf{F}(0) = \mathbf{r}_0$ be arbitrary. As we have already mentioned in **(C)** above, there exist a unique strong position $\mathcal{Q} = (Q_1, \dots, Q_d)$ and an element $\mathbf{u}$ of $\otimes\mathbf{U}_0$ (not necessarily unique) such that $\mathbf{F} = \mathbf{F}_{\mathbf{u},\mathcal{Q}}$. In order to show that $\mathbf{F} \in \mathcal{F}_{n+1}$ it is enough to prove that $\otimes\mathbf{U}_0 \subseteq \mathcal{G}_{\mathcal{Q}}$. To this end, we will argue by contradiction. So, assume that $\otimes\mathbf{U}_0 \cap \mathcal{G}_{\mathcal{Q}} = \emptyset$. We write $\mathbf{U}_0$ as $(U^i_j)_{i=1,j=1}^{d \ \ M_i}$ and for every $i \in \{1, \dots, d\}$ we set $S_i = \{r_i\} \cup \{U^i_j : j \in Q_i\}$. Let $\mathbf{S} = (S_1, \dots, S_d)$ and notice that $\mathbf{S}$ is a vector strong subtree of $\mathbf{T}$ with $\mathbf{S}(0) = \mathbf{r}_0$. Let $\mathbf{F}' \in \mathrm{Fan}(\mathbf{S})$ with $\mathbf{F}'(0) = \mathbf{r}_0$. Observe that there exists $k \in \mathbb{N}$ with $k \geqslant n+1$ such that $\mathbf{F}' \in \mathrm{Fan}(\mathbf{R}^{\mathbf{U}_0}, k)$. Hence, there exists an element $\mathbf{u}' \in \otimes\mathbf{U}_0$ (not necessarily unique) such that $\mathbf{F}' = \mathbf{F}_{\mathbf{u}',\mathcal{Q}}$. Since $\mathbf{u}' \in \otimes\mathbf{U}_0$ we see that $\mathbf{u}' \notin \mathcal{G}_{\mathcal{Q}}$ and so $\mathbf{F}' \notin \mathcal{F}_{n+1}$. In other words, for every $\mathbf{F}' \in \mathrm{Fan}(\mathbf{S})$ with $\mathbf{F}'(0) = \mathbf{S}(0)$ we have that $\mathbf{F}' \notin \mathcal{F}_{n+1}$. This contradicts property (P). Therefore, $\otimes\mathbf{U}_0 \subseteq \mathcal{G}_{\mathcal{Q}}$ and so the vector tree $\mathbf{R}_{n+1}$ has the desired properties.

This completes the recursive construction, and as we have already indicated, the proof of Proposition 6 is also completed. □

## 4. Dense level selections

4.1. **Definitions and statement of the main result.** We start by introducing the following definition.

**Definition 7.** *Let* $\mathbf{T} = (T_1, \dots, T_d)$ *be a vector homogeneous tree,* $W$ *a homogeneous tree and* $0 < \varepsilon \leqslant 1$. *We say that a map* $D\colon \otimes\mathbf{T} \to 2^W$ *is an* $\varepsilon$-dense level selection *if there exists a strictly increasing sequence* $(l_n)$ *in* $\mathbb{N}$ *such that for every* $n \in \mathbb{N}$ *and every* $\mathbf{t} \in \otimes\mathbf{T}(n)$ *we have* $D(\mathbf{t}) \subseteq W(l_n)$ *and* $\mathrm{dens}\big(D(\mathbf{t})\big) \geqslant \varepsilon$.

The next definition is a crucial conceptual step towards the proof of Theorem 2.

**Definition 8.** *Let* $\mathbf{T} = (T_1, \dots, T_d)$ *be a vector homogeneous tree,* $W$ *a homogeneous tree,* $0 < \varepsilon \leqslant 1$ *and* $D\colon \otimes\mathbf{T} \to 2^W$ *an* $\varepsilon$*-dense level selection. Also let* $\mathbf{R}$ *be a vector strong subtree of* $\mathbf{T}$*,* $w \in W$ *and* $0 < \theta \leqslant 1$. *We say that the pair* $(\mathbf{R}, w)$ *is* strongly $\theta$-correlated with respect to $D$ *if, setting* $\mathbf{r}_0 = \mathbf{R}(0)$*, the following conditions are satisfied.*

(C1) *We have* $w \in D(\mathbf{r}_0)$.

(C2) *For every* $\mathbf{F} \in \mathrm{Fan}(\mathbf{R})$ *with* $\mathbf{F}(0) = \mathbf{r}_0$ *and every* $p \in \{0, \dots, b_W - 1\}$ *we have*

$$\mathrm{dens}\Big(\bigcap_{\mathbf{r} \in \otimes\mathbf{F}(1)} D(\mathbf{r}) \,\Big|\, w^{\frown_W} p\Big) \geqslant \theta. \tag{14}$$

We are now ready to state the main result in this section.

**Theorem 9.** *Let $d \geqslant 1$ and assume that* $\mathrm{DHL}(d)$ *holds. Also let* $\mathbf{T} = (T_1, \dots, T_d)$ *be a vector homogeneous tree,* $W$ *a homogeneous tree,* $0 < \varepsilon \leqslant 1$ *and* $D \colon \otimes\mathbf{T} \to 2^W$ *an* $\varepsilon$*-dense level selection. Then there exist a vector strong subtree* $\mathbf{R}$ *of* $\mathbf{T}$*,* $w \in W$ *and* $0 < \theta \leqslant 1$ *such that the pair* $(\mathbf{R}, w)$ *is strongly* $\theta$*-correlated with respect to* $D$*.*

The proof of Theorem 9 will be given in §4.3. At this point, let us isolate the following consequence of Theorem 9. It will be of particular importance in §5.

**Corollary 10.** *Let $d \geqslant 1$ and assume that* $\mathrm{DHL}(d)$ *holds. Also let* $\mathbf{T} = (T_1, \dots, T_d)$ *be a vector homogeneous tree,* $W$ *a homogeneous tree,* $0 < \varepsilon \leqslant 1$ *and* $D \colon \otimes\mathbf{T} \to 2^W$ *an* $\varepsilon$*-dense level selection. Then there exist a vector strong subtree* $\mathbf{S}$ *of* $\mathbf{T}$ *and for every* $\mathbf{s} \in \otimes\mathbf{S}$ *a node* $w_{\mathbf{s}} \in W$ *and a constant* $0 < \theta_{\mathbf{s}} \leqslant 1$ *with the following property. For every* $\mathbf{s} \in \otimes\mathbf{S}$ *and every vector strong subtree* $\mathbf{Z}$ *of* $\mathrm{Succ}_{\mathbf{S}}(\mathbf{s})$ *with* $\mathbf{Z}(0) = \mathbf{s}$ *the pair* $(\mathbf{Z}, w_{\mathbf{s}})$ *is strongly* $\theta_{\mathbf{s}}$*-correlated with respect to* $D$*.*

*Proof.* We start with the following observation. Let $\mathbf{R}$ be a vector strong subtree of $\mathbf{T}$, $w \in W$ and $0 < \theta \leqslant 1$ and assume that the pair $(\mathbf{R}, w)$ is strongly $\theta$-correlated with respect to $D$. Then for every vector strong subtree $\mathbf{Z}$ of $\mathbf{R}$ with $\mathbf{Z}(0) = \mathbf{R}(0)$ the pair $(\mathbf{Z}, w)$ is also strongly $\theta$-correlated with respect to $D$.

Therefore, what we need to find is a vector strong subtree $\mathbf{S}$ of $\mathbf{T}$, a family $\{w_{\mathbf{s}} : \mathbf{s} \in \otimes\mathbf{S}\}$ in $W$ and a family $\{\theta_{\mathbf{s}} : \mathbf{s} \in \otimes\mathbf{S}\}$ of reals in $(0,1]$ such that for every $\mathbf{s} \in \otimes\mathbf{S}$ the pair $\big(\mathrm{Succ}_{\mathbf{S}}(\mathbf{s}), w_{\mathbf{s}}\big)$ is strongly $\theta_{\mathbf{s}}$-correlated with respect to $D$. This can be proved using $\mathrm{HL}\big(\sum_{i=1}^d b_{T_i}\big)$ as pigeonhole principle, Theorem 9 and arguing as in the proof of Proposition 6. We prefer, however, to give a very simple proof which is based on Theorem 9 and on the work of Milliken on Ramsey properties of strong subtrees. For every vector strong subtree $\mathbf{Z}$ of $\mathbf{T}$ let $[\mathbf{Z}]_{\mathrm{strong}}$ be the set of all vector strong subtrees of $\mathbf{Z}$ and notice that $[\mathbf{Z}]_{\mathrm{strong}}$ is $G_\delta$ (hence Polish) subspace of $2^{T_1} \times \dots \times 2^{T_d}$. Now let $\mathcal{C}$ be the subset of $[\mathbf{T}]_{\mathrm{strong}}$ defined by

$$\mathbf{R} \in \mathcal{C} \quad \Leftrightarrow \quad \begin{array}{l} \text{there exist } w \in W \text{ and } 0 < \theta \leqslant 1 \text{ such that the pair} \\ (\mathbf{R}, w) \text{ is strongly } \theta\text{-correlated with respect to } D. \end{array}$$

Notice that $\mathcal{C}$ is an $F_\sigma$ subset of $[\mathbf{T}]_{\mathrm{strong}}$. Moreover, by Theorem 9, we see that $\mathcal{C} \cap [\mathbf{Z}]_{\mathrm{strong}} \neq \emptyset$ for every vector strong subtree $\mathbf{Z}$ of $\mathbf{T}$. By [21, Theorem 2.1], there exists a vector strong subtree $\mathbf{S}$ of $\mathbf{T}$ such that $[\mathbf{S}]_{\mathrm{strong}} \subseteq \mathcal{C}$. Observing that $\mathrm{Succ}_{\mathbf{S}}(\mathbf{s}) \in [\mathbf{S}]_{\mathrm{strong}}$ for every $\mathbf{s} \in \otimes\mathbf{S}$, the result follows. □

4.2. **A consequence of** $\mathrm{DHL}(d)$**.** In this subsection we shall obtain a consequence of $\mathrm{DHL}(d)$ which is stated within the context of dense level selections. It will be used in the proof of Theorem 9.

**Proposition 11.** *Let $d \geqslant 1$ and assume that* $\mathrm{DHL}(d)$ *holds. Let* $\mathbf{Z} = (Z_1, \dots, Z_d)$ *be a vector homogeneous tree,* $W$ *a homogeneous tree,* $0 < \eta \leqslant 1$ *and* $B \colon \otimes\mathbf{Z} \to 2^W$

*an $\eta$-dense level selection. Then for every $p \in \{0,\dots,b_W-1\}$ there exist $\mathbf{F} \in \mathrm{Fan}(\mathbf{Z})$ and $w \in B(\mathbf{z}_0)$, where $\mathbf{z}_0 = \mathbf{F}(0)$, such that*

$$\bigcap_{\mathbf{z}\in\otimes\mathbf{F}(1)} B(\mathbf{z}) \cap \mathrm{Succ}_W(w^\frown{}^W p) \neq \emptyset.$$

*Proof.* We fix $p \in \{0,\dots,b_W-1\}$. Let $(l_n)$ be the strictly increasing sequence in $\mathbb{N}$ such that for every $n\in\mathbb{N}$ and every $\mathbf{z}\in\otimes\mathbf{Z}(n)$ we have $B(\mathbf{z}) \subseteq W(l_n)$ and $\mathrm{dens}\big(B(\mathbf{z})\big) \geqslant \eta$. For every $n \in \mathbb{N}$ we define $C_n \subseteq W(l_n)$ by the rule

$$w \in C_n \Leftrightarrow |\{\mathbf{z}\in\otimes\mathbf{Z}(n) : w \in B(\mathbf{z})\}| \geqslant \eta/2\ |\otimes\mathbf{Z}(n)|.$$

**Claim 12.** *For every $n\in\mathbb{N}$ we have $\mathrm{dens}(C_n) \geqslant \eta/2$.*

*Proof of Claim* 12. This is a rather standard estimate and follows using a Fubini-type argument. Indeed, let

$$E_n = \big\{(\mathbf{z},w) \in \otimes\mathbf{Z}(n)\times W(l_n) : w \in B(\mathbf{z})\big\}.$$

Since $\mathrm{dens}\big(B(\mathbf{z})\big) \geqslant \eta$ for every $\mathbf{z}\in\otimes\mathbf{Z}(n)$, we have

$$\eta\cdot|\otimes\mathbf{Z}(n)|\cdot|W(l_n)| \leqslant |E_n|.$$

On the other hand, by the definition of the set $C_n$, we obtain that

$$|E_n| \leqslant |C_n|\cdot|\otimes\mathbf{Z}(n)| + (\eta/2)\cdot|\otimes\mathbf{Z}(n)|\cdot|W(l_n)|.$$

Therefore, $\mathrm{dens}(C_n)\geqslant\eta/2$. The proof of Claim 12 is completed. □

By Claim 12 and [3, Theorem 2.3], we may find a strictly increasing sequence $(n_k)$ in $\mathbb{N}$ and a sequence $(w_k)$ in $W$ such that for every $k,m\in\mathbb{N}$ with $k<m$ we have

(a) $w_k \in C_{n_k}$ and
(b) $w_m \in \mathrm{Succ}_W(w_k^\frown{}^W p)$.

We define $B' \subseteq \otimes\mathbf{Z}$ by

$$B' = \bigcup_{k\in\mathbb{N}} \big\{\mathbf{z}\in\otimes\mathbf{Z}(n_k) : w_k \in B(\mathbf{z})\big\}$$

By (a) and the definition of the set $C_{n_k}$, we see that

$$\limsup_{n\to\infty}\frac{|B'\cap\otimes\mathbf{Z}(n)|}{|\otimes\mathbf{Z}(n)|} = \limsup_{k\to\infty}\frac{|B'\cap\otimes\mathbf{Z}(n_k)|}{|\otimes\mathbf{Z}(n_k)|} \geqslant \eta/2 > 0.$$

Therefore, using our hypothesis that $\mathrm{DHL}(d)$ holds, it is possible to find a vector strong subtree $\mathbf{R}$ of $\mathbf{Z}$ such that $\otimes\mathbf{R}\subseteq B'$. We set $\mathbf{F} = \mathbf{R}\upharpoonright 1 \in \mathrm{Fan}(\mathbf{Z})$ and $\mathbf{z}_0 = \mathbf{F}(0)$. Let $k_0$ and $k_1$ be the unique integers such that $\mathbf{z}_0 \in \otimes\mathbf{Z}(n_{k_0})$ and $\otimes\mathbf{F}(1)\subseteq\otimes\mathbf{Z}(n_{k_1})$. Clearly $k_0<k_1$. Notice that

$$w_{k_0} \in B(\mathbf{z}_0)$$

since $\mathbf{z}_0 \in \otimes\mathbf{R} \subseteq B'$. Moreover,

$$w_{k_1} \in \bigcap_{\mathbf{z}\in\otimes\mathbf{F}(1)} B(\mathbf{z})$$

since $\otimes\mathbf{F}(1) \subseteq \otimes\mathbf{R} \subseteq B'$. Using (b), we conclude that

$$w_{k_1} \in \bigcap_{\mathbf{z}\in\otimes\mathbf{F}(1)} B(\mathbf{z}) \cap \mathrm{Succ}_W(w_{k_0}^{\frown_W} p).$$

The proof of Proposition 11 is completed. □

4.3. **Proof of Theorem 9.** The proof is a quest of a contradiction. So, assume that there exist a vector homogeneous tree $\mathbf{T} = (T_1, \dots, T_d)$, a homogeneous tree $W$, a constant $0 < \varepsilon \leqslant 1$ and an $\varepsilon$-dense level selection $D\colon \otimes\mathbf{T} \to 2^W$ such that

(H) for every vector strong subtree $\mathbf{R}$ of $\mathbf{T}$, every $w \in W$ and every $0 < \theta \leqslant 1$ the pair $(\mathbf{R}, w)$ is not strongly $\theta$-correlated with respect to $D$.

*The vector homogeneous tree* $\mathbf{T}$ *and the* $\varepsilon$*-dense level selection* $D\colon \otimes\mathbf{T} \to 2^W$ *will be fixed throughout the proof.* We will use hypothesis (H) to derive a contradiction. Our strategy is to construct a vector strong subtree $\mathbf{Z}$ of $\mathbf{T}$ and an $(\varepsilon/2b_W)$-dense level selection $B\colon \otimes\mathbf{Z} \to 2^W$ that violates the conclusion of Proposition 11 for some $p_0 \in \{0, \dots, b_W - 1\}$. The construction will be done in several intermediate steps. For notational simplicity, for every $i \in \{1, \dots, d\}$ by $b_i$ we shall denote the branching number of the tree $T_i$.

**Step 1: selection of a rapidly decreasing sequence.** For every $n \in \mathbb{N}$ with $n \geqslant 1$ we define

$$\Theta_n = |\mathrm{Fan}(\mathbf{T}, n)|. \tag{15}$$

Setting $\beta = \prod_{i=1}^d b_i$ we see that $\Theta_n \leqslant \Theta_{n+1}$ and $\beta^{n-1} \leqslant \Theta_n \leqslant 2^{\beta^n}$ for every $n \in \mathbb{N}$ with $n \geqslant 1$. We will need the following elementary facts.

**Fact 13.** *For every vector strong subtree* $\mathbf{S}$ *of* $\mathbf{T}$ *and every* $n \in \mathbb{N}$ *with* $n \geqslant 1$ *we have* $|\mathrm{Fan}(\mathbf{S}, n)| = \Theta_n$.

**Fact 14.** *For every vector strong subtree* $\mathbf{R}$ *of* $\mathbf{T}$*, every vector strong subtree* $\mathbf{S}$ *of* $\mathbf{R}$ *and every* $n \in \mathbb{N}$ *with* $n \geqslant 1$ *there exists* $k \in \mathbb{N}$ *with* $k \geqslant n$ *such that* $\mathrm{Fan}(\mathbf{S}, n) \subseteq \mathrm{Fan}(\mathbf{R}, k)$.

We define a sequence $(\theta_n)$ in $\mathbb{R}$ by the rule $\theta_0 = 1$ and

$$\theta_n = \frac{\varepsilon}{2b_W\Theta_n} \tag{16}$$

for every $n \in \mathbb{N}$ with $n \geqslant 1$. Notice that for every $n \in \mathbb{N}$ we have

$$\theta_{n+1} \leqslant \theta_n. \tag{17}$$

**Step 2: a family $\{\mathcal{F}_n : n \geqslant 1\}$ of subsets of** $\mathrm{Fan}(\mathbf{T})$**.** Let $(\theta_n)$ be the sequence defined in Step 1. For every $n \in \mathbb{N}$ with $n \geqslant 1$ we define a subset $\mathcal{F}_n$ of $\mathrm{Fan}(\mathbf{T})$ by the rule

$$\begin{aligned} \mathbf{F} \in \mathcal{F}_n \quad \Leftrightarrow \quad & \text{there exists a map } \phi\colon D\big(\mathbf{F}(0)\big) \to \{0,\dots,b_W-1\} \\ & \text{such that for every } w \in D\big(\mathbf{F}(0)\big) \text{ if } p = \phi(w), \\ & \text{then } \mathrm{dens}\Big( \bigcap_{\mathbf{t}\in\otimes\mathbf{F}(1)} D(\mathbf{t}) \,\Big|\, w^{\frown_W} p \Big) \leqslant \theta_n. \end{aligned}$$

For every $\mathbf{F} \in \mathcal{F}_n$ there exists a canonical map $\phi^n_{\mathbf{F}}$ witnessing that $\mathbf{F}$ belongs to $\mathcal{F}_n$. It is defined by setting $\phi^n_{\mathbf{F}}(w)$ to be the least $p \in \{0,\dots,b_W-1\}$ for which the above inequality is satisfied. We will call the map $\phi^n_{\mathbf{F}}$ the *witness* of $\mathbf{F}$.

The next lemma reduces hypothesis (H) to certain properties of the sets in the family $\{\mathcal{F}_n : n \geqslant 1\}$.

**Lemma 15.** *Under hypothesis* (H)*, for every $n \in \mathbb{N}$ with $n \geqslant 1$ the following hold.*

(a) *We have $\mathcal{F}_{n+1} \subseteq \mathcal{F}_n$.*
(b) *For every vector strong subtree $\mathbf{S}$ of $\mathbf{T}$ there exists $\mathbf{F} \in \mathrm{Fan}(\mathbf{S}) \cap \mathcal{F}_n$ with $\mathbf{F}(0) = \mathbf{S}(0)$.*

*Proof.* Part (a) follows by (17) and the relevant definitions. For part (b) we will argue by contradiction. So, assume that there exist $n_0 \in \mathbb{N}$ with $n_0 \geqslant 1$ and a vector strong subtree $\mathbf{S}$ of $\mathbf{T}$ such that for every $\mathbf{F} \in \mathrm{Fan}(\mathbf{S})$ with $\mathbf{F}(0) = \mathbf{S}(0)$ we have that $\mathbf{F} \notin \mathcal{F}_{n_0}$. This implies that for every $\mathbf{F} \in \mathrm{Fan}(\mathbf{S})$ with $\mathbf{F}(0) = \mathbf{S}(0)$ there exists $w_{\mathbf{F}} \in D\big(\mathbf{S}(0)\big)$ such that for every $p \in \{0,\dots,b_W-1\}$ we have

$$\mathrm{dens}\Big( \bigcap_{\mathbf{s}\in\otimes\mathbf{F}(1)} D(\mathbf{s}) \,\Big|\, w_{\mathbf{F}}^{\frown_W} p \Big) \geqslant \theta_{n_0}.$$

The set $D\big(\mathbf{S}(0)\big)$ is finite. Therefore, by Proposition 5, there exist a vector strong subtree $\mathbf{R}$ of $\mathbf{S}$ with $\mathbf{R}(0) = \mathbf{S}(0)$ and $w_0 \in D\big(\mathbf{S}(0)\big)$ such that $w_{\mathbf{F}} = w_0$ for every $\mathbf{F} \in \mathrm{Fan}(\mathbf{R})$ with $\mathbf{F}(0) = \mathbf{R}(0)$. It follows that the pair $(\mathbf{R}, w_0)$ is strongly $\theta_{n_0}$-correlated with respect to $D$ and this contradicts hypothesis (H). The proof of Lemma 15 is completed. □

**Step 3: control of vector fans with a fixed root.** Let $\mathbf{R}$ be an arbitrary vector strong subtree of $\mathbf{T}$. Our goal in this step is to construct a vector strong subtree $\mathbf{S}$ of $\mathbf{R}$ with the same root as $\mathbf{R}$ such that for every vector fan $\mathbf{F}$ of $\mathbf{S}$ with $\mathbf{F}(0) = \mathbf{S}(0)$ we have significant control over the quantity appearing on the left side of inequality (14). Precisely, we have the following lemma.

**Lemma 16.** *Let $(\theta_n)$ be the sequence defined in Step 1. Also let $\mathbf{R}$ be a vector strong subtree of $\mathbf{T}$ and set $\mathbf{r}_0 = \mathbf{R}(0)$. Then there exist a vector strong subtree $\mathbf{S}$ of $\mathbf{R}$ with $\mathbf{S}(0) = \mathbf{r}_0$ and a map $\phi\colon D(\mathbf{r}_0) \to \{0,\dots,b_W-1\}$ such that the following* *is satisfied. For every* $n \in \mathbb{N}$ *with* $n \geqslant 1$*, every* $\mathbf{F} \in \mathrm{Fan}(\mathbf{S}, n)$ *with* $\mathbf{F}(0) = \mathbf{r}_0$ *and every* $w \in D(\mathbf{r}_0)$ *if* $p = \phi(w)$*, then*

$$\mathrm{dens}\Big( \bigcap_{\mathbf{s} \in \otimes \mathbf{F}(1)} D(\mathbf{s}) \,\Big|\, w^{\frown_W} p \Big) \leqslant \theta_n.$$

*Proof.* By part (b) of Lemma 15, we may apply Proposition 6 to the vector homogeneous tree $\mathbf{T}$, the family $\{\mathcal{F}_n : n \geqslant 1\}$ and the vector strong subtree $\mathbf{R}$ of $\mathbf{T}$. Therefore, there exists a vector strong subtree $\mathbf{Z}$ of $\mathbf{R}$ with $\mathbf{Z}(0) = \mathbf{r}_0$ such that for every $k \in \mathbb{N}$ with $k \geqslant 1$ and every $\mathbf{F} \in \mathrm{Fan}(\mathbf{Z}, k)$ with $\mathbf{F}(0) = \mathbf{r}_0$ we have that $\mathbf{F} \in \mathcal{F}_k$. Let $\phi^k_{\mathbf{F}} \colon D(\mathbf{r}_0) \to \{0, \dots, b_W - 1\}$ be the corresponding witness. The set $\{0, \dots, b_W - 1\}^{D(\mathbf{r}_0)}$ is finite. By Proposition 5, there exist a vector strong subtree $\mathbf{S}$ of $\mathbf{Z}$ with $\mathbf{S}(0) = \mathbf{r}_0$ and a map $\phi \colon D(\mathbf{r}_0) \to \{0, \dots, b_W - 1\}$ such that for every $\mathbf{F} \in \mathrm{Fan}(\mathbf{S})$ with $\mathbf{F}(0) = \mathbf{r}_0$ if $k$ is the unique integer such that $\mathbf{F} \in \mathrm{Fan}(\mathbf{Z}, k)$, then $\phi^k_{\mathbf{F}} = \phi$. The vector tree $\mathbf{S}$ and the map $\phi$ are as desired.

Indeed, let $n \in \mathbb{N}$ with $n \geqslant 1$ and $\mathbf{F} \in \mathrm{Fan}(\mathbf{S}, n)$ with $\mathbf{F}(0) = \mathbf{r}_0$ be arbitrary. By Fact 14, there exists $k \in \mathbb{N}$ with $k \geqslant n$ such that $\mathbf{F} \in \mathrm{Fan}(\mathbf{Z}, k)$. Let $w \in D(\mathbf{r}_0)$ be arbitrary. If $p = \phi(w)$, then $p = \phi^k_{\mathbf{F}}(w)$. Hence,

$$\mathrm{dens}\Big( \bigcap_{\mathbf{s} \in \otimes \mathbf{F}(1)} D(\mathbf{s}) \,\Big|\, w^{\frown_W} p \Big) \leqslant \theta_k \overset{(17)}{\leqslant} \theta_n.$$

The proof of Lemma 16 is completed. □

**Step 4: construction of an "asymptotically sparse" vector tree.** In this step we will refine the construction presented in Step 3. Our goal is to construct an "asymptotically sparse" vector tree, that is, a vector strong subtree $\mathbf{S}$ of $\mathbf{T}$ for which we have control over the behavior of *every* vector fan of $\mathbf{S}$. Specifically, we have the following lemma.

**Lemma 17.** *Let* $(\theta_n)$ *be the sequence defined in Step* 1*. Then there exists a vector strong subtree* $\mathbf{S}$ *of* $\mathbf{T}$ *with the following property. For every* $\mathbf{s}_0 \in \otimes \mathbf{S}$ *there exists a map* $\phi_{\mathbf{s}_0} \colon D(\mathbf{s}_0) \to \{0, \dots, b_W - 1\}$ *such that for every* $n \in \mathbb{N}$ *with* $\ell_{\mathbf{S}}(\mathbf{s}_0) < n$*, every* $\mathbf{F} \in \mathrm{Fan}(\mathbf{S}, n)$ *with* $\mathbf{F}(0) = \mathbf{s}_0$ *and every* $w \in D(\mathbf{s}_0)$ *if* $p = \phi_{\mathbf{s}_0}(w)$*, then*

$$\mathrm{dens}\Big( \bigcap_{\mathbf{s} \in \otimes \mathbf{F}(1)} D(\mathbf{s}) \,\Big|\, w^{\frown_W} p \Big) \leqslant \theta_n.$$

*Proof.* We say that a vector strong subtree of $\mathbf{T}$ is in a *good position* if it satisfies the conclusion of Lemma 16. That is, a vector strong subtree $\mathbf{Z}$ of $\mathbf{T}$ is in a good position if there exists a map $\phi \colon D\big(\mathbf{Z}(0)\big) \to \{0, \dots, b_W - 1\}$ such that for every $k \in \mathbb{N}$ with $k \geqslant 1$, every $\mathbf{F} \in \mathrm{Fan}(\mathbf{Z}, k)$ with $\mathbf{F}(0) = \mathbf{Z}(0)$ and every $w \in D\big(\mathbf{Z}(0)\big)$ if $p = \phi(w)$, then $\mathrm{dens}\big( \bigcap_{\mathbf{z} \in \otimes \mathbf{F}(1)} D(\mathbf{z}) \,|\, w^{\frown_W} p \big) \leqslant \theta_k$.

We notice two permanence properties of this notion. The first one is that it is *hereditary* when passing to vector subtrees. Precisely, if a vector strong subtree $\mathbf{Z}$ of $\mathbf{T}$ is in a good position and $\mathbf{Z}'$ is a vector strong subtree of $\mathbf{Z}$ with $\mathbf{Z}'(0) = \mathbf{Z}(0)$, then $\mathbf{Z}'$ is also in a good position. This can be easily checked arguing as in the proof of Lemma 16 and using the fact that the sequence $(\theta_n)$ is decreasing. The second property is that the family of vector strong subtrees of $\mathbf{T}$ which are in a good position is *dense*, that is, for every vector strong subtree $\mathbf{R}$ of $\mathbf{T}$ there exists a vector strong subtree $\mathbf{Z}$ of $\mathbf{R}$ with $\mathbf{Z}(0)=\mathbf{R}(0)$ such that $\mathbf{Z}$ is in a good position. This is, of course, the content of Lemma 16. Using these properties and a standard recursive construction, it is possible to find a vector strong subtree $\mathbf{V}$ of $\mathbf{T}$ such that for every $\mathbf{v}\in\otimes\mathbf{V}$ the vector strong subtree $\mathrm{Succ}_{\mathbf{V}}(\mathbf{v})$ of $\mathbf{T}$ is in a good position.

The desired vector tree $\mathbf{S}$ will be an appropriately chosen vector strong subtree of $\mathbf{V}$. Specifically, let $(m_j)$ be a sequence in $\mathbb{N}$ such that for every $j\in\mathbb{N}$ with $j\geqslant 1$ we have $m_j\geqslant m_{j-1}+j$. We select a vector strong subtree $\mathbf{S}$ of $\mathbf{V}$ such that $\otimes\mathbf{S}(j)\subseteq\otimes\mathbf{V}(m_j)$ for every $j\in\mathbb{N}$. We will show that $\mathbf{S}$ is as desired. To this end, let $\mathbf{s}_0\in\otimes\mathbf{S}$ be arbitrary and set $k=\ell_{\mathbf{S}}(\mathbf{s}_0)$. By the properties of $\mathbf{V}$, the vector tree $\mathrm{Succ}_{\mathbf{V}}(\mathbf{s}_0)$ is in a good position. We fix a map $\phi_{\mathbf{s}_0}\colon D(\mathbf{s}_0)\to\{0,\dots,b_W-1\}$ witnessing this fact. Let $n\in\mathbb{N}$ with $k<n$ and $\mathbf{F}\in\mathrm{Fan}(\mathbf{S},n)$ with $\mathbf{F}(0)=\mathbf{s}_0$. Observe that $\mathbf{s}_0\in\otimes\mathbf{S}(k)\subseteq\otimes\mathbf{V}(m_k)$ and $\otimes\mathbf{F}(1)\subseteq\otimes\mathbf{S}(n)\subseteq\otimes\mathbf{V}(m_n)$. By the choice of the sequence $(m_j)$, there exists $l\in\mathbb{N}$ with $l\geqslant n$ such that $\mathbf{F}\in\mathrm{Fan}\big(\mathrm{Succ}_{\mathbf{V}}(\mathbf{s}_0),l\big)$. It follows that for every $w\in D(\mathbf{s}_0)$ if $p=\phi_{\mathbf{s}_0}(w)$, then

$$\mathrm{dens}\Big(\bigcap_{\mathbf{s}\in\otimes\mathbf{F}(1)} D(\mathbf{s})\,\Big|\,w^\frown{}_W p\Big)\leqslant\theta_l\overset{(17)}{\leqslant}\theta_n.$$

The proof of Lemma 17 is completed. $\square$

**Step 5: fixing the "direction".** Let $\mathbf{S}$ be the vector strong subtree of $\mathbf{T}$ obtained by Lemma 17. For every $p\in\{0,\dots,b_W-1\}$ we define $C_p\colon\otimes\mathbf{S}\to 2^W$ by the rule

$$C_p(\mathbf{s})=\{w\in D(\mathbf{s}):\phi_{\mathbf{s}}(w)=p\}. \tag{18}$$

**Lemma 18.** *There exist a vector strong subtree* $\mathbf{Z}$ *of* $\mathbf{S}$ *and* $p_0\in\{0,\dots,b_W-1\}$ *such that for every* $\mathbf{z}\in\otimes\mathbf{Z}$ *we have*

$$\mathrm{dens}\big(C_{p_0}(\mathbf{z})\big)\geqslant\varepsilon/b_W.$$

*Proof.* Let $\mathbf{s}\in\otimes\mathbf{S}$ be arbitrary. Let $p_{\mathbf{s}}$ be the least $p\in\{0,\dots,b_W-1\}$ such that $\mathrm{dens}\big(C_p(\mathbf{s})\big)\geqslant\varepsilon/b_W$. Notice that, by the classical pigeonhole principle, $p_{\mathbf{s}}$ is well-defined. By $\mathrm{HL}(d)$, there exist a vector strong subtree $\mathbf{Z}$ of $\mathbf{S}$ and $p_0\in\{0,\dots,b_W-1\}$ such that $p_{\mathbf{z}}=p_0$ for every $\mathbf{z}\in\otimes\mathbf{Z}$. It is clear that $\mathbf{Z}$ and $p_0$ are as desired. $\square$

**Step 6: properties of $C_{p_0}$.** In this step we will not construct something new but rather summarize what we have achieved so far. Let $\mathbf{Z}$ and $p_0$ be as in Lemma 18. Then the following are satisfied.

($\mathcal{P}1$) For every $\mathbf{z}\in\otimes\mathbf{Z}$ we have $C_{p_0}(\mathbf{z})\subseteq D(\mathbf{z})$.
($\mathcal{P}2$) The map $C_{p_0}\colon\otimes\mathbf{Z}\to 2^W$ is an $(\varepsilon/b_W)$-dense level selection.

($\mathcal{P}3$) For every $k\in\mathbb{N}$ with $k\geqslant 1$, every $\mathbf{F}\in \mathrm{Fan}(\mathbf{Z},k)$ and every $w\in C_{p_0}(\mathbf{z}_0)$, where $\mathbf{z}_0=\mathbf{F}(0)$, we have $\mathrm{dens}\big(\bigcap_{\mathbf{z}\in\otimes\mathbf{F}(1)} D(\mathbf{z})\,|\,w^{\frown_W}p_0\big)\leqslant \theta_k$.

Property ($\mathcal{P}1$) follows immediately by (18). Property ($\mathcal{P}2$) is essentially the content of Lemma 18. Finally, property ($\mathcal{P}3$) follows by Lemma 17 and Fact 14.

**Step 7: a sequence of "forbidden" subsets of $W$.** Let $C_{p_0}\colon \otimes\mathbf{Z}\to 2^W$ be the $(\varepsilon/b_W)$-dense level selection obtained in Step 5. Let $(l_k)$ be the strictly increasing sequence in $\mathbb{N}$ such that for every $k\in\mathbb{N}$ and every $\mathbf{z}\in\otimes\mathbf{Z}(k)$ we have $C_{p_0}(\mathbf{z})\subseteq W(l_k)$.

For every $k\in\mathbb{N}$ with $k\geqslant 1$ we define a subset $G_k$ of $W(l_k)$ by the rule

$$\begin{aligned} w'\in G_k \quad\Leftrightarrow\quad & \text{there exist } \mathbf{F}\in\mathrm{Fan}(\mathbf{Z},k) \text{ and } w\in C_{p_0}(\mathbf{z}_0),\ \text{where } \mathbf{z}_0=\mathbf{F}(0),\\ & \text{such that } w'\in \bigcap_{\mathbf{z}\in\otimes\mathbf{F}(1)} D(\mathbf{z})\cap \mathrm{Succ}_W(w^{\frown_W}p_0). \end{aligned}$$

We view the sequence $(G_k)$ as a sequence of "forbidden" subsets of $W$. Specifically, we will modify the dense level selection $C_{p_0}$ in such a way that the range of the new one will be disjoint from every $G_k$. But in order to do so, we will need the following estimate on the size of each $G_k$.

**Lemma 19.** *For every $k\in\mathbb{N}$ with $k\geqslant 1$ we have*

$$\mathrm{dens}(G_k)\leqslant \frac{\varepsilon}{2b_W}.$$

*Proof.* For every $\mathbf{F}\in\mathrm{Fan}(\mathbf{Z},k)$ let

$$H_{\mathbf{F}}=\bigcap_{\mathbf{z}\in\otimes\mathbf{F}(1)} D(\mathbf{z})\cap \mathrm{Succ}_W\Big(\big\{w^{\frown_W}p_0 : w\in C_{p_0}\big(\mathbf{F}(0)\big)\big\}\Big)$$

and notice that

$$G_k=\bigcup_{\mathbf{F}\in\mathrm{Fan}(\mathbf{Z},k)} H_{\mathbf{F}}.$$

Since $\mathbf{Z}$ is a vector strong subtree of the vector homogeneous tree $\mathbf{T}$, by Fact 13, we have $|\mathrm{Fan}(\mathbf{Z},k)|=\Theta_k$. Therefore, it is enough to show that for every $\mathbf{F}\in\mathrm{Fan}(\mathbf{Z},k)$ we have $\mathrm{dens}(H_{\mathbf{F}})\leqslant \varepsilon/(2b_W\Theta_k)$.

To this end, let $\mathbf{F}\in\mathrm{Fan}(\mathbf{Z},k)$ be arbitrary and set $\mathbf{z}_0=\mathbf{F}(0)$. Also set $\lambda=\mathrm{dens}\big(C_{p_0}(\mathbf{z}_0)\big)/b_W$ and observe that $\lambda\leqslant 1$. The tree $W$ is homogeneous. Hence,

$$\begin{aligned} \mathrm{dens}(H_{\mathbf{F}}) &\leqslant \lambda\cdot\max\Big\{\mathrm{dens}\Big(\bigcap_{\mathbf{z}\in\otimes\mathbf{F}(1)} D(\mathbf{z})\,\Big|\,w^{\frown_W}p_0\Big) : w\in C_{p_0}(\mathbf{z}_0)\Big\}\\ &\leqslant \max\Big\{\mathrm{dens}\Big(\bigcap_{\mathbf{z}\in\otimes\mathbf{F}(1)} D(\mathbf{z})\,\Big|\,w^{\frown_W}p_0\Big) : w\in C_{p_0}(\mathbf{z}_0)\Big\}\\ &\overset{(\mathcal{P}3)}{\leqslant} \theta_k \overset{(16)}{=} \frac{\varepsilon}{2b_W\Theta_k}. \end{aligned}$$

The proof of Lemma 19 is completed. $\square$

**Step 8: definition of the dense level selection** $B$**.** Let $C_{p_0}\colon \otimes\mathbf{Z}\to 2^W$ be the $(\varepsilon/b_W)$-dense level selection obtained in Step 5. Also let $(G_k)$ be the sequence of subsets of $W$ defined in Step 7.

We define $B\colon \otimes\mathbf{Z}\to 2^W$ as follows. First we set $B\big(\mathbf{Z}(0)\big)=C_{p_0}\big(\mathbf{Z}(0)\big)$. If $\mathbf{z}\in\otimes\mathbf{Z}(k)$ for some $k\in\mathbb{N}$ with $k\geqslant 1$, then we set

$$B(\mathbf{z})=C_{p_0}(\mathbf{z})\setminus G_k. \tag{19}$$

We summarize, below, the main properties of the map $B$.

($\mathcal{P}4$) For every $\mathbf{z}\in\otimes\mathbf{Z}$ we have $B(\mathbf{z})\subseteq C_{p_0}(\mathbf{z})\subseteq D(\mathbf{z})$.
($\mathcal{P}5$) The map $B\colon\otimes\mathbf{Z}\to 2^W$ is an $(\varepsilon/2b_W)$-dense level selection.
($\mathcal{P}6$) For every $\mathbf{F}\in\mathrm{Fan}(\mathbf{Z})$ and every $w\in B(\mathbf{z}_0)$, where $\mathbf{z}_0=\mathbf{F}(0)$, we have

$$\bigcap_{\mathbf{z}\in\otimes\mathbf{F}(1)} B(\mathbf{z})\cap\mathrm{Succ}_W(w^{\frown_W}p_0)=\emptyset.$$

Property ($\mathcal{P}4$) follows by property ($\mathcal{P}1$) isolated in Step 6 and (19). Property ($\mathcal{P}5$) follows by property ($\mathcal{P}2$) and Lemma 19. To see that property ($\mathcal{P}6$) is satisfied, let $\mathbf{F}\in\mathrm{Fan}(\mathbf{Z})$ and $w\in B(\mathbf{z}_0)$, where $\mathbf{z}_0=\mathbf{F}(0)$. We set

$$A=\bigcap_{\mathbf{z}\in\otimes\mathbf{F}(1)} B(\mathbf{z})\cap\mathrm{Succ}_W(w^{\frown_W}p_0).$$

Let $k$ be the unique integer such that $\mathbf{F}\in\mathrm{Fan}(\mathbf{Z},k)$. By property ($\mathcal{P}4$) and the definition of $G_k$, we see that $A\subseteq G_k$. We fix $\mathbf{z}'\in\otimes\mathbf{F}(1)$ and we notice that $A\subseteq B(\mathbf{z}')$. Since $\mathbf{z}'\in\otimes\mathbf{Z}(k)$, by the previous inclusions and the definition of the dense level selection $B$, we conclude that $A\subseteq G_k\cap B(\mathbf{z}')=\emptyset$, as desired.

**Step 9: obtaining the contradiction.** We are finally in a position to derive the contradiction. Indeed, by property ($\mathcal{P}5$), the map $B\colon\otimes\mathbf{Z}\to 2^W$ is an $(\varepsilon/2b_W)$-dense level selection. Moreover, by our assumptions, we have that $\mathrm{DHL}(d)$ holds. Therefore, by Proposition 11 applied to "$B\colon\otimes\mathbf{Z}\to 2^W$" and "$p_0$", there must exist $\mathbf{F}\in\mathrm{Fan}(\mathbf{Z})$ and $w\in B(\mathbf{z}_0)$, where $\mathbf{z}_0=\mathbf{F}(0)$, such that

$$\bigcap_{\mathbf{z}\in\otimes\mathbf{F}(1)} B(\mathbf{z})\cap\mathrm{Succ}_W(w^{\frown_W}p_0)\neq\emptyset.$$

This contradicts property ($\mathcal{P}6$). The proof of Theorem 9 is thus completed.

4.4. **Comments.** We recall that a vector tree $\mathbf{T}=(T_1,\dots,T_d)$ is said to be *boundedly branching* if for every $i\in\{1,\dots,d\}$ there exists $b_i\in\mathbb{N}$ with $b_i\geqslant 1$ such that every $t\in T_i$ has at most $b_i$ immediate successors. By Theorem 1, for every vector boundedly branching, pruned tree $\mathbf{T}=(T_1,\dots,T_d)$ there exist a vector strong subtree $\mathbf{S}=(S_1,\dots,S_d)$ of $\mathbf{T}$ and $b_1,\dots,b_d\in\mathbb{N}$, with $b_i\geqslant 1$ for every $i\in\{1,\dots,d\}$, such that every $s\in S_i$ has exactly $b_i$ immediate successors in $S_i$ for every $i\in\{1,\dots,d\}$. Therefore, Theorem 9 is also valid for nontrivial vector boundedly branching trees simply by reducing the general case to the case of vector homogeneous trees.

The next natural class of vector trees for which Theorem 9 could be possibly true is that of vector *quasi-homogeneous* trees: a vector tree $\mathbf{T}=(T_1,\dots,T_d)$ is said to be quasi-homogeneous if for every $i\in\{1,\dots,d\}$ the number of immediate successors of a node in $T_i$ depends only on its length. We point out that all arguments in this section (as well as the recursive construction presented in §5) can be easily adapted to treat vector quasi-homogeneous trees *except* Fact 13. Indeed, by Fact 13, we have an a priori estimate for the cardinality of the set $\mathrm{Fan}(\mathbf{S},k)$ for every $k\in\mathbb{N}$ and every vector strong subtree $\mathbf{S}$ of a vector homogeneous tree $\mathbf{T}$. If the vector tree $\mathbf{T}$ is quasi-homogeneous but not boundedly branching, then no estimate can be obtained. As is shown in [3, Theorem 2.5] this obstacle is a necessity rather than a coincidence.

Finally we remark that, using essentially the same arguments as in the proof of Theorem 9, one can show that there exist a vector strong subtree $\mathbf{R}$ of $\mathbf{T}$ and a constant $0<\theta\leqslant 1$ such that for "almost all" nodes $w$ in $D\big(\mathbf{R}(0)\big)$ the pair $(\mathbf{R},w)$ is strongly $\theta$-correlated with respect to $D$. Precisely, we have the following theorem.

**Theorem 20.** *Let $d\geqslant 1$ and assume that* $\mathrm{DHL}(d)$ *holds. Also let $\mathbf{T}=(T_1,\dots,T_d)$ be a vector homogeneous tree, $W$ a homogeneous tree, $0<\varepsilon\leqslant 1$ and $D\colon\otimes\mathbf{T}\to 2^W$ an $\varepsilon$-dense level selection. Then for every $0<\delta<\varepsilon$ there exist a vector strong subtree $\mathbf{R}$ of $\mathbf{T}$ and a constant $0<\theta\leqslant 1$ such that, setting $\mathbf{r}_0=\mathbf{R}(0)$ and*

$$G=\big\{w\in D(\mathbf{r}_0):\ \textit{the pair } (\mathbf{R},w) \textit{ is strongly } \theta\textit{-correlated with respect to } D\big\},$$

*we have* $\mathrm{dens}(G)\geqslant\mathrm{dens}\big(D(\mathbf{r}_0)\big)-\delta$.

## 5. Proof of Theorem 2

As we have already mentioned in the introduction, the proof of Theorem 2 proceeds by induction. The case "$d=1$" is the content of [3, Theorem 2.3]. So, assume that we have proven $\mathrm{DHL}(d)$ for some $d\in\mathbb{N}$ with $d\geqslant 1$ and that we are given a vector homogeneous tree $(T_1,\dots,T_d,W)$, a constant $0<\varepsilon\leqslant 1$, a subset $D$ of the level product of $(T_1,\dots,T_d,W)$ and an infinite subset $L$ of $\mathbb{N}$ such that

$$|D\cap\big(T_1(n)\times\cdots\times T_d(n)\times W(n)\big)|\geqslant\varepsilon|\big(T_1(n)\times\cdots\times T_d(n)\times W(n)\big)| \tag{20}$$

for every $n\in L$. Our goal is to find a vector strong subtree $(Z_1,\dots,Z_d,V)$ of $(T_1,\dots,T_d,W)$ whose level product is contained in $D$. This will be done in several steps. We set $\mathbf{T}=(T_1,\dots,T_d)$. For notational simplicity, for every $i\in\{1,\dots,d\}$ by $b_i$ we shall denote the branching number of the homogeneous tree $T_i$. The vector homogeneous tree $\mathbf{B}_{\mathbf{T}}=(b_1^{<\mathbb{N}},\dots,b_d^{<\mathbb{N}})$ will be denoted simply by $\mathbf{B}$. Notice that if $\mathbf{p}\in\otimes\mathbf{B}(1)$, then $\mathbf{p}$ is a finite sequence $(p_1,\dots,p_d)$ with $p_i\in\{0,\dots,b_i-1\}$ for every $i\in\{1,\dots,d\}$. By $\mathbf{0}$ we shall denote the unique finite sequence in $\otimes\mathbf{B}(1)$ having zero entries. For every $n\in\mathbb{N}$, every $\mathbf{u}=(u_1,\dots,u_d)\in\otimes\mathbf{B}(n)$ and every $\mathbf{p}=(p_1,\dots,p_d)\in\otimes\mathbf{B}(1)$ we set $\mathbf{u}^\frown\mathbf{p}=(u_1^\frown p_1,\dots,u_d^\frown p_d)\in\otimes\mathbf{B}(n+1)$.

**Step 1: obtaining a dense level selection.** For every $n \in L$ let $C_n$ be the subset of $\otimes\mathbf{T}(n)$ defined by the rule

$$\mathbf{t} \in C_n \Leftrightarrow |\{w \in W(n) : (\mathbf{t}, w) \in D\}| \geqslant \varepsilon/2|W(n)|.$$

Using (20) and arguing as in the proof of Claim 12, we obtain the following fact.

**Fact 21.** *For every $n \in L$ we have $|C_n| \geqslant \varepsilon/2|\otimes\mathbf{T}(n)|$.*

We set $C \coloneqq \bigcup_{n\in L} C_n$. By Fact 21, we see that

$$\limsup_{n\to\infty} \frac{|C \cap \otimes\mathbf{T}(n)|}{|\otimes\mathbf{T}(n)|} = \limsup_{n\in L} \frac{|C_n \cap \otimes\mathbf{T}(n)|}{|\otimes\mathbf{T}(n)|} \geqslant \varepsilon/2 > 0.$$

Since DHL($d$) holds, there exists a vector strong subtree $\mathbf{S}$ of $\mathbf{T}$ such that $\otimes\mathbf{S} \subseteq C$. It follows that the section map

$$\otimes\mathbf{S} \ni \mathbf{s} \mapsto \{w \in W : (\mathbf{s}, w) \in D\} \in 2^W$$

is an $(\varepsilon/2)$-dense level selection. It will be denoted by $D\colon \otimes\mathbf{S} \to 2^W$.

**Step 2: defining certain vector fans.** Let $\mathbf{S} = (S_1, \dots, S_d)$ be the vector strong subtree of $\mathbf{T}$ obtained in Step 1. Also let $\mathbf{R}$ be an arbitrary vector strong subtree of $\mathbf{S}$. In this step, we will introduce a method to obtain vector fans of $\mathbf{R}$ from certain elements of $\otimes\mathbf{R}$. The method is based on the notion of a vector canonical isomorphism described in §2.5. The resulting vector fans will be used in the next step.

We will describe, first, the one-dimensional case in abstract setting. So, let $Z$ be a homogeneous tree and set $z_0 = Z(0)$. For every $p \in \{0, \dots, b_Z - 1\}$ let

$$Z[p] = \mathrm{Succ}_Z(z_0^\frown{}^{Z}p). \tag{21}$$

It is clear that $Z[p]$ is a strong subtree of $Z$, and so, it is homogeneous with branching number $b_Z$. This observation permits us to consider the canonical isomorphism $\mathrm{I}\big(Z[0], Z[p]\big)$ between $Z[0]$ and $Z[p]$ for every $p \in \{0, \dots, b_Z - 1\}$. Now, for every $z \in Z[0]$ we set

$$F_{z,Z} \coloneqq \{z_0\} \cup \Big\{\mathrm{I}\big(Z[0], Z[p]\big)(z) : p \in \{0, \dots, b_Z - 1\}\Big\}. \tag{22}$$

Notice that $F_{z,Z} \in \mathrm{Fan}(Z)$. The fan $F_{z,Z}$ will be called a $(z,Z)$-*directed fan*. We point that not every fan of $Z$ is $(z,Z)$-directed for some $z \in Z[0]$. Actually, the set of all $(z,Z)$-directed fans is a rather "thin" subset of $\mathrm{Fan}(Z)$.

After this preliminary discussion, we are ready to introduce the vector fans we mentioned above. Specifically, let $\mathbf{R} = (R_1, \dots, R_d)$ be an arbitrary vector strong subtree of $\mathbf{S}$. For every $\mathbf{p} = (p_1, \dots, p_d) \in \otimes\mathbf{B}(1)$ we set

$$\mathbf{R}[\mathbf{p}] \coloneqq \big(R_1[p_1], \dots, R_d[p_d]\big) \tag{23}$$

and we notice that $\mathbf{R}[\mathbf{p}]$ is a vector strong subtree of $\mathbf{R}$. Again we emphasize that this observation permits us to consider that vector canonical isomorphism $\mathbf{I}\big(\mathbf{R}[\mathbf{0}], \mathbf{R}[\mathbf{p}]\big)$ between $\mathbf{R}[\mathbf{0}]$ and $\mathbf{R}[\mathbf{p}]$ for every $\mathbf{p} \in \otimes\mathbf{B}(1)$. Observe that

$\mathbf{I}\big(\mathbf{R}[\mathbf{0}],\mathbf{R}[\mathbf{0}]\big)$ is the identity map on $\mathbf{R}[\mathbf{0}]$. For every $\mathbf{r}=(r_1,\dots,r_d)\in\otimes\mathbf{R}[\mathbf{0}]$ we define

$$\mathbf{F}_{\mathbf{r},\mathbf{R}}=(F_{r_1,R_1},\dots,F_{r_d,R_d}). \tag{24}$$

Notice that $\mathbf{F}_{\mathbf{r},\mathbf{R}}$ is well-defined since $r_i\in R_i[0]$ for every $i\in\{1,\dots,d\}$. Also observe that $\mathbf{F}_{\mathbf{r},\mathbf{R}}\in\mathrm{Fan}(\mathbf{R})$ and $\mathbf{F}_{\mathbf{r},\mathbf{R}}(0)=\mathbf{R}(0)$. The vector fan $F_{\mathbf{r},\mathbf{R}}$ will be called an $(\mathbf{r},\mathbf{R})$-*directed vector fan.* We isolate, for future use, the following representation of the set $\otimes\mathbf{F}_{\mathbf{r},\mathbf{R}}(1)$. It is a direct consequence of the relevant definitions.

**Fact 22.** *For every vector strong subtree* $\mathbf{R}$ *of* $\mathbf{S}$ *and every* $\mathbf{r}\in\mathbf{R}[\mathbf{0}]$ *we have*

$$\otimes\mathbf{F}_{\mathbf{r},\mathbf{R}}(1)=\Big\{\mathbf{I}\big(\mathbf{R}[\mathbf{0}],\mathbf{R}[\mathbf{p}]\big)(\mathbf{r}):\mathbf{p}\in\otimes\mathbf{B}(1)\Big\}.$$

*In particular, we have* $\mathbf{r}\in\otimes\mathbf{F}_{\mathbf{r},\mathbf{R}}(1)$.

**Step 3: a recursive construction.** This is the main step of the proof. Let $D\colon\otimes\mathbf{S}\to 2^W$ be the $(\varepsilon/2)$-dense level selection obtained in Step 1. Recursively, we shall construct

(a) two sequences $(\mathbf{S}_n)$ and $(\mathbf{R}_n)$ of vector strong subtrees of $\mathbf{S}$,
(b) two sequences $(\varepsilon_n)$ and $(\theta_n)$ of reals in $(0,1]$,
(c) a strictly increasing sequence $(l_n)$ in $\mathbb{N}$,
(d) for every $n\in\mathbb{N}$ a map $D_n\colon\otimes\mathbf{S}_n\to 2^W$ and
(e) a family $\{w_v:v\in b_W^{<\mathbb{N}}\}$ in $W$

such that for every $n\in\mathbb{N}$ the following conditions are satisfied.

(C1) $\mathbf{R}_n$ is a vector strong subtree of $\mathbf{S}_n$.
(C2) $\mathbf{S}_{n+1}=\mathbf{R}_n[\mathbf{0}]$.
(C3) For every $v\in b_W^n$ we have $\ell_W(w_v)=l_n$.
(C4) For every $v\in b_W^n$ and $p\in\{0,\dots,b_W-1\}$ we have $w_{v^\frown p}\in\mathrm{Succ}_W(w_v^{\frown_W}p)$.
(C5) The map $D_n\colon\otimes\mathbf{S}_n\to 2^W$ is an $\varepsilon_n$-dense level selection.
(C6) For every $\mathbf{s}\in\otimes\mathbf{S}_n$ we have $D_n(\mathbf{s})\subseteq D_0(\mathbf{s})=D(\mathbf{s})$.
(C7) For every $v\in b_W^n$ the pair $(\mathbf{R}_n,w_v)$ is strongly $\theta_n$-correlated with respect to the dense level selection $D_n$.
(C8) For every $\mathbf{r}\in\otimes\mathbf{R}_n[\mathbf{0}]$ we have

$$D_{n+1}(\mathbf{r})=\bigcap_{\mathbf{p}\in\otimes\mathbf{B}(1)}D_n\Big(\mathbf{I}\big(\mathbf{R}_n[\mathbf{0}],\mathbf{R}_n[\mathbf{p}]\big)(\mathbf{r})\Big). \tag{25}$$

We proceed to the construction. For $n=0$ we set "$\mathbf{S}_0=\mathbf{S}$", "$\varepsilon_0=\varepsilon/2$" and "$D_0=D$" and we notice that with these choices conditions (C5) and (C6) are satisfied. Recall that we have already proven DHL($d$). Therefore, by Theorem 9 applied to the $\varepsilon_0$-dense level selection $D_0\colon\otimes\mathbf{S}_0\to 2^W$, there exist a vector strong subtree $\mathbf{R}$ of $\mathbf{S}_0$, a node $w\in W$ and a constant $0<\theta\leqslant 1$ such that the pair $(\mathbf{R},w)$ is strongly $\theta$-correlated with respect to $D_0$. We set "$\mathbf{R}_0=\mathbf{R}$", "$\theta_0=\theta$", "$w_\emptyset=w$" and "$l_0=\ell_W(w)$" and we observe that with these choices conditions (C1), (C3) and (C7) are satisfied. Since conditions (C2), (C4) and (C8) are meaningless for $n=0$, the first step of the recursive construction is completed.

Let $n\in\mathbb{N}$ and assume that the construction has been carried out up to $n$. We set "$\mathbf{S}_{n+1}=\mathbf{R}_n[\mathbf{0}]$" and we notice that condition (C2) is satisfied. Let $\mathbf{r}\in\otimes\mathbf{S}_{n+1}$ be arbitrary and consider the $(\mathbf{r},\mathbf{R}_n)$-directed fan $\mathbf{F}_{\mathbf{r},\mathbf{R}_n}$ described in Step 2. We define $D_{n+1}\colon \otimes\mathbf{S}_{n+1}\to 2^W$ by the rule

$$D_{n+1}(\mathbf{r})=\bigcap_{\mathbf{s}\in\otimes\mathbf{F}_{\mathbf{r},\mathbf{R}_n}(1)} D_n(\mathbf{s}).$$

By Fact 22 and our inductive assumptions, we see that conditions (C6) and (C8) are satisfied. We set "$\varepsilon_{n+1}=\theta_n b_W^{n-l_n}$" and we claim that with this choice condition (C5) is satisfied. To this end, it is enough to show that $\mathrm{dens}\big(D_{n+1}(\mathbf{r})\big)\geqslant\varepsilon_{n+1}$ for every $\mathbf{r}\in\otimes\mathbf{S}_{n+1}$. So, let $\mathbf{r}\in\otimes\mathbf{S}_{n+1}$ be arbitrary. By our inductive assumptions, the pair $(\mathbf{R}_n,w_v)$ is strongly $\theta_n$-correlated with respect to $D_n$ for every $v\in b_W^n$. Recall that $\ell_W(w_v)=l_n$ and $\mathbf{F}_{\mathbf{r},\mathbf{R}_n}\in\mathrm{Fan}(\mathbf{R}_n)$. Since the tree $W$ is homogeneous, we obtain that

$$\begin{aligned}
\mathrm{dens}\big(D_{n+1}(\mathbf{r})\big) &\geqslant \frac{1}{b_W^{l_n+1}}\sum_{v\in b_W^n}\sum_{p=0}^{b_W-1}\mathrm{dens}\big(D_{n+1}(\mathbf{r})\,|\,w_v^\frown{}_W p\big)\\
&= \frac{1}{b_W^{l_n+1}}\sum_{v\in b_W^n}\sum_{p=0}^{b_W-1}\mathrm{dens}\Big(\bigcap_{\mathbf{s}\in\otimes\mathbf{F}_{\mathbf{r},\mathbf{R}_n}(1)} D_n(\mathbf{s})\,\Big|\,w_v^\frown{}_W p\Big)\\
&\geqslant \frac{b_W^{n+1}}{b_W^{l_n+1}}\cdot\theta_n=\varepsilon_{n+1}.
\end{aligned}$$

This shows that $D_{n+1}$ is an $\varepsilon_{n+1}$-dense level selection.

Now for every $v\in b_W^n$ and every $p\in\{0,\dots,b_W-1\}$ define $D_{v,p}\colon\otimes\mathbf{S}_{n+1}\to 2^W$ by the rule $D_{v,p}(\mathbf{r})=D_{n+1}(\mathbf{r})\cap\mathrm{Succ}_W(w_v^\frown{}_W p)$. Arguing as above, it is easy to check that $D_{v,p}$ is a $\delta_n$-dense level selection where $\delta_n=\theta_n/b_W^{l_n+1}$. Again we emphasize that we have already proven $\mathrm{DHL}(d)$. Therefore, by repeated applications of Corollary 10, we may select a vector strong subtree $\mathbf{R}$ of $\mathbf{S}_{n+1}$ and for every $v\in b_W^n$ and every $p\in\{0,\dots,b_W-1\}$ a node $w_{v,p}\in W$ and a constant $0<\theta_{v,p}\leqslant 1$ such that the pair $(\mathbf{R},w_{v,p})$ is strongly $\theta_{v,p}$-correlated with respect to $D_{v,p}$. We set "$\mathbf{R}_{n+1}=\mathbf{R}$", "$\theta_{n+1}=\min\big\{\theta_{v,p}: v\in b_W^n \text{ and } p\in\{0,\dots,b_W-1\}\big\}$" and "$w_{v^\frown p}=w_{v,p}$" for every $v\in b_W^n$ and every $p\in\{0,\dots,b_W-1\}$. Notice that with these choices conditions (C1), (C4) and (C7) are satisfied. Let $\mathbf{r}_{n+1}$ be the root of $\mathbf{R}_{n+1}$. Since the pair $(\mathbf{R}_{n+1},w_{v^\frown p})$ is strongly $\theta_{n+1}$-correlated with respect to $D_{v,p}$, we see that $w_{v^\frown p}\in D_{v,p}(\mathbf{r}_{n+1})\subseteq D_{n+1}(\mathbf{r}_{n+1})\subseteq D(\mathbf{r}_{n+1})$. Hence, there exists $l\in\mathbb{N}$ with $l>l_n$ such that $\ell_W(w_{v^\frown p})=l$ for every $v\in b_W^n$ and every $p\in\{0,\dots,b_W-1\}$. We set "$l_{n+1}=l$" and we observe that the last condition, condition (C3), is also satisfied. The recursive construction is completed.

**Step 4: a family of vector canonical embeddings.** Let $(\mathbf{S}_n)$ and $(\mathbf{R}_n)$ be the sequences of vector strong subtrees of $\mathbf{S}$ obtained in Step 3. Also let $(D_n)$ be the corresponding sequence of dense level selections. Recall that $\mathbf{S}_{n+1} = \mathbf{R}_n[\mathbf{0}]$. For every $n \in \mathbb{N}$ we write $\mathbf{S}_n = (S_1^n, \dots, S_d^n)$ and $\mathbf{R}_n = (R_1^n, \dots, R_d^n)$. Our goal in this step is to define a family $\{H_{\mathbf{u}} : \mathbf{u} \in \otimes\mathbf{B}\}$ of vector canonical embeddings. It will be used to "unravel" the recursive definition of the sequence $(D_n)$ and relate each $D_n$ with the $(\varepsilon/2)$-dense level selection $D\colon \otimes \mathbf{S} \to 2^W$ obtained in Step 1. This is the content of Fact 23 below.

To this end, we will describe first how these embeddings are acting in each coordinate. So, fix $i \in \{1, \dots, d\}$. Recursively, for every $n \in \mathbb{N}$ and every $u \in b_i^n$ we define a map $h_i^u\colon S_i^n \to T_i$ as follows. For $u = \emptyset$ let $h_i^{\emptyset}\colon S_i^0 \to T_i$ be the identity. Let $n \in \mathbb{N}$ and $u \in b_i^n$ and assume that the map $h_i^u\colon S_i^n \to T_i$ has been defined. For every $p \in \{0, \dots, b_i - 1\}$ we set

$$h_i^{u^\frown p} = h_i^u \circ \mathrm{I}\big(S_i^{n+1}, R_i^n[p]\big) = h_i^u \circ \mathrm{I}\big(R_i^n[0], R_i^n[p]\big) \tag{26}$$

where $\mathrm{I}\big(S_i^{n+1}, R_i^n[p]\big)$ is the canonical isomorphism between the homogeneous trees $S_i^{n+1}$ and $R_i^n[p]$. Inductively, it is easy to verify the following properties guaranteed by the above construction.

(P1) For every $u \in b_i^{<\mathbb{N}}$ the map $h_i^u$ is a canonical embedding.
(P2) For every $n \in \mathbb{N}$, every $u \in b_i^n$ and every $s \in S_i^n$ we have $\ell_{T_i}\big(h_i^u(s)\big) = \ell_{T_i}(s)$.

We are ready to introduce the desired family $\{H_{\mathbf{u}} : \mathbf{u} \in \otimes\mathbf{B}\}$ of vector canonical embeddings. So, let $n \in \mathbb{N}$ and $\mathbf{u} = (u_1, \dots, u_d) \in \otimes\mathbf{B}(n)$ be arbitrary. We define $H_{\mathbf{u}}\colon \otimes \mathbf{S}_n \to \otimes\mathbf{T}$ by the rule

$$H_{\mathbf{u}}\big((s_1, \dots, s_d)\big) = \big(h_1^{u_1}(s_1), \dots, h_d^{u_d}(s_d)\big). \tag{27}$$

By properties (P1) and (P2), it is clear that $H_{\mathbf{u}}$ is a well-defined vector canonical embedding. We will need a formula satisfied by these maps which follows by identities (23), (26) and (27). Specifically, for every $n \in \mathbb{N}$, every $\mathbf{u} = (u_1, \dots, u_d) \in \otimes\mathbf{B}(n)$ and every $\mathbf{p} = (p_1, \dots, p_d) \in \otimes\mathbf{B}(1)$ it holds that

$$H_{\mathbf{u}^\frown\mathbf{p}} = H_{\mathbf{u}} \circ \mathbf{I}\big(\mathbf{R}_n[\mathbf{0}], \mathbf{R}_n[\mathbf{p}]\big) \tag{28}$$

where $\mathbf{I}\big(\mathbf{R}_n[\mathbf{0}], \mathbf{R}_n[\mathbf{p}]\big)$ is the vector canonical isomorphism between $\mathbf{R}_n[\mathbf{0}]$ and $\mathbf{R}_n[\mathbf{p}]$. We will also need the following fact.

**Fact 23.** *For every $n \in \mathbb{N}$ and every $\mathbf{s} \in \otimes\mathbf{S}_n$ we have*

$$D_n(\mathbf{s}) = \bigcap_{\mathbf{u} \in \otimes\mathbf{B}(n)} D\big(H_{\mathbf{u}}(\mathbf{s})\big).$$

*Proof.* The proof proceeds by induction on $n$. For $n = 0$ the desired identity follows immediately by condition (C6) in Step 3 and the fact that $H_{\mathbf{B}(0)}$ is the identity map on $\otimes\mathbf{S}_0$. Assume that the result has been proved for some $n \in \mathbb{N}$. Let $\mathbf{s} \in \otimes\mathbf{S}_{n+1}$ be arbitrary. Recall that $\mathbf{S}_{n+1} = \mathbf{R}_n[\mathbf{0}]$ and that $\mathbf{R}_n$ is a vector strong subtree of $\mathbf{S}_n$.

Therefore,

$$\begin{aligned} D_{n+1}(\mathbf{s}) &\overset{(25)}{=} \bigcap_{\mathbf{p}\in\otimes\mathbf{B}(1)} D_n\Big(\mathbf{I}\big(\mathbf{R}_n[\mathbf{0}],\mathbf{R}_n[\mathbf{p}]\big)(\mathbf{s})\Big) \\ &= \bigcap_{\mathbf{p}\in\otimes\mathbf{B}(1)} \bigcap_{\mathbf{u}\in\otimes\mathbf{B}(n)} D\Big(H_{\mathbf{u}}\Big(\mathbf{I}\big(\mathbf{R}_n[\mathbf{0}],\mathbf{R}_n[\mathbf{p}]\big)(\mathbf{s})\Big)\Big) \\ &\overset{(28)}{=} \bigcap_{\mathbf{p}\in\otimes\mathbf{B}(1)} \bigcap_{\mathbf{u}\in\otimes\mathbf{B}(n)} D\big(H_{\mathbf{u}^\frown\mathbf{p}}(\mathbf{s})\big) = \bigcap_{\mathbf{u}'\in\otimes\mathbf{B}(n+1)} D\big(H_{\mathbf{u}'}(\mathbf{s})\big) \end{aligned}$$

where the second equality follows by our inductive assumption. The proof of Fact 23 is completed. □

**Step 5: an infinite chain of $(T_1,\dots,T_d)$ and an "unfolding" argument.** Let $(l_n)$ be the strictly increasing sequence in $\mathbb{N}$ and $(\mathbf{R}_n)$ the sequence of vector strong subtrees of $\mathbf{S}$ obtained in Step 3. For every $n\in\mathbb{N}$ we set

$$\mathbf{r}_n = \mathbf{R}_n(0) \tag{29}$$

and we write $\mathbf{r}_n=(r_1^n,\dots,r_d^n)$. Recall that a subset $C$ of a tree $(T,<)$ is said to be a *chain* if for every $s,t\in C$ we have that either $t\leqslant s$ or $s\leqslant t$.

**Lemma 24.** *For every $i\in\{1,\dots,d\}$ the family $\{r_i^n : n\in\mathbb{N}\}$ is an infinite chain of the tree $T_i$. Moreover, for every $n\in\mathbb{N}$ we have $\ell_{T_i}(r_i^n)=l_n$.*

*Proof.* Let $n\in\mathbb{N}$ be arbitrary. As in Step 4, we write $\mathbf{R}_n=(R_1^n,\dots,R_d^n)$. By conditions (C1) and (C2) in Step 3, we see that $r_i^{n+1}\in \mathrm{Succ}_{R_i^n}(r_i^n)\subseteq \mathrm{Succ}_{T_i}(r_i^n)$. This shows that the family $\{r_i^n : n\in\mathbb{N}\}$ is an infinite chain of $T_i$. Also notice that, by conditions (C6) and (C7), we have $(r_1^n,\dots,r_d^n,w_v)\in D$ for every $v\in b_W^n$. Invoking condition (C3), we conclude that $\ell_{T_i}(r_i^n)=l_n$. The proof of Lemma 24 is completed. □

Let $i\in\{1,\dots,d\}$ and consider the family $\{h_i^u : u\in b_i^{<\mathbb{N}}\}$ of canonical embeddings defined in Step 4. We define a map $\Phi_i\colon b_i^{<\mathbb{N}}\to T_i$ as follows. For every $n\in\mathbb{N}$ and every $u\in b_i^n$ we set

$$\Phi_i(u)=h_i^u(r_i^n). \tag{30}$$

That is, for every $n\in\mathbb{N}$ the family $\{\Phi_i(u) : u\in b_i^n\}$ is the "orbit" of the node $r_i^n$ under the family of maps $\{h_i^u : u\in b_i^n\}$.

**Lemma 25.** *For every $i\in\{1,\dots,d\}$ the map $\Phi_i\colon b_i^{<\mathbb{N}}\to T_i$ is a canonical embedding. Moreover, for every $n\in\mathbb{N}$ and every $u\in b_i^n$ we have*

$$\ell_{T_i}\big(\Phi_i(u)\big)=l_n. \tag{31}$$

*Proof.* Let $i\in\{1,\dots,d\}$ be arbitrary. First notice that, by property (P2) in Step 4 and Lemma 24, condition (a) in §2.5 and equality (31) are both satisfied. To show that condition (b) in §2.5 is satisfied we need to prove that for every $n\in\mathbb{N}$, every

$u \in b_i^n$ and every $p \in \{0,\dots,b_i-1\}$ we have that $\Phi_i(u^\frown p) \in \mathrm{Succ}_{T_i}\big(\Phi_i(u)^{\frown_{T_i}} p\big)$. To this end let

$$w = r_i^n \ \text{ and } \ r = \mathrm{I}\big(R_i^n[0], R_i^n[p]\big)(r_i^{n+1}).$$

By (26) and (30), we see that $\Phi_i(u) = h_i^u(w)$ and $\Phi_i(u^\frown p) = h_i^u(r)$. Also observe that $r \in \mathrm{Succ}_{T_i}(w^{\frown_{T_i}} p)$. By property (P1) in Step 4, the map $h_i^u$ is a canonical embedding. Therefore, $h_i^u(r) \in \mathrm{Succ}_{T_i}\big(h_i^u(w)^{\frown_{T_i}} p\big)$. The proof of Lemma 25 is completed. □

**Step 6: the end of the proof.** For every $i \in \{1,\dots,d\}$ we set

$$Z_i \coloneqq \{\Phi_i(u) : u \in b_i^{<\mathbb{N}}\} \tag{32}$$

where $\Phi_i$ is the canonical embedding defined in (30). Also, we set

$$V \coloneqq \{w_v : v \in b_W^{<\mathbb{N}}\} \tag{33}$$

where $\{w_v : v \in b_W^{<\mathbb{N}}\}$ is the family obtained in part (e) of the construction presented in Step 3. By conditions (C3) and (C4), we see that $V$ is a strong subtree of $W$ and $L_W(V) = \{l_n : n \in \mathbb{N}\}$. Moreover, by Lemma 25, $Z_i$ is a strong subtree of $T_i$ and $L_{T_i}(Z_i) = \{l_n : n \in \mathbb{N}\}$ for every $i \in \{1,\dots,d\}$. It follows that $(Z_1,\dots,Z_d,V)$ is a vector strong subtree of $(T_1,\dots,T_d,W)$. The proof will be completed once we show that the level product of $(Z_1,\dots,Z_d,V)$ is contained in $D$.

So, let $(z_1,\dots,z_d,v)$ be an element of the level product of $(Z_1,\dots,Z_d,V)$. There exist $n \in \mathbb{N}$, $\mathbf{u}_0 = (u_1,\dots,u_d) \in \otimes\mathbf{B}(n)$ and $v_0 \in b_W^n$ such that $v = w_{v_0}$ and $z_i = \Phi_i(u_i)$ for every $i \in \{1,\dots,d\}$. Notice that

$$\big(\Phi_1(u_1),\dots,\Phi_d(u_d)\big) \overset{(30)}{=} \big(h_1^{u_1}(r_1^n),\dots,h_d^{u_d}(r_d^n)\big) \overset{(27)}{=} H_{\mathbf{u}_0}(\mathbf{r}_n) \overset{(29)}{=} H_{\mathbf{u}_0}\big(\mathbf{R}_n(0)\big).$$

By condition (C7), the pair $(\mathbf{R}_n, w_{v_0})$ is strongly $\theta_n$-correlated with respect to $D_n$. Therefore, $w_{v_0} \in D_n\big(\mathbf{R}_n(0)\big)$. By Fact 23, we obtain that

$$w_{v_0} \in D_n\big(\mathbf{R}_n(0)\big) \subseteq D\big(H_{\mathbf{u}_0}\big(\mathbf{R}_n(0)\big)\big).$$

Summing up, we conclude that

$$(z_1,\dots,z_d,v) = \big(\Phi_1(u_1),\dots,\Phi_d(u_d),w_{v_0}\big) \in D.$$

The proof of Theorem 2 is thus completed.

## 6. Comments

Using a standard compactness argument we obtain the following finite version of Theorem 2.

**Theorem 26.** *For every integer $d \geqslant 1$, every $b_1,\dots,b_d \in \mathbb{N}$ with $b_i \geqslant 2$ for all $i \in \{1,\dots,d\}$, every integer $k \geqslant 1$, every real $0 < \varepsilon \leqslant 1$ and every infinite subset $M = \{m_0 < m_1 < \cdots\}$ of $\mathbb{N}$ there exists an integer $N$ with the following property.*

*If* $\mathbf{T} = (T_1, \dots, T_d)$ *is a vector homogeneous tree with* $b_{\mathbf{T}} = (b_1, \dots, b_d)$ *and* $D$ *is a subset of the level product of* $(T_1, \dots, T_d)$ *satisfying*

$$|D \cap \big(T_1(m_n) \times \cdots \times T_d(m_n)\big)| \geqslant \varepsilon |T_1(m_n) \times \cdots \times T_d(m_n)|$$

*for every* $n \leqslant N$*, there exists a finite vector strong subtree* $\mathbf{S}$ *of* $\mathbf{T}$ *of height* $k$ *such that the level product of* $\mathbf{S}$ *is a subset of* $D$*. The least integer* $N$ *with this property will be denoted by* $\mathrm{DHL}(b_1, \dots, b_d|k, \varepsilon, M)$*.*

Notice, however, that the reduction of Theorem 26 to Theorem 2 via compactness is noneffective and gives no estimate for the numbers $\mathrm{DHL}(b_1, \dots, b_d|k, \varepsilon, M)$. The natural problem of obtaining explicit upper bounds for the "density Halpern–Läuchli numbers" is studied in [9].

## References


[1] S. A. Argyros, P. Dodos and V. Kanellopoulos, *A classification of separable Rosenthal compacta and its applications*, Dissertationes Math. 449 (2008), 1–52.

[2] S. A. Argyros, P. Dodos and V. Kanellopoulos, *Unconditional families in Banach spaces*, Math. Annalen 341 (2008), 15–38.

[3] R. Bicker and B. Voigt, *Density theorems for finitistic trees*, Combinatorica 3 (1983), 305–313.

[4] A. Blass, *A partition theorem for perfect sets*, Proc. Amer. Math. Soc. 82 (1981), 271–277.

[5] T. J. Carlson, *Some unifying principles in Ramsey Theory*, Discr. Math. 68 (1988), 117–169.

[6] T. J. Carlson and S. G. Simpson, *A dual form of Ramsey's theorem*, Adv. Math. 53 (1984), 265–290.

[7] P. Dodos, *Operators whose dual has non-separable range*, J. Funct. Anal. 260 (2011), 1285–1303.

[8] P. Dodos and V. Kanellopoulos, *On filling families of finite subsets of the Cantor set*, Math. Proc. Cambridge Phil. Soc. 145 (2008), 165–175.

[9] P. Dodos, V. Kanellopoulos and K. Tyros, *Dense subsets of products of finite trees*, Int. Math. Res. Not. 4 (2013), 924–970.

[10] P. Dodos, V. Kanellopoulos and K. Tyros, *A simple proof of the density Hales–Jewett theorem*, Int. Math. Res. Not. 12 (2014), 3340–3352.

[11] H. Furstenberg and Y. Katznelson, *A density version of the Hales–Jewett theorem*, Journal d'Anal. Math. 57 (1991), 64–119.

[12] F. Galvin, *Partition theorems for the real line*, Notices Amer. Math. Soc. 15 (1968), 660.

[13] A. H. Hales and R. I. Jewett, *Regularity and positional games*, Trans. Amer. Math. Soc. 106 (1963), 222–229.

[14] J. D. Halpern, *Nonstandard combinatorics*, Proc. London Math. Soc. 30 (1975), 40–54.

[15] J. D. Halpern and H. Läuchli, *A partition theorem*, Trans. Amer. Math. Soc. 124 (1966), 360–367.

[16] J. D. Halpern and A. Lévy, *The boolean prime ideal theorem does not imply the axiom of choice*, Proc. Sympos. Pure Math., Vol. 13, part I, Amer. Math. Soc., Providence, R. I., 1967, 83–134.

[17] V. Kanellopoulos, *Ramsey families of subtrees of the dyadic tree*, Trans. Amer. Math. Soc. 357 (2005), 3865–3886.

[18] R. Laver, *Products of infinitely many perfect trees*, J. London Math. Soc. 29 (1984), 385–396.

[19] A. Louveau, S. Shelah and B. Veličković, *Borel partitions of infinite trees of a perfect tree*, Annals Pure and Applied Logic 63 (1993), 271–281.

[20] K. Milliken, *A Ramsey theorem for trees*, J. Comb. Theory, Ser. A 26 (1979), 215–237.

[21] K. Milliken, *A partition theorem for the infinite subtrees of a tree*, Trans. Amer. Math. Soc. 263 (1981), 137–148.

[22] D. Pincus and J. D. Halpern, *Partitions of products*, Trans. Amer. Math. Soc. 267 (1981), 549–568.

[23] S. Todorcevic, *Compact subsets of the first Baire class*, J. Amer. Math. Soc. 12 (1999), 1179–1212.

[24] S. Todorcevic, *Introduction to Ramsey Spaces*, Annals Math. Studies, No. 174, Princeton Univ. Press, 2010.

Department of Mathematics, University of Athens, Panepistimiopolis 157 84, Athens, Greece

*E-mail address*: pdodos@math.uoa.gr

National Technical University of Athens, Faculty of Applied Sciences, Department of Mathematics, Zografou Campus, 157 80, Athens, Greece

*E-mail address*: bkanel@math.ntua.gr

National Technical University of Athens, Faculty of Applied Sciences, Department of Mathematics, Zografou Campus, 157 80, Athens, Greece

*E-mail address*: nkaragiannis@math.ntua.gr