# BANACH SPACES AND RAMSEY THEORY: SOME OPEN PROBLEMS

PANDELIS DODOS, JORDI LOPEZ-ABAD AND STEVO TODORCEVIC

Abstract. We discuss some open problems in the geometry of Banach spaces having Ramsey-theoretic flavor. The problems are exposed together with well known results related to them.

## 1. Introduction

One of the most illuminating imports in Banach space theory is the field of Ramsey theory. There have been some remarkable achievements using combinatorial techniques; see, for instance, [7, 9, 38, 60] and the references therein. The purpose of this paper is to collect a number of *open problems* in the geometry of Banach spaces having Ramsey flavor. We have not tried to be encyclopedic in our selection. Instead, we have chosen to discuss the following list of open problems which shows, in particular, that Ramsey theory cuts across all parts of modern Banach space theory.

- *Distance from the cube.*
- *Elton unconditionality constant.*
- *Rosenthal basis problem.*
- *Unique spreading model problem.*
- *Block homogeneous basis problem.*
- *Fixing properties of operators on* $C[0,1]$.
- *Bounded distortion.*
- *Converse Aharoni problem.*
- *Separable quotient problem.*
- *Quotients with long Schauder bases and biorthogonal systems.*
- *Rolewicz problem on support sets.*
- *Unconditional basic sequences in non-separable spaces.*

In several cases, a problem naturally leads to a number of related questions which are interesting on their own. We discuss these issues in detail.

Our Banach space theoretic notation and terminology is standard and follows [51]. By $\mathbb{N} = \{0, 1, 2, \dots\}$ we denote the natural numbers. If $S$ is a set, then the

2010 *Mathematics Subject Classification*: 03E02, 03E55, 05D10, 46B03, 46B07, 46B15, 46B26, 46B80, 47B38.

*Key words*: Banach spaces, Ramsey theory, open problems.

cardinality of $S$ will be denoted by $|S|$. Finally, by $[\mathbb{N}]^\infty$ we denote the set of all infinite subsets of $\mathbb{N}$.

## 2. Distance from the cube

Recall that the *Banach–Mazur distance* between two isomorphic Banach spaces $X$ and $Y$ (not necessarily infinite-dimensional) is defined by

$$d_{BM}(X,Y) = \inf\big\{\|T\|\cdot\|T^{-1}\| : T\colon X\to Y \text{ is an isomorphism}\big\}.$$

For every $n\in\mathbb{N}$ define

$$R^n_\infty = \max\big\{d_{BM}(X,\ell^n_\infty) : \dim(X)=n\big\}.$$

**Problem 1.** (Distance from the cube—Pełczyński [65]) *Determine the asymptotic behavior of $R^n_\infty$ as $n$ tends to infinity.*

The best known lower bound for $R^n_\infty$ is due to Szarek [74]:

$$c\sqrt{n}\log n \leqslant R^n_\infty$$

where $c$ is an absolute constant. The problem of finding good upper estimates for $R^n_\infty$ turned out to be a very challenging problem and has attracted the attention of several researchers; see [14, 75]. The best known upper bound is due to Giannopoulos [32]:

$$R^n_\infty \leqslant Cn^{5/6}$$

where $C$ is an absolute constant.

This problem appears in our list because all proofs establishing a non-trivial upper bound for $R^n_\infty$ use some variant of the *Sauer–Shelah lemma*. This basic combinatorial fact asserts that if $S$ is a nonempty finite set and $\mathcal{F}$ is a "large" family of subsets of $S$, then there exists a "large" subset $A$ of $S$ such that the *trace $\mathcal{F}[A]$ of $\mathcal{F}$ on $A$*, that is, the family $\mathcal{F}[A] \coloneqq \{F\cap A : F\in\mathcal{F}\}$, is the powerset of $A$.

**Theorem 1.** (Sauer–Shelah lemma [70, 71]) *Let $S$ be a nonempty finite set, $k\in\mathbb{N}$ with $1\leqslant k\leqslant |S|$ and $\mathcal{F}$ a family of subsets of $S$ such that*

$$|\mathcal{F}| > \sum_{i=0}^{k-1}\binom{|S|}{i}.$$

*Then there exists a subset $A$ of $S$ with $|A|=k$ such that the trace $\mathcal{F}[A]$ of $\mathcal{F}$ on $A$ is the powerset of $A$.*

The Sauer–Shelah lemma turned out to be a very important tool in the "local theory" of Banach spaces and has found significant applications. It was first used in Banach space theory by Elton [25].

## 3. Elton unconditionality constant

The Sauer–Shelah lemma has an infinite version. To state it let us recall that a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is said to be *compact* (respectively, *hereditary*) if $\mathcal{F}$ is a closed subset of $2^{\mathbb{N}}$ (respectively, for every $F \in \mathcal{F}$ and every $G \subseteq F$ we have that $G \in \mathcal{F}$).

**Theorem 2.** (Elton—see [7]) *Let $\mathcal{F}$ be a compact family of finite subsets of $\mathbb{N}$. Then there exists $L \in [\mathbb{N}]^\infty$ such that $\mathcal{F}[L]$ is a hereditary family.*

This result can be seen as the discrete analogue of a heavily investigated phenomenon in Banach space theory known as *partial unconditionality*.

Recall that a normalized sequence $(x_n)$ is a Banach space $X$ is said to be *unconditional* if there exists a constant $C \geqslant 1$ such that for every $k \in \mathbb{N}$, every $a_0, \dots, a_k \in \mathbb{R}$ and every $G \subseteq \{0, \dots, k\}$ we have

$$\Big\| \sum_{n \in G} a_n x_n \Big\| \leqslant C \Big\| \sum_{n=0}^{k} a_n x_n \Big\|. \tag{1}$$

A classical discovery due to Maurey and Rosenthal [56] provides an example of a normalized weakly null sequence $(e_n)$ in a Banach space $E$ with no unconditional subsequence. A surprising fact is, however, that for *every* normalized weakly null sequence $(x_n)$ there exist a subsequence $(y_n)$ of $(x_n)$ and a constant $C \geqslant 1$ such that inequality (1) is valid provided that we restrict our attention to certain sequences of coefficients (or if we project to certain finite subsets of $\mathbb{N}$); that is, the sequence $(y_n)$ is *partially* unconditional.

By now several partial unconditionality results have been obtained; see, for instance, [7, 9, 21, 53, 60] and the references therein. The first (and undoubtedly the most influential) one was discovered by Elton. To state it let us recall the following notion.

**Definition 3.** *Let $0 < \delta \leqslant 1$ and $K \geqslant 1$. A normalized basic sequence $(x_n)$ is said to be $\delta$-*near unconditional with constant $K$ *if the basis constant of $(x_n)$ is at most $K$ and is such that for every $k \in \mathbb{N}$, every $a_0, \dots, a_k \in \mathbb{R}$ and every $G \subseteq \big\{ i \in \{0, \dots, k\} : \delta \leqslant |a_i| \leqslant 1 \big\}$ we have*

$$\Big\| \sum_{n \in G} a_n x_n \Big\| \leqslant K \Big\| \sum_{n=0}^{k} a_n x_n \Big\|. \tag{2}$$

We can now state Elton's theorem.

**Theorem 4.** (Elton [24]) *Let $0 < \delta \leqslant 1$. Then there exists a constant $K \geqslant 1$, possibly depending on $\delta$, such that for every normalized weakly null sequence $(e_n)$ in a Banach space $E$ there exists a subsequence $(x_n)$ of $(e_n)$ such that $(x_n)$ is $\delta$-near unconditional with constant $K$.*

For each $0 < \delta \leqslant 1$ let $K(\delta)$ be the infimum of the set of real numbers $K$ such that every normalized weakly null sequence has a $\delta$-near unconditional subsequence with constant $K$.

**Problem 2.** (Elton unconditionality constant—Dilworth, Odell, Schlumprecht and Zsák [21]) *Does there exist $M \geqslant 1$ such that $K(\delta) \leqslant M$ for every $0 < \delta \leqslant 1$?*

It is known (see [19, 60]) that there exists an absolute constant $C$ such that

$$K(\delta) \leqslant C \log(1/\delta)$$

for every $0 < \delta \leqslant 1$. It is also known (see [21]) that $5/4 \leqslant \sup\{K(\delta) : 0 < \delta \leqslant 1\}$.

## 4. Rosenthal basis problem

A normalized basic sequence $(x_n)$ in a Banach space $X$ is said to be *perfectly homogeneous* if every normalized block sequence $(v_n)$ of $(x_n)$ is equivalent to $(x_n)$. It is easy to see that the standard unit vector bases of the classical sequence spaces $c_0$ and $\ell_p$ $(1 \leqslant p < +\infty)$ are perfectly homogeneous. Remarkably, the converse is also true.

**Theorem 5.** (Zippin [84]) *Every perfectly homogeneous basic sequence $(x_n)$ is equivalent to the standard unit vector basis of $c_0$ or $\ell_p$ for some $1 \leqslant p < +\infty$.*

There is a slightly weaker notion: a normalized basic sequence $(x_n)$ in a Banach space $X$ is said to be a *Rosenthal basis* if every normalized block sequence $(v_n)$ of $(x_n)$ has a subsequence which is equivalent to $(x_n)$.

**Problem 3.** (Rosenthal basis problem—Rosenthal [26]) *Is it true that every Rosenthal basis $(x_n)$ is equivalent to the standard unit vector basis of $c_0$ or $\ell_p$ for some $1 \leqslant p < +\infty$?*

It is known (see [26]) that the problem has an affirmative answer provided that there exists a constant $C \geqslant 1$ such that every normalized block sequence $(v_n)$ of $(x_n)$ has a subsequence which is $C$-equivalent to $(x_n)$.

## 5. Unique spreading model problem

Let $(x_n)$ be a normalized basic sequence in a Banach space $X$ and let $(y_n)$ be a sequence in a Banach space $Y$. The sequence $(x_n)$ is said to *generate the sequence $(y_n)$ as spreading model* if there exists a sequence $(\varepsilon_n)$ of positive reals with $\varepsilon_n \downarrow 0$ such that for every $n \in \mathbb{N}$, every $a_0, \dots, a_n$ in $[-1, 1]$ and every $n \leqslant k_0 < \dots < k_n$ in $\mathbb{N}$ we have

$$(1+\varepsilon_n)^{-1}\Big\| \sum_{i=0}^{n} a_i y_i \Big\| \leqslant \Big\| \sum_{i=0}^{n} a_i x_{k_i} \Big\| \leqslant (1+\varepsilon_n)\Big\| \sum_{i=0}^{n} a_i y_i \Big\|.$$

It is well-known (see [16, 17]) that every normalized basic sequence has a subsequence generating a spreading model. Actually, this fact is one of the earliest applications of Ramsey theory in the geometry of Banach spaces.

For every Banach space $X$ let $\mathrm{SP}_{\mathrm{w}}(X)$ be the class of all spreading models generated by a normalized weakly null sequence in $X$, and denote by $\sim$ the usual equivalence relation of equivalence of basic sequences.

**Problem 4.** (Unique spreading model problem—Argyros [2]) *Let $X$ be a reflexive Banach space such that $|\mathrm{SP}_{\mathrm{w}}(X)/\sim| = 1$ (that is, all normalized weakly null sequences in $X$ generate a unique spreading model up to equivalence). Is it true that the unique spreading model is equivalent to the standard unit vector basis of $c_0$ or $\ell_p$ for some $1 \leqslant p < +\infty$?*

It is known (see [20]) that the reflexivity assumption is essential. It is also known (and easy to see) that an affirmative answer to Problem 4 yields an affirmative answer to Problem 3.

The following stronger version of Problem 4 has been also considered.

**Problem 5.** (Countable spreading models problem—Dilworth, Odell and Sari [20]) *Let $X$ be a reflexive Banach space such that $|\mathrm{SP}_{\mathrm{w}}(X)/\sim| \leqslant \aleph_0$. Does there exist a normalized weakly null sequence in $X$ generating a spreading model equivalent to the standard unit vector basis of $c_0$ or $\ell_p$ for some $1 \leqslant p < +\infty$?*

Some partial answers have been obtained. For instance, it is known (see [20]) that if $X$ is reflexive, $|\mathrm{SP}_{\mathrm{w}}(X)/\sim| \leqslant \aleph_0$ and $|\mathrm{SP}_{\mathrm{w}}(Y^*)/\sim| \leqslant \aleph_0$ for every infinite-dimensional subspace $Y$ of $X$, then there exists a normalized weakly null sequence in $X$ generating a spreading model equivalent to the standard unit vector basis of $c_0$ or $\ell_p$ for some $1 \leqslant p < +\infty$.

## 6. Block homogeneous basis problem

A normalized Schauder basis $(x_n)$ of a Banach space $X$ is said to be *block homogeneous* if every normalized block sequence $(v_n)$ of $(x_n)$ spans a subspace isomorphic to $X$.

**Problem 6.** (Block homogeneous basis problem—see [27]) *Let $X$ be a Banach space with a normalized block homogeneous Schauder basis. Is it true that $X$ is isomorphic to $c_0$ or $\ell_p$ for some $1 \leqslant p < +\infty$?*

Notice that an affirmative answer to the above problem would considerably strengthen Theorem 5 and, combined with the deep results of Szankowski [73], would yield an alternative proof to the *homogeneous Banach space problem* that avoids the use of Gowers' dichotomy [37].

Although it is known that every separable non-Hilbertian Banach space contains two non-isomorphic subspaces, we point out that there are several related open problems. The following one is probably the most natural.

**Problem 7.** (Godefroy [33]) *Let $X$ be a separable non-Hilbertian Banach space. Is it true that there exists a sequence $(X_n)$ of subspaces of $X$ which are pairwise non-isomorphic?*

It is likely that Problem 7 has an affirmative answer. Indeed, several partial results have been obtained (see [27] and the references therein).

We would like to isolate an instance of Problem 7 exposing, in particular, its connection with the *basis problem*, a famous problem in the geometry of Banach spaces that has received considerable attention during the 1970s and 1980s. Recall that, by the results in [55, 73], if a separable Banach space $X$ fails to have type $2-\varepsilon$ (or fails to have co-type $2+\varepsilon$) for some $\varepsilon>0$, then $X$ contains a subspace without a Schauder basis (actually, $X$ contains a subspace without the compact approximation property).

**Problem 8.** *Let $X$ be a separable Banach space which fails to have type $2-\varepsilon$ (or fails to have co-type $2+\varepsilon$) for some $\varepsilon>0$. Is it true that $X$ contains continuum many pairwise non-isomorphic subspaces?*

It is known (see [3]) that the above problem has an affirmative answer if $X$ is a separable weak Hilbert space not isomorphic to $\ell_2$.

## 7. Fixing properties of operators on $C[0,1]$

A central open problem in Banach space theory is the classification, up to isomorphism, of all complemented subspaces of the classical function space $C[0,1]$.

**Problem 9.** (Complemented subspace problem for $C[0,1]$—see [12, 52, 63]) *Is it true that every complemented subspace of $C[0,1]$ is isomorphic to a $C(K)$ space for some closed subset $K$ of $[0,1]$.*

The *complemented subspace problem* is discussed in detail in [69]. The strongest result to date is due to Rosenthal [68] and asserts that every complemented subspace of $C[0,1]$ with non-separable dual is isomorphic to $C[0,1]$. Actually, this fact is a consequence of the following surprising discovery concerning general operators on $C[0,1]$.

**Theorem 6.** (Rosenthal [68]) *Let $T\colon C[0,1]\to C[0,1]$ be a bounded linear operator such that its dual $T^*$ has non-separable range. Then $T$ fixes a copy of $C[0,1]$; that is, there exists a subspace $Y$ of $C[0,1]$ which is isomorphic to $C[0,1]$ and is such that $T|_Y$ is an isomorphic embedding.*

The following problem, asked in the 1970s, is motivated by the *complemented subspace problem* and Theorem 6.

**Problem 10.** (Fixing properties of operators on $C[0,1]$—Rosenthal [69]) *Let $T\colon C[0,1]\to C[0,1]$ be a bounded linear operator and let $X$ be a separable Banach space not containing a copy $c_0$. Assume that the operator $T$ fixes a copy of $X$. Is it true that $T$ fixes a copy of $C[0,1]$? Equivalently, is it true that the dual operator $T^*$ of $T$ has non-separable range?*

Notice that an affirmative answer to Problem 10 yields that every complemented subspace $X$ of $C[0,1]$ with separable dual is hereditarily $c_0$ (that is, every infinite-dimensional subspace of $X$ contains a copy of $c_0$). This property is predicted by the *complemented subspace problem* (see [66]) though it has not been decided yet. It is known, however, that every complemented subspace of $C[0,1]$ contains a copy $c_0$ (see [63]).

There are some partial positive answers to Problem 10.

**Theorem 7.** (Bourgain [13]) *If an operator $T\colon C[0,1]\to C[0,1]$ fixes a copy of a co-type space, then $T$ fixes a copy of $C[0,1]$.*

**Theorem 8.** (Gasparis [31]) *If an operator $T\colon C[0,1]\to C[0,1]$ fixes a copy of an asymptotic $\ell_1$ space, then $T$ fixes a copy of $C[0,1]$.*

Also, recently, a trichotomy was obtained characterizing, by means of fixing properties, the class of operators on general separable Banach spaces whose dual has non-separable range. To state it let us say that a sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ in a Banach space $X$ is said to be a *tree basis* provided that the following are satisfied.

(1) If $(t_n)$ is the canonical enumeration[1] of $2^{<\mathbb{N}}$, then $(x_{t_n})$ is a seminormalized basic sequence.
(2) For every infinite antichain $A$ of $2^{<\mathbb{N}}$ the sequence $(x_t)_{t\in A}$ is weakly null.
(3) For every $\sigma\in 2^{\mathbb{N}}$ the sequence $(x_{\sigma|n})$ is weak* convergent to an element $x^{**}_\sigma\in X^{**}\setminus X$. Moreover, if $\sigma,\tau\in 2^{\mathbb{N}}$ with $\sigma\neq\tau$, then $x^{**}_\sigma\neq x^{**}_\tau$.

The archetypical example of such a sequence is the standard unit vector basis of James tree space (see [42]).

**Theorem 9.** (Dodos [22]) *Let $X$ and $Y$ be separable Banach spaces. Also let $T\colon X\to Y$ be an operator.*

(a) *If $X$ does not contain a copy of $\ell_1$, then $T^*$ has non-separable range if and only if there exists a sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ in $X$ such that both $(x_t)_{t\in 2^{<\mathbb{N}}}$ and $(T(x_t))_{t\in 2^{<\mathbb{N}}}$ are tree bases.*
(b) *If $X$ contains a copy of $\ell_1$, then $T^*$ has non-separable range if and only if either the operator $T$ fixes a copy of $\ell_1$, or there exists a bounded sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ in $X$ such that its image $(T(x_t))_{t\in 2^{<\mathbb{N}}}$ is a tree basis.*

The proofs of the aforementioned results have a strong combinatorial flavor. In particular, in [31] heavy use is made of the infinite-dimensional Ramsey theorem for Borel partitions (see [30]) while the main result in [22] is based on the deep Halpern–Läuchli theorem and its consequences (see [40, 46, 81]).

[1] This means that for every $n,m\in\mathbb{N}$ we have $n<m$ if and only if either $|t_n|<|t_m|$ or $|t_n|=|t_m|$ and $t_n<_{\text{lex}} t_m$.

## 8. Bounded distortion

Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space and $\lambda > 1$. The space $X$ is said to be *$\lambda$-distortable* if there exists an equivalent norm $|\cdot|$ on $X$ such that for every infinite-dimensional subspace $Y$ of $X$ we have

$$\sup\Big\{ \frac{|y|}{|z|} : y, z \in Y \text{ and } \|y\| = \|z\| = 1 \Big\} \geqslant \lambda.$$

The space $X$ is said to be *distortable* if it is $\lambda$-distortable for some $\lambda > 1$. A distortable Banach space $X$ is said to be *arbitrarily distortable* if it is $\lambda$-distortable for every $\lambda > 1$; otherwise, $X$ is said to be *boundedly distortable.*

The first result on distortion is due to James [41]: *$c_0$ and $\ell_1$ are not distortable.* Milman [57] proved that a non-distortable Banach space must contain almost isometric copies of $c_0$ or $\ell_p$ for some $1 \leqslant p < +\infty$. The first example of a distortable Banach space was Tsirelson's space $T$ [83]. The natural problem whether the spaces $\ell_p$ $(1 < p < +\infty)$ are distortable became known as the *distortion problem* and it was solved by the following famous result.

**Theorem 10.** (Odell and Schlumprecht [61]) *For any $1 < p < +\infty$ the space $\ell_p$ is arbitrarily distortable.*

The *distortion problem* is discussed in detail in [62]. A surprising phenomenon observed so far is that "most" distortable Banach spaces are arbitrarily distortable.

**Problem 11.** (Bounded distortion—see [62]) *Does there exist a boundedly distortable Banach space?*

Tsirelson's space is a candidate for being boundedly distortable. Indeed, we have already mentioned that $T$ is distortable; actually, $T$ is $(2-\varepsilon)$-distortable for every $\varepsilon > 0$ (see [11]). But it not known whether $T$ is arbitrarily distortable.

**Problem 12.** (Distortion of Tsirelson's space—Rosenthal) *Is $T$ (or its dual) boundedly distortable?*

The *distortion problem* has a non-linear analogue which is also closely related to Ramsey theory. Recall that a real-valued Lipschitz map $f$ defined on the unit sphere $S_X$ of an infinite-dimensional Banach space $X$ is said to be *oscillation stable* if for every $\varepsilon > 0$ there exists an infinite-dimensional subspace $Y$ of $X$ such that the oscillation of $f$ on $S_Y$ is at most $\varepsilon$, that is,

$$\sup\{|f(y) - f(z)| : y, z \in Y \text{ and } \|y\| = \|z\| = 1\} \leqslant \varepsilon.$$

Notice that if a Banach space $X$ has the property that every real-valued Lipschitz map $f$ on $S_X$ is oscillation stable, then $X$ is not distortable. This is, however, a stronger property and is possessed only by hereditarily $c_0$ spaces (see [11]). That the space $c_0$ has this property follows by the following remarkable result.

**Theorem 11.** (Gowers [35]) *Every real-valued uniformly continuous map on the unit sphere of* $c_0$ *is oscillation stable.*

All known proofs of Theorem 11 use methods of topological dynamics on $\beta\mathbb{N}$. It would be very interesting to have a purely combinatorial proof avoiding the use of ultrafilters.

## 9. Converse Aharoni problem

The space $c_0$ is admittedly "small" in the linear category: *if a Banach space* $X$ *is isomorphic to a subspace of* $c_0$*, then* $X$ *contains a copy of* $c_0$. The situation, however, is dramatically different in the non-linear category.

**Theorem 12.** (Aharoni [1]) *Every separable metric space is Lipschitz isomorphic to a subset of* $c_0$.

The best constant of the Lipschitz embedding has been computed (see [45]). Theorem 12 can be rephrased to say that $c_0$ is universal in the non-linear category. But it is not known whether $c_0$ is the *minimal* Banach space with this property.

**Problem 13.** (Converse Aharoni problem—see [11, 44]) *Let* $X$ *be a separable Banach space with the property that every separable Banach space is Lipschitz isomorphic to a subset of* $X$*. Is it true that* $X$ *contains a copy of* $c_0$*?*

It is known (see [34]) that the class of subspaces of $c_0$ is stable under Lipschitz isomorphisms (moreover, if a Banach space $X$ is Lipschitz isomorphic to $c_0$, then $X$ is linearly isomorphic to $c_0$). It is possible that Theorem 11 could be used to attack Problem 13.

## 10. Separable quotient problem

In the last four sections we will present some problems about general Banach spaces where we do not assume separability. It turns out that this area has its own Ramsey theory to which it is related, the rich area of Ramsey theory of uncountable structure with a strong set-theoretic flavor (see, for example, [76, 77, 80]).

Recall the very old result due to Banach and Mazur saying that every infinite-dimensional Banach space has an infinite basic sequence. It turns out that the dual version of this *basic sequence problem* is still widely open.

**Problem 14.** (Banach [10]—Pełczyński [64]) *Does every infinite-dimensional Banach space* $X$ *has an infinite-dimensional quotient with a Schauder basis?*

The following result represents the first major progress made on this problem.

**Theorem 13.** (Johnson and Rosenthal [43]) *Every infinite-dimensional separable space $X$, and in fact every infinite-dimensional space of density $< \mathfrak{b}$*[2]*, has an infinite-dimensional quotient with a Schauder basis.*

So the problem of Banach and Pełczyński now becomes the following.

**Problem 15.** (Separable quotient problem) *Does every infinite-dimensional Banach space have a non-trivial infinite-dimensional separable quotient?*

There are several partial results worth mentioning (see [58] for a rather complete historical review of the early work on this problem).

**Theorem 14.** (Johnson and Rosenthal [43]—Hagler and Johnson [39]) *If $X^*$ contains an infinite-dimensional subspace with separable dual, then $X$ has a non-trivial separable quotient. If $X^*$ has an unconditional basic sequence, then $X$ has a quotient with an unconditional basis.*

**Theorem 15.** (Argyros, Dodos and Kanellopoulos [5, 6]) *Every dual Banach space and every representable Banach space has a separable non-trivial quotient.*

Concerning the two sufficient conditions given in Theorem 14 above, we mention that Gowers [36] has constructed a *separable* dual space which contains neither a copy of $\ell_1$ nor an infinite-dimensional subspace with a separable dual.

## 11. Quotients with long Schauder bases and biorthogonal systems

Note that Problem 14 has also the following natural variant.

**Problem 16.** *Given an infinite-dimensional Banach space $X$, what is the longest Schauder basis a quotient of $X$ can have?*

Note that this in particular asks if every non-separable Banach space has an uncountable biorthogonal system, a problem that has been first explicitly addressed by Davis and Johnson [18]. Today we know that the problem is really about additional set-theoretic methods that can be relevant in building uncountable biorthogonal systems and quotient maps. The first and today well-known examples showing this were constructed by Kunen (see [49, 50, 59]) assuming CH, and Shelah [72] assuming $\diamondsuit$. Recently a systematic study of generic Banach spaces is given in [54] and many of these examples are related to the uncountable biorthogonal system problem. The following old example (see [76, Chapter 1]) is not that well-known and we mention it here because of its relevance to the discussion below.

**Theorem 16.** (Todorcevic [76]) *If $\mathfrak{b} = \aleph_1$, then there is an Asplund $C(K)$ space without uncountable biorthogonal systems.*

[2]Recall that $\mathfrak{b}$ is the minimal cardinality of a subset of $\mathbb{N}^\mathbb{N}$ which is unbounded in the ordering of eventual dominance.

The second version of Problem 14 (Problem 16) was analyzed in [79] where the problem was connected with the following well-known set-theoretic and Ramsey-theoretic principle.

**P-ideal dichotomy (PID)**: For every P-ideal[3] $\mathcal{I}$ consisting of countable subsets of some index-set $S$, either

(1) there is an uncountable set $T \subseteq S$ such that every countable subset of $T$ belongs to $\mathcal{I}$, or
(2) the set $S$ can be decomposed into countably many subsets that have no infinite subsets belonging to $\mathcal{I}$.

**Theorem 17.** (Todorcevic [79]) *Assume* PID. *Every nonseparable Banach space $X$ of density $< \mathfrak{m}$ has a quotient with a Schauder basis of length $\omega_1$.*

Recall that $\mathfrak{m}$ is the minimal cardinality of a family $\mathcal{F}$ of nowhere dense subsets of a compact ccc space $K$ such that $\bigcup \mathcal{F} = K$ (see, for example, [28]).

**Corollary 18.** (Todorcevic [79]) *Assume* PID. *If $\mathfrak{m} > \aleph_1$[4], then every non-separable Banach space has an uncountable biorthogonal system.*

The reader is referred to [78] and [82] which list known applications of this principle many of which have a strong Ramsey-theoretic flavor. It turns out that PID is an additional set-theoretic principle which, while independent from CH, has a power of reducing problems about general sets to problems about reals. Finding out what is the actual problem about reals that corresponds to the quotient basis problem or the biorthogonal sequence problem is now in order. The following is a particular instance of what one may expect from such analysis.

**Problem 17.** *Are the following equivalent assuming the P-ideal dichotomy?*

(1) *Every non-separable Banach space has an uncountable biorthogonal system.*
(2) $\mathfrak{b} = \aleph_2$.

## 12. Rolewicz's problem on support sets

Recall that a *support set* in a Banach space $X$ is a nonempty convex set $C$ with the property that every point $x$ of $C$ is its support point, that is, there is a functional $f$ of $X$ such that

$$f(x) = \inf\{f(y) : y \in C\} < \sup\{f(y) : y \in C\}.$$

[3] Recall that an ideal $\mathcal{I}$ is a P-ideal if for every sequence $(X_n)$ of elements of $\mathcal{I}$ there is $Y$ such that $X_n \setminus Y$ is finite for all $n$.

[4] Under PID, the statement $\mathfrak{m} > \aleph_1$ and the statement $\mathfrak{m} = \aleph_2$ are equivalent. This also holds for other cardinals such as $\mathfrak{b}$ and $\mathfrak{p}$ (see [78, 82]).

The existence of a support set in every non-separable Banach space is an old problem of Rolewicz [67] who showed that separable Banach spaces do not admit support sets and who noticed that many of the non-separable Banach spaces do admit support sets.

**Problem 18.** (Rolewicz [67]) *Which Banach spaces admit support sets? Do all non-separable Banach spaces have support sets?*

It is easily seen that if $X$ has an uncountable biorthogonal system, then $X$ has a support set, so the second part of this question has a positive answer assuming PID and $\mathfrak{m} > \aleph_1$. It is also known that some assumptions are necessary as it is consistence to have a $C(K)$ space of density $\aleph_1$ without support sets (see [54] and [48]). It turns out, however, that finding uncountable biorthogonal systems is a different problem from finding support sets in Banach spaces.

**Theorem 19.** (Todorcevic [79]) *Every $C(K)$ space of density $> \aleph_1$ has a support set.*

Combining this with the following result we see the difference between the Rolewicz problem and the uncountable biorthogonal sequence problem.

**Theorem 20.** (Brech and Koszmider [15]) *It is consistent to have $C(K)$ spaces of density $> \aleph_1$ without uncountable biorthogonal systems.*

This leaves however open the following interesting variant of Rolewicz's problem.

**Problem 19.** (Todorcevic [79]) *Does every Banach space of density $> \aleph_1$ have a support set?*

## 13. Unconditional basic sequences in non-separable spaces

We now turn our attention to another well-known problem in this area.

**Problem 20.** (Unconditional basic sequences in non-separable spaces) *Does every Banach space $X$ of density $> \mathfrak{c}$ have an infinite unconditional basic sequence? Does additional properties, like reflexivity, make a difference?*

We mention some results related to this problem.

**Theorem 21.** (Argyros, Arvanitakis and Tolias [4]) *There is a Banach space $X$ of density $\mathfrak{c}$ with no infinite unconditional sequences* (*in fact, $X$ is hereditarily indecomposable*) *nor an infinite-dimensional reflexive subspace.*

**Theorem 22.** (Argyros, Lopez-Abad and Todorcevic [8]) *There exists a reflexive Banach space of density $\aleph_1$ without infinite unconditional sequences.*

So, the unconditional basic sequence problem might have a different answers for different classes of Banach spaces. This has been hinted also in a recent article by the authors where the following result is proven.

**Theorem 23.** (Dodos, Lopez-Abad and Todorcevic [23]) *It is consistent to assume that every Banach space of density* $\geqslant \aleph_\omega$ *has an infinite unconditional sequence.*

**Corollary 24.** (Dodos, Lopez-Abad and Todorcevic [23]) *It is consistent to assume that every Banach space of density* $\geqslant \aleph_\omega$ *has a quotient with an unconditional basis.*

There are related questions about the existence of infinite sub-symmetric sequences which seem to call for a different approach. First of all let us recall the following well-known result of Ketonen in a slightly different form.

**Theorem 25.** (Ketonen [47]) *Every Banach space of density at least the first Erdős cardinal*[5] *has an infinite sub-symmetric sequence.*

On the other hand no consistent bounds, like the one of [23] mentioned above, is known in this context. So the following problem is wide open.

**Problem 21.** *Does every* (*respectively, every reflexive*) *Banach space of density* $> \mathfrak{c}$ (*respectively, of density* $> \aleph_1$) *has an infinite sub-symmetric basic sequence?*

While analyzing this problem one is naturally led to the following variation.

**Problem 22.** *Is there a non-separable Tsirelson-like space?*

This problem seems related to the existence of a nonseparable analogue of the Schreier family on which the Tsirelson construction could be based. This is of course related to the following well-known problem.

**Problem 23.** (Fremlin [29]) *Is there a compact hereditary family of finite subsets of* $\omega_1$ *containing the singletons and having the property that every finite subset of* $\omega_1$ *has a subset half of its size belonging to the family* $\mathcal{F}$*?*

**Acknowledgment.** We would like to thank Gilles Godefroy for his valuable comments and remarks.

## References

[1] I. Aharoni, *Every separable metric space is Lipschitz equivalent to a subset of* $c_0$, Israel J. Math. 19 (1974), 284–291.

[2] G. Androulakis, E. Odell, Th. Schlumprecht and N.Tomczak-Jaegermann, *On the structure of the spreading models of a Banach space*, Canad. J. Math. 57 (2005), 673–707.

[3] R. Anisca, *The ergodicity of weak Hilbert spaces*, Proc. Amer. Math. Soc. 138 (2010), 1405–1413.

[4] S. A. Argyros, A. D. Arvanitakis and A. G. Tolias, *Saturated extensions, the attractors method and hereditarily James tree spaces*, Methods in Banach spaces, edited by J. M. Castillo and W. B. Johnson, LMS Lecture Notes, 337, Cambridge Univ. Press, 2006.

[5] S. A. Argyros, P. Dodos and V. Kanellopoulos, *A classification of separable Rosenthal compacta and its applications*, Dissertationes Math. 449 (2008), 1–52.

[5] Recall that $\theta$ is said to be an *Erdős cardinal* if for every function $f\colon [\theta]^{<\omega} \to \omega$ there is an infinite set $\Gamma \subseteq \theta$ such that $f$ is constant on $[\Gamma]^n$ for every $n < \omega$. This is a large cardinal which is larger than the first inaccessible cardinal and, in particular, much larger than the continuum.

[6] S. A. Argyros, P. Dodos and V. Kanellopoulos, *Unconditional families in Banach spaces*, Math. Annalen 341 (2008), 15–38.

[7] S. A. Argyros, G. Godefroy and H. P. Rosenthal, *Descriptive Set Theory and Banach spaces*, Handbook of the Geometry of Banach Spaces, Volume 2, Edited by W. B. Johnson and J. Lindenstrauss, Elsevier, 2003.

[8] S. A. Argyros, J. Lopez-Abad and S. Todorcevic, *A class of Banach spaces with few non-strictly singular operators*, J. Funct. Anal. 222 (2005), 306–384.

[9] S. A. Argyros and S. Todorcevic, *Ramsey Methods in Analysis*, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser, Verlag, Basel, 2005.

[10] S. Banach, *Théorie des opérations linéaires*, Warszawa, 1932.

[11] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, AMS Colloquium Publications, 2000.

[12] C. Bessaga and A. Pełczyński, *Spaces of continuous functions IV*, Studia Math. 19 (1960), 53–62.

[13] J. Bourgain, *A result on operators on* $C[0,1]$, J. Operator Theory 3 (1980), 275–289.

[14] J. Bourgain and S. J. Szarek, *The Banach–Mazur distance to the cube and the Dvoretsky–Rogers factorization*, Israel J. Math. 62 (1988), 169–180.

[15] C. Brech and P. Koszmider, *Thin very-tall compact scattered spaces which are hereditarily separable*, Trans. Amer. Math. Soc. 363 (2011), 501–519.

[16] A. Brunel and L. Sucheston, *On B convex Banach spaces*, Math. Systems Th. 7 (1974), 294–299.

[17] A. Brunel and L. Sucheston, *On J-convexity and some ergodic super properties of Banach spaces*, Trans. Amer. Math. Soc. 204 (1975), 79–90.

[18] W. J. Davis and W. B. Johnson, *On the existence of fundamental and total bounded biorthogonal systems in Banach spaces*, Studia Math. 45 (1972) 173–179.

[19] S. J. Dilworth, N. J. Kalton and D. Kutzarova, *On the existence of almost greedy bases in Banach spaces*, Studia Math. 159 (2003), 67–101.

[20] S. J. Dilworth, E. Odell and B. Sari, *Lattice structures and spreading models*, Israel J. Math. 161 (2007), 387–410.

[21] S. J. Dilworth, E. Odell, Th. Schlumprecth and A. Zsák, *Partial unconditionality*, Houston J. Math. 35 (2009), 1251–1311.

[22] P. Dodos, *Operators whose dual has non-separable range*, J. Funct. Anal. 260 (2011), 1285–1303.

[23] P. Dodos, J. Lopez-Abad and S. Todorcevic, *Unconditional basic sequences in spaces of large density*, Adv. Math. 226 (2011), 3297–3308.

[24] J. Elton, *Weakly null normalized sequences in Banach spaces*, Ph.D. Thesis, Yale University, 1978.

[25] J. Elton, *Sign-embeddings of* $\ell_1^n$, Trans. Amer. Math. Soc. 279 (1983), 113–124.

[26] V. Ferenczi, A. M. Pelczar and C. Rosendal, *On a question of Haskell P. Rosenthal concerning a characterization of* $c_0$ *and* $\ell_p$, Bull. London Math. Soc. 36 (2004), 396–406.

[27] V. Ferenczi and C. Rosendal, *Complexity and homogeneity in Banach spaces*, Banach Spaces and their Applications in Mathematics, Edited by B. Randrianantoanina and N. Randrianantoanina, Walter de Gruyter, Berlin, 83–110.

[28] D. H. Fremlin, *Consequences of Martin's axiom*, Cambridge University Press, 1984.

[29] D. H. Fremlin, *Problems; Version of 11.1.10*, University of Essex, Colchester, England.

[30] F. Galvin and K. Prikry, *Borel Sets and Ramsey's Theorem*, J. Symbolic Logic 38 (1973), 193–198.

[31] I. Gasparis, *Operators on* $C[0,1]$ *preserving copies of asymptotic* $\ell_1$ *spaces*, Math. Annalen 333 (2005), 831–858.

[32] A. Giannopoulos, *A note on the Banach–Mazur distance to the cube*, Operator Theory: Advances and Applications 77 (1995), 67–73.

[33] G. Godefroy, *Descriptive set theory and the geometry of Banach spaces*, Statistical Science and Interdisciplinary research - Vol. 8, Perspectives in Mathematical Sciences II, Pure Mathematics; Editors N. S. N. Sastry, T. S. S. R. K. Rao, Mohan Delampady, B. Rajeev (2009).

[34] G. Godefroy, N. J. Kalton and G. Lancien, *Subspaces of* $c_0(\mathbb{N})$ *and Lipschitz isomorphisms*, Geom. Funct. Anal. 10 (2000), 798–820.

[35] W. T. Gowers, *Lipschitz functions on classical spaces*, European J. Comb. 13 (1992), 141–151.

[36] W. T. Gowers, *A space not containing* $c_0$, $\ell_1$ *or a reflexive subspace*, Trans. Amer. Math. Soc. 344 (1994), 407–420.

[37] W. T. Gowers, *An infinite Ramsey theorem and some Banach space dichotomies*, Ann. Math. 156 (2002), 797–833.

[38] W. T. Gowers, *Ramsey methods in Banach spaces*, Handbook of the Geometry of Banach Spaces, Volume 2, Edited by W. B. Johnson and J. Lindenstrauss, Elsevier, 2003.

[39] J. N. Hagler and W. B. Johnson, *On Banach spaces whose dual balls are not weak* sequentially compact*, Israel J. Math. 28 (1977), 325–330.

[40] J. D. Halpern and H. Läuchli, *A partition theorem*, Trans. Amer. Math. Soc. 124 (1966), 360–367.

[41] R. C. James, *Uniformly non-square Banach spaces*, Ann. Math. 69 (1972), 409–419.

[42] R. C. James, *A separable somewhat reflexive Banach space with non-separable dual*, Bull. Amer. Math. Soc. 80 (1974), 738–743.

[43] W. B. Johnson and H. P. Rosenthal, *On weak* basic sequences and their applications to the study of Banach spaces*, Studia Math. 43 (1972), 77–92.

[44] N. J. Kalton, *The nonlinear geometry of Banach spaces*, Rev. Mat. Complut. 21 (2008), 7–60.

[45] N. J. Kalton and G. Lancien, *Best constants for Lipschitz embeddings of metric spaces into* $c_0$, Fund. Math. 199 (2008), 249–272.

[46] V. Kanellopoulos, *Ramsey families of subtrees of the dyadic tree*, Trans. Amer. Math. Soc. 357 (2005), 3865–3886.

[47] J. Ketonen, *Banach spaces and large cardinals*, Fund. Math. 81 (1974), 291–303.

[48] P. Koszmider, *On a problem of Rolewicz about Banach spaces that admit support sets*, J. Funct. Anal. 257 (2009), 2723–2741.

[49] K. Kunen, *Products of S-spaces*, note of May 1975.

[50] K. Kunen, *On hereditarily Lindelöf Banach spaces*, note of July 1980.

[51] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces vol. I: Sequence spaces*, Ergebnisse, 92, Springer, 1977.

[52] J. Lindenstrauss and D. E. Wulbert, *On the classification of the Banach spaces whose duals are* $L_1$ *spaces*, J. Funct. Anal. 4 (1969), 332–349.

[53] J. Lopez-Abad and S. Todorcevic, *Partial unconditionality of weakly null sequences*, RACSAM Rev. R. Acad. Cien. Serie A. Mat. 100 (2006), 237–277.

[54] J. Lopez-Abad and S. Todorcevic, *Generic Banach spaces and generic simplexes*, J. Funct. Anal. 261 (2011), 300–386.

[55] B. Maurey and G. Pisier, *Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach*, Studia Math. 58 (1976), 45–90.

[56] B. Maurey and H. P. Rosenthal, *Normalized weakly null sequence with no unconditional subsequence*, Studia Math. 61 (1977), 77–98.

[57] V. D. Milman, *Geometric theory of Banach spaces part II: Geometry of the unit sphere of a Banach space*, Usp. Mat. Nauk 26 (1971), 73–149.

[58] J. Mujica, *Separable quotients of Banach spaces*, Revista Mat. Univ. Complut. 10 (1997), 299–330.

[59] S. Negrepontis, *Banach spaces and topology,* Handbook of Set-Theoretic topology, Edited by K. Kunen and J. E. Vaughan, North-Holland, 1984, 1045–1142.

[60] E. Odell, *Applications of Ramsey theorems to Banach space theory*, Longhorn Notes, University of Texas, Texas Functional Analysis Seminar, 1980, 379–404.

[61] E. Odell and Th. Schlumprecht, *The distortion problem*, Acta Math. 173 (1994), 259–281.

[62] E. Odell and Th. Schlumprecht, *Distortion and asymptotic structure*, Handbook of the Geometry of Banach spaces, Volume 2, Edited by W. B. Johnson and J. Lindenstrauss, Elsevier, 2003.

[63] A. Pełczyński, *Projections in certain Banach spaces*, Studia Math. 32 (1960), 209–228.

[64] A. Pełczyński, *Some problems on basis in Banach and Fréchet spaces*, Israel J. Math. 2 (1964), 132–138.

[65] A. Pełczyński, *Structural theory of Banach spaces and its interplay with analysis and probability*, Proceedings of the ICM 1983, PWN-North Holland 1984, 237–269.

[66] A. Pełczyński and Z. Semadeni, *Spaces of continuous functions III. Spaces* $C(\Omega)$ *for* $\Omega$ *without perfect subsets*, Studia Math. 18 (1959), 211–222.

[67] S. Rolewicz, *On convex sets containing only points of support*, Comment. Math., Prace Math. Special issue 1 (1978), 279–281.

[68] H. P. Rosenthal, *On factors of* $C([0,1])$ *with non-separable dual*, Israel J. Math. 13 (1972), 361–378.

[69] H. P. Rosenthal, *The Banach spaces* $C(K)$, Handbook of the Geometry of Banach Spaces, Volume 2, Edited by W. B. Johnson and J. Lindenstrauss, Elsevier, 2003.

[70] N. Sauer, *On the density of families of sets*, J. Comb. Theory, Ser. A 13 (1972), 145–147.

[71] S. Shelah, *A combinatorial problem: stability and order for models and theories in infinitary languages*, Pacific J. Math. 41 (1972), 247–261.

[72] S. Shelah, *Uncountable constructions for BA, e.g. groups and Banach spaces*, Israel J. Math. 51 (1985), 273–297.

[73] A. Szankowski, *Subspaces without the approximation property*, Israel J. Math. 30 (1978), 123–129.

[74] S. J. Szarek, *Spaces with large distance to* $\ell_\infty^n$ *and random matrices*, Amer. J. Math. 112 (1990), 899–942.

[75] S. J. Szarek and M. Talagrand, *An isomorphic version of the Sauer–Shelah lemma and the Banach–Mazur distance to the cube*, GAFA Seminar 87-88, Lecture Notes in Mathematics, Springer, 1376 (1989), 105–112.

[76] S. Todorcevic, *Partition Problems in Topology*, Contemp. Math., Vol. 84, American Mathematical Society, Providence 1989.

[77] S. Todorcevic, *Compact subsets of the first Baire class*, J. Amer. Math. Soc. 12 (1999), 1179–1212.

[78] S. Todorcevic, *A dichotomy for P-ideals of countable sets*, Fund. Math. 166 (2000), 251–267.

[79] S. Todorcevic, *Biorthogonal systems and quotient spaces via Baire category methods*, Math. Annalen 335 (2006), 687–715.

[80] S. Todorcevic, *Walks on ordinals and their characteristics*, Progress in Mathematics, Vol. 263, Birkhäuser, 2007.

[81] S. Todorcevic, *Introduction to Ramsey Spaces*, Annals Math. Studies, No. 174, Princeton Univ. Press, 2010.

[82] S. Todorcevic, *Combinatorial Dichotomies in Set Theory*, Bull. Symbolic Logic 17 (2011), 1–72.

[83] B. S. Tsirelson, *Not every Banach space contains an imdedding of* $\ell_p$ *or* $c_0$, Funct. Anal. Appl. 8 (1974), 138–141.

[84] M. Zippin, *On perfectly homogeneous bases in Banach spaces*, Israel J. Math. 4 (1966), 265–272.

Department of Mathematics, University of Athens, Panepistimiopolis 157 84, Athens, Greece.

*E-mail address*: pdodos@math.uoa.gr

Instituto de Ciencias Matematicas (ICMAT), CSIC-UAM-UC3M-UCM, Madrid, Spain.

*E-mail address*: abad@logique.jussieu.fr

Université Denis Diderot - Paris 7, C.N.R.S., UMR 7056, 2 place Jussieu - Case 7012, 72521 Paris Cedex 05, France

and

Department of Mathematics, University of Toronto, Toronto, Canada, M5S 2E4.

*E-mail address*: stevo@math.toronto.edu