ON STRICTLY SINGULAR OPERATORS BETWEEN SEPARABLE BANACH SPACES

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ABSTRACT. Let X and Y be separable Banach spaces and denote by $\mathcal{SS}(X,Y)$ the subset of $\mathcal{L}(X,Y)$ consisting of all strictly singular operators. We study various ordinal ranks on the set $\mathcal{SS}(X,Y)$. Our main results are summarized as follows. Firstly, we define a new rank $\mathrm{r}_{\mathcal{S}}$ on $\mathcal{SS}(X,Y)$. We show that $\mathrm{r}_{\mathcal{S}}$ is a co-analytic rank and that dominates the rank ϱ introduced by Androulakis, Dodos, Sirotkin and Troitsky [Israel J. Math. 169 (2009), 221–250]. Secondly, for every $1 \leqslant p < +\infty$ we construct a Banach space Y_p with an unconditional basis such that $\mathcal{SS}(\ell_p, Y_p)$ is a co-analytic non-Borel subset of $\mathcal{L}(\ell_p, Y_p)$ yet every strictly singular operator $T\colon \ell_p \to Y_p$ satisfies $\varrho(T) \leqslant 2$. This answers a question of Argyros.

1. Introduction

An operator $T\colon X \to Y$ between two infinite-dimensional Banach spaces X and Y is said to be *strictly singular* if the restriction of T on every infinite-dimensional subspace Z of X is not an isomorphic embedding (throughout the paper by the term *operator* we mean bounded linear operator; all Banach spaces are over the real field). This is a wide class of operators between Banach spaces that includes the compact ones. A number of discoveries, especially after the work of Gowers and Maurey [17], have revealed the critical role of strictly singular operators on the structure theory of general Banach spaces.

Notice that an operator $T\colon X \to Y$ is strictly singular if and only if for every normalized basic sequence (x_n) in X and every $\varepsilon > 0$ there exist a nonempty finite subset F of $\mathbb{N}$ and a norm-one vector $x \in \mathrm{span}\{x_n : n \in F\}$ such that $\|T(x)\| \leqslant \varepsilon$. This equivalence gives us no hint of where the set F is located and, in particular, of how "difficult" it is to find it. Recently, the notion of a strictly singular operator was refined in order to measure this difficulty. The refinement was achieved with the use of the Schreier families $\mathcal{S}_\xi$ $(1 \leqslant \xi < \omega_1)$ introduced in [1].

Definition 1 ([4])**.** *Let X, Y be infinite-dimensional Banach spaces, $T \in \mathcal{L}(X,Y)$ and $1 \leqslant \xi < \omega_1$. The operator T is said to be* $\mathcal{S}_\xi$-strictly singular *if for every*

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normalized basic sequence (x_n) *in* X *and every* $\varepsilon > 0$ *there exist a nonempty set* $F \in \mathcal{S}_\xi$ *and a norm-one vector* $x \in \mathrm{span}\{x_n : n \in F\}$ *such that* $\|T(x)\| \leqslant \varepsilon$.

For every $T \in \mathcal{L}(X,Y)$ *we set*

$$\varrho(T) = \inf\{\xi : T \text{ is } \mathcal{S}_\xi\text{-strictly singular}\} \tag{1}$$

if T *is* $\mathcal{S}_\xi$*-strictly singular for some* $1 \leqslant \xi < \omega_1$*; otherwise, we set* $\varrho(T) = \omega_1$.

A basic fact, proved in [4], is that if X and Y are separable, then an operator $T\colon X \to Y$ is strictly singular if and only if $\varrho(T) < \omega_1$ (this equivalence fails if X and Y are non-separable; see [4]). In particular, the map $T \mapsto \varrho(T)$ is an ordinal rank[1] on the set $\mathcal{SS}(X,Y)$ of all strictly singular operators from X to Y. It was further studied in [3, 8, 11, 23].

In the present paper we continue the study of the rank ϱ by focusing on its global properties. These kind of questions are naturally studied within the framework of descriptive set theory (we briefly recall in Subsections 2.1, 2.2 and 2.3 all concepts from descriptive set theory related to our work). To put things in a proper perspective, let us first notice that if X and Y are separable Banach spaces, then the set $\mathcal{L}(X,Y)$ carries a natural structure of a standard Borel space (see Subsection 2.2) and it is easy to see that $\mathcal{SS}(X,Y)$ is a co-analytic subset of $\mathcal{L}(X,Y)$. Once the proper framework has been set up, a basic problem is to decide whether the rank ϱ is actually a co-analytic rank on the set $\mathcal{SS}(X,Y)$. Co-analytic ranks are fundamental tools in descriptive set theory and have proven to be extremely useful in studying the geometry of Banach spaces (see, for instance, [5, 6, 9, 12, 13, 14]).

As we shall see, the rank ϱ is not, in general, a co-analytic rank. Our first main result shows, however, that it is always sufficiently well-behaved.

Theorem 2. *Let* X *and* Y *be separable Banach spaces. Then there exists a co-analytic rank* $\mathrm{r}_\mathcal{S}\colon \mathcal{SS}(X,Y) \to \omega_1$ *such that*

$$\varrho(T) \leqslant \mathrm{r}_\mathcal{S}(T) \tag{2}$$

for every strictly singular operator $T\colon X \to Y$.

In particular, the rank ϱ *satisfies boundedness; that is, if* $\mathcal{A}$ *is an analytic subset of* $\mathcal{SS}(X,Y)$*, then* $\sup\{\varrho(T) : T \in \mathcal{A}\} < \omega_1$.

As a consequence we obtain the following corollary.

Corollary 3 ([8])**.** *If* X *and* Y *are separable Banach spaces and* $\mathcal{SS}(X,Y)$ *is a Borel subset of* $\mathcal{L}(X,Y)$*, then* $\sup\{\varrho(T) : T \in \mathcal{SS}(X,Y)\} < \omega_1$.

A natural problem, originally asked by Argyros, is whether the converse of Corollary 3 is true. In particular, it was conjectured in [8, Subsection 4.5] that if X and Y are separable Banach spaces and $\sup\{\varrho(T) : T \in \mathcal{SS}(X,Y)\} < \omega_1$, then $\mathcal{SS}(X,Y)$ is a Borel subset of $\mathcal{L}(X,Y)$. Our second main result answers this question in the negative.

[1] Ordinal ranks are standard tools in Banach space theory; see, for instance, [6, 10, 22, 25].

Theorem 4. *Let $1 \leqslant p < +\infty$. Then there exists a Banach space Y_p with an unconditional basis such that the following are satisfied.*

(i) *The set $\mathcal{SS}(\ell_p, Y_p)$ is a complete co-analytic (in particular, non-Borel) subset of $\mathcal{L}(\ell_p, Y_p)$.*
(ii) *If $T\colon \ell_p \to Y_p$ is strictly singular, then $\varrho(T) \leqslant 2$.*

In particular, the rank ϱ is not a co-analytic rank on $\mathcal{SS}(\ell_p, Y_p)$.

The paper is organized as follows. In Section 2 we gather some background material. In Section 3 we give a criterion for checking that $\varrho(T) \leqslant \xi$ when the spaces X and Y have Schauder bases. In Section 4 we give the proof of Theorem 2. In Section 5 we introduce a class of spaces $Z_{p,q}$ $(1 \leqslant p \leqslant q < +\infty)$ which are needed in the proof of Theorem 4. Finally, the proof of Theorem 4 is given in Section 6.

2. Background material

Our general notation and terminology is standard as can be found, for instance, in [21] and [20]. By $\mathbb{N} = \{1, 2, \dots\}$ we denote the natural numbers. For every infinite subset L of $\mathbb{N}$ by $[L]^\infty$ we denote the set of all infinite subsets of L. If F and G are two nonempty finite subsets of $\mathbb{N}$ we write $F < G$ if $\max(F) < \min(G)$. Finally, for every set A by $|A|$ we denote the cardinality of A.

2.1. **Trees.** By $\mathbb{N}^{<\mathbb{N}}$ we denote the set of all finite sequences of natural numbers while by $[\mathbb{N}]^{<\mathbb{N}}$ we denote the subset of $\mathbb{N}^{<\mathbb{N}}$ consisting of all strictly increasing finite sequences (the empty sequence is denoted by $\emptyset$ and is included in both $\mathbb{N}^{<\mathbb{N}}$ and $[\mathbb{N}]^{<\mathbb{N}}$). We will use the letters s and t to denote elements of $\mathbb{N}^{<\mathbb{N}}$. By $\sqsubset$ we shall denote the (strict) partial order on $\mathbb{N}^{<\mathbb{N}}$ of end-extension. If $\sigma \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$, then we set $\sigma|k = \big(\sigma(1), \dots, \sigma(k)\big)$; by convention $\sigma|0 = \emptyset$.

A *tree on* $\mathbb{N}$ is a subset of $\mathbb{N}^{<\mathbb{N}}$ which is closed under initial segments. By Tr we denote the set of all trees on $\mathbb{N}$. Hence

$$S \in \mathrm{Tr} \Leftrightarrow \forall s, t \in \mathbb{N}^{<\mathbb{N}} \ (s \sqsubseteq t \text{ and } t \in S \Rightarrow s \in S).$$

Notice that Tr is a closed subset of the compact metrizable space $2^{\mathbb{N}^{<\mathbb{N}}}$. Also notice that $[\mathbb{N}]^{<\mathbb{N}} \in \mathrm{Tr}$. We will reserve the letters S and R to denote trees. The *body* of a tree S on $\mathbb{N}$ is defined to be the set $\{\sigma \in \mathbb{N}^{\mathbb{N}} : \sigma|n \in S \ \forall n \in \mathbb{N}\}$ and is denoted by $[S]$. A tree S is said to be *well-founded* if $[S] = \emptyset$. By WF we denote the set of all well-founded trees on $\mathbb{N}$. For every $S \in \mathrm{WF}$ we set

$$S' \coloneqq \{s \in S : \exists t \in S \text{ with } s \sqsubset t\} \in \mathrm{WF}.$$

By transfinite recursion, we define the iterated derivatives S^ξ $(\xi < \omega_1)$ of S. The *order* $o(S)$ of S is defined to be the least ordinal ξ such that $S^\xi = \emptyset$. By convention, we set $o(S) = \omega_1$ if $S \notin \mathrm{WF}$.

Let S and R be trees on $\mathbb{N}$. A map $\psi\colon S \to R$ is said to be *monotone* if for every $s, s' \in S$ with $s \sqsubset s'$ we have $\psi(s) \sqsubset \psi(s')$. We notice that if there exists a monotone map $\psi : S \to R$ and R is well-founded, then S is well-founded and $o(S) \leqslant o(R)$.

2.2. **Standard Borel spaces.** Let (X, Σ) be a standard Borel space; that is, X is a set, Σ is a σ-algebra on X and the measurable space (X, Σ) is Borel isomorphic to the reals. A subset A of X is said to be *analytic* if there exists a Borel map $f\colon \mathbb{N}^{\mathbb{N}} \to X$ with $f(\mathbb{N}^{\mathbb{N}}) = A$. A subset of X is said to be *co-analytic* if its complement is analytic.

A natural, and relevant for our purposes, example of a standard Borel space is the following. Let X and Y be separable Banach spaces and denote by Σ the σ-algebra on $\mathcal{L}(X,Y)$ of all Borel subsets of $\mathcal{L}(X,Y)$ where $\mathcal{L}(X,Y)$ is equipped with the strong operator topology. It is well-known and easy to prove that the measurable space $\big(\mathcal{L}(X,Y), \Sigma\big)$ is standard (see [20, page 80] for more details).

2.3. **Complete co-analytic sets and co-analytic ranks.** Let B be a co-analytic subset of a standard Borel space X. The set B is said to be *co-analytic complete* if for every co-analytic subset C of a standard Borel space Y there exists a Borel map $f : Y \to X$ such that $f^{-1}(B) = C$. It is well-known that a complete co-analytic set is not Borel. We will need the following well-known fact. Its proof is based on the classical result that the set WF is co-analytic complete (see [20, Theorem 27.1]).

Fact 5. *Let B be a co-analytic subset of a standard Borel space X. Assume that there exists a Borel map $h\colon \mathrm{Tr} \to X$ such that $h^{-1}(B) = \mathrm{WF}$. Then B is complete.*

As above, let B be a co-analytic subset of a standard Borel space X. A map $\varphi\colon B \to \omega_1$ is said to be a *co-analytic rank on B* if there exist two binary relations $\leqslant_\Sigma$ and $\leqslant_\Pi$ on X, which are analytic and co-analytic respectively, such that for every $y \in B$ we have

$$\varphi(x) \leqslant \varphi(y) \Leftrightarrow (x \in B) \text{ and } \varphi(x) \leqslant \varphi(y) \Leftrightarrow x \leqslant_\Sigma y \Leftrightarrow x \leqslant_\Pi y.$$

A basic property of co-analytic ranks is that they satisfy *boundedness*; that is, if A is an analytic subset of B, then $\sup\{\varphi(x) : x \in A\} < \omega_1$. For a proof as well as for a thorough presentation of rank theory we refer to [20, Section 34].

We will also need the following fact.

Fact 6. *Let X be a standard Borel space and let $\mathcal{P}$ be an analytic subset of $X \times \mathrm{Tr}$. Then the set $\mathcal{P}^\sharp \subseteq X$ defined by*

$$x \in \mathcal{P}^\sharp \Leftrightarrow \forall S \in \mathrm{Tr}\ [(x,S) \in \mathcal{P} \Rightarrow S \in \mathrm{WF}]$$

is co-analytic. Moreover, there exists a co-analytic rank $\varphi\colon \mathcal{P}^\sharp \to \omega_1$ such that for every $x \in \mathcal{P}^\sharp$ we have $\sup\{o(S) : S \in \mathrm{Tr}$ and $(x,S) \in \mathcal{P}\} \leqslant \varphi(x)$.

Proof. First notice that $\mathcal{P}^\sharp$ is co-analytic since

$$x \notin \mathcal{P}^\sharp \Leftrightarrow \exists S \in \mathrm{Tr} \text{ such that } [(x,S) \in \mathcal{P} \text{ and } S \notin \mathrm{WF}].$$

The existence of the rank φ follows from the parameterized version of Lusin's boundedness theorem for WF. Indeed, by [20, page 365] (see also [5, Theorem 11]), there exists a Borel map $f\colon X \to \mathrm{Tr}$ such that

(a) $f(x) \notin \mathrm{WF}$ if $x \notin \mathcal{P}^\sharp$, while
(b) $f(x) \in \mathrm{WF}$ if $x \in \mathcal{P}^\sharp$ and $\sup\{o(S) : S \in \mathrm{Tr} \text{ and } (x,S) \in \mathcal{P}\} \leqslant o\big(f(x)\big)$.

We set $\varphi(x) = o\big(f(x)\big)$ for every $x \in \mathcal{P}^\sharp$. It is easy to check that φ is as desired. □

2.4. **Regular families.** Notice that every subset of $\mathbb{N}$ is naturally identified with an element of $2^{\mathbb{N}}$. We recall the following notions.

Definition 7. *Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$.*

(1) *The family $\mathcal{F}$ is said to be* compact *if $\mathcal{F}$ is a compact subset of $2^{\mathbb{N}}$.*
(2) *The family $\mathcal{F}$ is said to be* hereditary *if for every $F \in \mathcal{F}$ and every $G \subseteq F$ we have that $G \in \mathcal{F}$.*
(3) *The family $\mathcal{F}$ is said to be* spreading *if for every $F = \{n_1 < \cdots < n_k\} \in \mathcal{F}$ and every $G = \{m_1 < \cdots < m_k\}$ with $n_i \leqslant m_i$ for all $i \in \{1, \ldots, k\}$ we have that $G \in \mathcal{F}$.*
(4) *The family $\mathcal{F}$ is said to be* regular *if $\mathcal{F}$ is compact, hereditary and spreading.*

Regular families are basic combinatorial objects. They have been widely used in combinatorics and functional analysis (see [7] for a detailed exposition). Notice that every regular family $\mathcal{F}$ is a well-founded tree on $\mathbb{N}$, and so, its order $o(\mathcal{F})$ can be defined as in Subsection 2.1.

Let $L = \{l_1 < l_2 < \cdots\} \in [\mathbb{N}]^\infty$. For every nonempty finite subset F of $\mathbb{N}$ let $L(F) = \{l_i : i \in F\}$; also let $L(\emptyset) = \emptyset$. For every regular family $\mathcal{F}$ we set

$$\mathcal{F}[L] \coloneqq \{F \in \mathcal{F} : F \subseteq L\} \quad \text{and} \quad \mathcal{F}(L) \coloneqq \{L(F) : F \in \mathcal{F}\}. \tag{3}$$

We will need the following fact.

Fact 8. *Let $L \in [\mathbb{N}]^\infty$ and let $\mathcal{F}$ be a regular family of finite subsets of $\mathbb{N}$. Then the following are satisfied.*

(i) *We have $\mathcal{F}(L) \subseteq \mathcal{F}[L] \subseteq \mathcal{F}$.*
(ii) *We have $o\big(\mathcal{F}(L)\big) = o\big(\mathcal{F}[L]\big) = o(\mathcal{F})$.*

Proof. Part (i) follows readily from the relevant definitions. To see part (ii), notice that the map $\mathcal{F} \ni F \mapsto L(F) \in \mathcal{F}(L)$ is monotone. Therefore, $o(\mathcal{F}) \leqslant o\big(\mathcal{F}(L)\big)$ and the result follows. □

2.5. **Schreier families.** The Schreier families $\mathcal{S}_\xi$ $(1 \leqslant \xi < \omega_1)$ are important examples of regular families. We recall the definition of the first two families $\mathcal{S}_1$ and $\mathcal{S}_2$ which are more relevant to the rest of the paper (for more details we refer to [1, 6, 7]). The first Schreier family is defined by

$$\mathcal{S}_1 \coloneqq \{F \subseteq \mathbb{N} : |F| \leqslant \min(F)\} \tag{4}$$

while the second one is defined by

$$\mathcal{S}_2 := \Big\{ \bigcup_{i=1}^{n} F_i : n \in \mathbb{N}, n \leqslant \min(F_1), F_1 < \cdots < F_n \text{ and } F_i \in \mathcal{S}_1 \ \forall i = 1, \dots, n \Big\}. \tag{5}$$

We will need the following facts.

Fact 9. *For every $1 \leqslant \xi < \omega_1$ we have $o(\mathcal{S}_\xi) = \omega^\xi$.*

Fact 10. *Let $d \in \mathbb{N}$ and $N = \{n_1 < n_2 < \cdots\} \in [\mathbb{N}]^\infty$. Let $1 \leqslant \xi < \omega_1$ and let $F \in \mathcal{S}_\xi$ be nonempty. Then we have*

$$\big\{ n_{dk+i-1} : k \in F \text{ and } i \in \{1, \dots, d\} \big\} \in \mathcal{S}_\xi.$$

Facts 9 and 10 are proved using transfinite induction. We leave the details to the interested reader.

3. A criterion for checking that $\varrho(T) \leqslant \xi$

Let X and Y be two Banach spaces with Schauder bases, $T \in \mathcal{L}(X, Y)$ and $1 \leqslant \xi < \omega_1$. The main result of this section is a simple criterion for checking that $\varrho(T) \leqslant \xi$. To state it, we need to introduce the following definition.

Definition 11. *Let X and Y be two Banach spaces with normalized bases (e_n) and (z_n) respectively, and $T \in \mathcal{L}(X, Y)$. We say that two sequences (x_n) and (y_n), in X and Y respectively, are T-*compatible with respect to (e_n) and (z_n) *if the following are satisfied.*

1. *The sequence (x_n) is a normalized block sequence of (e_n).*
2. *The sequence (y_n) is a seminormalized block sequence of (z_n).*
3. *We have $\|T(x_n) - y_n\| \leqslant 2^{-n}$ for every $n \in \mathbb{N}$.*

If the bases (e_n) and (z_n) are understood, then we simply say that (x_n) and (y_n) are T-compatible.

We can now state the main result of this section.

Lemma 12. *Let X and Y be two Banach spaces with normalized bases (e_n) and (z_n) respectively, $T \in \mathcal{L}(X, Y)$ and $1 \leqslant \xi < \omega_1$. Then the following are equivalent.*

(i) *We have $\varrho(T) \leqslant \xi$.*
(ii) *For every pair (x_n) and (y_n) of T-compatible sequences with respect to (e_n) and (z_n) and every $\delta > 0$ there exist a nonempty set $F \in \mathcal{S}_\xi$ and reals $(a_n)_{n \in F}$ such that*

$$\Big\| \sum_{n \in F} a_n x_n \Big\| = 1 \quad \textit{and} \quad \Big\| \sum_{n \in F} a_n y_n \Big\| \leqslant \delta.$$

For the proof of Lemma 12 we will need the following simple fact. It was also observed in [4].

Fact 13. *Let X and Y be separable Banach spaces, $T \in \mathcal{L}(X,Y)$ and $1 \leqslant \xi < \omega_1$. Also let (x_n) be a normalized basic sequence in X and $\varepsilon > 0$. Then there exist a nonempty set $F \in \mathcal{S}_\xi$ and a norm-one vector $x \in \mathrm{span}\{x_n : n \in F\}$ such that $\|T(x)\| \leqslant \varepsilon$ if and only if there exist a subsequence (x_{n_k}) of (x_n), a nonempty set $H \in \mathcal{S}_\xi$ and a norm-one vector $x' \in \mathrm{span}\{x_{n_k} : k \in H\}$ such that $\|T(x')\| \leqslant \varepsilon$.*

We proceed to the proof of Lemma 12.

Proof of Lemma 12. It is clear that (i) implies (ii). We work to prove the converse implication. The arguments are fairly standard, and so, we will be rather sketchy.

Let (e^*_n) and (z^*_n) be the bi-orthogonal functionals associated to (e_n) and (z_n) respectively. Let (v_n) be a normalized basic sequence in X and $\varepsilon > 0$. We need to find a nonempty set $G \in \mathcal{S}_\xi$ and a norm-one vector $v \in \mathrm{span}\{v_n : n \in G\}$ such that $\|T(v)\| \leqslant \varepsilon$. To this end, by Fact 13, we are allowed to pass to subsequences of (v_n). Therefore, we may assume that for every $k \in \mathbb{N}$ the sequences $\big(e^*_k(v_n)\big)$ and $\big(z^*_k(T(v_n))\big)$ are both convergent. Let (d_n) be the difference sequence of (v_n); that is, $d_n \coloneqq v_{2n} - v_{2n-1}$ for every $n \in \mathbb{N}$. Notice that

(a) the sequence (d_n) is seminormalized,
(b) $e^*_k(d_n) \to 0$ for every $k \in \mathbb{N}$, and
(c) $z^*_k\big(T(d_n)\big) \to 0$ for every $k \in \mathbb{N}$.

By (a) and (b) and by passing to a subsequence of (v_n), it is possible to find a seminormalized block sequence (b^0_n) of (e_n) such that $\|b^0_n - d_n\| \leqslant 2^{-n}$ for every $n \in \mathbb{N}$. Now we distinguish the following (mutually exclusive) cases.

Case 1: *There exists a subsequence of $\big(T(d_n)\big)$ which is norm convergent to* 0. In this case it is easy to see that there exist $G \in \mathcal{S}_\xi$ with $|G| = 2$ and a norm-one vector $v \in \mathrm{span}\{v_n : n \in G\}$ such that $\|T(v)\| \leqslant \varepsilon$.

Case 2: *There exists a subsequence of $\big(T(d_n)\big)$ which is seminormalized.* In this case, by (c) above and by passing to a further subsequence of (v_n), we may find a seminormalized block sequence (b^1_n) of (z_n) such that $\|b^1_n - T(d_n)\| \leqslant 2^{-n}$ for every $n \in \mathbb{N}$. Summing up, we see that it is possible to select an infinite subset $N = \{n_1 < n_2 < \cdots\}$ of $\mathbb{N}$ such that, setting

$$w_k \coloneqq v_{n_{2k+1}} - v_{n_{2k}}$$

for every $k \in \mathbb{N}$, the sequences (w_k) and $\big(T(w_k)\big)$ are both seminormalized and "almost block". Hence, using our hypotheses, we may find a nonempty set $F \in \mathcal{S}_\xi$ and a norm-one vector $v \in \mathrm{span}\{w_k : k \in F\}$ such that $\|T(v)\| \leqslant \varepsilon$. By Fact 10, we see that $G \coloneqq \big\{n_{2k+i-1} : k \in F \text{ and } i \in \{1,2\}\big\} \in \mathcal{S}_\xi$ and the result follows. □

4. Proof of Theorem 2

Let X and Y be separable Banach spaces. Let $\mathcal{B}$ be the subset of $X^{\mathbb{N}}$ defined by

$$(x_n) \in \mathcal{B} \Leftrightarrow (x_n) \text{ is a normalized basic sequence.}$$

It is easy to see that $\mathcal{B}$ is an F_σ subset of $X^{\mathbb{N}}$. Hence, the set $\mathcal{B}$ equipped with the relative Borel σ-algebra of $X^{\mathbb{N}}$ is a standard Borel space (see [20]).

For every $T \in \mathcal{L}(X,Y)$, every $(x_n) \in \mathcal{B}$ and every $m \in \mathbb{N}$ we introduce a tree $S(T,(x_n),m)$ on $\mathbb{N}$ defined by the rule

$$\text{(6)}\quad s \in S(T,(x_n),m) \quad\Leftrightarrow\quad \begin{array}{l} \text{either } s=\emptyset \text{ or } s=(n_1<\cdots<n_k)\in[\mathbb{N}]^{<\mathbb{N}} \text{ and} \\ \forall d\in\mathbb{N} \text{ with } d\leqslant k,\ \forall (l_1<\cdots<l_d)\in[\mathbb{N}]^{<\mathbb{N}} \text{ with} \\ n_i\leqslant l_i \text{ for every } i\in\{1,\dots,d\} \text{ and } \forall a_1,\dots,a_d\in\mathbb{Q} \\ \text{we have } \big\|T\big(\sum_{i=1}^d a_i x_{l_i}\big)\big\| \geqslant \frac{1}{m}\big\|\sum_{i=1}^d a_i x_{l_i}\big\|. \end{array}$$

We notice the following simple facts. The proofs are left to the reader.

Fact 14. *The map $\mathcal{L}(X,Y)\times\mathcal{B}\times\mathbb{N}\ni(T,(x_n),m)\mapsto S(T,(x_n),m)\in\mathrm{Tr}$ is Borel.*

Fact 15. *Let $T\in\mathcal{L}(X,Y)$. If T is not strictly singular, then there exist $(x_n)\in\mathcal{B}$ and $m\in\mathbb{N}$ such that the tree $S(T,(x_n),m)$ is not well-founded.*

We proceed to analyze the above defined trees when the operator T is strictly singular.

Claim 16. *Let $T\in\mathcal{SS}(X,Y)$ with $\varrho(T)=\xi$. Also let $(x_n)\in\mathcal{B}$ and $m\in\mathbb{N}$. Then the tree $S(T,(x_n),m)$ is a regular family. Moreover,*

$$o\big(S(T,(x_n),m)\big)\leqslant\omega^{\xi+1}. \tag{7}$$

Proof. For notational simplicity, let us denote by $\mathcal{F}$ the tree $S(T,(x_n),m)$. It is clear from the definition that $\mathcal{F}$ is a hereditary and spreading family of finite subsets of $\mathbb{N}$. It is easy to see that $\mathcal{F}$ is in addition well-founded. This implies that $\mathcal{F}$ is compact in $2^{\mathbb{N}}$. Hence, $\mathcal{F}$ is a regular family.

We work now to prove that $o(\mathcal{F})\leqslant\omega^{\xi+1}$. We argue by contradiction. So, assume that $o(\mathcal{F})>\omega^{\xi+1}$. A result of Gasparis [18] asserts that if $\mathcal{G}$ and $\mathcal{H}$ are two hereditary families of finite subsets of $\mathbb{N}$, then there exists $L\in[\mathbb{N}]^\infty$ such that either $\mathcal{G}[L]\subseteq\mathcal{H}$ or $\mathcal{H}[L]\subseteq\mathcal{G}$. Applying this dichotomy to the families $\mathcal{F}$ and $\mathcal{S}_{\xi+1}$, we find $L=\{l_1<l_2<\cdots\}\in[\mathbb{N}]^\infty$ such that either $\mathcal{F}[L]\subseteq\mathcal{S}_{\xi+1}$ or $\mathcal{S}_{\xi+1}[L]\subseteq\mathcal{F}$. We claim that the first case is impossible. Indeed, assume on the contrary that $\mathcal{F}[L]\subseteq\mathcal{S}_{\xi+1}$. By part (ii) of Fact 8 and Fact 9, we see that

$$\omega^{\xi+1}<o(\mathcal{F})=o\big(\mathcal{F}[L]\big)\leqslant o(\mathcal{S}_{\xi+1})=\omega^{\xi+1}$$

which is clearly impossible. Hence, $\mathcal{S}_{\xi+1}[L]\subseteq\mathcal{F}$.

Introduce now the sequence (z_n) defined by the rule that $z_n=x_{l_n}$ for every $n\in\mathbb{N}$. Clearly (z_n) is a normalized basic sequence in X. Let $F\in\mathcal{S}_{\xi+1}$ be arbitrary and nonempty. The family $\mathcal{S}_{\xi+1}$ is regular. Hence, by part (i) of Fact 8, we obtain that

$$L(F)=\{l_n:n\in F\}\in\mathcal{S}_{\xi+1}(L)\subseteq\mathcal{S}_{\xi+1}[L]\subseteq\mathcal{F}.$$

By the definition of $\mathcal{F}$ and the continuity of the operator T, we see that for every choice $(a_n)_{n\in F}$ of reals we have

$$\|T\big(\sum_{n\in F} a_n z_n\big)\| = \|T\big(\sum_{n\in F} a_n x_{l_n}\big)\| \geqslant \frac{1}{m}\|\sum_{n\in F} a_n x_{l_n}\| = \frac{1}{m}\|\sum_{n\in F} a_n z_n\|.$$

In other words, we conclude that for every nonempty set $F \in \mathcal{S}_{\xi+1}$ and every norm-one vector $z \in \mathrm{span}\{z_n : n \in F\}$ we have $\|T(z)\| \geqslant m^{-1}$. This implies that T is not $\mathcal{S}_{\xi+1}$-strictly singular. By [4, Proposition 2.4], the operator T is not $\mathcal{S}_\zeta$-strictly singular for every $1 \leqslant \zeta \leqslant \xi+1$, and so, $\varrho(T) > \xi+1$. This is a contradiction. Therefore, $o(\mathcal{F}) \leqslant \omega^{\xi+1}$ and the proof is completed. □

As a consequence we obtain the following result which shows that the family of trees $\{S(T,(x_n),m) : (x_n) \in \mathcal{B} \text{ and } m \in \mathbb{N}\}$ can be used to compute the ordinal $\varrho(T)$ quite accurately.

Corollary 17. *Let $T \in \mathcal{SS}(X,Y)$ with $\varrho(T) = \xi$. Then*

$$\sup\{\omega^\zeta : \zeta < \xi\} \leqslant \sup\big\{o\big(S(T,(x_n),m)\big) : (x_n) \in \mathcal{B} \text{ and } m \in \mathbb{N}\big\} \leqslant \omega^{\xi+1}. \tag{8}$$

Proof. The second inequality follows immediately by Claim 16. We work to prove the first inequality. Clearly we may assume that $\xi > 1$. Let ζ be an arbitrary countable ordinal with $1 \leqslant \zeta < \xi$. Since $\varrho(T) > \zeta$, the operator T is not $\mathcal{S}_\zeta$-strictly singular. Therefore, we may find $(x_n) \in \mathcal{B}$ and $\varepsilon > 0$ such that for every nonempty set $F \in \mathcal{S}_\zeta$ and every $x \in \mathrm{span}\{x_n : n \in F\}$ we have $\|T(x)\| \geqslant \varepsilon\|x\|$. We select $m \in \mathbb{N}$ such that $\varepsilon \geqslant m^{-1}$. The family $\mathcal{S}_\zeta$ is spreading and hereditary. Hence, by the definition of the tree $S(T,(x_n),m)$, we see that $F \in S(T,(x_n),m)$ for every $F \in \mathcal{S}_\zeta$. In particular, the identity map $\mathrm{Id}\colon \mathcal{S}_\zeta \to S(T,(x_n),m)$ is a well-defined monotone map. Therefore, by Fact 9, we see that

$$\omega^\zeta = o(\mathcal{S}_\zeta) \leqslant o\big(S(T,(x_n),m)\big)$$

and the result follows. □

Now, define $\mathcal{P} \subseteq \mathcal{L}(X,Y) \times \mathrm{Tr}$ by the rule

$$(T,R) \in \mathcal{P} \Leftrightarrow \exists (x_n) \in \mathcal{B} \text{ and } \exists m \in \mathbb{N} \text{ such that } R = S(T,(x_n),m). \tag{9}$$

By Fact 14, we see that the set $\mathcal{P}$ is analytic. As in Fact 6, let $\mathcal{P}^\sharp \subseteq \mathcal{L}(X,Y)$ be defined by

$$T \in \mathcal{P}^\sharp \Leftrightarrow \forall R \in \mathrm{Tr}\ [(T,R) \in \mathcal{P} \Rightarrow R \in \mathrm{WF}].$$

By Fact 15 and Claim 16, we see that $\mathcal{P}^\sharp = \mathcal{SS}(X,Y)$. Let $\varphi\colon \mathcal{P}^\sharp \to \omega_1$ be the co-analytic rank on $\mathcal{P}^\sharp$ obtained in Fact 6.

We define

$$\mathrm{r}_{\mathcal{S}}(T) = \varphi(T) + 1 \tag{10}$$

for every $T \in \mathcal{SS}(X,Y)$, and we claim that $\mathrm{r}_{\mathcal{S}}$ is the desired rank. Clearly $\mathrm{r}_{\mathcal{S}}$ is a co-analytic rank on $\mathcal{SS}(X,Y)$. It remains to check that $\varrho(T) \leqslant \mathrm{r}_{\mathcal{S}}(T)$ for every $T \in \mathcal{SS}(X,Y)$. To this end, fix $T \in \mathcal{SS}(X,Y)$. By Fact 6, we have

$$\sup\{o(R) : (T,R) \in \mathcal{P}\} \leqslant \varphi(T) \tag{11}$$

while, by the definition of the set $\mathcal{P}$, we obtain that

$$\sup\big\{o\big(S(T,(x_n),m)\big) : (x_n) \in \mathcal{B} \text{ and } m \in \mathbb{N}\big\} = \sup\{o(R) : (T,R) \in \mathcal{P}\}. \tag{12}$$

Finally, notice that

$$\xi \leqslant \sup\{\omega^{\zeta} : \zeta < \xi\} + 1 \tag{13}$$

for every countable ordinal ξ. Combining inequalities (8), (11), (12) and (13), we conclude that $\varrho(T) \leqslant \mathrm{r}_{\mathcal{S}}(T)$ as desired.

Finally, to see that the rank ϱ satisfies boundedness, let $\mathcal{A}$ be an analytic subset of $\mathcal{SS}(X,Y)$. The rank $\mathrm{r}_{\mathcal{S}}$ is a co-analytic rank. Therefore, there exists a countable ordinal ξ such that $\mathrm{r}_{\mathcal{S}}(T) \leqslant \xi$ for every $T \in \mathcal{A}$. Hence, $\sup\{\varrho(T) : T \in \mathcal{A}\} \leqslant \xi < \omega_1$.

The proof of Theorem 2 is completed.

5. The spaces $Z_{p,q}$ $(1 \leqslant p \leqslant q < +\infty)$

This section contains some results which are needed for the proof of Theorem 4. It is organized as follows. In Subsection 5.1 we introduce some pieces of notation. In Subsection 5.2 we define the space $Z_{p,q}$ $(1 \leqslant p \leqslant q < +\infty)$ and we gather some of its basic properties. Finally, in Subsection 5.3 we present a result concerning a class of sequences in $Z_{p,q}$ which we call "asymptotically sparse".

5.1. **Notation.** For the rest of this paper we fix a bijection $\chi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ satisfying $\chi(s) < \chi(t)$ for every $s,t \in \mathbb{N}^{<\mathbb{N}}$ with $s \sqsubset t$.

Let $s,t \in \mathbb{N}^{<\mathbb{N}}$. The nodes s and t are said to be *comparable* if either $s \sqsubseteq t$ or $t \sqsubseteq s$; otherwise s and t are said to be *incomparable*. A subset of $\mathbb{N}^{<\mathbb{N}}$ consisting of pairwise comparable nodes is said to be a *chain* while a subset of $\mathbb{N}^{<\mathbb{N}}$ consisting of pairwise incomparable nodes is said to be an *antichain*. A *branch* of $\mathbb{N}^{<\mathbb{N}}$ is a maximal chain of $\mathbb{N}^{<\mathbb{N}}$. The branches of $\mathbb{N}^{<\mathbb{N}}$ are naturally identified with the elements of $\mathbb{N}^{\mathbb{N}}$; indeed, a subset A of $\mathbb{N}^{<\mathbb{N}}$ is a branch if and only if there exists $\sigma \in \mathbb{N}^{\mathbb{N}}$ (unique) such that $A = \{\sigma|n : n \geqslant 0\}$. Two subsets A and B of $\mathbb{N}^{<\mathbb{N}}$ are said to be *incomparable* if for every $s \in A$ and every $t \in B$ the nodes s and t are incomparable. A *segment* $\mathfrak{s}$ of $\mathbb{N}^{<\mathbb{N}}$ is a chain of $\mathbb{N}^{<\mathbb{N}}$ satisfying

$$\forall s,t,s' \in \mathbb{N}^{<\mathbb{N}} \ (s \sqsubseteq t \sqsubseteq s' \text{ and } s,s' \in \mathfrak{s} \Rightarrow t \in \mathfrak{s}). \tag{14}$$

If $\mathfrak{s}$ is a segment of $\mathbb{N}^{<\mathbb{N}}$, then by $\min(\mathfrak{s})$ we denote the $\sqsubseteq$-minimal node of $\mathfrak{s}$. Notice that two segments $\mathfrak{s}$ and $\mathfrak{s}'$ are incomparable if and only if the nodes $\min(\mathfrak{s})$ and $\min(\mathfrak{s}')$ are incomparable. If σ is a branch of $\mathbb{N}^{<\mathbb{N}}$ and $k \geqslant 0$, then the set $\{\sigma|n : n \geqslant k\}$ is said to be a *final segment* of σ while the set $\{\sigma|n : n \leqslant k\}$ is said to be an *initial segment* of σ.

5.2. **Definitions and basic properties.** We start with the following definition.

Definition 18. *Let $1 \leqslant p \leqslant q < +\infty$. We define the space $Z_{p,q}$ to be the completion of $c_{00}(\mathbb{N}^{<\mathbb{N}})$ equipped with the norm*

$$\|z\|_{Z_{p,q}} \coloneqq \sup\Big\{ \Big(\sum_{i=1}^{d} \big(\sum_{t\in\mathfrak{s}_i} |z(t)|^p \big)^{q/p} \Big)^{1/q} \Big\} \tag{15}$$

where the above supremum is taken over all families $(\mathfrak{s}_i)_{i=1}^d$ of pairwise incomparable nonempty segments of $\mathbb{N}^{<\mathbb{N}}$.

The space $Z_{p,q}$ is a variant of James tree space JT [19]. We notice that spaces of this form have found significant applications and have been extensively studied by several authors (see, for instance, [5, 9, 10, 13, 15, 16]). We gather, below, some elementary properties of the space $Z_{p,q}$.

Let $\{z_t : t \in \mathbb{N}^{<\mathbb{N}}\}$ be the standard Hamel basis of $c_{00}(\mathbb{N}^{<\mathbb{N}})$ and let (t_n) be the enumeration of $\mathbb{N}^{<\mathbb{N}}$ according to the bijection χ (see Subsection 5.1). The sequence (z_{t_n}) defines an 1-unconditional basis of $Z_{p,q}$. For every node t of $\mathbb{N}^{<\mathbb{N}}$ by z_t^* we shall denote the bi-orthogonal functional associated to z_t. For every vector z in $Z_{p,q}$ the *support* $\mathrm{supp}(z)$ of z is defined to be the set $\{t \in \mathbb{N}^{<\mathbb{N}} : z_t^*(z) \neq 0\}$.

For every nonempty $A \subseteq \mathbb{N}^{<\mathbb{N}}$ set

$$Z^A_{p,q} \coloneqq \overline{\mathrm{span}}\{z_t : t \in A\}. \tag{16}$$

The subspace $Z^A_{p,q}$ of $Z_{p,q}$ is complemented via the natural projection

$$P_A \colon Z_{p,q} \to Z^A_{p,q}. \tag{17}$$

Notice that $\|P_A\| = 1$. Observe that for every nonempty chain $\mathfrak{c}$ of $\mathbb{N}^{<\mathbb{N}}$ and every vector z in $Z_{p,q}$ we have

$$\|P_{\mathfrak{c}}(z)\| = \Big(\sum_{t\in\mathfrak{c}} |z_t^*(z)|^p \Big)^{1/p}. \tag{18}$$

In particular, for every branch σ of $\mathbb{N}^{<\mathbb{N}}$ the subspace $Z^\sigma_{p,q}$ of $Z_{p,q}$ is isometric to ℓ_p and complemented via the norm-one projection $P_\sigma \colon Z_{p,q} \to Z^\sigma_{p,q}$.

Let X and E be infinite-dimensional Banach spaces. Recall that the space X is said to be *hereditarily* E if every infinite-dimensional subspace of X contains an isomorphic copy of E. We will need the following easy (and essentially known) fact concerning the structure of the space $Z^S_{p,q}$ when S is a well-founded tree. The proof is sketched for completeness.

Fact 19. *Let $1 \leqslant p \leqslant q < +\infty$ and $S \in \mathrm{WF}$. Then the space $Z^S_{p,q}$ is either finite-dimensional or hereditarily ℓ_q.*

Proof. We proceed by induction on the order of the tree S. If $o(S) = 1$, then the space $Z^S_{p,q}$ is one-dimensional. Let $S \in \mathrm{WF}$ with $o(S) > 1$ and assume that the result has been proved for every $R \in \mathrm{WF}$ with $o(R) < o(S)$. We set

$$L_S \coloneqq \{n \in \mathbb{N} : (n) \in S\}.$$

For every $n \in L_S$ let $S_n = \{t \in \mathbb{N}^{<\mathbb{N}} : n^\frown t \in S\}$ and notice that $S_n \in \mathrm{WF}$ and $o(S_n) < o(S)$. Therefore, by our induction hypothesis, the space $Z_{p,q}^{S_n}$ is either finite-dimensional or hereditarily ℓ_q. Noticing that the space $Z_{p,q}^{S}$ is isomorphic to the space

$$\mathbb{R} \oplus \Big(\sum_{n \in L_S} \oplus Z_{p,q}^{S_n} \Big)_{\ell_q}$$

the result follows. □

5.3. **Asymptotically sparse sequences in $Z_{p,q}$.** We start by introducing the following definition.

Definition 20. *Let $1 \leqslant p \leqslant q < +\infty$. We say that a bounded block sequence (y_n) in $Z_{p,q}$ is* asymptotically sparse *if for every $k \in \mathbb{N}$ and every $\sigma \in \mathbb{N}^{\mathbb{N}}$ we have*

$$|\{n \geqslant k : \|P_\sigma(y_n)\| \geqslant 2^{-k}\}| \leqslant 1. \tag{19}$$

Notice that if (y_n) is an asymptotically sparse sequence, then $\lim \|P_\sigma(y_n)\| = 0$ for every $\sigma \in \mathbb{N}^{\mathbb{N}}$. The main result of this subsection asserts that (essentially) the converse is also true. Precisely, we have the following lemma.

Lemma 21. *Let $1 \leqslant p \leqslant q < +\infty$ and let (y_n) be a bounded block sequence in $Z_{p,q}$ such that $\lim \|P_\sigma(y_n)\| = 0$ for every $\sigma \in \mathbb{N}^{\mathbb{N}}$. Then (y_n) has an asymptotically sparse subsequence.*

Lemma 21 is a Ramsey-theoretical result and the arguments in its proof can be traced in the work of Amemiya and Ito [2] concerning the structure of normalized weakly null sequences in the space JT. We proceed to the proof.

Proof of Lemma 21. Let $1 \leqslant p \leqslant q < +\infty$ and fix a bounded block sequence (y_n) in $Z_{p,q}$ such that $\lim \|P(y_n)\| = 0$ for every $\sigma \in \mathbb{N}^{\mathbb{N}}$. We select $C > 0$ such that $\|y_n\| \leqslant C$ for every $n \in \mathbb{N}$.

Claim 22. *For every $\theta > 0$ and every $M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that for every $\sigma \in \mathbb{N}^{\mathbb{N}}$ we have $|\{n \in L : \|P_\sigma(y_n)\| \geqslant \theta\}| \leqslant 1$.*

Granting Claim 22 the proof of Lemma 21 is completed. Indeed, by repeated applications of Claim 22, it is possible to select a sequence (L_k) of infinite subsets of $\mathbb{N}$ such that for every $k \in \mathbb{N}$ the following are satisfied.

(a) We have $\min(L_k) < \min(L_{k+1})$.
(b) We have $L_{k+1} \subseteq L_k$.
(c) We have $|\{n \in L_k : \|P_\sigma(y_n)\| \geqslant 2^{-k}\}| \leqslant 1$ for every $\sigma \in \mathbb{N}^{\mathbb{N}}$.

Introduce the sequence (w_k) in $Z_{p,q}$ defined by $w_k = y_{\min(L_k)}$ for every $k \in \mathbb{N}$. By (a) above, we see that (w_k) is a subsequence of (y_n) while, by (b) and (c), the sequence (w_k) is asymptotically sparse.

It remains to prove Claim 22. We will argue by contradiction. So, assume that there exist $\theta > 0$ and $M \in [\mathbb{N}]^\infty$ such that for every $L \in [M]^\infty$ there exist $m, k \in L$

with $m < k$ and $\sigma \in \mathbb{N}^{\mathbb{N}}$ such that $\|P_\sigma(y_m)\| \geqslant \theta$ and $\|P_\sigma(y_k)\| \geqslant \theta$. Therefore, applying the classical Ramsey theorem [24] and by passing to a subsequence of (y_n), we may assume that for every $m, k \in \mathbb{N}$ with $m < k$ there exists $\sigma_{m,k} \in \mathbb{N}^{\mathbb{N}}$ such that $\|P_{\sigma_{m,k}}(y_m)\| \geqslant \theta$ and $\|P_{\sigma_{m,k}}(y_k)\| \geqslant \theta$.

Fix $k \in \mathbb{N}$ with $k \geqslant 2$. For every $m \in \mathbb{N}$ with $m < k$ let $\mathfrak{s}^-_{m,k}$ be the maximal initial segment of $\sigma_{m,k}$ which is disjoint from $\mathrm{supp}(y_k)$. As the sequence (y_n) is block, we see that $\|P_{\mathfrak{s}^-_{m,k}}(y_m)\| \geqslant \theta$. Let $\mathfrak{s}^+_{m,k} = \sigma_{m,k} \setminus \mathfrak{s}^-_{m,k}$ and notice that $\mathfrak{s}^+_{m,k}$ is a final segment of $\sigma_{m,k}$ and $\|P_{\mathfrak{s}^+_{m,k}}(y_k)\| \geqslant \theta$. Moreover, $\min(\mathfrak{s}^+_{m,k}) \in \mathrm{supp}(y_k)$. For every $r > 0$ let $\lceil r \rceil$ be the least $k \in \mathbb{N}$ such that $r \leqslant k$. Now we observe that

$$|\{\mathfrak{s}^-_{m,k} : m < k\}| \leqslant \lceil C^q/\theta^q \rceil. \tag{20}$$

Indeed, let $\mathfrak{s}_1, \dots, \mathfrak{s}_d$ be an enumeration of the set $\{\mathfrak{s}^-_{m,k} : m < k\}$. Then for every $i \in \{1, \dots, d\}$ there exists $m_i < k$ such that $\mathfrak{s}_i = \mathfrak{s}^-_{m_i,k}$. Since the segments $(\mathfrak{s}^-_{m_i,k})_{i=1}^d$ are mutually different, the final segments $(\mathfrak{s}^+_{m_i,k})_{i=1}^d$ are pairwise incomparable. To see this assume, towards a contradiction, that there exist $i, j \in \{1, \dots, d\}$ such that $\min(\mathfrak{s}^+_{m_i,k})$ is a proper initial segment of $\min(\mathfrak{s}^+_{m_j,k})$. Since $\min(\mathfrak{s}^+_{m_i,k}) \in \mathrm{supp}(y_k)$, we see that $\min(\mathfrak{s}^+_{m_i,k}) \in \mathrm{supp}(y_k) \cap \mathfrak{s}^-_{m_j,k}$ contradicting the fact that $\mathfrak{s}^-_{m_j,k}$ is disjoint from $\mathrm{supp}(y_k)$. Therefore, the final segments $(\mathfrak{s}^+_{m_i,k})_{i=1}^d$ are pairwise incomparable, and so, $C \geqslant \|y_k\| \geqslant \theta \cdot d^{1/q}$ which gives the desired estimate.

Set $D := \lceil C^q/\theta^q \rceil$. By the previous discussion, for every $k \in \mathbb{N}$ with $k \geqslant 2$ there exists a family $\{\mathfrak{s}_{i,k} : i = 1, \dots, D\}$ of initial segments of $\mathbb{N}^{<\mathbb{N}}$ such that for every $m \in \mathbb{N}$ with $m < k$ there exists $i \in \{1, \dots, D\}$ such that $\|P_{\mathfrak{s}_{i,k}}(y_m)\| \geqslant \theta$. The space $2^{\mathbb{N}^{<\mathbb{N}}}$ is compact. Therefore, by passing to subsequences, we may select a family $\{\mathfrak{s}_1, \dots, \mathfrak{s}_D\}$ of initial segments of $\mathbb{N}^{<\mathbb{N}}$ such that $\mathfrak{s}_{i,k} \to \mathfrak{s}_i$ in $2^{\mathbb{N}^{<\mathbb{N}}}$ for every $i \in \{1, \dots, D\}$.

Let $m, k \in \mathbb{N}$ with $m < k$ and $i \in \{1, \dots, D\}$. We say that k is *i-good for m* if $\|P_{\mathfrak{s}_{i,k}}(y_m)\| \geqslant \theta$. Notice that for every $m \in \mathbb{N}$ there exists $i \in \{1, \dots, D\}$ such that the set $H^i_m := \{k > m : k \text{ is } i\text{-good for } m\}$ is infinite. Hence, there exist $j \in \{1, \dots, D\}$ and $N \in [\mathbb{N}]^\infty$ such that H^j_m is infinite for every $m \in N$. We select $\tau \in \mathbb{N}^{\mathbb{N}}$ such that $\mathfrak{s}_j$ is an initial segment of τ. Since $\mathfrak{s}_{j,k} \to \mathfrak{s}_j$ in $2^{\mathbb{N}^{<\mathbb{N}}}$ and $\|P_{\mathfrak{s}_{j,k}}(y_m)\| \geqslant \theta$ for every $m \in N$ and every $k \in H^j_m$, we obtain that

$$\limsup_{m \in N} \|P_\tau(y_m)\| \geqslant \limsup_{m \in N} \|P_{\mathfrak{s}_j}(y_m)\| = \limsup_{m \in N} \lim_k \|P_{\mathfrak{s}_{j,k}}(y_m)\| \geqslant \theta.$$

This is clearly a contradiction. The proof of Lemma 21 is completed. $\square$

6. Proof of Theorem 4

Let $1 \leqslant p < +\infty$. We set

$$q := 2p \tag{21}$$

and we define Y_p to be the space $Z_{p,q}$. By Subsection 5.2, the space Y_p has a normalized 1-unconditional basis (z_{t_n}). Let (e_n) be the standard unit vector basis of ℓ_p. By $\mathrm{I}\colon \ell_p \to Y_p$ we shall denote the unique norm-one operator satisfying

$$\mathrm{I}(e_n) = z_{t_n} \tag{22}$$

for every $n \in \mathbb{N}$. We proceed to show that the space Y_p is the desired one.

6.1. **The set $\mathcal{SS}(\ell_p, Y_p)$ is a complete co-analytic subset of $\mathcal{L}(\ell_p, Y_p)$.** As we have already mentioned in the introduction, the set $\mathcal{SS}(X,Y)$ is a co-analytic subset of $\mathcal{L}(X,Y)$ for every pair X and Y of separable Banach space. Hence, what remains is to show that the set $\mathcal{SS}(\ell_p, Y_p)$ is actually complete. By Fact 5, it is enough to find a Borel map $H\colon \mathrm{Tr} \to \mathcal{L}(\ell_p, Y_p)$ such that for every $S \in \mathrm{Tr}$ we have

$$S \in \mathrm{WF} \Leftrightarrow H(S) \in \mathcal{SS}(\ell_p, Y_p).$$

To this end, let $S \in \mathrm{Tr}$ be arbitrary. Let $Z^S_{p,q}$ be the subspace of Y_p defined in (16) and $P_S\colon Y_p \to Z^S_{p,q}$ be the natural norm-one projection. We define

$$H(S) = P_S \circ \mathrm{I} \in \mathcal{L}(\ell_p, Y_p). \tag{23}$$

Notice that $\|H(S)\| = 1$.

Claim 23. *The map $H\colon \mathrm{Tr} \to \mathcal{L}(\ell_p, Y_p)$ is continuous when $\mathcal{L}(\ell_p, Y_p)$ is equipped with the strong operator topology.*

Proof. Let (S_n) be a sequence in Tr and $S \in \mathrm{Tr}$ such that $S_n \to S$. Notice that for every $s \in \mathbb{N}^{<\mathbb{N}}$ we have $s \in S$ if and only if $s \in S_n$ for every $n \in \mathbb{N}$ large enough. Let $x \in \ell_p$ be arbitrary and set $y = \mathrm{I}(x)$. It follows from the above remarks that for every $r > 0$ there exists $k \in \mathbb{N}$ such that $\|P_S(y) - P_{S_n}(y)\| \leqslant r$ for every $n \in \mathbb{N}$ with $n \geqslant k$ and the result follows. □

Claim 24. *Let $S \in \mathrm{Tr}$. Then $S \in \mathrm{WF}$ if and only if $H(S) \in \mathcal{SS}(\ell_p, Y_p)$.*

Proof. First assume that $S \in \mathrm{WF}$. Notice that the operator $H(S)$ maps ℓ_p onto $Z^S_{p,q}$. By Fact 19, the space $Z^S_{p,q}$ is either finite-dimensional or hereditarily ℓ_q. Since $p \neq q$, the operator $H(S)$ is strictly singular.

Now assume that $S \notin \mathrm{WF}$ and let $\sigma \in [S]$. Let $\chi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ be the bijection described in Subsection 5.1 and for every $k \in \mathbb{N}$ set $n_k \coloneqq \chi\big(\sigma(k)\big)$. Notice that $n_k < n_{k+1}$ for every $k \in \mathbb{N}$. Let E be the subspace of ℓ_p spanned by the subsequence (e_{n_k}) of the basis (e_n). We claim that the operator $H(S)$ restricted on E is an isometric embedding. Indeed, let $d \in \mathbb{N}$ and $a_1, \dots, a_d \in \mathbb{R}$, and observe that

$$\Big\|\sum_{k=1}^{d} a_k e_{n_k}\Big\|_{\ell_p} = \Big(\sum_{k=1}^{d} |a_k|^p\Big)^{1/p} = \Big\|\sum_{k=1}^{d} a_k z_{\sigma(k)}\Big\|_{Y_p} = \Big\|(P_S \circ \mathrm{I})\Big(\sum_{k=1}^{d} a_k e_{n_k}\Big)\Big\|_{Y_p}.$$

The claim is proved. □

By Fact 5 and Claims 23 and 24, we conclude that $\mathcal{SS}(\ell_p, Y_p)$ is a complete co-analytic subset of $\mathcal{L}(\ell_p, Y_p)$.

6.2. **For every $T \in \mathcal{SS}(\ell_p, Y_p)$ we have $\varrho(T) \leqslant 2$.** We fix a strictly singular operator $T\colon \ell_p \to Y_p$. We need to prove that $\varrho(T) \leqslant 2$. To this end, we may assume that

(P1) $\|T\| = 1$.

By Lemma 12, it is enough to show that for every pair (x_n) and (y_n) of T-compatible sequences (with respect to the bases (e_n) and (z_{t_n}) of ℓ_p and Y_p respectively) and for every $\delta > 0$ there exist a nonempty set $F \in \mathcal{S}_2$ and reals $(a_n)_{n\in F}$ such that

$$\Big\| \sum_{n\in F} a_n x_n \Big\|_{\ell_p} = 1 \ \text{ and } \ \Big\| \sum_{n\in F} a_n y_n \Big\|_{Y_p} \leqslant \delta.$$

So, fix a pair (x_n) and (y_n) of T-compatible sequences and $\delta > 0$. By Definition 11 and (P1) above, we see that the following are satisfied.

(P2) The sequence (x_n) is 1-equivalent to the standard unit vector basis of ℓ_p.

(P3) The sequence (y_n) is block and satisfies $\|y_n\| \leqslant 2$ for every $n \in \mathbb{N}$.

We are going to refine the sequences (x_n) and (y_n) in order to achieve further properties. Observe that we are allowed to do so since the family $\mathcal{S}_2$ is spreading. First we notice that, by Definition 11 and by passing to common subsequences of (x_n) and (y_n) if necessary, we may find a constant $\Theta \geqslant 1$ such that

(P4) the sequences (y_n) and $\big(T(x_n)\big)$ are Θ-equivalent.

Now we make the following simple (but crucial) observation.

Lemma 25. *For every $\sigma \in \mathbb{N}^{\mathbb{N}}$ we have* $\lim \|P_\sigma(y_n)\| = 0$.

Proof. Assume, towards a contradiction, that there exist $\sigma \in \mathbb{N}^{\mathbb{N}}$, a constant $\theta > 0$ and $L = \{l_1 < l_2 < \cdots\} \in [\mathbb{N}]^\infty$ such that $\|P_\sigma(y_{l_n})\| \geqslant \theta$ for every $n \in \mathbb{N}$. Since the sequence (y_n) is block and $\|P_\sigma\| = 1$, this implies that for every $d \in \mathbb{N}$ and every $a_1, \dots, a_d \in \mathbb{R}$ we have

$$\theta\Big(\sum_{n=1}^{d} |a_n|^p\Big)^{1/p} \leqslant \Big\|P_\sigma\Big(\sum_{n=1}^{d} a_n y_{l_n}\Big)\Big\|_{Y_p} \leqslant \Big\|\sum_{n=1}^{d} a_n y_{l_n}\Big\|_{Y_p}. \tag{24}$$

Let E be the subspace of ℓ_p spanned by the subsequence (x_{l_n}) of (x_n). We claim that the operator T restricted on E is an isomorphic embedding. Indeed, let $d \in \mathbb{N}$ and $a_1, \dots, a_d \in \mathbb{R}$ and notice that

$$\begin{aligned} \theta\Big\|\sum_{n=1}^{d} a_n x_{l_n}\Big\|_{\ell_p} &\overset{\text{(P2)}}{=} \theta\Big(\sum_{n=1}^{d} |a_n|^p\Big)^{1/p} \overset{(24)}{\leqslant} \Big\|\sum_{n=1}^{d} a_n y_{l_n}\Big\|_{Y_p} \\ &\overset{\text{(P4)}}{\leqslant} \Theta\Big\|\sum_{n=1}^{d} a_n T(x_{l_n})\Big\|_{Y_p} = \Theta\Big\|T\Big(\sum_{n=1}^{d} a_n x_{l_n}\Big)\Big\|_{Y_p}. \end{aligned}$$

Therefore, for every $x \in E$ with $\|x\| = 1$ we have $\|T(x)\| \geqslant \theta \cdot \Theta^{-1}$. This is clearly a contradiction and the proof is completed. $\square$

By (P3) above, Lemmas 21 and 25 and by passing to further common subsequences of (x_n) and (y_n), we may additionally assume that

(P5) the sequence (y_n) is asymptotically sparse.

We fix $N \in \mathbb{N}$ with $N \geqslant 2$ and such that

$$N^{1/q-1/p} \leqslant \delta \cdot (2\Theta)^{-1}. \tag{25}$$

Such a natural number can be found since $q = 2p$ and $p \geqslant 1$. Recursively, for every $i \in \{1, \dots, N\}$ we will select

(a) a natural number k_i,
(b) a positive real ε_i, and
(c) a nonempty finite subset F_i of $\mathbb{N}$

such that, setting $\mu_1 \coloneqq 1$ and

$$\mu_i \coloneqq \sum_{m=1}^{i-1} \Big(\sum_{n \in F_m} |\mathrm{supp}(y_n)| \Big)^{1/q} \tag{26}$$

for every $i \in \{2, \dots, N\}$, the following conditions are satisfied.

(C1) We have $F_1 < \dots < F_N$ and $N \leqslant \min(F_1)$.
(C2) We have $|F_i| = k_i$ and $k_i \leqslant \min(F_i)$ for every $i \in \{1, \dots, N\}$.
(C3) We have $|\{n \in F_i : \|P_\sigma(y_n)\| \geqslant \varepsilon_i\}| \leqslant 1$ for every $\sigma \in \mathbb{N}^{\mathbb{N}}$ and every $i \in \{1, \dots, N\}$.
(C4) We have $(k_i \varepsilon_i + 2) \cdot k_i^{-1/p} \leqslant \mu_i^{-1} \cdot 2^{-i}$ for every $i \in \{1, \dots, N\}$.

We proceed to the recursive selection. As the first step is identical to the general one, we may assume that for some $i \in \{1, \dots, N-1\}$ the natural numbers $k_1, \dots, k_i$, the positive reals $\varepsilon_1, \dots, \varepsilon_i$ and the sets $F_1, \dots, F_i$ have been selected so that conditions (C1)–(C4) are satisfied. In particular, the number μ_{i+1} can be defined. (For the first step of the recursive selection, recall that we have already set $\mu_1 = 1$.) First, we select $k_{i+1} \in \mathbb{N}$ such that $k_{i+1} \geqslant N$ and

$$2 \cdot k_{i+1}^{-1/p} \leqslant 2^{-1} \cdot \mu_{i+1}^{-1} \cdot 2^{-(i+1)}.$$

Next, we select $\varepsilon_{i+1} > 0$ such that

$$k_{i+1}^{1-1/p} \cdot \varepsilon_{i+1} \leqslant 2^{-1} \cdot \mu_{i+1}^{-1} \cdot 2^{-(i+1)}$$

and we notice that with these choices condition (C4) is satisfied. By (P5), the sequence (y_n) is asymptotically sparse. Therefore, it is possible to find $l \in \mathbb{N}$ such that $|\{n \geqslant l : \|P_\sigma(y_n)\| \geqslant \varepsilon_{i+1}\}| \leqslant 1$ for every $\sigma \in \mathbb{N}^{\mathbb{N}}$. We select a nonempty finite subset F_{i+1} of $\mathbb{N}$ such that $F_i < F_{i+1}$, $|F_{i+1}| = k_{i+1}$ and $\min(F_{i+1}) \geqslant \max\{k_{i+1}, l\}$ and we observe that with these choices conditions (C1), (C2) and (C3) are satisfied. The recursive selection is completed.

We define

$$F \coloneqq F_1 \cup \dots \cup F_N \tag{27}$$

Notice that for every $n \in F$ there exists a unique $i(n) \in \{1,\dots,N\}$ such that $n \in F_{i(n)}$. For every $n \in F$ we define

$$a_n \coloneqq N^{-1/p} \cdot k_{i(n)}^{-1/p}. \tag{28}$$

We will show that the set F and the reals $(a_n)_{n\in F}$ are as desired.

Claim 26. *We have $F \in \mathcal{S}_2$.*

Proof. Follows immediately by (C1) and (C2). □

Claim 27. *We have*

$$\Big\| \sum_{n\in F} a_n x_n \Big\|_{\ell_p} = 1.$$

Proof. By (P2), the sequence (x_n) is 1-equivalent to the standard unit vector basis of ℓ_p. Therefore,

$$\begin{aligned} \Big\| \sum_{n\in F} a_n x_n \Big\|_{\ell_p} &= N^{-1/p} \Big\| \sum_{i=1}^{N} \sum_{n\in F_i} k_i^{-1/p} x_n \Big\|_{\ell_p} = N^{-1/p} \Big(\sum_{i=1}^{N} \sum_{n\in F_i} k_i^{-1} \Big)^{1/p} \\ &= N^{-1/p} \Big(\sum_{i=1}^{N} |F_i| \cdot k_i^{-1} \Big)^{1/p} \overset{\text{(C2)}}{=} N^{-1/p} \cdot N^{1/p} = 1. \end{aligned}$$

The claim is proved. □

The final claim is the following.

Claim 28. *We have*

$$\Big\| \sum_{n\in F} a_n y_n \Big\|_{Y_p} \leqslant \delta.$$

For the proof of Claim 28 we need to do some preparatory work. For every $i \in \{1,\dots,N\}$ we introduce the vector z_i in Y_p defined by

$$z_i \coloneqq k_i^{-1/p} \sum_{n\in F_i} y_n. \tag{29}$$

Notice that

$$\sum_{n\in F} a_n y_n = N^{-1/p} \sum_{i=1}^{N} z_i \tag{30}$$

and

$$|\mathrm{supp}(z_i)| = \sum_{n\in F_i} |\mathrm{supp}(y_n)| \tag{31}$$

for every $i \in \{1,\dots,N\}$.

Subclaim 29. *Let $i \in \{1,\dots,N\}$. Then the following are satisfied.*

(i) *We have $\|z_i\| \leqslant \Theta$.*
(ii) *For every segment $\mathfrak{s}$ of $\mathbb{N}^{<\mathbb{N}}$ we have $\|P_{\mathfrak{s}}(z_i)\| \leqslant \mu_i^{-1} \cdot 2^{-i}$.*

Proof. (i) By (P4), the sequences (y_n) and $\big(T(x_n)\big)$ are Θ-equivalent. Hence,

$$\begin{aligned}\|z_i\| &\leqslant \Theta\Big\|T\Big(\sum_{n\in F_i} k_i^{-1/p} x_n\Big)\Big\|_{Y_p} \overset{\text{(P1)}}{\leqslant} \Theta\Big\|\sum_{n\in F_i} k_i^{-1/p} x_n\Big\|_{\ell_p} \\ &\overset{\text{(P2)}}{=} \Theta\Big(\sum_{n\in F_i} k_i^{-1}\Big)^{1/p} = \Theta\big(|F_i|\cdot k_i^{-1}\big)^{1/p} \overset{\text{(C2)}}{=} \Theta.\end{aligned}$$

(ii) Fix a segment $\mathfrak{s}$ of $\mathbb{N}^{<\mathbb{N}}$. Clearly we may assume that $\mathfrak{s}$ is nonempty. We select $\sigma\in\mathbb{N}^{\mathbb{N}}$ such that $\mathfrak{s}\subseteq\sigma$ and we notice that $\|P_{\mathfrak{s}}(y)\|\leqslant\|P_\sigma(y)\|$ for every vector y in Y_p. Therefore, it is enough to show that $\|P_\sigma(z_i)\|\leqslant\mu_i^{-1}\cdot 2^{-i}$. By (P3), we see that $\|P_\sigma(y_n)\|\leqslant 2$ for every $n\in\mathbb{N}$. Hence,

$$\Big\|P_\sigma\Big(k_i^{-1/p}\sum_{n\in F_i} y_n\Big)\Big\| \overset{\text{(C3)}}{\leqslant} \frac{(|F_i|-1)\varepsilon_i+2}{k_i^{1/p}} \overset{\text{(C2)}}{=} \frac{(k_i-1)\varepsilon_i+2}{k_i^{1/p}} \overset{\text{(C4)}}{\leqslant} \mu_i^{-1}\cdot 2^{-i}$$

and the result follows. □

We are ready to proceed to the proof of Claim 28.

Proof of Claim 28. We set

$$z \coloneqq z_1+\cdots+z_N. \tag{32}$$

By the choice of N in (25) and equality (30), it is enough to show that

$$\|z\|\leqslant N^{1/q}\cdot(2\Theta). \tag{33}$$

By (18), we see that for every vector y in Y_p we have

$$\|y\| = \sup\Big\{\Big(\sum_{j=1}^{d}\|P_{\mathfrak{s}_j}(y)\|^q\Big)^{1/q}\Big\}$$

where the above supremum is taken over all families of pairwise incomparable nonempty segments of $\mathbb{N}^{<\mathbb{N}}$. Therefore, it is enough to consider an arbitrary family $(\mathfrak{s}_j)_{j=1}^d$ of pairwise incomparable nonempty segments of $\mathbb{N}^{<\mathbb{N}}$ and show that

$$\sum_{j=1}^{d}\|P_{\mathfrak{s}_j}(z)\|^q\leqslant N\cdot(2\Theta)^q. \tag{34}$$

So, fix such a family $(\mathfrak{s}_j)_{j=1}^d$. We may assume that for every $j\in\{1,\dots,d\}$ there exists $i\in\{1,\dots,N\}$ such that $\mathfrak{s}_j\cap\mathrm{supp}(z_i)\neq\emptyset$. We define recursively a partition $(\Delta_i)_{i=1}^N$ of $\{1,\dots,d\}$ by the rule $\Delta_1\coloneqq\big\{j\in\{1,\dots,d\}:\mathfrak{s}_j\cap\mathrm{supp}(z_1)\neq\emptyset\big\}$ and

$$\Delta_i\coloneqq\Big\{j\in\{1,\dots,d\}\setminus\Big(\bigcup_{m=1}^{i-1}\Delta_m\Big):\mathfrak{s}_j\cap\mathrm{supp}(z_i)\neq\emptyset\Big\}.$$

for every $i\in\{2,\dots,N\}$. The segments $(\mathfrak{s}_j)_{j=1}^d$ are pairwise incomparable and a fortiori disjoint. Therefore, by equality (31), for every $i\in\{1,\dots,N\}$ we have

$$|\Delta_i|\leqslant\sum_{n\in F_i}|\mathrm{supp}(y_n)|. \tag{35}$$

Fix $i \in \{1,\dots,N\}$. Let $j \in \Delta_i$ be arbitrary. Notice that if $m \in \{1,\dots,N\}$ with $m < i$, then $P_{\mathfrak{s}_j}(z_m) = 0$. Therefore,

$$\|P_{\mathfrak{s}_j}(z)\| = \|P_{\mathfrak{s}_j}(z_i + \dots + z_N)\| \leqslant \|P_{\mathfrak{s}_j}(z_i)\| + \sum_{l=i+1}^{N} \|P_{\mathfrak{s}_j}(z_l)\|. \tag{36}$$

Let $l \in \{i+1,\dots,N\}$ be arbitrary. By (26) and (35), we have

$$|\Delta_i|^{1/q} \leqslant \Big(\sum_{n\in F_i} |\mathrm{supp}(y_n)|\Big)^{1/q} \leqslant \mu_l$$

while, by part (ii) of Subclaim 29, we have $\|P_{\mathfrak{s}_j}(z_l)\| \leqslant \mu_l^{-1} \cdot 2^{-l}$. By plugging the previous two estimates into (36), we obtain that

$$\|P_{\mathfrak{s}_j}(z)\| \leqslant \|P_{\mathfrak{s}_j}(z_i)\| + |\Delta_i|^{-1/q} \cdot 2^{-i}. \tag{37}$$

Using the fact that $q \geqslant 2$ and that $(a+b)^q \leqslant 2^{q-1}a^q + 2^{q-1}b^q$ for every pair a and b of positive reals, inequality (37) yields that

$$\sum_{j\in\Delta_i} \|P_{\mathfrak{s}_j}(z)\|^q \leqslant 2^{q-1}\Big(\sum_{j\in\Delta_i} \|P_{\mathfrak{s}_j}(z_i)\|^q\Big) + 2^{q-1}\cdot 2^{-i}.$$

The family $(\mathfrak{s}_j)_{j\in\Delta_i}$ consists of pairwise incomparable nonempty segments of $\mathbb{N}^{<\mathbb{N}}$. Therefore, by part (i) of Subclaim 29, we obtain that

$$\sum_{j\in\Delta_i} \|P_{\mathfrak{s}_j}(z)\|^q \leqslant 2^{q-1}\|z_i\|^q + 2^{q-1}\cdot 2^{-i} \leqslant 2^{q-1}\cdot\Theta^q + 2^{q-1}\cdot 2^{-i}. \tag{38}$$

Summing up, we conclude that

$$\sum_{j=1}^{d} \|P_{\mathfrak{s}_j}(z)\|^q = \sum_{i=1}^{N}\sum_{j\in\Delta_i} \|P_{\mathfrak{s}_j}(z)\|^q \overset{(38)}{\leqslant} 2^{q-1}\cdot(N\cdot\Theta^q + 1) \leqslant N\cdot(2\Theta)^q$$

and the result follows. □

As we have already indicated, having completed the proof of Claim 28, the proof of part (ii) of Theorem 4 is completed.

Finally, we notice that the map ϱ is not a co-analytic rank on $\mathcal{SS}(\ell_p, Y_p)$. For if not, by part (ii) of Theorem 4 and [20, Theorem 35.23], we would obtain that $\mathcal{SS}(\ell_p, Y_p)$ is a Borel subset of $\mathcal{L}(\ell_p, Y_p)$. This contradicts part (i) of Theorem 4.

The proof of Theorem 4 is completed.

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