UNCONDITIONAL BASIC SEQUENCES IN SPACES OF LARGE DENSITY

PANDELIS DODOS, JORDI LOPEZ-ABAD AND STEVO TODORCEVIC

Abstract. We study the problem of the existence of unconditional basic sequences in Banach spaces of high density. We show in particular the relative consistency of the statement that every Banach space of density $\aleph_\omega$ contains an unconditional basic sequence.

1. Introduction

In this paper we study particular instances of the general unconditional basic sequence problem asking under which conditions a given Banach space must contain an infinite unconditional basic sequence (see [LT, page 27]). We chose to study instances of the problem for Banach spaces of large densities exposing thus its connections with large-cardinal axioms of set theory. The first paper on this line of research is a well-known paper of Ketonen [Ke] which shows that if a density of a given Banach space E is greater than or equal to the ω-Erdős cardinal (usually denoted as $\kappa(\omega)$, see Section 2.2), then E contains an infinite unconditional basic sequence. More precisely, let $\mathfrak{nc}$ be the minimal cardinal λ such that every Banach space of density at least λ contains an infinite unconditional basic sequence. Then Ketonen's result can be restated as follows.

Theorem 1 ([Ke]). $\kappa(\omega) \geqslant \mathfrak{nc}$.

Since $\kappa(\omega)$ is a considerably large cardinal (strongly inaccessible and more) one would like to determine is $\mathfrak{nc}$ really a large cardinal or not, and, of course at some point one would also like to determine the exact value of this cardinal. Unfortunately, there are not too many results in the literature that would point out towards lower bounds for this cardinal. In fact, the largest known lower bound for $\mathfrak{nc}$ is given by Argyros and Tolias [AT] who showed that $\mathfrak{nc} > 2^{\aleph_0}$. So in particular the following problem is widely open.

Question 1. Is $\exp_\omega(\aleph_0)$, any of the finite-tower exponents $\exp_n(\aleph_0)$, or any of their ω-successors $\exp_n(\aleph_0)^{+\omega}$ an upper bound of $\mathfrak{nc}$? In particular, does the inequality $(2^{\aleph_0})^{+\omega} \geqslant \mathfrak{nc}$ hold?

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The cardinals $\exp_n(\aleph_0)$ are listed here because of their strong Ramsey-theoretic properties. In fact our results here show that their supremum $\exp_\omega(\aleph_0)$ is not such a bad candidate for an upper bound of $\mathfrak{nc}$. We prove this using a variation of a partition property originally appearing in the problem lists of Erdős and Hajnal [EH1], [EH2, Problem 29] (see also [Sh2]). Let κ be a cardinal and $d \in \omega$ with $d \geqslant 1$. By $\mathrm{Pl}_d(\kappa)$ we denote the combinatorial principle asserting that for every coloring

$$c\colon \big[[\kappa]^d \big]^{<\omega} \to \omega$$

there exists a sequence $(\mathbf{x}_n)$ of infinite disjoint subsets of κ such that for every $m \in \omega$ the restriction

$$c \restriction \prod_{n=0}^{m} [\mathbf{x}_n]^d$$

is constant. Clearly property $\mathrm{Pl}_d(\kappa)$ implies property $\mathrm{Pl}_{d'}(\kappa)$ for any cardinal κ and any pair $d, d' \in \omega$ with $d \geqslant d' \geqslant 1$. From known results one can easily deduce that the principle $\mathrm{Pl}_d(\exp_{d-1}(\aleph_0)^{+n})$ is false for every $n \in \omega$ and every integer $d \geqslant 1$ (see, for instance, [EHMR, CDPM, DT]). Thus, the minimal cardinal κ for which $\mathrm{Pl}_d(\kappa)$ could possibly be true is $\exp_{d-1}(\aleph_0)^{+\omega}$. Indeed, Di Prisco and Todorcevic [DT] have established the consistency of $\mathrm{Pl}_1(\aleph_\omega)$ relative to the consistency of a single measurable cardinal, an assumption that also happens to be optimal. On the other hand, Shelah [Sh2] was able to establish that GCH and principles $\mathrm{Pl}_d(\aleph_\omega)$ $(d \geqslant 1)$ are jointly consistent, relative to the consistency of GCH and the existence of an infinite sequence of strongly compact cardinals. In a previous version of our paper we have presented an improvement of this result to the effect that the partition relations $\mathrm{Pl}_d(\aleph_\omega)$ were replaced by natural conditions on ideals of subsets of $\aleph_k$'s. While we find these improvements easy to use and therefore of potential interested to experts not familiar with consistency results in set theory, the referee and the editor of this journal were of the opinion that this part of our paper should be published in a separate form (see [DLT]).

Theorem 2. *Let κ be a cardinal for which the partition property $\mathrm{Pl}_2(\kappa)$ holds. Then every Banach space E not containing a copy of ℓ_1 and of density κ contains an 1-unconditional basic sequence*[1].

In particular, if E is any Banach space of some density κ for which $\mathrm{Pl}_2(\kappa)$ holds, then for every $\varepsilon > 0$ the space E contains an $(1+\varepsilon)$-unconditional basic sequence. Recall that the separable Hilbert space $\ell_2 = \ell_2(\omega)$ is *arbitrarily distortable*, that is, for every $\lambda > 1$ there is an equivalent norm $|\cdot|$ on ℓ_2 with the property that for every infinite-dimensional subspace E of ℓ_2 there exist $x, y \in E$ such that $\|x\|_2 = \|y\|_2 = 1$ and $|x| > \lambda \cdot |y|$ (see [OS]). The referee remarks that Theorem 2

[1] Recall that a sequence (x_n) in a Banach space E is said to be *C-unconditional*, where $C \geqslant 1$, if for every pair F and G of nonempty finite subsets of ω with $F \subseteq G$ and every choice $(a_n)_{n\in G}$ of scalars we have $\|\sum_{n\in F} a_n x_n\| \leqslant C \cdot \|\sum_{n\in G} a_n x_n\|$.

might imply that large Hilbert spaces are not even distortable. We don't know how to verify this nor whether this kind of result was known before.

A well-known consequence of a result due to Hagler and Johnson [HJ] asserts that if E is a Banach space such that E^* has an unconditional basic sequence, then E has a separable quotient with an unconditional basis (see also [ADK, Proposition 16]). Noticing that the density of the dual E^* of a Banach space E is at least as big as the density of E, we obtain the following consequence of Theorem 2.

Corollary 3. *If a cardinal κ satisfies $\mathrm{Pl}_2(\kappa)$, then every Banach space of density at least κ has a separable quotient with an unconditional basis.*

Since $\mathrm{Pl}_2(\aleph_\omega)$ is a consistent statement, we have the following corollary.

Corollary 4. *It is consistent that every Banach space of density at least $\aleph_\omega$ has a separable quotient with an unconditional basis.*

Recall that under an appropriate Baire category assumption every Banach space E of density $\aleph_1$ has a separable quotient (see [To1]). In fact one can combine the work of [To1] with the consistency proof from [DLT] (which was originally a part of the present paper) and show that for every positive integer k there is a generic extension of the universe of sets in which every Banach space E of density at most $\aleph_k$ or at least $\aleph_\omega$ has a separable quotient.

Theorem 2 together with the fact that $\mathrm{Pl}_2(\aleph_\omega)$ is a consistent statement shows that the inequality $\aleph_\omega \geqslant \mathfrak{nc}$ is consistent with the usual axioms of set theory. A close examination of this consistency proof and some known results from Banach space theory suggest that by restricting the class of Banach spaces to, say, reflexive, or more generally weakly compactly generated Banach spaces, one might obtain different answers about the size of the corresponding cardinal numbers $\mathfrak{nc}_{\mathrm{rfl}}$ and $\mathfrak{nc}_{\mathrm{wcg}}$ respectively. To describe this difference it is convenient to introduce yet another natural cardinal characteristic $\mathfrak{nc}_{\mathrm{seq}}$, the minimal cardinal θ such that every normalized weakly null[2] sequence $(x_\alpha : \alpha < \kappa)$ in some Banach space E has a subsequence which is unconditional. Clearly $\mathfrak{nc}_{\mathrm{rfl}} \leqslant \mathfrak{nc}_{\mathrm{wcg}}$ while, by the Amir–Lindenstrauss theorem [AL], we see that $\mathfrak{nc}_{\mathrm{wcg}} \leqslant \mathfrak{nc}_{\mathrm{seq}}$. The first known lower bound on these cardinal is due to Maurey and Rosenthal [MR] who showed that $\mathfrak{nc}_{\mathrm{seq}} > \aleph_0$, though considerably deeper is the lower bound of Gowers and Maurey [GM] who showed that in fact $\mathfrak{nc}_{\mathrm{rfl}} > \aleph_0$. The largest known lower bound on these cardinals is given in [ALT] who showed that $\mathfrak{nc}_{\mathrm{rfl}} > \aleph_1$. This suggests the following question.

Question 2. Is $\aleph_\omega$ or any of the finite successors $\aleph_n$ $(n \geqslant 2)$ an upper bound on any of the three cardinals $\mathfrak{nc}_{\mathrm{seq}}$, $\mathfrak{nc}_{\mathrm{rfl}}$ or $\mathfrak{nc}_{\mathrm{wcg}}$?

[2]Recall that $(x_\alpha : \alpha < \lambda)$ is *weakly null* in E if for every $f \in E^*$ and every $\varepsilon > 0$ the set $\{\alpha < \kappa : |f(x_\alpha)| \geqslant \varepsilon\}$ is finite.

That $\aleph_\omega$ is not such a bad choice for an upper bound of $\mathfrak{nc}_{\mathrm{seq}}$ may be seen from our second result and the fact that $\mathrm{Pl}_1(\aleph_\omega)$ is a consistent statement.

Theorem 5. *If a cardinal κ has the property $\mathrm{Pl}_1(\kappa)$ then every weakly null normalized sequence $(x_\alpha : \alpha < \kappa)$ in some Banach space E has an infinite 1-unconditional subsequence $(x_{\alpha_n} : n < \omega)$.*

It turns out that the consistency proof that $\mathrm{Pl}_1(\aleph_\omega)$ uses a considerably weaker assumption than the consistency proof of $\mathrm{Pl}_2(\aleph_\omega)$. It relies on two Ramsey-theoretic principles, one established by Koepke [Ko] and the other by Di Prisco and Todorcevic [DT]. It also gives the joint consistency of the GCH and the cardinal inequality $\aleph_\omega \geqslant \mathfrak{nc}_{\mathrm{seq}}$ relative to the consistency of a single measurable cardinal.

The rest of the paper is organized as follows. In Section 2 we present some preliminaries. In Subection 3.1 we give the proof of Theorem 2, while in Subsection 3.2 we present its "sequential" version. Two proofs of this version are given, each of which is based on a different combinatorial principle.

2. Preliminaries

Our Banach space theoretic and set theoretic terminology and notation are standard and follow [LT] and [Ku] respectively. We will consider only real Banach spaces though, using essentially the same arguments, one notices that all our results are valid for complex Banach spaces as well.

2.1. Banach space cardinals.

Since in this note we are concerned with the problem of the existence of unconditional basic sequences in Banach spaces of high density, let us introduce the following cardinal invariants related to the version of the unconditional basic sequence problem that we study here.

Definition 6. *Let $\mathfrak{nc}$, $\mathfrak{nc}_{\mathrm{wcg}}$, $\mathfrak{nc}_{\mathrm{rfl}}$ and $\mathfrak{nc}_{\mathrm{seq}}$ be defined as follows.*

(1) *$\mathfrak{nc}$ is the minimal cardinal λ such that every Banach space of density λ contains an unconditional basic sequence.*
(2) *$\mathfrak{nc}_{\mathrm{wcg}}$ (respectively, $\mathfrak{nc}_{\mathrm{rfl}}$) is the minimal cardinal λ such that every weakly compactly generated (respectively, reflexive) Banach space of density λ contains an unconditional basic sequence.*
(3) *$\mathfrak{nc}_{\mathrm{seq}}$ is the minimal cardinal λ such that every normalized weakly null sequence $(x_\alpha : \alpha < \lambda)$ in a Banach space E has a subsequence $(x_{\alpha_n} : n < \omega)$ which is unconditional.*

Let us now recall some standard set theoretic notions.

2.2. Large Cardinals.

Let θ be a cardinal.

(a) θ is said to be *inaccessible* if it is regular and strong limit; that is, $2^\lambda < \theta$ for every $\lambda < \theta$.

(b) θ is said to be 0-*Mahlo* if it is inaccessible. In general, for an ordinal α, θ is said to be α-*Mahlo* if for every $\beta < \alpha$ and every closed and unbounded subset C of θ there is a β-Mahlo cardinal λ in C.

(c) An α-*Erdős* cardinal, usually denoted by $\kappa(\alpha)$ if exists, is the minimal cardinal λ such that $\lambda \to (\alpha)^{<\omega}_2$; that is, λ is the least cardinal with the property that for every coloring $c\colon [\lambda]^{<\omega} \to 2$ there is $H \subseteq \lambda$ of order-type α such that c is constant on $[H]^n$ for every $n < \omega$. A cardinal λ that is λ-Erdős (in other words, a cardinal λ which has the partition property $\lambda \to (\lambda)^{<\omega}_2$) is called a *Ramsey* cardinal.

(d) θ is said to be *measurable* if there exists a θ-complete normal ultrafilter $\mathcal{U}$ on θ. Looking at the ultrapower of the universe using $\mathcal{U}$ one can observe that the set $\{\lambda < \theta : \lambda \text{ is inaccessible}\}$ belongs to $\mathcal{U}$. Similarly, one shows that sets $\{\lambda < \theta : \lambda \text{ is } \lambda\text{-Mahlo}\}$ and $\{\lambda < \theta : \lambda \text{ is Ramsey}\}$ belong to $\mathcal{U}$.

(e) θ is said to be *strongly compact* if every θ-complete filter can be extended to a θ-complete ultrafilter.

Finally, for every cardinal κ and every $n \in \omega$ we define recursively the cardinal $\exp_n(\kappa)$ by $\exp_0(\kappa) = \kappa$ and $\exp_{n+1}(\kappa) = 2^{\exp_n(\kappa)}$. For more details see [Ka].

3. Banach space implications of polarized partition relations

Recall that a sequence (x_n) in a Banach space E is said to be C-*unconditional*, where $C \geqslant 1$, if for every pair F and G of nonempty finite subsets of ω with $F \subseteq G$ and every choice $(a_n)_{n\in G}$ of scalars we have

$$\Big\| \sum_{n\in F} a_n x_n \Big\| \leqslant C \cdot \Big\| \sum_{n\in G} a_n x_n \Big\|.$$

Also recall the following partition property, a variation of a partition property originally appearing in the problem lists of Erdős and Hajnal [EH1], [EH2, Problem 29] (see also [Sh2]).

Definition 7. *Let κ be a cardinal and $d \in \omega$ with $d \geqslant 1$. By $\mathrm{Pl}_d(\kappa)$ we shall denote the combinatorial principle asserting that for every coloring $c\colon \big[[\kappa]^d \big]^{<\omega} \to \omega$ there exists a sequence $(\mathbf{x}_n)$ of infinite disjoint subsets of κ such that for every $m \in \omega$ the restriction $c \restriction \prod_{n=0}^{m} [\mathbf{x}_n]^d$ is constant.*

The main result in this section is the following theorem.

Theorem 8. *Let κ be a cardinal and assume that property $\mathrm{Pl}_2(\kappa)$ holds. Then every Banach space E not containing ℓ_1 and of density κ contains an 1-unconditional basic sequence.*

In particular, if E is any Banach space of density κ, then for every $\varepsilon > 0$ the space E contains an $(1+\varepsilon)$-unconditional basic sequence.

3.1. **Proof of Theorem 8.** We start with the following lemma, which is essentially a multi-dimensional version of Odell's Schreier unconditionality theorem [O2].

Lemma 9. *Let E be a Banach space, $m \in \omega$ with $m \geqslant 1$ and $\varepsilon > 0$. For every $i \in \{0,\dots,m\}$ let (x^i_n) be a normalized weakly null sequence in the space E. Then, there exists an infinite subset L of ω such that for every $\{n_0 < \cdots < n_m\} \subseteq L$ the sequence $(x^i_{n_i})_{i=0}^m$ is $(1+\varepsilon)$-unconditional.*

Proof. The first step towards the proof of the lemma is included in the following claim. It shows that, by passing to an infinite subset of ω, we may assume that for every $\{n_0 < \cdots < n_m\} \in [\mathbb{N}]^{m+1}$ the finite sequence $(x^i_{n_i})_{i=0}^m$ is a particularly well behaved basic sequence.

Claim 10. *For every $\varepsilon > 0$ there exists an infinite subset M of ω such that for every $\{n_0 < \cdots < n_m\} \subseteq M$ the sequence $(x^i_{n_i})_{i=0}^m$ is an $(1+\varepsilon)$-basic sequence.*

Proof of Claim 10. First, we define a coloring $\mathcal{B}\colon [\mathbb{N}]^{m+1} \to 2$ as follows. Let $s = \{n_0 < \cdots < n_m\} \in [\mathbb{N}]^{m+1}$ be arbitrary. We set $\mathcal{B}(s) = 0$ if $(x^i_{n_i})_{i=0}^m$ is an $(1+\varepsilon)$-basic sequence; otherwise, we set $\mathcal{B}(s) = 1$. By Ramsey's theorem, there exist an infinite subset M of ω and $c \in \{0,1\}$ such that $\mathcal{B} \upharpoonright [M]^{m+1}$ is constantly equal to c. Next, using Mazur's classical procedure for selecting basic sequences (see [LT, Lemma 1.a.6]), we may select $t = \{k_0 < \cdots < k_m\} \in [M]^{m+1}$ such that the sequence $(x^i_{k_i})_{i=0}^m$ is basic with basis constant $(1+\varepsilon)$. Therefore, $\mathcal{B}(t) = 0$ and so, by homogeneity, $\mathcal{B} \upharpoonright [M]^{m+1} = 0$. The claim is proved. □

Applying Claim 10 for $\varepsilon = 1$, we obtain an infinite subset M of ω as described above. Observe that for every $\{n_0 < \cdots < n_m\} \in [M]^{m+1}$ and every choice $(a_i)_{i=0}^m$ of scalars we have

$$\|\sum_{i=0}^m a_i x^i_{n_i}\| \geqslant \frac{1}{4}\max\{|a_i| : i = 0,\dots,m\}. \tag{1}$$

The desired subset L of ω will be an infinite subset of M obtained by another application of Ramsey's theorem. Specifically, consider the coloring $\mathcal{U}\colon [M]^{m+1} \to 2$ defined as follows. Let $s = \{n_0 < \cdots < n_m\} \in [M]^{m+1}$ and assume that the sequence $(x^i_{n_i})_{i=0}^m$ is $(1+\varepsilon)$-unconditional. In such a case, we set $\mathcal{U}(s) = 0$; otherwise we set $\mathcal{U}(s) = 1$. Let L be an infinite subset of M such that $\mathcal{U}$ is constant on $[L]^{m+1}$. It is enough to find some $s \in [L]^{m+1}$ such that $\mathcal{U}(s) = 0$.

To this end, fix $\delta > 0$ such that $(1+\delta)\cdot(1-\delta)^{-1} \leqslant (1+\varepsilon)$. Notice that there exists a finite family $\mathcal{D}$ of normalized basic sequences of length $m+1$ such that any normalized basic sequence $(y_i)_{i=0}^m$ in some Banach space Y, is $\sqrt{1+\delta}$-equivalent to some sequence in the family $\mathcal{D}$. Hence, by another application of Ramsey's theorem and by passing to an infinite subset of L if necessary, we may assume that

(∗) for every $\{n_0 < \cdots < n_m\}, \{k_0 < \cdots < k_m\} \in [L]^{m+1}$ the sequences $(x^i_{n_i})_{i=0}^m$ and $(x^i_{k_i})_{i=0}^m$ are $(1+\delta)$-equivalent.

Now, for every $i \in \{0,\dots,m\}$ and every $\rho > 0$ set

$$\mathcal{K}_i(\rho) \coloneqq \big\{\{n \in \omega : |x^*(x^i_n)| \geqslant \rho\} : x^* \in B_{E^*}\big\}.$$

Every sequence (x^i_n) is weakly null, and so, each $\mathcal{K}_i(\rho)$ is a pre-compact[3] family of finite subsets of ω. Hence, we may select a sequence $(F_i)_{i=0}^m$ of finite subsets of L such that

(a) $\max(F_i) < \min(F_{i+1})$ for every $i \in \{0,\dots,m-1\}$, and
(b) $F_i \notin \mathcal{K}_i(\delta \cdot 8^{-1} \cdot (m+1)^{-1})$ for every $i \in \{0,\dots,m\}$.

We set $n \coloneqq \min(F_i)$ for every $i \in \{0,\dots,m\}$. Property (a) above implies that $n_0 < \dots < n_m$. We claim that the sequence $(x^i_{n_i})_{i=0}^m$ is $(1+\varepsilon)$-unconditional. Indeed, let $F \subseteq \{0,\dots,m\}$ and let $(a_i)_{i=0}^m$ be a choice of scalars. We need to prove that

$$\Big\| \sum_{i\in F} a_i x^i_{n_i} \Big\| \leqslant (1+\varepsilon) \Big\| \sum_{i=0}^m a_i x^i_{n_i} \Big\|.$$

Clearly we may assume that $\| \sum_{i\in F} a_i x^i_{n_i}\| = 1$. If $\| \sum_{i\notin F} a_i x^i_{n_i}\| \geqslant 2$, then

$$\Big\| \sum_{i=0}^m a_i x^i_{n_i} \Big\| \geqslant \Big\| \sum_{i\notin F} a_i x^i_{n_i} \Big\| - \Big\| \sum_{i\in F} a_i x^i_{n_i} \Big\| \geqslant 1 = \Big\| \sum_{i\in F} a_i x^i_{n_i} \Big\|.$$

So, suppose that $\| \sum_{i\notin F} a_i x^i_{n_i}\| \leqslant 2$. By (1), we see that

$$\max\{|a_i| \,:\, i \notin F\} \leqslant 8. \tag{2}$$

We select $x^*_0 \in S_{E^*}$ such that $x^*_0\big(\sum_{i\in F} a_i x^i_{n_i}\big) = \| \sum_{i\in F} a_i x^i_{n_i}\|$. We define a sequence $(k_i)_{i=0}^m$ in L as follows. If $i \notin F$, then let k_i be any member of F_i satisfying $|x^*_0(x^i_{k_i})| < \delta \cdot 8^{-1} \cdot (m+1)^{-1}$ (such a selection is possible by (b) above); if $i \in F$, then we set $k_i \coloneqq n_i$. By (a), we have $k_0 < \dots < k_m$. Moreover,

$$\begin{aligned} \Big\| \sum_{i=0}^m a_i x^i_{k_i} \Big\| &\geqslant x^*_0\Big(\sum_{i=0}^m a_i x^i_{k_i}\Big) = x^*_0\Big(\sum_{i\in F} a_i x^i_{k_i}\Big) + x^*_0\Big(\sum_{i\notin F} a_i x^i_{k_i}\Big) \\ &\geqslant x^*_0\Big(\sum_{i\in F} a_i x^i_{k_i}\Big) - \sum_{i\notin F} |a_i| \cdot |x^*_0(x^i_{k_i})| \geqslant 1-\delta. \end{aligned}$$

Invoking $(*)$, we conclude that

$$\Big\| \sum_{i=0}^m a_i x^i_{n_i} \Big\| \geqslant \frac{1}{1+\delta} \Big\| \sum_{i=0}^m a_i x^i_{k_i} \Big\| \geqslant \frac{1-\delta}{1+\delta} \geqslant \frac{1}{1+\varepsilon} \Big\| \sum_{i\in F} a_i x^i_{n_i} \Big\|.$$

The proof is completed. □

We are ready to proceed to the proof of Theorem 8.

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[3]Recall that a family $\mathcal{F}$ of finite subsets of ω is said to be *pre-compact* if, identifying $\mathcal{F}$ with a subset of the Cantor set 2^ω, the closure $\overline{\mathcal{F}}$ of $\mathcal{F}$ in 2^ω consists only of finite sets.

Proof of Theorem 8. Let κ be a cardinal such that $\mathrm{Pl}_2(\kappa)$ holds. By a classical result of James (see [LT, Proposition 2.e.3]), it is enough to show that if E is a Banach space of density κ not containing an isomorphic copy of ℓ_1, then E has an 1-unconditional basic sequence. So, let E be one. By Rosenthal's ℓ_1 theorem [Ro] and our assumptions on the space E, we see that every bounded sequence in E has a weakly Cauchy subsequence. Let $(x_\alpha : \alpha < \kappa)$ be a normalized sequence such that $\|x_\alpha - x_\beta\| \geqslant 1$ for every $\alpha < \beta < \kappa$. We define a coloring $c_{\mathrm{un}} \colon \big[[\kappa]^2 \big]^{<\omega} \to \omega$ as follows. Let $s = (\{\alpha_0 < \beta_0\}, \dots, \{\alpha_m < \beta_m\}) \in \big[[\kappa]^2 \big]^{<\omega}$ be arbitrary. Assume that there exists $l \in \omega$ with $l > 0$ and such that the sequence $(x_{\beta_i} - x_{\alpha_i})_{i=0}^m$ is *not* $(1 + 1/l)$-unconditional. In such a case, setting l_s to be the least $l \in \omega$ with the above property, we define $c_{\mathrm{un}}(s) = l_s$. If such an l does not exist, then we set $c_{\mathrm{un}}(s) = 0$. By $\mathrm{Pl}_2(\kappa)$, there exist a sequence $(\mathbf{x}_i)$ of infinite subsets of κ and a sequence (l_m) in ω such that for every $m \in \omega$ the restriction $c_{\mathrm{un}} \upharpoonright \prod_{i=0}^m [\mathbf{x}_i]^2$ of the coloring c_{un} on the product $\prod_{i=0}^m [\mathbf{x}_i]^2$ is constant with value l_m.

Claim 11. *For every $m \in \omega$ we have $l_m = 0$.*

Granting the claim, the proof of the theorem is completed. Indeed, observe that for every infinite sequence of pairs $\big(\{\alpha_i < \beta_i\}\big) \in \prod_{i\in\omega} [\mathbf{x}_i]^2$ the sequence $(x_{\beta_i} - x_{\alpha_i})$ is a semi-normalized 1-unconditional basic sequence in the Banach space E.

It only remains to prove Claim 11. To this end we argue by contradiction. So, assume that there exists $m \in \omega$ such that $l_m > 0$. Our definition of the coloring c_{un} implies that $m \geqslant 1$. For every $i \in \{0, \dots, m\}$ we may select an infinite subset $\{\alpha^i_0 < \alpha^i_1 < \cdots\}$ of $\mathbf{x}_i$ such that the sequence (x_{α_i}) is weakly Cauchy. We set

$$y^i_n \coloneqq \frac{x_{\alpha^i_{2n}} - x_{\alpha^i_{2n+1}}}{\|x_{\alpha^i_{2n}} - x_{\alpha^i_{2n+1}}\|}$$

for every $i \in \{0, \dots, m\}$ and every $n \in \omega$. Then each (y^i_n) is a normalized weakly null sequence in E. Moreover, for every $\{n_0 < \cdots < n_m\} \subseteq [\mathbb{N}]^{m+1}$ the sequence $(y^i_{n_i})_{i=0}^m$ is not $(1 + 1/l_m)$-unconditional. This clearly contradicts Lemma 9. The proof is completed. □

3.2. **Unconditional subsequences of weakly null sequences.** This subsection is devoted to the proof of the following "sequential" version of Theorem 8.

Theorem 12. *Let κ be a cardinal and assume that property* $\mathrm{Pl}_1(\kappa)$ *holds. Then* $\mathfrak{nc}_{\mathrm{seq}} \leqslant \kappa$. *In fact, every normalized weakly null sequence $(x_\alpha : \alpha < \kappa)$ has an infinite* 1*-unconditional subsequence $(x_{\alpha_n} : n < \omega)$.*

Proof. It is similar to the proof of Theorem 8. Indeed, consider the coloring $c_{\mathrm{un}} \colon [\kappa]^{<\omega} \to \omega$ defined as follows. Let $s = (\alpha_0 < \cdots < \alpha_m) \in [\kappa]^{<\omega}$ be arbitrary and assume that there exists $l \in \omega$ with $l > 0$ such that the sequence $(x_{\alpha_i})_{i=0}^m$ is *not* $(1 + 1/l)$-unconditional. In such a case, let $c_{\mathrm{un}}(s)$ be the least l with this property. Otherwise, we set $c_{\mathrm{un}}(s) = 0$. Using $\mathrm{Pl}_1(\kappa)$ and Lemma 9, the result follows. □

It follows that is consistent relative to the existence of just a single measurable cardinal that every normalized weakly null sequence $(x_\alpha : \alpha < \aleph_\omega)$ has an 1-unconditional subsequence. Moreover, this statement is consistent with GCH.

There is another well-known combinatorial property of a cardinal κ which is implied by $\mathrm{Pl}_1(\kappa)$ and which is in turn sufficient for the estimate $\mathfrak{nc}_{\mathrm{seq}} \leqslant \kappa$. This property is in the literature called the *free set property* of κ (see [Sh1, Ko, DT] and the references therein).

Definition 13. *By the term* structure *on a cardinal κ we mean a first order structure $\mathcal{M} = (\kappa, (f_i)_{i\in\omega})$ where $n_i \in \omega$ and $f_i \colon \kappa^{n_i} \to \kappa$ for every $i \in \omega$. The* free set property *of κ, denoted by $\mathrm{Fr}_\omega(\kappa, \omega)$, is the assertion that every structure $\mathcal{M} = (\kappa, (f_i)_{i\in\omega})$ has a free infinite set, that is, there exists an infinite set $L \subseteq \kappa$ such that every $x \in L$ does not belong to the substructure of $\mathcal{M}$ generated by $L\setminus\{x\}$.*

We need the following fact (its proof is left to the interested reader).

Fact 14. *Let κ be a cardinal. Then the following are equivalent.*

(a) *We have that $\mathrm{Fr}_\omega(\kappa, \omega)$ holds true.*

(b) *For every structure $\mathcal{M} = (\kappa, (f_i)_{i\in\omega})$ there exists an infinite subset L of κ such that for every $x \in L$ we have that*

$$x \notin \big\{ f_i(s) : s \in (L \setminus \{x\})^{n_i} \text{ and } i \in \omega \big\}.$$

(c) *Every* extended *structure $\mathcal{N} = (\kappa, (g_i)_{i\in\omega})$, where $g_i \colon \kappa^{<\omega} \to [\kappa]^{\leqslant\omega}$ for every $i \in \omega$, has an infinite free subset. That is, there exists an infinite subset L of κ such that for every $x \in L$ we have*

$$x \notin \bigcup_{i\in\omega} \bigcup_{s\in(L\setminus\{x\})^{<\omega}} g_i(s).$$

As we have already indicated above, one can use the property $\mathrm{Fr}_\omega(\kappa, \omega)$ to derive the conclusion of Theorem 12. More precisely, we have the following theorem.

Theorem 15. *Let κ be a cardinal and assume that $\mathrm{Fr}_\omega(\kappa, \omega)$ holds. Then every normalized weakly null sequence $(x_\alpha : \alpha < \kappa)$ has an 1-unconditional subsequence.*

Proof. Let $(x_\alpha : \alpha < \kappa)$ be a normalized weakly null sequence in a Banach space E. For every $s \in [\kappa]^{<\omega}$ we select a subset F_s of S_{E^*} which is countable and 1-norming for the finite-dimensional subspace $E_s \coloneqq \mathrm{span}\{x_\alpha : \alpha \in s\}$ of E. That is, for every $x \in E_s$ we have

$$\|x\| = \sup\{x^*(x) : x \in F_s\}. \tag{3}$$

Define $g \colon [\kappa]^{<\omega} \to [\kappa]^{\leqslant\omega}$ by

$$g(s) = \{\alpha < \kappa : \text{there is some } x^* \in F_s \text{ such that } x^*(x_\alpha) \neq 0\}. \tag{4}$$

Since $(x_\alpha : \alpha < \kappa)$ is weakly null and F_s is countable, we see that $g(s)$ is also countable; that is, g is well-defined. Consider the extended structure $\mathcal{N} = (\kappa, g)$.

Since $\mathrm{Fr}_\omega(\kappa,\omega)$ holds, there exists an infinite free subset L of κ. We claim that the sequence $(x_\alpha:\alpha\in L)$ is 1-unconditional.

Indeed, let s and t be finite subsets of L with $s\subseteq t$. Fix a sequence $(a_\alpha:\alpha\in t)$ of scalars and let $\varepsilon>0$ be arbitrary. By equality (3) above, we may select $y^*\in F_s$ such that

$$\Big\|\sum_{\alpha\in s}a_\alpha x_\alpha\Big\|\leqslant(1+\varepsilon)\cdot y^*\Big(\sum_{\alpha\in s}a_\alpha x_\alpha\Big). \tag{5}$$

The set L is free, and so, for every $\alpha\in t\setminus s$ we have $\alpha\notin g(s)$. This implies, in particular, that $y^*(x_\alpha)=0$ for every $\alpha\in t\setminus s$. Hence,

$$\begin{aligned}\Big\|\sum_{\alpha\in s}a_\alpha x_\alpha\Big\| &\leqslant (1+\varepsilon)\cdot y^*\Big(\sum_{\alpha\in s}a_\alpha x_\alpha\Big)=(1+\varepsilon)\cdot y^*\Big(\sum_{\alpha\in t}a_\alpha x_\alpha\Big)\\ &\leqslant (1+\varepsilon)\cdot\Big\|\sum_{\alpha\in t}a_\alpha x_\alpha\Big\|.\end{aligned}$$

Since $\varepsilon>0$ was arbitrary, the result follows. □

4. Concluding remarks

In this section we would like to discuss the possible refinements of our results presented above. First of all we notice that Ketonen's arguments actually give that if the density of a given Banach space E is greater than or equal to the ω-Erdős cardinal, then E contains a normalized basic sequence which is equivalent to all of its subsequences, that is, a basic sequence which is in the literature usually called a *sub-symmetric basic sequence*. Note that this is stronger than saying that the space E contains an unconditional basic sequence which can be easily seen using Rosenthal's ℓ_1 theorem [Ro].

On the other hand, we notice that our proof of the existence of an unconditional basic sequence in every Banach space of density $\exp_\omega(\aleph_0)$ *does not* guarantee the existence of a sub-symmetric basic sequence. This is mainly due to the fact that the principle $\mathrm{Pl}_2(\kappa)$ is a rectangular Ramsey property while all attempts that we have in mind for getting sub-symmetric basic sequences seem to require more classical Ramsey-type principles such as these given, for example, by the ω-Erdős cardinal. Since ω-Erdős is a large-cardinal property one might expect that there are Banach spaces of large density not containing a sub-symmetric basic sequence. So let us discuss some difficulties one encounters when trying to build such spaces.

The first example of an infinite dimensional Banach space not containing a sub-symmetric basic sequence is Tsirelson's space [Ts]. Tsirelson's space is separable; however, there do exist non-separable Banach spaces with the same property. The first such example is due to Odell [O1]. Odell's space is the dual of a separable one, and so, it has density $2^{\aleph_0}$. There even exist non-separable *reflexive* spaces not containing a sub-symmetric basic sequence. For example, one such a space is the space constructed in [ALT] which has density $\aleph_1$. We note that both spaces in [O1] and [ALT] are connected in some way to Tsirelson's space. So one is led to explore generalizations of the Tsirelson construction to larger densities.

Let us comment on difficulties encountered when trying to generalize Tsirelson's construction to densities bigger than the continuum, keeping in mind that we would like to get a space not containing a sub-symmetric basic sequence. The first natural move is to provide, for a given cardinal κ, a compact hereditary family $\mathcal{F}$ of finite subsets of κ which is sufficiently rich in the sense that for every infinite subset M of κ the restriction $\mathcal{F} \restriction M$ of the family on M has infinite rank. Notice that such a family cannot exist if κ is greater than or equal to the ω-Erdős cardinal. On the other hand, using a characterization of n-Mahlo cardinals due to Schmerl (see [Sch] or [To2, Theorem 6.1.8]), we were able to show that if κ is smaller that the first ω-Mahlo cardinal, then κ carries such a family $\mathcal{F}$.

Given a compact hereditary family $\mathcal{F}$ as above, the next step is to construct the Tsirelson-like space $T(\mathcal{F})$ on $c_{00}(\kappa)$ in the natural way. Such a space always fails to contain c_0 and ℓ_p for any $1 < p < \infty$. However, there are examples of such families for which the corresponding space contains a copy of ℓ_1. The reason is that the family $\mathcal{F}$ cannot be *spreading* relative the natural well-ordering of ordinals if κ is uncountable. Recall that spreading is a crucial property of the Schreier family on ω used in the original Tsirelson construction for preventing isomorphic copies of ℓ_1. So we finish this section with the following natural question.

Question 3. Does there exists a compact hereditary family $\mathcal{F}$ of finite subsets of some uncountable cardinal κ such that the corresponding Tsirelson-like space $T(\mathcal{F})$ fails to contain a copy of c_0 and ℓ_p for any $1 \leqslant p < \infty$?

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Department of Mathematics, University of Athens, Panepistimiopolis 157 84, Athens, Greece.

E-mail address: pdodos@math.uoa.gr

Université Denis Diderot - Paris 7, Equipe de Logique Mathématiques, 2 place Jussieu, 72521 Paris Cedex 05, France.

E-mail address: abad@logique.jussieu.fr

Université Denis Diderot - Paris 7, C.N.R.S., UMR 7056, 2 place Jussieu - Case 7012, 72521 Paris Cedex 05, France and

Department of Mathematics, University of Toronto, Toronto, Canada, M5S 2E4.
E-mail address: stevo@math.toronto.edu