QUOTIENTS OF BANACH SPACES AND SURJECTIVELY UNIVERSAL SPACES

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Abstract. We characterize those classes $\mathcal{C}$ of separable Banach spaces for which there exists a separable Banach space Y not containing ℓ_1 and such that every space in the class $\mathcal{C}$ is a quotient of Y.

1. Introduction

There are two classical universality results in Banach space theory. The first one, known to Banach [5], asserts that the space $C(2^{\mathbb{N}})$, where $2^{\mathbb{N}}$ stands for the Cantor set, is isometrically universal for all separable Banach spaces; that is, every separable Banach space is isometric to a subspace of $C(2^{\mathbb{N}})$. The second result, also known to Banach, is "dual" to the previous one and asserts that every separable Banach space is isometric to a quotient of ℓ_1.

By now, it is well understood that there are natural classes of separable Banach spaces for which one cannot obtain something better from what it is quoted above (see [1, 8, 16, 31]). For instance, if a separable Banach space Y is universal for the separable reflexive Banach spaces, then Y must contain an isomorphic copy of $C(2^{\mathbb{N}})$, and so, it is universal for all separable Banach spaces. However, there are non-trivial classes of separable Banach spaces which do admit "smaller" universal spaces (see [2, 11, 12, 13, 15, 23, 24, 27]).

Recently, in [11], a characterization was obtained of those classes of separable Banach spaces admitting a universal space which is not universal for all separable Banach spaces. One of the goals of the present paper is to obtain the corresponding characterization for the "dual" problem concerning quotients instead of embeddings. To proceed with our discussion it is useful to introduce the following definition.

Definition 1. *We say that a Banach space Y is a* surjectively universal space *for a class $\mathcal{C}$ of Banach spaces if every space in the class $\mathcal{C}$ is a quotient*[1] *of Y.*

We can now state the main problem addressed in this paper.

2000 *Mathematics Subject Classification*: Primary 46B03; Secondary 03E15, 05D10.

Key words: quotients of Banach spaces, Schauder bases, universal spaces.

Research supported by NSF grant DMS-0903558.

[1] If X and Y are Banach spaces, then we say that X is a *quotient* of Y if there exists a bounded, linear and onto operator $Q\colon Y \to X$.

(P) Let $\mathcal{C}$ be a class of separable Banach space. When can we find a separable Banach space Y which is surjectively universal for the class $\mathcal{C}$ and does not contain a copy of ℓ_1?

We notice that if a separable Banach space Y does not contain a copy of ℓ_1, then ℓ_1 is not a quotient of Y (see [21, Proposition 2.f.7]) and therefore Y is not surjectively universal for all separable Banach spaces.

To state our results we recall the following (more or less standard) notation and terminology. By SB we denote the standard Borel space of separable Banach spaces defined by Bossard [7], by NC_{ℓ_1} we denote the subset of SB consisting of all $X \in \mathrm{SB}$ not containing an isomorphic copy of ℓ_1 and, finally, by $\phi_{\mathrm{NC}_{\ell_1}}$ we denote Bourgain's ℓ_1 index [8] (these concepts are properly defined in §2). We show the following theorem.

Theorem 2. *Let $\mathcal{C} \subseteq \mathrm{SB}$. Then the following are equivalent.*

(i) *There exists a separable Banach space Y which is surjectively universal for the class $\mathcal{C}$ and does not contain a copy of ℓ_1.*
(ii) *We have* $\sup\{\phi_{\mathrm{NC}_{\ell_1}}(X) : X \in \mathcal{C}\} < \omega_1$.
(iii) *There exists an analytic subset $\mathcal{A}$ of NC_{ℓ_1} with $\mathcal{C} \subseteq \mathcal{A}$.*

We notice that stronger versions of Theorem 2 are valid provided that all spaces in the class $\mathcal{C}$ have some additional property (see §5).

A basic ingredient of the proof of Theorem 2 (an ingredient which is probably of independent interest) is the construction for every separable Banach space X of a Banach space E_X with special properties. Specifically we show the following theorem.

Theorem 3. *Let X be a separable Banach space. Then there exists a separable Banach space E_X such that the following are satisfied.*

(i) (Existence of a Schauder basis) *The space E_X has a normalized monotone Schauder basis (e_n^X).*
(ii) (Existence of a quotient map) *There exists a norm-one linear and onto operator $Q_X : E_X \to X$.*
(iii) (Subspace structure) *If Y is an infinite-dimensional subspace of E_X and the operator $Q_X : Y \to X$ is strictly singular, then Y contains a copy of c_0.*
(iv) (Representability of X) *For every normalized basic sequence (w_k) in X there exists a subsequence $(e_{n_k}^X)$ of (e_n^X) such that $(e_{n_k}^X)$ is equivalent to (w_k).*
(v) (Uniformity) *The set $\mathcal{E} \subseteq \mathrm{SB} \times \mathrm{SB}$ defined by*

$$(X, Y) \in \mathcal{E} \Leftrightarrow Y \text{ is isometric to } E_X$$

is analytic.
(vi) (Preservation of separability of the dual) *E_X^* is separable if and only if X^* is separable.*

We notice that there exists a large number of related results found in the literature; see, for instance, [9, 14, 15, 19, 23, 24, 32]. The novelty in Theorem 3 is that, beside functional analytic tools, its proof is enriched with descriptive set theory and the combinatorial machinery developed in [3] and [4].

The paper is organized as follows. In §2 we gather some background material. In §3 we define the space E_X and we give the proof of Theorem 3. The proof of Theorem 2 (actually of a more detailed version of it) is given in §4. Finally, in §5 we present some related results and we discuss open problems.

2. Background material

Our general notation and terminology is standard as can be found, for instance, in [21] and [20]. By $\mathbb{N} = \{0, 1, 2, \dots\}$ we denote the natural numbers.

We will frequently need to compute the descriptive set-theoretic complexity of various sets and relations. To this end, we will use the "Kuratowski–Tarski algorithm". We will assume that the reader is familiar with this classical method. For more details we refer to [20, page 353].

2.1. **Trees.** Let Λ be a nonempty set. By $\Lambda^{<\mathbb{N}}$ we denote the set of all finite sequences in Λ while by $\Lambda^{\mathbb{N}}$ we denote the set of all infinite sequences in Λ (the empty sequence is denoted by $\emptyset$ and is included in $\Lambda^{<\mathbb{N}}$). We view $\Lambda^{<\mathbb{N}}$ as a tree equipped with the (strict) partial order $\sqsubset$ of extension. Two nodes $s, t \in \Lambda^{<\mathbb{N}}$ are said to be *comparable* if either $s \sqsubseteq t$ or $t \sqsubseteq s$. Otherwise, s and t are said to be *incomparable*. A subset of $\Lambda^{<\mathbb{N}}$ consisting of pairwise comparable nodes is said to be a *chain*, while a subset of $\Lambda^{<\mathbb{N}}$ consisting of pairwise incomparable nodes is said to be an *antichain*.

A *tree* T on Λ is a subset of $\Lambda^{<\mathbb{N}}$ which is closed under initial segments. By $\mathrm{Tr}(\Lambda)$ we denote the set of all trees on Λ. Hence,

$$T \in \mathrm{Tr}(\Lambda) \Leftrightarrow \forall s, t \in \Lambda^{<\mathbb{N}} \ (s \sqsubseteq t \text{ and } t \in T \Rightarrow s \in T).$$

The *body* of a tree T on Λ is defined to be the set $\{\sigma \in \Lambda^{\mathbb{N}} : \sigma|n \in T \ \forall n \in \mathbb{N}\}$ and is denoted by $[T]$. A tree T is said to be *well-founded* if $[T] = \emptyset$. By $\mathrm{WF}(\Lambda)$ we denote the set of all well-founded trees on Λ. For every $T \in \mathrm{WF}(\Lambda)$ we set $T' \coloneqq \{s \in T : \exists t \in T \text{ with } s \sqsubset t\} \in \mathrm{WF}(\Lambda)$. By transfinite recursion, we define the iterated derivatives T^{ξ} $(\xi < \kappa^{+})$ of T, where κ stands for the cardinality of Λ. The *order* $o(T)$ of T is defined to be the least ordinal ξ such that $T^{\xi} = \emptyset$.

Let S and T be trees on two nonempty sets Λ_1 and Λ_2 respectively. A map $\psi\colon S \to T$ is said to be *monotone* if for every $s_0, s_1 \in S$ with $s_0 \sqsubset s_1$ we have $\psi(s_0) \sqsubset \psi(s_1)$. We notice that if there exists a monotone map $\psi\colon S \to T$ and T is well-founded, then S is well-founded and $o(S) \leqslant o(T)$.

2.2. **Dyadic subtrees and related combinatorics.** Let $2^{<\mathbb{N}}$ be the Cantor tree; that is, $2^{<\mathbb{N}}$ is the set of all finite sequences of 0's and 1's. For every $s, t \in 2^{<\mathbb{N}}$ we let $s \wedge t$ be the $\sqsubset$-maximal node w of $2^{<\mathbb{N}}$ with $w \sqsubseteq s$ and $w \sqsubseteq t$. If $s, t \in 2^{<\mathbb{N}}$ are incomparable with respect to $\sqsubseteq$, then we write $s \prec t$ provided that $(s \wedge t)^\frown 0 \sqsubseteq s$ and $(s \wedge t)^\frown 1 \sqsubseteq t$. We say that a subset D of $2^{<\mathbb{N}}$ is a *dyadic subtree* of $2^{<\mathbb{N}}$ if D can be written in the form $\{d_t : t \in 2^{<\mathbb{N}}\}$ so that for every $t_0, t_1 \in 2^{<\mathbb{N}}$ we have $t_0 \sqsubset t_1$ (respectively, $t_0 \prec t_1$) if and only if $d_{t_0} \sqsubset d_{t_1}$ (respectively, $d_{t_0} \prec d_{t_1}$). It is easy to see that such a representation of D as $\{d_t : t \in 2^{<\mathbb{N}}\}$ is unique. In the sequel when we write $D = \{d_t : t \in 2^{<\mathbb{N}}\}$, where D is a dyadic subtree, we will assume that this is the canonical representation of D described above.

For every dyadic subtree D of $2^{<\mathbb{N}}$ by $[D]_{\mathrm{chains}}$ we denote the set of all infinite chains of D. Notice that $[D]_{\mathrm{chains}}$ is a G_δ, hence Polish, subspace of $2^{2^{<\mathbb{N}}}$. We will need the following partition theorem due to Stern (see [30]).

Theorem 4. *Let D be a dyadic subtree of $2^{<\mathbb{N}}$ and let $\mathcal{X}$ be an analytic subset of $[D]_{\mathrm{chains}}$. Then there exists a dyadic subtree S of $2^{<\mathbb{N}}$ with $S \subseteq D$ and such that either $[S]_{\mathrm{chains}} \subseteq \mathcal{X}$ or $[S]_{\mathrm{chains}} \cap \mathcal{X} = \emptyset$.*

2.3. **Separable Banach spaces with non-separable dual.** We will need a structural result concerning separable Banach spaces with non-separable dual. To state this result and to facilitate future references to it, it is convenient to introduce the following definition.

Definition 5. *Let X be a Banach space and let $(x_t)_{t\in 2^{<\mathbb{N}}}$ be a sequence in X indexed by the Cantor tree. We say that $(x_t)_{t\in 2^{<\mathbb{N}}}$ is* topologically equivalent to the basis of James tree *if the following are satisfied.*

(1) *The sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ is semi-normalized.*
(2) *For every infinite antichain A of $2^{<\mathbb{N}}$ the sequence $(x_t)_{t\in A}$ is weakly null.*
(3) *For every $\sigma \in 2^{\mathbb{N}}$ the sequence $(x_{\sigma|n})$ is weak* convergent to an element $x^{**}_\sigma \in X^{**} \setminus X$. Moreover, if $\sigma, \tau \in 2^{\mathbb{N}}$ with $\sigma \neq \tau$, then $x^{**}_\sigma \neq x^{**}_\tau$.*

The archetypical example of such a sequence is the standard Schauder basis of the space JT (see [17]). There are also classical Banach spaces having a natural Schauder basis which is topologically equivalent to the basis of James tree; the space $C(2^{\mathbb{N}})$ is an example. We isolate, for future use, the following fact.

Fact 6. *Let X be a Banach space and let $(x_t)_{t\in 2^{<\mathbb{N}}}$ be a sequence in X which is topologically equivalent to the basis of James tree. Then for every dyadic subtree $D = \{d_t : t \in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$ the sequence $(x_{d_t})_{t\in 2^{<\mathbb{N}}}$ is topologically equivalent to the basis of James tree.*

We notice that if a Banach space X contains a sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ which is topologically equivalent to the basis of James tree, then X^* is not separable. The following theorem establishes the converse for separable Banach spaces not containing a copy of ℓ_1 (see [3, Theorem 40] or [4, Theorem 17]).

Theorem 7. *Let X be a separable Banach space not containing a copy of ℓ_1 and with non-separable dual. Then X contains a sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ which is topologically equivalent to the basis of James tree.*

2.4. **Co-analytic ranks.** Let (X,Σ) be a standard Borel space; that is, X is a set, Σ is a σ-algebra on X and the measurable space (X,Σ) is Borel isomorphic to the reals. A subset A of X is said to be *analytic* if there exists a Borel map $f\colon \mathbb{N}^{\mathbb{N}}\to X$ with $f(\mathbb{N}^{\mathbb{N}})=A$. A subset of X is said to be *co-analytic* if its complement is analytic. Now let B be a co-analytic subset of X. A map $\phi\colon B\to\omega_1$ is said to be a *co-analytic rank on B* if there exist relations $\leqslant_\Sigma$ and $\leqslant_\Pi$ in $X\times X$ which are analytic and co-analytic respectively and are such that for every $y\in B$ we have

$$x\in B \text{ and } \phi(x)\leqslant\phi(y)\Leftrightarrow x\leqslant_\Sigma y\Leftrightarrow x\leqslant_\Pi y.$$

For our purposes, the most important property of co-analytic ranks is that they satisfy *boundedness*. This means that if $\phi\colon B\to\omega_1$ is a co-analytic rank on a co-analytic set B and $A\subseteq B$ is analytic, then $\sup\{\phi(x):x\in A\}<\omega_1$. For a proof as well as for a thorough presentation of rank theory we refer to [20, §34].

2.5. **The standard Borel space of separable Banach spaces.** Let $F\big(C(2^{\mathbb{N}})\big)$ denote the set of all closed subsets of $C(2^{\mathbb{N}})$ and let Σ be the Effors–Borel structure on $F\big(C(2^{\mathbb{N}})\big)$; that is, Σ is the σ-algebra generated by the sets

$$\big\{F\in F\big(C(2^{\mathbb{N}})\big) : F\cap U\neq\emptyset\big\}$$

where U ranges over all open subsets of $C(2^{\mathbb{N}})$. Consider the set

$$\mathrm{SB}\coloneqq\big\{X\in F\big(C(2^{\mathbb{N}})\big) : X \text{ is a linear subspace}\big\}.$$

It is easy to see that the set SB equipped with the relative Effors–Borel structure is a standard Borel space (see [7] for more details). The space SB is referred in the literature as the *standard Borel space of separable Banach spaces*. We will need the following consequence of the Kuratowski–Ryll-Nardzewski selection theorem (see [20, Theorem 12.13]).

Proposition 8. *There exists a sequence $S_n\colon \mathrm{SB}\to C(2^{\mathbb{N}})$ $(n\in\mathbb{N})$ of Borel maps such that for every $X\in\mathrm{SB}$ with $X\neq\{0\}$ we have $S_n(X)\in S_X$ and the sequence $(S_n(X))$ is norm dense in S_X where S_X stands for the unit sphere of X.*

2.6. **The class NC_Z and Bourgain's indices.** Let Z be a Banach space with a Schauder basis[2]. We fix a normalized Schauder basis (z_n) of Z. If Z is one of the classical sequence spaces c_0 and ℓ_p $(1\leqslant p<+\infty)$, then by (z_n) we denote the standard unit vector basis. We consider the set

$$\mathrm{NC}_Z\coloneqq\{X\in\mathrm{SB} : X \text{ does not contain an isomorphic copy of } Z\}.$$

[2]Throughout the paper when we say that a Banach space X has a Schauder basis or the bounded approximation property, then we implicitly assume that X is infinite-dimensional.

Let $\delta \geqslant 1$ and let Y be an arbitrary separable Banach space. Following Bourgain [8], we introduce a tree $\mathbf{T}(Y, Z, (z_n), \delta)$ on Y defined by the rule

$$(y_n)_{n=0}^k \in \mathbf{T}(Y, Z, (z_n), \delta) \;\Leftrightarrow\; (y_n)_{n=0}^k \text{ is } \delta\text{-equivalent to } (z_n)_{n=0}^k.$$

In particular, if Z is the space ℓ_1, then for every $\delta \geqslant 1$ and every finite sequence $(y_n)_{n=0}^k$ in Y we have $(y_n)_{n=0}^k \in \mathbf{T}(Y, \ell_1, (z_n), \delta)$ if and only if for every $a_0, \dots, a_k \in \mathbb{R}$ it holds that

$$\frac{1}{\delta} \sum_{n=0}^{k} |a_n| \leqslant \Big\| \sum_{n=0}^{k} a_n y_n \Big\| \leqslant \delta \sum_{n=0}^{k} |a_n|.$$

We notice that $Y \in \mathrm{NC}_Z$ if and only if for every $\delta \geqslant 1$ the tree $\mathbf{T}(Y, Z, (z_n), \delta)$ is well-founded. We set $\phi_{\mathrm{NC}_Z}(Y) = \omega_1$ if $Y \notin \mathrm{NC}_Z$, while if $Y \in \mathrm{NC}_Z$ we define

$$\phi_{\mathrm{NC}_Z}(Y) = \sup\big\{ o\big(\mathbf{T}(Y, Z, (z_n), \delta)\big) : \delta \geqslant 1 \big\}. \tag{1}$$

In [8], Bourgain proved that for every Banach space Z with a Schauder basis and every $Y \in \mathrm{SB}$ we have $Y \in \mathrm{NC}_Z$ if and only if $\phi_{\mathrm{NC}_Z}(Y) < \omega_1$. We need the following refinement of this result (see [7, Theorem 4.4]).

Theorem 9. *Let Z be a Banach space with a Schauder basis. Then the set NC_Z is co-analytic and the map $\phi_{\mathrm{NC}_Z} \colon \mathrm{NC}_Z \to \omega_1$ is a co-analytic rank on NC_Z.*

We will also need the following quantitative strengthening of the classical fact that ℓ_1 has the lifting property.

Lemma 10. *Let X and Y be separable Banach spaces and assume that X is a quotient of Y. Then $\phi_{\mathrm{NC}_{\ell_1}}(X) \leqslant \phi_{\mathrm{NC}_{\ell_1}}(Y)$.*

Proof. Clearly we may assume that Y does not contain a copy of ℓ_1. We fix a quotient map $Q \colon Y \to X$. There exists a constant $C \geqslant 1$ such that

(a) $\|Q\| \leqslant C$, and
(b) for every $x \in X$ there exists $y \in Y$ with $Q(y) = x$ and $\|y\| \leqslant C\|x\|$.

For every $x \in X$ we select $y_x \in Y$ such that $Q(y_x) = x$ and $\|y_x\| \leqslant C\|x\|$. We define a map $\psi \colon X^{<\mathbb{N}} \to Y^{<\mathbb{N}}$ as follows. We set $\psi(\emptyset) = \emptyset$. If $s = (x_n)_{n=0}^k \in X^{<\mathbb{N}} \setminus \{\emptyset\}$, then we set

$$\psi(s) = (y_{x_n})_{n=0}^k.$$

We notice that the map ψ is monotone. Denote by (z_n) the standard unit vector basis of ℓ_1.

Claim 11. *For every $\delta \geqslant 1$ if $s \in \mathbf{T}(X, \ell_1, (z_n), \delta)$, then $\psi(s) \in \mathbf{T}(Y, \ell_1, (z_n), C\delta)$.*

Granting Claim 11, the proof of the lemma is completed. Indeed, by Claim 11, we have that for every $\delta \geqslant 1$ the map ψ is a monotone map from the tree $\mathbf{T}(X, \ell_1, (z_n), \delta)$ into the tree $\mathbf{T}(Y, \ell_1, (z_n), C\delta)$. Therefore,

$$o\big(\mathbf{T}(X, \ell_1, (z_n), \delta)\big) \leqslant o\big(\mathbf{T}(Y, \ell_1, (z_n), C\delta)\big).$$

The above estimate clearly implies that $\phi_{\mathrm{NC}_{\ell_1}}(X) \leqslant \phi_{\mathrm{NC}_{\ell_1}}(Y)$.

It remains to prove Claim 11. To this end let $s = (x_n)_{n=0}^k \in \mathbf{T}(X, \ell_1, (z_n), \delta)$. Also let $a_0, \dots, a_k \in \mathbb{R}$ be arbitrary. Notice that

$$Q(a_0 y_{x_0} + \cdots + a_k y_{x_k}) = a_0 x_0 + \cdots + a_k x_k.$$

Hence, by (a), we obtain that

$$\frac{1}{\delta} \sum_{n=0}^{k} |a_n| \leqslant \Big\| \sum_{n=0}^{k} a_n x_n \Big\| = \Big\| Q\Big(\sum_{n=0}^{k} a_n y_{x_n} \Big) \Big\| \leqslant C \Big\| \sum_{n=0}^{k} a_n y_{x_n} \Big\|. \tag{2}$$

Observe that $\|x_n\| \leqslant \delta$ for every $n \in \{0, \dots, k\}$. Therefore,

$$\Big\| \sum_{n=0}^{k} a_n y_{x_n} \Big\| \leqslant \sum_{n=0}^{k} |a_n| \cdot \|y_{x_n}\| = \sum_{n=0}^{k} |a_n| \cdot \|Q(x_n)\| \leqslant C\delta \sum_{n=0}^{k} |a_n|. \tag{3}$$

Since the coefficients $a_0, \dots, a_k \in \mathbb{R}$ were arbitrary, inequalities (2) and (3) imply that $\psi(s) = (y_{x_n})_{n=0}^k \in \mathbf{T}(Y, \ell_1, (z_n), C\delta)$. This completes the proof of Claim 11, and as we have indicated above, the proof of the lemma is also completed. □

2.7. **Separable spaces with the B. A. P. and Lusky's theorem.** By the results in [18] and [26], a separable Banach space X has the bounded approximation property (in short B. A. P.) if and only if X is isomorphic to a complemented subspace of a Banach space Y with a Schauder basis. Lusky found an effective way to produce the space Y. To state his result we need, first, to recall the definition of the space C_0 due to Johnson. Let (F_n) be a sequence of finite-dimensional spaces dense in the Banach–Mazur distance in the class of all finite-dimensional spaces. We set

$$C_0 := \Big(\sum_{n \in \mathbb{N}} \oplus F_n \Big)_{c_0} \tag{4}$$

and we notice that C_0 is hereditarily c_0 (that is, every infinite-dimensional subspace of C_0 contains a copy of c_0). We can now state Lusky's theorem (see [22]).

Theorem 12. *Let X be a separable Banach space with the bounded approximation property. Then $X \oplus C_0$ has a Schauder basis.*

Theorem 12 will be used in the following parameterized form.

Lemma 13. *Let Z be a minimal*[3] *Banach space not containing a copy of c_0. Let $\mathcal{A}$ be an analytic subset of $\mathrm{NC}_Z \cap \mathrm{NC}_{\ell_1}$. Then there exists a (possibly empty) subset $\mathcal{D}$ of $\mathrm{NC}_Z \cap \mathrm{NC}_{\ell_1}$ with the following properties.*

(i) *The set $\mathcal{D}$ is analytic.*
(ii) *Every $Y \in \mathcal{D}$ has a Schauder basis.*

[3] We recall that an infinite-dimensional Banach space Z is said to be minimal if every infinite-dimensional subspace of Z contains an isomorphic copy of Z; e.g., the classical sequence spaces c_0 and ℓ_p ($1 \leqslant p < +\infty$) are minimal spaces.

(iii) *For every $X \in \mathcal{A}$ with the bounded approximation property there exists $Y \in \mathcal{D}$ such that X is isomorphic to a complemented subspace of Y.*

Proof. The result is essentially known, and so, we will be rather sketchy. First we consider the set $\mathcal{B} \subseteq \mathrm{SB}$ defined by

$$X \in \mathcal{B} \Leftrightarrow X \text{ has the bounded approximation property.}$$

Using the characterization of B. A. P. mentioned above, it is easy to check that the set $\mathcal{B}$ is analytic. Next, consider the set $\mathcal{C} \subseteq \mathrm{SB} \times \mathrm{SB}$ defined by

$$(X, Y) \in \mathcal{C} \Leftrightarrow Y \text{ is isomorphic to } X \oplus C_0.$$

It is also easy to see that $\mathcal{C}$ is analytic (see [2] for more details). Define $\mathcal{D} \subseteq \mathrm{SB}$ by the rule

$$Y \in \mathcal{D} \Leftrightarrow \exists X \; [X \in \mathcal{A} \cap \mathcal{B} \text{ and } (X, Y) \in \mathcal{C}]$$

and notice that $\mathcal{D}$ is analytic. By Theorem 12, the set $\mathcal{D}$ is as desired. □

2.8. **Amalgamated spaces.** A recurrent theme in the proof of various universality results found in the literature (a theme that goes back to the classical results of Pełczyński [25]) is the use at a certain point of a "gluing" procedure. A number of different "gluing" procedures have been proposed by several authors. We will need the following result (see [2, Theorem 71]).

Theorem 14. *Let $1 < p < +\infty$ and let $\mathcal{C}$ be an analytic subset of* SB *such that every $Y \in \mathcal{C}$ has a Schauder basis. Then there exists a Banach space V with a Schauder basis that contains a complemented copy of every space in the class $\mathcal{C}$.*

Moreover, if W is an infinite-dimensional subspace of V, then either

(i) *W contains a copy of ℓ_p, or*
(ii) *there exists a finite sequence $Y_0, \dots, Y_n$ in $\mathcal{C}$ such that W is isomorphic to a subspace of $Y_0 \oplus \cdots \oplus Y_n$.*

The space V obtained above is called in [2] as the *p-amalgamation space* of the class $\mathcal{C}$. The reader can find in [2, §8] an extensive study of its properties.

3. Quotients of Banach spaces

3.1. **Definitions.** We start with the following definition.

Definition 15. *Let X be a separable Banach space and let (x_n) be a sequence (with possible repetitions) in the unit sphere of X which is norm dense in S_X. By E_X we shall denote the completion of $c_{00}(\mathbb{N})$ under the norm*

$$\|z\|_{E_X} := \sup\Big\{ \Big\| \sum_{n=0}^{m} z(n) x_n \Big\|_X : m \in \mathbb{N} \Big\}. \tag{5}$$

By (e_n^X) we shall denote the standard Hamel basis of $c_{00}(\mathbb{N})$ regarded as a sequence in E_X. If $X = \{0\}$, then by convention we set $E_X = c_0$.

The construction of the space E_X is somehow "classical" and the motivation for the above definition can be traced in the proof of the fact that every separable Banach space is a quotient of ℓ_1 (see [21, page 108]). A similar construction was presented by Schechtman in [29] for different, though related, purposes.

We isolate two elementary properties of the space E_X. First, we observe that the sequence (e_n^X) defines a normalized monotone Schauder basis of E_X. It is also easy to see that the map $E_X \ni e_n^X \mapsto x_n \in X$ is extended to a norm-one linear operator. This operator will be denoted as follows.

Definition 16. *By $Q_X \colon E_X \to X$ we shall denote the (unique) bounded linear operator satisfying $Q_X(e_n^X) = x_n$ for every $n \in \mathbb{N}$.*

Let us make at this point two comments about the above definitions. Let (y_n) be a basic sequence in a Banach space Y and assume that the map

$$\overline{\mathrm{span}}\{y_n : n \in \mathbb{N}\} \ni y_n \mapsto x_n \in X$$

is extended to a bounded linear operator. Then it is easy to see that there exists a constant $C \geqslant 1$ such that the sequence (e_n^X) is C-dominated[4] by the sequence (y_n). In other words, among all basic sequences (y_n) with the property that the map $\overline{\mathrm{span}}\{y_n : n \in \mathbb{N}\} \ni y_n \mapsto x_n \in X$ is extended to a bounded linear operator, the sequence (e_n^X) is the *minimal* one with respect to domination.

Also notice that the space E_X depends on the choice of the sequence (x_n). For our purposes, however, the dependence is not important as can be seen from the simple following observation. Let (d_n) be another sequence in the unit sphere of X which is norm dense in S_X and let E'_X be the completion of $c_{00}(\mathbb{N})$ under the norm

$$\|z\|_{E'_X} = \sup\Big\{ \big\| \sum_{n=0}^{m} z(n) d_n \big\|_X : m \in \mathbb{N} \Big\}.$$

Then it is easy to check that E_X embeds isomorphically into E'_X and vice versa. Actually, it is possible to modify the construction to obtain a different space sharing most of the properties of the space E_X and not depending on the choice of the dense sequence. We could not find, however, any application of this construction and since it is involved and conceptually less natural to grasp we prefer not to bother the reader with it.

The rest of the section is organized as follows. In §3.2 we present some preliminary tools needed for the proof of Theorem 3. The proof of Theorem 3 is given in §3.3 while in §3.4 we present some its consequences. Finally, in §3.5 we make some comments.

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[4]We recall that if (v_n) and (y_n) are two basic sequences in two Banach spaces V and Y respectively, then (v_n) is said to be C-dominated by (y_n) if for every $k \in \mathbb{N}$ and every $a_0, \dots, a_k \in \mathbb{R}$ we have $\| \sum_{n=0}^{k} a_n v_n \|_V \leqslant C \| \sum_{n=0}^{k} a_n y_n \|_Y$.

3.2. **Preliminary tools.** We start by introducing some pieces of notation that will be used only in this section. Let F and G be two nonempty finite subsets of $\mathbb{N}$. We write $F<G$ if $\max(F)<\min(G)$. Let (e_n) be a basic sequence in a Banach space E and let v be a vector in $\overline{\mathrm{span}}\{e_n : n\in\mathbb{N}\}$. There exists a (unique) sequence (a_n) of reals such that $v=\sum_{n\in\mathbb{N}} a_n e_n$. The *support* of the vector v, denoted by $\mathrm{supp}(v)$, is defined to be the set $\{n\in\mathbb{N} : a_n\neq 0\}$. The *range* of the vector v, denoted by $\mathrm{range}(v)$, is defined to be the minimal interval of $\mathbb{N}$ that contains $\mathrm{supp}(v)$.

In what follows X will be a separable Banach space and (x_n) will be the sequence in X which is used to define the space E_X. The following propositions will be basic tools for the analysis of the space E_X.

Proposition 17. *Let (v_k) be a semi-normalized block sequence of (e^X_n) and assume that $\|Q_X(v_k)\|_X \leqslant 2^{-k}$ for every $k\in\mathbb{N}$. Then the sequence (v_k) is equivalent to the standard unit vector basis of c_0.*

Proof. We select a constant $C>0$ such that $\|v_k\|_{E_X}\leqslant C$ for every $k\in\mathbb{N}$. Let $F=\{k_0<\cdots<k_j\}$ be a finite subset of $\mathbb{N}$. We will show that

$$\Big\|\sum_{i=0}^{j} v_{k_i}\Big\|_{E_X} \leqslant 2+C.$$

This will finish the proof. To this end we argue as follows. First we set

(a) $G_i \coloneqq \mathrm{supp}(v_{k_i})$ and $m_i\coloneqq \min(G_i)$ for every $i\in\{0,\dots,j\}$.

Let (a_n) be the unique sequence of reals such that

(b) $a_n=0$ if $n\notin G_0\cup\cdots\cup G_j$, and
(c) $v_{k_i}=\sum_{n\in G_i} a_n e^X_n$ for every $i\in\{0,\dots,j\}$.

Notice that for every $l\in\{0,\dots,j\}$ and every $m\in\mathbb{N}$ with $m\in\mathrm{range}(v_{k_l})$ we have

$$\Big\|\sum_{n=m_l}^{m} a_n x_n\Big\|_X \leqslant \|v_{k_l}\|_{E_X}\leqslant C. \tag{6}$$

We select $p\in\mathbb{N}$ such that

$$\Big\|\sum_{i=0}^{j} v_{k_i}\Big\|_{E_X} = \Big\|\sum_{n=0}^{p} a_n x_n\Big\|_X$$

and we distinguish the following cases.

Case 1: $p\in\mathrm{range}(v_{k_0})$. Using (6), we see that

$$\Big\|\sum_{i=0}^{j} v_{k_i}\Big\|_{E_X} = \Big\|\sum_{n=m_0}^{p} a_n x_n\Big\|_X \leqslant C.$$

Case 2: *there exists* $l \in \{1,\dots,j\}$ *such that* $p \in \mathrm{range}(v_{k_l})$. Using our hypotheses on the sequence (v_k) and inequality (6), we obtain that

$$\begin{aligned}\Big\|\sum_{i=0}^{j} v_{k_i}\Big\|_{E_X} &= \Big\|\sum_{i=0}^{l-1}\sum_{n\in G_i} a_n x_n + \sum_{n=m_l}^{p} a_n x_n\Big\|_X\\ &\leqslant \sum_{i=0}^{l-1}\Big\|\sum_{n\in G_i} a_n x_n\Big\|_X + \Big\|\sum_{n=m_l}^{p} a_n x_n\Big\|_X\\ &= \sum_{i=0}^{l-1}\|Q_X(v_{k_i})\|_X + \Big\|\sum_{n=m_l}^{p} a_n x_n\Big\|_X \leqslant 2+C.\end{aligned}$$

Case 3: *for every* $i \in \{0,\dots,j\}$ *we have that* $p \notin \mathrm{range}(v_{k_i})$. In this case we notice that there exists $l \in \{0,\dots,j\}$ such that $\mathrm{range}(v_{k_i}) < \{p\}$ for every $i \in \{0,\dots,l\}$ while $\{p\} < \mathrm{range}(v_{k_i})$ otherwise. Using this observation we see that

$$\Big\|\sum_{i=0}^{j} v_{k_i}\Big\|_{E_X} = \Big\|\sum_{i=0}^{l}\sum_{n\in G_i} a_n x_n\Big\|_X \leqslant \sum_{i=0}^{l}\|Q_X(v_{k_i})\|_X \leqslant 2.$$

The above cases are exhaustive, and so, the proof is completed. □

Proposition 18. *Let* (v_k) *be a bounded block sequence of* (e_n^X). *If* $(Q_X(v_k))$ *is weakly null, then* (v_k) *is also weakly null.*

For the proof of Proposition 18 we will need the following "unconditional" version of Mazur's theorem.

Lemma 19. *Let* (v_k) *be a weakly null sequence in a Banach space* V. *Then for every* $\varepsilon > 0$ *there exist* $k_0 < \dots < k_j$ *in* $\mathbb{N}$ *and* $\lambda_0,\dots,\lambda_j \in \mathbb{R}^+$ *with* $\sum_{i=0}^{j}\lambda_i = 1$ *and such that*

$$\max\{\lambda_i : 0 \leqslant i \leqslant j\} \leqslant \varepsilon$$

and

$$\max\Big\{\Big\|\sum_{i\in F}\lambda_i v_{k_i}\Big\| : F \subseteq \{0,\dots,j\}\Big\} \leqslant \varepsilon.$$

Proof. Clearly we may assume that $V = \overline{\mathrm{span}}\{v_k : k \in \mathbb{N}\}$, and so, we may also assume that V is a subspace of $C(2^{\mathbb{N}})$. Therefore, each v_k is a continuous function on $2^{\mathbb{N}}$ and the norm of V is the usual $\|\cdot\|_\infty$ norm. By Lebesgue's dominated convergence theorem, a sequence (f_k) in $C(2^{\mathbb{N}})$ is weakly null if and only if (f_k) is bounded and pointwise convergent to 0. Hence, setting $y_k := |v_k|$ for every $k \in \mathbb{N}$, we see that the sequence (y_k) is weakly null. Therefore, using Mazur's theorem, it is possible to find $k_0 < \dots < k_j$ in $\mathbb{N}$ and $\lambda_0,\dots,\lambda_j \in \mathbb{R}^+$ with $\sum_{i=0}^{j}\lambda_i = 1$ and such that

$$\max\{\lambda_i : 0 \leqslant i \leqslant j\} \leqslant \varepsilon$$

and $\| \sum_{i=0}^{j} \lambda_i y_{k_i} \|_\infty \leqslant \varepsilon$. Noticing that

$$\max\Big\{ \big\| \sum_{i\in F} \lambda_i v_{k_i} \big\|_\infty : F \subseteq \{0,\dots,j\}\Big\} \leqslant \big\| \sum_{i=0}^{j} \lambda_i y_{k_i} \big\|_\infty \leqslant \varepsilon$$

the proof is completed. □

We proceed to the proof of Proposition 18.

Proof of Proposition 18. We will argue by contradiction. So, assume that the sequence $(Q_X(v_k))$ is weakly null while the sequence (v_k) is not. We select $C \geqslant 1$ such that $\|v_k\|_{E_X} \leqslant C$ for every $k \in \mathbb{N}$. By passing to a subsequence of (v_k) if necessary, we find $e^* \in E_X^*$ and $\delta > 0$ such that $e^*(v_k) \geqslant \delta$ for every $k \in \mathbb{N}$. This property implies that

(a) for every vector $z \in \mathrm{conv}\{v_k : k \in \mathbb{N}\}$ we have $\|z\|_{E_X} \geqslant \delta$.

We apply Lemma 19 to the weakly null sequence $(Q_X(v_k))$ and $\varepsilon = \delta \cdot (4C)^{-1}$ and we find $k_0 < \dots < k_j$ in $\mathbb{N}$ and $\lambda_0, \dots, \lambda_j \in \mathbb{R}^+$ with $\sum_{i=0}^{j} \lambda_i = 1$ and such that

$$\max\{\lambda_i : 0 \leqslant i \leqslant j\} \leqslant \frac{\delta}{4C} \tag{7}$$

and

$$\max\Big\{ \big\| \sum_{i\in F} \lambda_i Q_X(v_{k_i}) \big\|_X : F \subseteq \{0,\dots,j\}\Big\} \leqslant \frac{\delta}{4C}. \tag{8}$$

Since $\|v_k\|_{E_X} \leqslant C$ for every $k \in \mathbb{N}$, inequality (7) implies that

(b) $\|\lambda_i v_{k_i}\|_{E_X} \leqslant \delta/4$ for every $i \in \{0,\dots,j\}$.

We define

$$w = \sum_{i=0}^{j} \lambda_i v_{k_i} \in \mathrm{conv}\{v_k : k \in \mathbb{N}\}.$$

We will show that $\|w\|_{E_X} \leqslant \delta/2$. This estimate contradicts property (a) above.

To this end we will argue as in the proof of Proposition 17. First set

(c) $G_i \coloneqq \mathrm{supp}(v_{k_i})$ and $m_i \coloneqq \min(G_i)$ for every $i \in \{0,\dots,j\}$

and let (a_n) be the unique sequence of reals such that

(d) $a_n = 0$ if $n \notin G_0 \cup \dots \cup G_j$, and
(e) $\lambda_i v_{k_i} = \sum_{n\in G_i} a_n e_n^X$ for every $i \in \{0,\dots,j\}$.

Using (b), we see that if $l \in \{0,\dots,j\}$ and $m \in \mathbb{N}$ with $m \in \mathrm{range}(v_{k_l})$, then

$$\big\| \sum_{n=m_l}^{m} a_n x_n \big\|_X \leqslant \|\lambda_l v_{k_l}\|_{E_X} \leqslant \frac{\delta}{4}. \tag{9}$$

We select $p \in \mathbb{N}$ such that

$$\|w\|_{E_X} = \big\| \sum_{n=0}^{p} a_n x_n \big\|_X$$

and, as in the proof of Proposition 17, we consider the following three cases.

CASE 1: $p \in \mathrm{range}(v_{k_0})$. Using (9), we see that

$$\|w\|_{E_X} = \Big\| \sum_{n=m_0}^{p} a_n x_n \Big\|_X \leqslant \frac{\delta}{4}.$$

CASE 2: *there exists* $l \in \{1, \ldots, j\}$ *such that* $p \in \mathrm{range}(v_{k_l})$. In this case the desired estimate will be obtained combining inequalities (8) and (9). Specifically, let $F = \{0, \ldots, l-1\}$ and notice that

$$\begin{aligned}
\|w\|_{E_X} &= \Big\| \sum_{i=0}^{l-1} \sum_{n \in G_i} a_n x_n + \sum_{n=m_l}^{p} a_n x_n \Big\|_X \\
&\leqslant \Big\| \sum_{i \in F} \sum_{n \in G_i} a_n x_n \Big\|_X + \Big\| \sum_{n=m_l}^{p} a_n x_n \Big\|_X \\
&= \Big\| \sum_{i \in F} \lambda_i Q_X(v_{k_i}) \Big\|_X + \Big\| \sum_{n=m_l}^{p} a_n x_n \Big\|_X \\
&\overset{(8)}{\leqslant} \frac{\delta}{4C} + \Big\| \sum_{n=m_l}^{p} a_n x_n \Big\|_X \overset{(9)}{\leqslant} \frac{\delta}{4C} + \frac{\delta}{4} \leqslant \frac{\delta}{2}.
\end{aligned}$$

CASE 3: *for every* $i \in \{0, \ldots, j\}$ *we have that* $p \notin \mathrm{range}(v_{k_i})$. In this case we will use only inequality (8). Indeed, there exists $l \in \{0, \ldots, j\}$ such that $\mathrm{range}(v_{k_i}) < \{p\}$ if $i \in \{0, \ldots, l\}$ while $\{p\} < \mathrm{range}(v_{k_i})$ otherwise. Setting $H := \{0, \ldots, l\}$, we see that

$$\|w\|_{E_X} = \Big\| \sum_{i=0}^{l} \sum_{n \in G_i} a_n x_n \Big\|_X = \Big\| \sum_{i \in H} \lambda_i Q_X(v_{k_i}) \Big\|_X \overset{(8)}{\leqslant} \frac{\delta}{4C} \leqslant \frac{\delta}{4}.$$

The above cases are exhaustive, and so, $\|w\|_{E_X} \leqslant \delta/2$. As we have already pointed out, this estimate yields a contradiction. The proof is completed. □

3.3. **Proof of Theorem 3.** Let X be a separable Banach space. In what follows (x_n) will be the sequence in X which is used to define the space E_X.

(i) It is straightforward.

(ii) We have already noticed that $\|Q_X\| = 1$. To see that Q_X is onto, observe that the image of the closed unit ball of E_X under the operator Q_X contains the set $\{x_n : n \in \mathbb{N}\}$ and therefore it is dense in the closed unit ball of X.

(iii) Let Y be an infinite-dimensional subspace of E_X and assume that the operator $Q_X : Y \to X$ is strictly singular. Using a standard sliding hump argument we find a block subspace V of E_X and a subspace Y' of Y with V isomorphic to Y' and such that the operator $Q_X : V \to X$ is strictly singular. Hence, we may select a normalized block sequence (v_k) of (e_n^X) with $v_k \in V$ and $\|Q_X(v_k)\|_X \leqslant 2^{-k}$ for every $k \in \mathbb{N}$. By Proposition 17, the sequence (v_k) is equivalent to the standard unit vector basis of c_0 and the result follows.

(iv) This part was essentially observed in [29]. We reproduce the argument for completeness. So, let (w_k) be a normalized basic sequence in X. The sequence (x_n) is dense in the unit sphere of X. Therefore it is possible to select an infinite subset $N=\{n_0<n_1<\cdots\}$ of $\mathbb{N}$ such that the subsequence (x_{n_k}) of (x_n) determined by N is basic and equivalent to (w_k) (see [21]). Let $K\geqslant 1$ be the basis constant of (x_{n_k}). Also let $(e^X_{n_k})$ be the subsequence of (e^X_n) determined by N. Let $j\in\mathbb{N}$ and let $a_0,\dots,a_j\in\mathbb{R}$ be arbitrary, and notice that

$$\big\|\sum_{k=0}^{j}a_kx_{n_k}\big\|_X\leqslant\big\|\sum_{k=0}^{j}a_ke^X_{n_k}\big\|_{E_X}=\max_{0\leqslant i\leqslant j}\big\|\sum_{k=0}^{i}a_kx_{n_k}\big\|_X\leqslant K\big\|\sum_{k=0}^{j}a_kx_{n_k}\big\|_X.$$

Therefore, the sequence (x_{n_k}) is K-equivalent to the sequence $(e^X_{n_k})$ and the result follows.

(v) First we consider the relation $\mathcal{S}\subseteq C(2^{\mathbb{N}})^{\mathbb{N}}\times\mathrm{SB}$ defined by

$$\big((y_n),Y\big)\in\mathcal{S}\Leftrightarrow(\forall n\ y_n\in Y)\text{ and }\overline{\mathrm{span}}\{y_n:n\in\mathbb{N}\}=Y.$$

The relation $\mathcal{S}$ is analytic (see [7, Lemma 2.6]). Next, we apply Proposition 8 and we obtain a sequence $S_n\colon\mathrm{SB}\to C(2^{\mathbb{N}})$ $(n\in\mathbb{N})$ of Borel maps such that for every $X\in\mathrm{SB}$ with $X\neq\{0\}$ the sequence $(S_n(X))$ is norm dense in the unit sphere of X. Now notice that

$$\begin{aligned}(X,Y)\in\mathcal{E}\quad\Leftrightarrow\quad&\exists(y_n)\in C(2^{\mathbb{N}})^{\mathbb{N}}\text{ with }\big((y_n),Y\big)\in\mathcal{S}\text{ and either}\\&\Big(X=\{0\}\text{ and }\forall k\in\mathbb{N}\ \forall a_0,\dots,a_k\in\mathbb{Q}\text{ we have}\\&\quad\big\|\sum_{n=0}^{k}a_ny_n\big\|_\infty=\max_{0\leqslant n\leqslant k}|a_n|\Big)\text{ or}\\&\Big(X\neq\{0\}\text{ and }\forall k\in\mathbb{N}\ \forall a_0,\dots,a_k\in\mathbb{Q}\text{ we have}\\&\quad\big\|\sum_{n=0}^{k}a_ny_n\big\|_\infty=\max_{0\leqslant m\leqslant k}\big\|\sum_{n=0}^{m}a_nS_n(X)\big\|_\infty\Big).\end{aligned}$$

The above formula implies that the set $\mathcal{E}$ is analytic.

(vi) By part (ii), the space X is a quotient of E_X. Therefore, if E^*_X is separable, then X^* is also separable. For the converse implication we argue by contradiction. So, assume that there exists a Banach space X with separable dual such that E^*_X is non-separable. Our strategy is to show that there exists a sequence $(w_t)_{t\in 2^{<\mathbb{N}}}$ in E_X which is topologically equivalent to the basis of James tree (see Definition 5) and is such that its image under the operator Q_X has the same property; that is, the sequence $(Q_X(w_t))_{t\in 2^{<\mathbb{N}}}$ will also be topologically equivalent to the basis of James tree. As we have already indicated in §2.3, this implies that X^* is non-separable and yields a contradiction.

To this end we argue as follows. First we notice that the space X does not contain a copy of ℓ_1. Therefore, by part (iii), the space E_X does not contain a copy of ℓ_1 either. Hence, we may apply Theorem 7 to the space E_X and we obtain a sequence $(e_t)_{t\in 2^{<\mathbb{N}}}$ in E_X which is topologically equivalent to the basis of James tree. We need to replace the sequence $(e_t)_{t\in 2^{<\mathbb{N}}}$ with another sequence having an additional property. Specifically, let us say that a sequence $(v_t)_{t\in 2^{<\mathbb{N}}}$ in E_X is a *tree-block* if for every $\sigma \in 2^{\mathbb{N}}$ the sequence $(v_{\sigma|n})$ is a block sequence of (e_n^X). Notice that the notion of a tree-block is hereditary with respect to dyadic subtrees; that is, if $(v_t)_{t\in 2^{<\mathbb{N}}}$ is a tree-block and $D = \{d_t : t \in 2^{<\mathbb{N}}\}$ is a dyadic subtree of $2^{<\mathbb{N}}$, then the sequence $(v_{d_t})_{t\in 2^{<\mathbb{N}}}$ is also a tree-block.

Claim 20. *There exists a sequence $(v_t)_{t\in 2^{<\mathbb{N}}}$ in E_X which is topologically equivalent to the basis of James tree and a tree-block.*

Proof of Claim 20. We select $C \geqslant 1$ such that $C^{-1} \leqslant \|e_t\|_{E_X} \leqslant C$ for all $t \in 2^{<\mathbb{N}}$. Let $s \in 2^{<\mathbb{N}}$ be arbitrary. There exists an infinite antichain A of $2^{<\mathbb{N}}$ such that $s \sqsubset t$ for every $t \in A$. The sequence $(e_t)_{t\in 2^{<\mathbb{N}}}$ is topologically equivalent to the basis of James tree, and so, the sequence $(e_t)_{t\in A}$ is weakly null. Using this observation, we may recursively construct a dyadic subtree $R = \{r_t : t \in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$ and a tree-block sequence $(v_t)_{t\in 2^{<\mathbb{N}}}$ in E_X such that $\|e_{r_t} - v_t\|_{E_X} \leqslant (2C)^{-|t|+1}$ for every $t \in 2^{<\mathbb{N}}$. Clearly the sequence $(v_t)_{t\in 2^{<\mathbb{N}}}$ is as desired. □

Claim 21. *There exist a dyadic subtree S_0 of $2^{<\mathbb{N}}$ and a constant $\Theta \geqslant 1$ such that $\Theta^{-1} \leqslant \|Q_X(v_t)\|_X \leqslant \Theta$ for every $t \in S_0$.*

Proof of Claim 21. Let $K \geqslant 1$ be such that $\|v_t\|_{E_X} \leqslant K$ for every $t \in 2^{<\mathbb{N}}$. We will show that there exist $s_0 \in 2^{<\mathbb{N}}$ and $\theta > 0$ such that for every $t \in 2^{<\mathbb{N}}$ with $s_0 \sqsubseteq t$ we have $\|Q_X(v_t)\|_X \geqslant \theta$. In such a case, set $S_0 := \{s_0^\frown t : t \in 2^{<\mathbb{N}}\}$ and $\Theta := \max\{\theta^{-1}, K\}$ and notice that S_0 and Θ satisfy the requirements of the claim.

To find the node s_0 and the constant θ we will argue by contradiction. So, assume that for every $s \in 2^{<\mathbb{N}}$ and every $\theta > 0$ there exists $t \in 2^{<\mathbb{N}}$ with $s \sqsubseteq t$ and such that $\|Q_X(v_t)\|_X \leqslant \theta$. Hence, we may select a sequence (t_k) in $2^{<\mathbb{N}}$ such that for every $k \in \mathbb{N}$ we have

(a) $t_k \sqsubset t_{k+1}$, and
(b) $\|Q_X(v_{t_k})\|_X \leqslant 2^{-k}$.

By (a) above, the set $\{t_k : k \in \mathbb{N}\}$ is a chain while, by Claim 20, the sequence $(v_t)_{t\in 2^{<\mathbb{N}}}$ is semi-normalized and a tree block. Therefore, the sequence (v_{t_k}) is a semi-normalized block sequence of (e_n^X). By Proposition 17 and (b) above, we see that the sequence (v_{t_k}) is equivalent to the standard unit vector basis of c_0, and so, it is weakly null. By Claim 20, however, the sequence $(v_t)_{t\in 2^{<\mathbb{N}}}$ is topologically equivalent to the basis of James tree. Since the set $\{t_k : k \in \mathbb{N}\}$ is a chain, the sequence (v_{t_k}) must be non-trivial weak* Cauchy. This yields a contradiction. □

Claim 22. *There exists a dyadic subtree S_1 of $2^{<\mathbb{N}}$ with $S_1 \subseteq S_0$ and such that for every infinite chain $\{t_0 \sqsubset t_1 \sqsubset \cdots\}$ of S_1 the sequence $(Q_X(v_{t_n}))$ is basic.*

Proof of Claim 22. By Claims 20 and 21, we see that for every $s \in S_0$ there exists an infinite antichain A of S_0 with $s \sqsubset t$ for every $t \in A$ and such that the sequence $(Q_X(v_t))_{t\in A}$ is semi-normalized and weakly null. Using this observation and the classical procedure of Mazur for selecting basic sequences (see [21]), the claim follows. $\square$

Claim 23. *There exists a dyadic subtree S_2 of $2^{<\mathbb{N}}$ with $S_2 \subseteq S_1$ and such that for every infinite chain $\{t_0 \sqsubset t_1 \sqsubset \cdots\}$ of S_2 the sequence $(Q_X(v_{t_n}))$ is weak* Cauchy.*

Proof of Claim 23. Set

$$\mathcal{X} := \big\{c \in [S_1]_{\text{chains}} : \text{the sequence } (Q_X(v_t))_{t\in c} \text{ is weak* Cauchy}\big\}.$$

The set $\mathcal{X}$ is co-analytic (see [30] for more details). Therefore, by Theorem 4, there exists a dyadic subtree S_2 of $2^{<\mathbb{N}}$ with $S_2 \subseteq S_1$ and such that $[S_2]_{\text{chains}}$ is monochromatic. It is enough to show that $[S_2]_{\text{chains}} \cap \mathcal{X} \neq \emptyset$. Recall that the space X does not contain a copy of ℓ_1. Therefore, by Rosenthal's dichotomy [28], we may find an infinite chain c of S_2 such that the sequence $(Q_X(v_t))_{t\in c}$ is weak* Cauchy and the result follows. $\square$

Let S_2 be the dyadic subtree of $2^{<\mathbb{N}}$ obtained by Claim 23 and let $\{s_t : t \in 2^{<\mathbb{N}}\}$ be the canonical representation of S_2. We are in the position to define the sequence $(w_t)_{t\in 2^{<\mathbb{N}}}$ we mentioned in the beginning of the proof. Specifically, set

$$w_t := v_{s_t}$$

for every $t \in 2^{<\mathbb{N}}$. By Claim 20 and Fact 6, the sequence $(w_t)_{t\in 2^{<\mathbb{N}}}$ is topologically equivalent to the basis of James tree and a tree block. The final claim is the following.

Claim 24. *The sequence $(Q_X(w_t))_{t\in 2^{<\mathbb{N}}}$ is topologically equivalent to the basis of James tree.*

Proof of Claim 24. By Claim 21, the sequence $(Q_X(w_t))_{t\in 2^{<\mathbb{N}}}$ is semi-normalized. Notice also that for every infinite antichain A of $2^{<\mathbb{N}}$ the sequence $(Q_X(w_t))_{t\in A}$ is weakly null.

Let $\sigma \in 2^{\mathbb{N}}$. By Claim 23, the sequence $(Q_X(w_{\sigma|n}))$ is weak* convergent to an element $x^{**}_\sigma \in X^{**}$. First we notice that $x^{**}_\sigma \neq 0$. Indeed, the sequence $(w_{\sigma|n})$ is a semi-normalized block sequence of (e^X_n) which is weak* convergent to an element $w^{**}_\sigma \in E^{**}_X \setminus E_X$. If $x^{**}_\sigma = 0$, then by Proposition 18 we would have that $(w_{\sigma|n})$ is weakly null. Hence $x^{**}_\sigma \neq 0$. Next we observe that $x^{**}_\sigma \in X^{**} \setminus X$. Indeed, by Claim 22, the sequence $(Q_X(w_{\sigma|n}))$ is basic. Therefore, if the sequence $(Q_X(w_{\sigma|n}))$ was weakly convergent to an element $x \in X$, then necessarily we would have that $x = 0$. This possibility, however, is ruled out by the previous reasoning, and so, $x^{**}_\sigma \in X^{**} \setminus X$.

Finally suppose, towards a contradiction, that there exist $\sigma,\tau\in 2^{\mathbb{N}}$ with $\sigma\neq\tau$ and such that $x^{**}_\sigma = x^{**}_\tau$. In such a case it is possible to select two sequences (s_n) and (t_n) in $2^{<\mathbb{N}}$ such that the following are satisfied.

(a) $s_n \sqsubset s_{n+1} \sqsubset \sigma$ for every $n\in\mathbb{N}$.
(b) $t_n \sqsubset t_{n+1} \sqsubset \tau$ for every $n\in\mathbb{N}$.
(c) Setting $z_n \coloneqq w_{s_n} - w_{t_n}$ for every $n\in\mathbb{N}$, we have that the sequence (z_n) is a semi-normalized block sequence of (e^X_n).

Our assumption that $x^{**}_\sigma = x^{**}_\tau$ reduces to the fact that the sequence $(Q_X(z_n))$ is weakly null. By (c) above, we may apply Proposition 18 to infer that the sequence (z_n) is also weakly null. Therefore, the sequences $(w_{\sigma|n})$ and $(w_{\tau|n})$ are weak* convergent to the same element of E^{**}_X. This contradicts the fact that the sequence $(w_t)_{t\in 2^{<\mathbb{N}}}$ is topologically equivalent to the basis of James tree. The proof is completed. □

As we have already indicated, Claim 24 yields a contradiction. This completes the proof of part (vi) of Theorem 3, and so, the entire proof is completed.

3.4. **Consequences.** We isolate below three corollaries of Theorem 3. The second one will be of particular importance in the next section.

Corollary 25. *Let Z be a minimal Banach space not containing a copy of c_0. If X is a separable Banach space not containing a copy of Z, then E_X does not contain a copy of Z either.*

Proof. Follows immediately by part (iii) of Theorem 3. □

Corollary 26. *Let Z be a minimal Banach space not containing a copy of c_0 and let $\mathcal{A}$ be an analytic subset of $\mathrm{NC}_Z\cap\mathrm{NC}_{\ell_1}$. Then there exists a subset $\mathcal{B}$ of $\mathrm{NC}_Z\cap\mathrm{NC}_{\ell_1}$ with the following properties.*

(i) *The set $\mathcal{B}$ is analytic.*
(ii) *Every $Y\in\mathcal{B}$ has a Schauder basis.*
(iii) *For every $X\in\mathcal{A}$ there exists $Y\in\mathcal{B}$ such that X is a quotient of Y.*

Proof. Let $\mathcal{E}$ be the set defined in part (v) of Theorem 3. We define $\mathcal{B}\subseteq\mathrm{SB}$ by the rule

$$Y\in\mathcal{B}\Leftrightarrow \exists X\ [X\in\mathcal{A} \text{ and } (X,Y)\in\mathcal{E}].$$

The set $\mathcal{B}$ is clearly analytic. Invoking parts (i) and (ii) of Theorem 3 and Corollary 25, we see that $\mathcal{B}$ is as desired. □

Corollary 27. *There exists a map $f\colon\omega_1\to\omega_1$ such that for every countable ordinal ξ and every separable Banach space X with $\phi_{\mathrm{NC}_{\ell_1}}(X)\leqslant\xi$ the space X is a quotient of a Banach space Y with a Schauder basis satisfying $\phi_{\mathrm{NC}_{\ell_1}}(Y)\leqslant f(\xi)$.*

Proof. We define the map $f\colon \omega_1 \to \omega_1$ as follows. Fix a countable ordinal ξ and consider the set

$$\mathcal{A}_\xi := \{X \in \mathrm{SB} : \phi_{\mathrm{NC}_{\ell_1}}(X) \leqslant \xi\}.$$

By Theorem 9, the map $\phi_{\mathrm{NC}_{\ell_1}} : \mathrm{NC}_{\ell_1} \to \omega_1$ is a co-analytic rank on NC_{ℓ_1}. Hence, the set $\mathcal{A}_\xi$ is analytic (in fact Borel—see [20]). We apply Corollary 26 to the space $Z = \ell_1$ and the analytic set $\mathcal{A}_\xi$ and we obtain an analytic subset $\mathcal{B}$ of NC_{ℓ_1} such that for every $X \in \mathcal{A}_\xi$ there exists $Y \in \mathcal{B}$ with a Schauder basis and having X as quotient. By boundedness, there exists a countable ordinal ζ such that

$$\sup\{\phi_{\mathrm{NC}_{\ell_1}}(Y) : Y \in \mathcal{B}\} = \zeta.$$

We define $f(\xi) = \zeta$. Clearly the map f is as desired. □

3.5. **Comments.** By a well-known result due to Davis, Fiegel, Johnson and Pełczyński [9], if X is a Banach space with separable dual, then X is a quotient of a Banach space V_X with a *shrinking* Schauder basis. By Theorem 3, the space E_X has a Schauder basis, separable dual and admits X as quotient. We point out, however, that the natural Schauder basis (e_n^X) of E_X is *not* shrinking. On the other hand, the subspace structure of E_X is very well understood. The space V_X mentioned above is defined using the interpolation techniques developed in [9] and it is not clear which are the isomorphic types of its subspaces.

We would also like to make some comments about the proof of the separability of the dual of E_X. As we have already indicated, our strategy was to construct a sequence $(w_t)_{t\in 2^{<\mathbb{N}}}$ in E_X which is topologically equivalent to the basis of James tree and is such that its image under the operator Q_X has the same property; in other words, the operator Q_X fixes a copy of this basic object. This kind of reasoning can be applied to a more general framework. Specifically, let Y and Z be separable Banach spaces and let $T\colon Y \to Z$ be a bounded linear operator. There are a number of problems in Functional Analysis which boil down to understand when the dual operator T^* of T has non-separable range. Using the combinatorial tools developed in [3] and an analysis similar to the one in the present paper, it can be shown that if Y does not contain a copy of ℓ_1, then the operator T^* has non-separable range if and only if T fixes a copy of a sequence which is topologically equivalent to the basis of James tree.

4. Proof of the main result

In this section we will give the proof of Theorem 2 stated in the introduction. The proof will be based on the following, more detailed, result.

Theorem 28. *Let Z be a minimal Banach space not containing a copy of c_0 and let $\mathcal{A}$ be an analytic subset of $\mathrm{NC}_Z \cap \mathrm{NC}_{\ell_1}$. Then there exists a Banach space $V \in \mathrm{NC}_Z \cap \mathrm{NC}_{\ell_1}$ with a Schauder basis which is surjectively universal for the* *class $\mathcal{A}$. Moreover, if $X \in \mathcal{A}$ has the bounded approximation property, then X is isomorphic to a complemented subspace of V.*

Let us point out that the assumption on the complexity of the set $\mathcal{A}$ in Theorem 28 is optimal. Notice also that if E is any Banach space with a Schauder basis, then the set of all $X \in \mathcal{A}$ which are isomorphic to a complemented subspace of E is contained in the set of all $X \in \mathcal{A}$ having the bounded approximation property. Therefore, the "moreover" part of the above result is optimal too.

Proof of Theorem 28. Since Z is minimal, there exists $1 < p < +\infty$ such that Z does not contain a copy of ℓ_p. We fix such a p. We apply Lemma 13 to the space Z and the analytic set $\mathcal{A}$ and we obtain a subset $\mathcal{D}$ of $\mathrm{NC}_Z \cap \mathrm{NC}_{\ell_1}$ such that the following are satisfied.

(a) The set $\mathcal{D}$ is analytic.
(b) Every $Y \in \mathcal{D}$ has a Schauder basis.
(c) For every $X \in \mathcal{A}$ with the bounded approximation property there exists $Y \in \mathcal{D}$ such that X is isomorphic to a complemented subspace of Y.

Next, we apply Corollary 26 to the space Z and the analytic set $\mathcal{A}$ and we obtain a subset $\mathcal{B}$ of $\mathrm{NC}_Z \cap \mathrm{NC}_{\ell_1}$ with the following properties.

(d) The set $\mathcal{B}$ is analytic.
(e) Every $Y \in \mathcal{B}$ has a Schauder basis.
(f) For every $X \in \mathcal{A}$ there exists $Y \in \mathcal{B}$ such that X is a quotient of Y.

We set $\mathcal{C} := \mathcal{B} \cup \mathcal{D}$ and we notice that $\mathcal{C} \subseteq \mathrm{NC}_Z \cap \mathrm{NC}_{\ell_1}$. By (a) and (d), the set $\mathcal{C}$ is analytic while, by (b) and (e), every $Y \in \mathcal{C}$ has a Schauder basis. The desired space V is the p-amalgamation space of the class $\mathcal{C}$ obtained by Theorem 14. It remain to check that V has the desired properties. Notice, first, that V has a Schauder basis.

Claim 29. *The space V is surjectively universal for the class $\mathcal{A}$.*

Proof of Claim 29. Let $X \in \mathcal{A}$ arbitrary. By (f), there exists a space $Y \in \mathcal{B}$ such that X is a quotient of Y. We fix a quotient map $Q\colon Y \to X$. Next we observe that the space V contains a complemented copy of Y. Therefore, it is possible to find a subspace E of V, a projection $P\colon V \to E$ and an isomorphism $T\colon E \to Y$. Let $Q'\colon V \to X$ be the operator defined by $Q' = Q \circ T \circ P$ and notice that Q' is onto. Hence, X is a quotient of V and the result follows. □

Claim 30. *We have $V \in \mathrm{NC}_Z \cap \mathrm{NC}_{\ell_1}$.*

Proof of Claim 30. We will show that V does not contain a copy of Z (the proof of the fact that V does not contain a copy of ℓ_1 is identical). We will argue by contradiction. So, assume that there exists a subspace W of V which is isomorphic to Z. By the choice of p, we see that W does not contain a copy of ℓ_p. Therefore, by Theorem 14, there exists a finite sequence $Y_0, \dots, Y_n$ in $\mathcal{C}$ such that W is isomorphic to a subspace of $Y_0 \oplus \cdots \oplus Y_n$. There exist an infinite-dimensional subspace W' of W and $i_0 \in \{0, \ldots, n\}$ such that W' is isomorphic to a subspace of Y_{i_0}. Since Z is minimal, we obtain that Y_{i_0} must contain a copy of Z. This contradicts the fact that $\mathcal{C} \subseteq \mathrm{NC}_Z$, and so, the claim is proved. □

Finally, we notice that if $X \in \mathcal{A}$ has the bounded approximation property, then, by (c) above and Theorem 14, the space X is isomorphic to a complemented subspace of V. This shows that the space V has the desired properties. The proof of Theorem 28 is completed. □

We proceed to the proof of Theorem 2.

Proof of Theorem 2. Let $\mathcal{C} \subseteq \mathrm{SB}$.

(i)⇒(ii) Assume that there exists a separable Banach space Y not containing a copy of ℓ_1 which is surjectively universal for the class $\mathcal{C}$. The space Y does not contain a copy of ℓ_1, and so, $\phi_{\mathrm{NC}_{\ell_1}}(Y) < \omega_1$. Moreover, every space in the class $\mathcal{C}$ is a quotient of Y. Therefore, by Lemma 10, we obtain that

$$\sup\{\phi_{\mathrm{NC}_{\ell_1}}(X) : X \in \mathcal{C}\} \leqslant \phi_{\mathrm{NC}_{\ell_1}}(Y) < \omega_1.$$

(ii)⇒(iii) Let ξ be a countable ordinal such that $\sup\{\phi_{\mathrm{NC}_{\ell_1}}(X) : X \in \mathcal{C}\} = \xi$. By Theorem 9, the map $\phi_{\mathrm{NC}_{\ell_1}} : \mathrm{NC}_{\ell_1} \to \omega_1$ is a co-analytic rank on the set NC_{ℓ_1}. It follows that the set

$$\mathcal{A} := \{V \in \mathrm{SB} : \phi_{\mathrm{NC}_{\ell_1}}(V) \leqslant \xi\}$$

is a Borel subset of NC_{ℓ_1} (see [20]) and clearly $\mathcal{C} \subseteq \mathcal{A}$.

(iii)⇒(i) Assume that there exists an analytic subset $\mathcal{A}$ of NC_{ℓ_1} with $\mathcal{C} \subseteq \mathcal{A}$. We apply Theorem 28 for $Z = \ell_1$ and the class $\mathcal{A}$ and we obtain a Banach space V with a Schauder basis which does not contain a copy of ℓ_1 and is surjectively universal for the class $\mathcal{A}$. A fortiori, the space V is surjectively universal for the class $\mathcal{C}$ and the result follows. □

5. A related result and open problems

Let us recall the following notion (see [2, Definition 90]).

Definition 31. *A class $\mathcal{C} \subseteq \mathrm{SB}$ is said to be* strongly bounded *if for every analytic subset $\mathcal{A}$ of $\mathcal{C}$ there exists $Y \in \mathcal{C}$ which is universal for the class $\mathcal{A}$.*

This is a quite strong structural property. It turned out, however, that many natural classes of separable Banach spaces are strongly bounded.

Part of the research in this paper grew out from our attempt to find natural instances of the "dual" phenomenon. The "dual" phenomenon is described in abstract form in the following definition.

Definition 32. *A class $\mathcal{C} \subseteq \mathrm{SB}$ is said to be* surjectively strongly bounded *if for every analytic subset $\mathcal{A}$ of $\mathcal{C}$ there exists $Y \in \mathcal{C}$ which is surjectively universal for the class $\mathcal{A}$.*

So, according to Definition 32, Theorem 28 has the following consequence.

Corollary 33. *Let Z be a minimal Banach space not containing a copy of c_0. Then the class $\mathrm{NC}_{\ell_1} \cap \mathrm{NC}_Z$ is surjectively strongly bounded.*

The following proposition provides two more natural examples.

Proposition 34. *The class* REFL *of separable reflexive Banach spaces and the class* SD *of Banach spaces with separable dual are surjectively strongly bounded.*

Proposition 34 follows combining a number of results already existing in the literature, and so instead of giving a formal proof we will only give a guideline. To see that the class REFL is surjectively strongly bounded, let $\mathcal{A}$ be an analytic subset of REFL and consider the *dual class* $\mathcal{A}^*$ of $\mathcal{A}$ defined by

$$Y \in \mathcal{A}^* \Leftrightarrow \exists X \in \mathcal{A} \text{ with } Y \text{ isomorphic to } X^*.$$

The set $\mathcal{A}^*$ is analytic (see [10]) and $\mathcal{A}^* \subseteq \mathrm{REFL}$. Since the class REFL is strongly bounded (see [12]), there exists a separable reflexive Banach space Z which is universal for the class $\mathcal{A}^*$. Therefore, every space X in $\mathcal{A}$ is a quotient of Z^*. The referee suggested that, alternatively, one can use the universality results obtained in [24].

The argument for the class SD is somewhat different and uses the parameterized version of the Davis–Fiegel–Johnson-Pełczyński construction due to Bossard, as well as, an idea already employed in the proof of Theorem 28. Specifically, let $\mathcal{A}$ be an analytic subset of SD. By the results in [9] and [6], there exists an analytic subset $\mathcal{B}$ of Banach spaces with a shrinking Schauder basis such that for every $X \in \mathcal{A}$ there exists $Y \in \mathcal{B}$ having X as quotient. It is then possible to apply the machinery developed in [2] to obtain a Banach space E with a shrinking Schauder basis that contains a complemented copy of every space in the class $\mathcal{B}$. By the choice of $\mathcal{B}$, we see that the space E is surjectively universal for the class $\mathcal{A}$.

Although, by Theorem 2, we know that the class NC_{ℓ_1} is surjectively strongly bounded, we should point out that it is not known whether the class NC_{ℓ_1} is strongly bounded. We close this section by mentioning the following related problems.

Problem 1. *Is it true that every separable Banach space X not containing a copy of ℓ_1 embeds into a space Y with a Schauder basis and not containing a copy of ℓ_1?*

Problem 2. *Does there exist a map $g\colon \omega_1 \to \omega_1$ such that for every countable ordinal ξ and every separable Banach space X with $\phi_{\mathrm{NC}_{\ell_1}}(X) \leqslant \xi$ the space X embeds into a Banach space Y with a Schauder basis satisfying $\phi_{\mathrm{NC}_{\ell_1}}(Y) \leqslant g(\xi)$?*

Problem 3. *Is the class NC_{ℓ_1} strongly bounded?*

We notice that an affirmative answer to Problem 2 can be used to provide an affirmative answer to Problem 3 (to see this combine Theorem 9 and Theorem 14 stated in §2).

It seems reasonable to conjecture that the above problems have an affirmative answer. Our optimism is based on the following facts. Firstly, Problem 3 is known to be true within the category of Banach spaces with a Schauder basis (see [2]). Secondly, it is known that for every minimal Banach space Z not containing a copy ℓ_1 the class NC_Z is strongly bounded (see [11]).

Acknowledgments. This work was done during my visit at Texas A&M University the academic year 2008-2009. I would like to thank the Department of Mathematics for the financial support and the members of the Functional Analysis group for providing excellent working conditions.

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