THE STEINHAUS PROPERTY AND HAAR-NULL SETS

PANDELIS DODOS

ABSTRACT. It is shown that if G is an uncountable Polish group and $A \subseteq G$ is a universally measurable set such that $A^{-1}A$ is meager, then the set $T_l(A) = \{\mu \in P(G) : \mu(gA) = 0 \text{ for all } g \in G\}$ is co-meager. In particular, if A is analytic and not left Haar-null, then $1 \in \mathrm{Int}(A^{-1}AA^{-1}A)$.

1. INTRODUCTION

The purpose of this paper is to show that there exists a satisfactory extension of the classical Steinhaus theorem for an arbitrary Polish group. In order to obtain the extension one needs, first, to isolate the appropriate σ-ideal on which the result will be applied. For the class of abelian Polish groups this is the σ-ideal of Haar-null sets defined by Christensen [C]. However, in non-abelian (and non-locally-compact) Polish groups this σ-ideal is no longer well-behaved. Actually, by the results of Solecki in [S2], the Steinhaus property of Haar-null sets fails in "most" non-abelian Polish groups. Also notice that the conclusion of the Steinhaus theorem is rather strong: if $A \subseteq \mathbb{R}$ is of positive Lebesgue measure, then $A - A$ contains a neighborhood of 0. If we relax the conclusion to $A - A$ is not meager[1], then this is valid in every abelian Polish group.

To state our result we need some definitions. Let G be a Polish group and let $A \subseteq G$ be a universally measurable set. The set A is said to be *Haar-null* if there exists $\mu \in P(G)$ (that is, μ is a Borel probability measure on G) such that $\mu(g_1 A g_2) = 0$ for every $g_1, g_2 \in G$. It is said to be *left Haar-null* if there exists $\mu \in P(G)$ such that $\mu(gA) = 0$ for every $g \in G$. By the results in [ST, S2], the notions of Haar-null and left Haar-null set are distinct (however, they obviously agree on abelian groups). We set

$$T(A) \coloneqq \{\mu \in P(G) : \mu(g_1 A g_2) = 0 \text{ for every } g_1, g_2 \in G\}$$

and

$$T_l(A) \coloneqq \{\mu \in P(G) : \mu(gA) = 0 \text{ for every } g \in G\}.$$

2000 *Mathematics Subject Classification*: 54H11, 28C10.

Key words: Haar-null sets, Polish groups, Steinhaus theorem, Borel measures.

[1]We recall that a subset A of a topological space X is said to be *meager* (or *of first category*) if A is covered by a countable union of closed nowhere dense sets. The complement of a meager set is usually referred as *co-meager*.

It is easy to see that if A is analytic[2], then both $T(A)$ and $T_l(A)$ are faces (that is, extreme convex subsets) of $P(G)$ with the Baire property. It follows by [D2, Theorem 4] that the sets $T(A)$ and $T_l(A)$ are either meager, or co-meager. A set A is said to be *generically Haar-null* if $T(A)$ is co-meager. Respectively, the set A is said to be *generically left Haar-null* if $T_l(A)$ is co-meager.

For every Polish group G the class of generically left Haar-null subsets of G forms a σ-ideal. Notice that if A is not generically left Haar-null, then A should not be considered as a small set (it is null only for a relatively small set of measures). This is indeed true as the following theorem demonstrates.

Theorem A. *Let G be an uncountable Polish group and let A be a universally measurable subset of G. Assume that $A^{-1}A$ is meager. Then $T_l(A)$ is co-meager.*

Thus, if A is analytic and not generically left Haar-null (in particular, not left Haar-null), then $A^{-1}A$ is non-meager.

The locally-compact abelian case of Theorem A can be also derived by the results of Laczkovich in [La] who proved that if A is not covered by an F_σ Haar-measure zero set, then $A^{-1}A$ is co-meager in a neighborhood of the identity. To see that this implies Theorem A one invokes [D1, Proposition 5] which states that if G is locally-compact and $A \subseteq G$ is covered by an F_σ Haar-null set, then $T_l(A)$ is co-meager. Both Laczkovich's result as well as the result of Christensen [C] that Haar-null sets satisfy the Steinhaus property in abelian Polish groups, are heavily depended on the classical Steinhaus theorem. The proof of Theorem A follows quite different arguments. It is based on the fact that if $\mathcal{H}$ is a dense G_δ and hereditary subset of $K(G)$, then this is witnessed in the probabilities of G.

1.1. **Preliminaries.** Our general notation and terminology follows [Ke]. By $\mathbb{N} = \{0, 1, 2, \dots\}$ we denote the natural numbers. For any Polish space X by $K(X)$ we denote the hyperspace of all compact subsets of X with the Vietoris topology and by $P(X)$ the space of all Borel probability measures on X with the weak* topology. Both are Polish (see [Ke]). If d is a compatible complete metric of X, then by d_H we denote the Hausdorff metric on $K(X)$ associated with d defined by

$$d_H(K, C) = \inf\{\varepsilon > 0 : K \subseteq C_\varepsilon \text{ and } C \subseteq K_\varepsilon\}$$

where $A_\varepsilon = \{x \in X : d(x, A) \leqslant \varepsilon\}$ for every $A \subseteq X$. All balls in $K(X)$ are taken with respect to d_H and are denoted by B_H. In $P(X)$ we consider the so called Lévy metric ρ defined by

$$\rho(\mu, \nu) = \inf\big\{\varepsilon > 0 \ : \ \mu(A) \leqslant \nu(A_\varepsilon) + \varepsilon \text{ and } \nu(A) \leqslant \mu(A_\varepsilon) + \varepsilon \\ \text{for every compact (or Borel) subset } A \text{ of } X\big\}$$

[2]We recall that a subset A of a Polish space X is said to be *analytic* if there exists a continuous map $f\colon \mathbb{N}^{\mathbb{N}} \to X$ with $f(\mathbb{N}^{\mathbb{N}}) = A$. It is a classical result that every Borel subset of a Polish space is analytic. It is also well-known that an analytic set which is not meager is actually co-meager in a nonempty open set.

(see [BL] for more details). All balls in $P(X)$ are taken with respect to ρ and are denoted by B_P. If G is a Polish group and $\mu, \nu \in P(G)$, then by $\mu * \nu$ we denote their convolution, defined by

$$\mu * \nu(A) = \int_G \mu(Ax^{-1})\, d\nu(x).$$

A subset $\mathcal{H}$ of $K(X)$ is said to be *hereditary* if for every $K \in \mathcal{H}$ and every $C \in K(X)$ with $C \subseteq K$ we have that $C \in \mathcal{H}$. All the other pieces of notation we use are standard.

2. Hereditary, dense G_δ sets and measures

Throughout this section X will be a Polish space and $\mathcal{H}$ a hereditary, dense G_δ subset of $K(X)$. By d we denote a compatible complete metric of X.

Lemma 1. *Let X and $\mathcal{H}$ be as above. Then there exists a sequence $(\mathcal{U}_n)$ of open, dense and hereditary subsets of $K(X)$ such that $\mathcal{H} = \bigcap_n \mathcal{U}_n$.*

Proof. Write $\mathcal{H} = \bigcap_n \mathcal{V}_n$ where each $\mathcal{V}_n$ is open and dense but not necessarily hereditary. Fix n and define

$$\mathcal{C}_n \coloneqq \{K \in K(X) : \exists C \subseteq K \text{ compact with } C \notin \mathcal{V}_n\}.$$

It is easy to check that $\mathcal{C}_n$ is closed and $\mathcal{C}_n \cap \mathcal{H} = \emptyset$. So if we set $\mathcal{U}_n = K(X) \setminus \mathcal{C}_n$ we see that the sequence $(\mathcal{U}_n)$ has all the desired properties. □

In the sequel we will say that the sequence $(\mathcal{U}_n)$ obtained by Lemma 1, is the *normal form* of $\mathcal{H}$. We need the following lemmas.

Lemma 2. *Let $\mathcal{U} \subseteq K(X)$ be open, dense and hereditary. Also let $x_0, \dots, x_n$ be distinct points in X and $r_1 > 0$. Then there exist $y_0, \dots, y_n$ distinct points in X such that $d(x_i, y_i) < r_1$ for every $i \in \{0, \dots, n\}$ and, moreover, $\{y_0, \dots, y_n\} \in \mathcal{U}$.*

Proof. We may assume that $B(x_i, r_1) \cap B(x_j, r_1) = \emptyset$ for every $i, j \in \{0, \dots, n\}$ with $i \neq j$. We set

$$\mathcal{V} \coloneqq \Big\{K : K \subseteq \bigcup_{i=0}^{n} B(x_i, r_1) \text{ and } K \cap B(x_i, r_1) \neq \emptyset \ \ \forall i = 0, \dots, n\Big\}.$$

Notice that $\mathcal{V}$ is open. Since $\mathcal{U}$ is open and dense, there exists $K \in \mathcal{V} \cap \mathcal{U}$. For every $i \in \{0, \dots, n\}$ we select $y_i \in K \cap B(x_i, r_1)$. As $\mathcal{U}$ is hereditary, we see that $\{y_0, \dots, y_n\} \in \mathcal{U}$. Clearly $y_0, \dots, y_n$ are as desired. □

Lemma 3. *Let $\mathcal{U} \subseteq K(X)$ be open, dense and hereditary. Also let $\varepsilon > 0$. Then the set*

$$G_{\mathcal{U},\varepsilon} \coloneqq \{\mu \in P(X) : \exists K \in \mathcal{U} \text{ with } \mu(K) \geqslant 1 - \varepsilon\}$$

is co-meager in $P(X)$.

Proof. Fix $\mathcal{U}$ and $\varepsilon > 0$. We will show that for every open subset V of $P(X)$ there exists an open subset W of V such that $W \subseteq G_{\mathcal{U},\varepsilon}$. This will finish the proof (actually, it implies that $G_{\mathcal{U},\varepsilon}$ contains a dense open set). So let $V \subseteq P(X)$ be open. Since finitely supported measures are dense in $P(X)$, we may select $\nu = \sum_{i=0}^{n} a_i \delta_{x_i}$ and $r > 0$ such that

(1) $a_i > 0$ for every $i \in \{0, \dots, n\}$ and $\sum_{i=0}^{n} a_i = 1$,
(2) $B_P(\nu, r) \subseteq V$.

By Lemma 2, there exist $y_0, \dots, y_n$ distinct points in X with $\{y_0, \dots, y_n\} \in \mathcal{U}$ and such that $d(x_i, y_i) < \frac{r}{2}$ for every $i \in \{0, \dots, n\}$. We set $\mu = \sum_{i=0}^{n} a_i \delta_{y_i}$. Then it is easy to see that

(3) $\rho(\mu, \nu) \leqslant \frac{r}{2}$.

Let $F = \{y_0, \dots, y_n\}$. As $\mathcal{U}$ is open and $F \in \mathcal{U}$ there exists $\theta > 0$ such that

(4) $\theta < \min\{\frac{\varepsilon}{3}, \frac{r}{3}\}$, and
(5) $B_H(F, 2\theta) \subseteq \mathcal{U}$.

Then $W = B_P(\mu, \theta)$ is as desired. Indeed, by (2), (3) and (4), it is clear that W is a subset of V. We only need to check that W is a subset of $G_{\mathcal{U},\varepsilon}$. Let $\lambda \in W$ be arbitrary. Then $\rho(\lambda, \mu) < \theta$ and so

$$1 = \mu(F) \leqslant \lambda(F_\theta) + \theta$$

which yields that $\lambda(F_\theta) \geqslant 1 - \frac{\varepsilon}{3}$ by the choice of θ. By the inner regularity of λ, there exists $C \subseteq F_\theta$ compact such that $\lambda(C) \geqslant 1 - \varepsilon$. We set $K = C \cup F$. Then $d_H(K, F) \leqslant \theta$ and so, by (5), $K \in \mathcal{U}$. Moreover, $\lambda(K) \geqslant \lambda(C) \geqslant 1 - \varepsilon$. This implies that $\lambda \in G_{\mathcal{U},\varepsilon}$ and the proof is completed. □

Our goal in this section is to prove the following proposition.

Proposition 4. *Let $\mathcal{H}$ be a hereditary, dense G_δ subset of $K(X)$. Then the set*

$$G_{\mathcal{H}} \coloneqq \{\mu \in P(X) : \forall \varepsilon > 0 \ \exists K \in \mathcal{H} \text{ with } \mu(K) \geqslant 1 - \varepsilon\}$$

is co-meager in $P(X)$.

Proof. Let $(\mathcal{U}_n)$ be the normal form of $\mathcal{H}$. For every $n, m \in \mathbb{N}$ we set

$$G_{n,m} \coloneqq \big\{\mu \in P(X) : \exists K \in \mathcal{U}_n \text{ with } \mu(K) \geqslant 1 - \frac{1}{m+1}\big\}.$$

By Lemma 3, we have that $G_{n,m}$ is co-meager. Hence, so is $\bigcap_{n,m} G_{n,m}$. We claim that $G_{\mathcal{H}} = \bigcap_{n,m} G_{n,m}$. This will finish the proof. It is clear that $G_{\mathcal{H}} \subseteq \bigcap_{n,m} G_{n,m}$. Conversely, fix $\mu \in \bigcap_{n,m} G_{n,m}$ and let $\varepsilon > 0$ be arbitrary. We select a sequence (ε_n) of positive reals such that

$$\sum_{n\in\mathbb{N}} \varepsilon_n < \frac{\varepsilon}{2}.$$

We also select a sequence (m_n) of natural numbers with $\frac{1}{m_n+1} \leqslant \varepsilon_n$ for every $n \in \mathbb{N}$. Since

$$\mu \in \bigcap_{n,m} G_{n,m} \subseteq \bigcap_n G_{n,m_n}$$

we may select a sequence (K_n) in $K(X)$ such that

(1) $K_n \in \mathcal{U}_n$, and
(2) $\mu(K_n) \geqslant 1 - \frac{1}{m_n+1} \geqslant 1 - \varepsilon_n$.

For every $n \in \mathbb{N}$ let $F_n = \bigcap_{i=0}^n K_i$ and set $F = \bigcap_n K_n$. Then $F_n \downarrow F$. Notice that $F \in \mathcal{U}_n$ as $F \subseteq F_n \subseteq K_n \in \mathcal{U}_n$ and $\mathcal{U}_n$ is hereditary. Hence, $F \in \bigcap_n \mathcal{U}_n = \mathcal{H}$. Moreover, by (2), we have

$$\mu(F_n) = \mu(K_0 \cap \cdots \cap K_n) \geqslant 1 - \sum_{k=0}^n \varepsilon_k.$$

Since $F_n \downarrow F$, we obtain that

$$\mu(F) = \lim_{n \in \mathbb{N}} \mu(F_n) \geqslant 1 - \sum_{n \in \mathbb{N}} \varepsilon_n \geqslant 1 - \varepsilon.$$

This shows that $\mu \in G_{\mathcal{H}}$ as desired. $\square$

3. Left Haar-null sets in Polish groups

Our aim is to give the proof of Theorem A stated in the introduction.

Proof of Theorem A. Let G be an uncountable Polish group and let A be a universally measurable subset of G such that $A^{-1}A$ is meager. We select a sequence (C_n) of closed, nowhere dense subsets of G with the following properties.

(i) $1 \notin C_n$ for all $n \in \mathbb{N}$.
(ii) $A^{-1}A \setminus \{1\} \subseteq \bigcup_n C_n$.

For every $n \in \mathbb{N}$ set

$$\mathcal{U}_n \coloneqq \{K \in K(G) : K^{-1}K \cap C_n = \emptyset\}.$$

Clearly every $\mathcal{U}_n$ is hereditary. Moreover, as the function $f \colon K(G) \to K(G)$ defined by $f(K) = K^{-1}K$ is continuous, we see that every $\mathcal{U}_n$ is open.

Claim 5. *For every $n \in \mathbb{N}$ the set $\mathcal{U}_n$ is dense in $K(G)$.*

Proof of Claim 5. As finite sets are dense in $K(G)$, it is enough to show that for every finite subset $\{x_0, \ldots, x_l\}$ of G and every $r > 0$ there exist $y_0, \ldots, y_l$ distinct points in G with

$$\big\{y_i^{-1} y_j : i, j \in \{0, \ldots, l\} \text{ with } i \neq j\big\} \cap C_n = \emptyset$$

and such that $d(x_i, y_i) \leqslant r$ for every $i \in \{0, \ldots, l\}$ (here d is simply a compatible complete metric of G). The points $y_0, \ldots, y_l$ will be chosen recursively. We set $y_0 = x_0$. Assume that $y_0, \ldots, y_k$ have been chosen for some $k < l$ so as $\big\{y_i^{-1} y_j : i, j \in \{0, \ldots, k\} \text{ with } i \neq j\big\} \cap C_n = \emptyset$. For every $g \in G$ the functions $x \mapsto gx^{-1}$ and

$x \mapsto gx$ are homeomorphisms. It follows that the set $F_k \coloneqq \bigcup_{i=0}^{k}(y_i C_n^{-1} \cup y_i C_n)$ is a closed set with empty interior. Hence, there exists $y_{k+1} \in B(x_{k+1}, r)$ such that $y_{k+1} \notin F_k \cup \{y_0, \dots, y_k\}$. This implies that for every $i \in \{0, \dots, k\}$ we have $y_{k+1}^{-1} y_i \notin C_n$ and $y_i^{-1} y_{k+1} \notin C_n$. This completes the recursive selection and the proof of the claim is completed. □

By the above claim, it follows that the set $\mathcal{H} = \bigcap_n \mathcal{U}_n$ is a hereditary, dense G_δ subset of $K(G)$ and that the sequence $(\mathcal{U}_n)$ is a normal form of $\mathcal{H}$. Notice that if $K \in \mathcal{H}$, then $K^{-1}K \cap A^{-1}A = \{1\}$. By Proposition 4, the set

$$B_1 \coloneqq \{\mu \in P(G) : \forall \varepsilon > 0 \ \exists K \in \mathcal{H} \text{ with } \mu(K) \geqslant 1 - \varepsilon\}$$

is co-meager. Our assumption that G is uncountable implies that the Polish group G viewed as a topological space is perfect. Hence, the set of all non-atomic Borel probability measures on G is co-meager in $P(G)$ (see [Kn], or [PRV]). It follows that the set

$$B_2 \coloneqq \{\mu \in P(G) : \mu \text{ is non-atomic and } \mu \in B_1\}$$

is co-meager in $P(G)$. We will show that $B_2 \subseteq T_l(A)$. This will finish the proof. We need the following fact (its easy proof is left to the reader).

Fact 6. *Let $\mu \in P(G)$. Then $\mu \in T_l(A)$ if and only if for every $\nu \in P(G)$ we have $\nu * \mu(A) = 0$.*

Fix $\mu \in B_2$. By the above fact, in order to verify that $\mu \in T_l(A)$ we have to show that $\nu * \mu(A) = 0$ for every $\nu \in P(G)$. So, let $\nu \in P(G)$ be arbitrary. Also let $\varepsilon > 0$ be arbitrary. Since $\mu \in B_2 \subseteq B_1$, there exists $K \in \mathcal{H}$ with $\mu(K) \geqslant 1 - \varepsilon$. Then

$$\begin{aligned} \nu * \mu(A) &= \int_G \nu(Ay^{-1})\, d\mu(y) \leqslant \int_K \nu(Ay^{-1})\, d\mu(y) + \mu(G \setminus K) \\ &\leqslant \int_K \nu(Ay^{-1})\, d\mu(y) + \varepsilon. \end{aligned}$$

We set $I \coloneqq \{y \in K : \nu(Ay^{-1}) > 0\}$.

Claim 7. *The set I is countable.*

Proof of Claim 7. Notice that if $y, z \in I$ with $y \neq z$, then $Ay^{-1} \cap Az^{-1} = \emptyset$. For if not, we would have that $1 \neq y^{-1}z \in K^{-1}K \cap A^{-1}A$ which contradicts the fact that $K \in \mathcal{H}$. It follows that the family $\{Ay^{-1} : y \in I\}$ is a family of pairwise disjoint sets of positive ν-measure. Hence, I is countable as claimed. □

The measure μ is non-atomic as $\mu \in B_2$. Therefore, by Claim 7, we see that $\mu(I) = 0$. It follows that

$$\int_K \nu(Ay^{-1})\, d\mu(y) = \int_I \nu(Ay^{-1})\, d\mu(y) \leqslant \mu(I) = 0$$

and so $\nu * \mu(A) \leqslant \varepsilon$. Since ε was arbitrary, this implies that $\nu * \mu(A) = 0$. The proof of Theorem A is completed. □

Combining Theorem A with Pettis' theorem (see [Ke, Theorem 9.9]) we obtain the following corollary.

Corollary 8. *Let G be an uncountable Polish group and let A be an analytic subset of G. If A is not generically left Haar-null* (*in particular, if A is not left Haar-null*), *then $1 \in \mathrm{Int}(A^{-1}AA^{-1}A)$.*

Clearly Theorem A implies that in non-locally-compact groups, compact sets are generically left Haar-null. Another application of this form concerns the size of analytic subgroups of Polish groups. Specifically we have the following corollary which may be considered as the non-locally-compact analogue of Laczkovich's theorem [La].

Corollary 9. *Let G be an uncountable Polish group and let H be an analytic subgroup of G with empty interior. Then H is generically left Haar-null.*

What about Haar-null sets? We would like to remark on the possibility of extending Theorem A to Haar-null sets instead of merely left Haar-null. As it has been shown by Solecki in [S2], the Steinhaus property of the σ-ideal of Haar-null sets fails in a large number of Polish groups (in a sense, it fails for most non-abelian Polish groups). Precisely, by [S2, Theorem 6.1], if (H_n) is a sequence of countable groups such that infinitely many of them are not FC (see [S2] for the definition of FC groups), then one can find a closed set $A \subseteq \prod_n H_n$ which is not Haar-null and $A^{-1}A$ is meager. So, there is no analogue of Theorem A for Haar-null sets in arbitrary Polish groups. Yet there is one if we further assume that the group G satisfies the following non-singularity condition.

(C) For every analytic and meager subset A of G, the conjugate saturation $[A] \coloneqq \{x : \exists g \in G\ \exists a \in A \text{ with } x = gag^{-1}\}$ of A is meager.

Clearly every abelian Polish group satisfies (C). Moreover we have the following proposition.

Proposition 10. *Let G_1 and G_2 be Polish groups. If both G_1 and G_2 satisfy* (C), *then so does $G_1 \times G_2$.*

Proof. Let $A \subseteq G_1 \times G_2$ be analytic and meager. By the Kuratowski–Ulam theorem (see [Ke, Theorem 8.41]), we have

$$\forall^* x \in G_1 \text{ the section } A_x = \{y \in G_2 : (x, y) \in A\} \text{ of } A \text{ is meager.}$$

Since G_2 satisfies (C), by another application of the Kuratowski–Ulam theorem, we see that the set

$$A_1 \coloneqq \{(x, z) : \exists g_2, y \in G_2 \text{ with } (x, y) \in A \text{ and } y = g_2 z g_2^{-1}\}$$

is analytic and meager. With the same reasoning we see that the set

$$A_2 \coloneqq \{(w, z) : \exists g_1, x \in G_1 \text{ with } (x, z) \in A_1 \text{ and } x = g_1 w g_1^{-1}\}$$

is analytic and meager too. Noticing that $A_2 = [A]$ the result follows. □

For groups that satisfy (C) we have the following strengthening of Theorem A.

Proposition 11. *Let G be an uncountable Polish group which satisfies* (C)*. If A is an analytic subset of G such that $A^{-1}A$ is meager, then $T(A)$ is co-meager.*

Proof. The proof is similar to the proof of Theorem A, and so, we shall only indicate the necessary changes. Let $A \subseteq G$ be analytic such that $A^{-1}A$ is meager. Notice that $A^{-1}A$ is analytic. The group G satisfies (C). It follows that the set $[A^{-1}A]$ is meager too. Arguing as in the proof of Theorem A this implies that there exists a co-meager set B_2 of non-atomic Borel probability measures on G such that for every $\mu \in B_2$ and every $\varepsilon > 0$ there exists a compact subset K of G with $\mu(K) \geqslant 1 - \varepsilon$ and $K^{-1}K \cap [A^{-1}A] = \{1\}$. We claim that $B_2 \subseteq T(A)$. To this end, it is enough to show that for every $\mu \in B_2$, every $\nu \in P(G)$ and every $x \in G$ we have $\nu * \mu(Ax) = 0$. Let $\varepsilon > 0$ be arbitrary, and a select compact set $K \subseteq G$ as described above. Then

$$\nu * \mu(Ax) \leqslant \int_K \nu(Axy^{-1})\, d\mu(y) + \varepsilon.$$

We set $I \coloneqq \{y \in K : \nu(Axy^{-1}) > 0\}$. Observe that if $y, z \in I$ with $y \neq z$, then $(Axy^{-1}) \cap (Axz^{-1}) = \emptyset$ (for if not, we would have that $1 \neq y^{-1}z \in K^{-1}K \cap [A^{-1}A]$). By the countable chain condition of ν, we obtain that I is countable and the result follows. □

Remark 1. The σ-ideal of generically left Haar-null sets is a quite satisfactory σ-ideal of measure-theoretic small sets in arbitrary Polish groups. Besides Theorem A, this is also supported by the results in [D1] asserting that every analytic and generically left Haar-null subset A of G can be covered by a *Borel* set B with the same property. The fact that this ideal is well-behaved is also reflected in the complexity of the collection of all closed generically left Haar-null sets (in the Effros-Borel structure). It is much better than the one of closed Haar-null sets, at least in abelian Polish groups. Specifically, it follows by the results of Solecki in [S1], that in non-locally-compact abelian Polish groups the σ-ideal of closed generically Haar-null sets is $\mathbf{\Pi}^1_1$-complete. The corresponding collection of closed Haar-null sets is much more complicated (it is both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$-hard).

References

[BL] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, American Mathematical Society Colloquium Publications, 2000.

[C] J. P. R. Christensen, *On sets of Haar measure zero in abelian Polish groups*, Israel J. Math. 13 (1972), 255–260.

[D1] P. Dodos, *On certain regularity properties of Haar-null sets*, Fund. Math. 181 (2004), 97–109.

[D2] P. Dodos, *Dichotomies of the set of test measures of a Haar-null set*, Israel J. Math. 144 (2004), 15–28.

[Ke] A. S. Kechris, *Classical Descriptive Set Theory*, Grad. Texts in Math., Vol. 156, Springer-Verlag, 1995.

[Kn] J. D. Knowles, *On the existence of non-atomic measures*, Mathematika 14 (1967), 62–67.

[La] M. Laczkovich, *Analytic subgroups of the reals*, Proc. Amer. Math. Soc. 126 (1998), 1783–1790.

[PRV] K. R. Parthasarathy, R. R. Rao and S. R. S. Varadhan, *On the category of indecomposable distributions on topological groups*, Trans. Amer. Math. Soc. 102 (1962), 200–217.

[ST] H. Shi and B. S. Thomson, *Haar null sets in the space of automorphisms of* $[0,1]$, Real Anal. Exchange 24 (1998/99), 337–350.

[S1] S. Solecki, *Haar null and non-dominating sets*, Fund. Math. 170 (2001), 197–217.

[S2] S. Solecki, *Size of subsets of groups and Haar null sets*, Geom. Funct. Anal. 15 (2005), 246–273.

Texas A&M University, Department of Mathematics, 3368 TAMU, College Station, TX 77843-3368.

E-mail address: pdodos@math.ntua.gr