ON PAIRS OF DEFINABLE ORTHOGONAL FAMILIES

PANDELIS DODOS AND VASSILIS KANELLOPOULOS

Abstract. We introduce the notion of an M-family of infinite subsets of $\mathbb{N}$ which is implicitly contained in the work of Mathias. We study the structure of a pair of orthogonal hereditary families $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ is analytic and $\mathcal{B}$ is C-measurable and an M-family.

1. Introduction

Two families $\mathcal{A}$ and $\mathcal{B}$ of infinite subsets of $\mathbb{N}$ are said to be *orthogonal* if $A \cap B$ is finite for every $A \in \mathcal{A}$ and every $B \in \mathcal{B}$. The study of the structure of a pair $(\mathcal{A}, \mathcal{B})$ of orthogonal families is a classical topic ([Hau]) which has found numerous applications (see, for instance, [DW, To4]). Among all pairs $(\mathcal{A}, \mathcal{B})$ of orthogonal families of particular importance is the study of the *definable* ones. Here the word definable refers to the descriptive set theoretic complexity of $\mathcal{A}$ and $\mathcal{B}$ as subsets of $\mathcal{P}(\mathbb{N})$. A fundamental result in this direction is the "perfect Lusin gap" theorem of Todorčević [To2] which deals with a pair of analytic and orthogonal families.

In this paper we study the structure of a pair $(\mathcal{A}, \mathcal{B})$ of hereditary and orthogonal families where $\mathcal{A}$ is analytic and $\mathcal{B}$ is C-measurable[1] and "large". Our notion of largeness is the following which is implicitly contained in the work of Mathias [Ma].

Definition 1. *We say that a hereditary family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$ is an* M-family *if for every sequence (A_n) in $\mathcal{A}$ there exists $A \in \mathcal{A}$ whose all but finitely many elements are in $\bigcup_{i \geqslant n} A_i$ for every $n \in \mathbb{N}$.*

We should point out that there are several other notions appearing in the literature, such as P-ideals (see [So, To2]) or semi-selective co-ideals (see [Fa]), involving the existence of diagonal sequences. We should also point out that the notion of an M-family is closely related to the weak diagonal sequence property of topological spaces and, in fact, it can be considered as its combinatorial analogue.

Using Ellentuck's theorem [El] we show that the class of C-measurable M-families possesses strong stability properties. It is closed, for instance, under intersection and "diagonal" products. As a consequence we prove that if (X, τ_1) and (Y, τ_2) are two countable analytic spaces with the weak diagonal sequence property, then the

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[1] We recall that a subset of a Polish space is C-measurable if it belongs to the smallest σ-algebra that contains the open sets and is closed under the Souslin operation.

product $(X \times Y, \tau_1 \times \tau_2)$ has the weak diagonal sequence property. This answers Question 5.4 from [TU].

Our first result, concerning the structure of a pair $(\mathcal{A}, \mathcal{B})$ as described above, is the following (see §2 for the relevant definitions).

Theorem I. *Let $\mathcal{A}$ and $\mathcal{B}$ be two hereditary, orthogonal families of infinite subsets of $\mathbb{N}$. Assume that $\mathcal{A}$ is analytic and that $\mathcal{B}$ is an M-family and C-measurable. Then, either*

(i) *$\mathcal{A}$ is countably generated in $\mathcal{B}^\perp$, or*
(ii) *there exists a perfect Lusin gap inside $(\mathcal{A}, \mathcal{B})$.*

Theorem I shows that the assumption of being an M-family can successfully replace analyticity in the perfect Lusin gap theorem of [To2]. We should point out that the phenomenon of replacing analyticity by a structural property and still getting the same conclusion as in Theorem I has already appeared in the literature (see [To4, TU]). As a matter of fact Theorem I was motivated by these applications.

Our second result, concerning the structure of a pair $(\mathcal{A}, \mathcal{B})$ as in Theorem I, extends a result of Krawczyk from [Kr]. To state it, it is useful to look at the second orthogonal $\mathcal{B}^{\perp\perp}$ of $\mathcal{B}$. In a sense the family $\mathcal{B}^{\perp\perp}$ is the "completion" of $\mathcal{B}$, as an infinite subset L of $\mathbb{N}$ belongs to $\mathcal{B}^{\perp\perp}$ if (and only if) every infinite subset of L contains an element of $\mathcal{B}$. To proceed with our discussion, let $\mathcal{C}$ be the family of all infinite chains of $\mathbb{N}^{<\mathbb{N}}$ (we recall that a subset of $\mathbb{N}^{<\mathbb{N}}$ is called a chain if it is linearly ordered under the order of end-extension). Also let $\mathcal{I}_{\mathrm{wf}}$ be the ideal on $\mathbb{N}^{<\mathbb{N}}$ generated by the set WF of all downwards closed, well-founded, infinite subtrees of $\mathbb{N}^{<\mathbb{N}}$. The following theorem shows that if $\mathcal{A}, \mathcal{B}$ are as above and $\mathcal{A}$ is not countably generated in $\mathcal{B}^\perp$, then the pair $(\mathcal{C}, \mathcal{I}_{\mathrm{wf}})$ "embeds" into the pair $(\mathcal{A}, \mathcal{B}^{\perp\perp})$ in a very canonical way.

Theorem II. *Let $\mathcal{A}$ and $\mathcal{B}$ be two hereditary, orthogonal families of infinite subsets of $\mathbb{N}$. Assume that $\mathcal{A}$ is analytic and that $\mathcal{B}$ is an M-family and C-measurable. Then, either*

(i) *$\mathcal{A}$ is countably generated in $\mathcal{B}^\perp$, or*
(ii) *there exists a one-to-one map $\psi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ such that*

$$\mathcal{C} \subseteq \{\psi^{-1}(A) : A \in \mathcal{A}\} \quad \textit{and} \quad \mathcal{I}_{\mathrm{wf}} \subseteq \{\psi^{-1}(B) : B \in \mathcal{B}^{\perp\perp}\}.$$

One of the main ingredients of the proofs of Theorems I and II is the infinite dimensional extension of Hindman's theorem [Hi] due to Milliken [Mil]. It is used in a spirit similar as in [ADK].

The paper is organized as follows. In §2 we gather some preliminaries needed in the rest of the paper. In §3 we study the connection of M-families with other related notions and we give some examples. In §4 we present some of their structural properties. The proof of Theorem I is given in §5 while the proof of Theorem II is given in §6. Our general notation and terminology is standard, as can be found, e.g., in [Ke] and [To3].

2. Preliminaries

It is a common fact that once one is willing to present some results about trees, ideals and related combinatorics, then one has to set up a, rather large, notational system. Below we gather all the conventions that we need and which are, more or less, standard. In what follows X will be a countable (infinite) set.

2.1. **Ideals.** By $\mathcal{P}_\infty(X)$ we denote the set of all infinite subsets of X (which is clearly a Polish subspace of 2^X). A family $\mathcal{A} \subseteq \mathcal{P}_\infty(X)$ is *hereditary* if for every $A \in \mathcal{A}$ and every $A' \in \mathcal{P}_\infty(A)$ we have $A' \in \mathcal{A}$. A subfamily $\mathcal{B}$ of a family $\mathcal{A}$ is *cofinal* in $\mathcal{A}$ if for every $A \in \mathcal{A}$ there exists $B \in \mathcal{P}_\infty(A)$ with $B \in \mathcal{B}$.

Given $A, B \in \mathcal{P}_\infty(X)$ we write $A \subseteq^* B$ if the set $A \setminus B$ is finite, while we write $A \perp B$ if the set $A \cap B$ is finite. Two families $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}_\infty(X)$ are said to be *orthogonal*, in symbols $\mathcal{A} \perp \mathcal{B}$, if $A \perp B$ for every $A \in \mathcal{A}$ and every $B \in \mathcal{B}$. For every $\mathcal{A} \subseteq \mathcal{P}_\infty(X)$ we set $\mathcal{A}^\perp \coloneqq \{B \in \mathcal{P}_\infty(X) : B \perp A \text{ for every } A \in \mathcal{A}\}$ and $\mathcal{A}^* \coloneqq \{X \setminus A : A \in \mathcal{A}\}$. The family $\mathcal{A}^\perp$ is called the *orthogonal* of $\mathcal{A}$. Notice that $\mathcal{A}^\perp$ is an ideal.

Two families $\mathcal{A}$ and $\mathcal{B}$ are *countably separated* if there exists a sequence (C_n) in $\mathcal{P}_\infty(X)$ such that for every $A \in \mathcal{A}$ and every $B \in \mathcal{B}$ there exists $n \in \mathbb{N}$ with $A \subseteq C_n$ and $C_n \perp B$. A family $\mathcal{A}$ is *countably generated* in a family $\mathcal{B}$, if there exists a sequence (B_n) in $\mathcal{B}$ such that for every $A \in \mathcal{A}$ there exists $n \in \mathbb{N}$ with $A \subseteq^* B_n$. An ideal $\mathcal{I}$ on X is said to be *bisequential* if for every ultrafilter p on X with $\mathcal{I} \subseteq p^*$ the family $\mathcal{I}$ is countably generated in p^*.

Given $\mathcal{A} \subseteq \mathcal{P}_\infty(X)$ we set

$$\text{(1)} \qquad \mathrm{co}(\mathcal{A}) \coloneqq \{B \in \mathcal{P}_\infty(X) : \exists A \in \mathcal{A} \text{ with } B \cap A \text{ infinite}\} = \mathcal{P}_\infty(X) \setminus \mathcal{A}^\perp.$$

Notice that $\mathrm{co}(\mathcal{A})$ is a co-ideal. We call $\mathrm{co}(\mathcal{A})$ as the *co-ideal generated by* $\mathcal{A}$. Observe that if $\mathcal{A}$ is hereditary, then $\mathrm{co}(\mathcal{A}) = \{B \in \mathcal{P}_\infty(X) : \exists A \in \mathcal{A} \text{ with } A \subseteq B\}$.

The following elementary, well-known, fact provides the description of the second orthogonal $\mathcal{A}^{\perp\perp}$ of a hereditary family $\mathcal{A}$.

Fact 1. *Let $\mathcal{A} \subseteq \mathcal{P}_\infty(X)$ hereditary. Also let $B \in \mathcal{P}_\infty(X)$. Then $B \in \mathcal{A}^{\perp\perp}$ if and only if for every $C \in \mathcal{P}_\infty(B)$ there exists $A \in \mathcal{P}_\infty(C)$ with $A \in \mathcal{A}$.*

An ideal $\mathcal{I}$ is said to have the *Fréchet property* if $\mathcal{I} = \mathcal{I}^{\perp\perp}$. We notice that if $\mathcal{A}$ is a hereditary family, then both $\mathcal{A}^\perp$ and $\mathcal{A}^{\perp\perp}$ have the Fréchet property. The following fact is also well-known. We sketch its proof for completeness.

Fact 2. *A bisequential ideal $\mathcal{I}$ on X has the Fréchet property.*

Proof. In light of Fact 1, it is enough to show that for every $A \notin \mathcal{I}$ there exists $C \in \mathcal{P}_\infty(A)$ with $C \in \mathcal{I}^\perp$. So, let $A \notin \mathcal{I}$. The family $\{A \setminus L : L \in \mathcal{I}\}$ has the finite intersection property. Hence, we may find $p \in \beta X$, non-principal, with $\mathcal{I} \subseteq p^*$ and $A \in p$. By the bisequentiality of $\mathcal{I}$, there exists a sequence (B_n) in p^* such that for every $L \in \mathcal{I}$ there exists $n \in \mathbb{N}$ with $L \subseteq^* B_n$. Clearly, we may assume that the sequence (B_n) is increasing. Let C be an infinite diagonalization of the decreasing sequence $(A \setminus B_n)$. Then $C \in \mathcal{P}_\infty(A)$ and $C \in \mathcal{I}^\perp$. The proof is completed. $\square$

2.2. **Trees and block sequences.** By $X^{<\mathbb{N}}$ we shall denote the set of all finite sequences in X. We view $X^{<\mathbb{N}}$ as a tree under the (strict) partial order $\sqsubset$ of end-extension. For every $s, t \in X^{<\mathbb{N}}$ by $s^\frown t$ we denote their concatenation. If T is a downwards closed subtree of $X^{<\mathbb{N}}$, then by $[T]$ we shall denote its body, that is, the set $\{\sigma \in X^{\mathbb{N}} : \sigma|n \in T \ \forall n \in \mathbb{N}\}$. Two nodes $s, t \in T$ are said to be *comparable* if either $t \sqsubseteq s$ or $s \sqsubseteq t$; otherwise they are said to be *incomparable*. A subset of T consisting of pairwise comparable nodes is said to be a *chain*, while a subset of T consisting of pairwise incomparable nodes is said to be an *antichain*.

By Σ we shall denote the downwards closed subtree of $\mathbb{N}^{<\mathbb{N}}$ consisting of all strictly increasing finite sequences. We view, however, every $t \in \Sigma$ not only as a finite increasing sequence but also as finite subset of $\mathbb{N}$. Given $s, t \in \Sigma \setminus \{\emptyset\}$ we write $s < t$ if $\max(s) < \min(t)$. By convention, we have $\emptyset < t$ for every $t \in \Sigma$ with $t \neq \emptyset$. If $s, t \in \Sigma$ with $s < t$, then we will frequently denote by $s \cup t$ the concatenation of s and t.

By $\mathbf{B}$ we shall denote the closed subset of $\Sigma^{\mathbb{N}}$ (Σ equipped with the discrete topology) consisting of all sequences (b_n) with $b_n \neq \emptyset$ and $b_n < b_{n+1}$ for every $n \in \mathbb{N}$. We call a sequence $\mathbf{b} = (b_n) \in \mathbf{B}$ a *block* sequence. For every block sequence $\mathbf{b} = (b_n)$ we set

$$(2) \qquad \langle \mathbf{b} \rangle := \Big\{ \bigcup_{n \in F} b_n : \emptyset \neq F \subseteq \mathbb{N} \text{ finite} \Big\} \ \text{ and } \ [\mathbf{b}] := \big\{ (c_n) \in \mathbf{B} : c_n \in \langle \mathbf{b} \rangle \ \forall n \big\}.$$

(Notice, in particular, that $\langle \mathbf{b} \rangle \subseteq \Sigma$.) We will need the following consequence of Milliken's theorem [Mil].

Theorem 2. *Let $\mathcal{X}$ be a C-measurable subset of $\mathbf{B}$. Then there exists $\mathbf{b} \in \mathbf{B}$ such that either $[\mathbf{b}] \subseteq \mathcal{X}$ or $\mathcal{X} \cap [\mathbf{b}] = \emptyset$.*

We recall that the class of C-measurable sets is strictly bigger than the σ-algebra generated by the analytic sets (see, for instance, [Ke]).

2.3. **Lusin gaps and related results.** Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}_\infty(X)$. A *perfect Lusin gap* inside $(\mathcal{A}, \mathcal{B})$ is a continuous, one-to-one map $2^{\mathbb{N}} \ni x \mapsto (A_x, B_x) \in \mathcal{A} \times \mathcal{B}$ such that the following are satisfied.

(a) For every $x \in 2^{\mathbb{N}}$ we have $A_x \cap B_x = \emptyset$.
(b) For every $x, y \in 2^{\mathbb{N}}$ with $x \neq y$ we have $(A_x \cap B_y) \cup (A_y \cap B_x) \neq \emptyset$.

The notion of a perfect Lusin gap was introduced by Todorčević. We notice that if there exists a perfect Lusin gap inside $(\mathcal{A}, \mathcal{B})$, then $\mathcal{A}$ and $\mathcal{B}$ are not countably separated. The following result of Todorčević [To2] shows that this the only case for a pair of analytic and orthogonal families.

Theorem 3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two analytic, hereditary and orthogonal families of infinite subsets of $\mathbb{N}$. Then, either*

(i) *$\mathcal{A}$ and $\mathcal{B}$ are countably separated, or*
(ii) *there exists a perfect Lusin gap inside $(\mathcal{A},\mathcal{B})$.*

Theorem 3 is a consequence of the open coloring axiom for $\mathbf{\Sigma}^1_1$ sets (see [Fe, To1]). We should point out that it is the perfectness of the gap which is essential in many applications. We refer the reader to [To2, To4] for more information.

We will also need the following slight reformulation of [To2, Theorem 3].

Theorem 4. *Let $\mathcal{A},\mathcal{B}\subseteq\mathcal{P}_\infty(\mathbb{N})$ be two hereditary orthogonal families. Assume that $\mathcal{A}$ is analytic and not countably generated in $\mathcal{B}^\perp$. Then there exists a one-to-one map $\phi\colon\Sigma\to\mathbb{N}$ such that, setting*

$$\mathcal{E}\coloneqq\{\phi^{-1}(A):A\in\mathcal{A}\}\quad\textit{and}\quad\mathcal{H}\coloneqq\{\phi^{-1}(B):B\in\mathcal{B}\},$$

the following are satisfied.

(i) *For every $\sigma\in[\Sigma]$ the set $\{\sigma|n:n\in\mathbb{N}\}$ belongs to $\mathcal{E}$.*
(ii) *For every $t\in\Sigma$ the set $\big\{t\cup\{n\}:n\in\mathbb{N}\text{ and }t<\{n\}\big\}$ of immediate successors of t in Σ belongs to $\mathcal{H}$.*

Proof. Assume that $\mathcal{A}$ is analytic, hereditary and not countably generated in $\mathcal{B}^\perp$. By [To2, Theorem 3], there exists a downwards closed subtree T of Σ such that the following are satisfied.

(B1) For every $\sigma\in[T]$ we have $\{\sigma(n):n\in\mathbb{N}\}\in\mathcal{A}$.
(B2) For every $t\in T$ the set $\{n\in\mathbb{N}:t<\{n\}\text{ and }t\cup\{n\}\in T\}$ is infinite and is included in an element of $\mathcal{B}$.

Recursively and using property (B2) above, we may select a downwards closed subtree S of T such that the following hold.

(a) For every $s\in S$ the set $\{n\in\mathbb{N}:s<\{n\}\text{ and }s\cup\{n\}\in S\}$ is infinite.
(b) For every $s,w\in S\setminus\{\emptyset\}$ with $s\neq w$ we have $\max(s)\neq\max(w)$.

Fix $m\in\mathbb{N}$ such that $(m)\in S$ and set $S_m\coloneqq\{t\in\Sigma:(m)^\frown t\in S\}$. By (a) above, S_m is an infinitely splitting, downwards closed subtree of Σ. Hence, there exists a bijection $h\colon\Sigma\to S_m$ such that $|t|=|h(t)|$ for every $t\in\Sigma$ and, moreover, $s\sqsubset t$ if and only if $h(s)\sqsubset h(t)$ for all $s,t\in\Sigma$. Now define $\phi\colon\Sigma\to\mathbb{N}$ as follows. We set $\phi(\emptyset)=m$, and for every $t\in\Sigma$ with $t\neq\emptyset$ let $\phi(t)=\max\big(h(t)\big)$. Notice that, by (b), the map ϕ is one-to-one. It is easy to check that ϕ is as desired. $\square$

3. Connections with related notions and examples

In this section we present the relation between M-families and other notions already studied in the literature. Let us start with the following fact which provides characterizations of M-families. The proof is left to the interested reader.

Fact 3. *Let X be a countable set and let $\mathcal{A} \subseteq \mathcal{P}_\infty(X)$ be a hereditary family. Then the following are equivalent.*

(i) *The family $\mathcal{A}$ is an M-family.*
(ii) *For every decreasing sequence (D_n) in $\mathrm{co}(\mathcal{A})$ there exists $A \in \mathcal{A}$ with $A \subseteq^* D_n$ for every $n \in \mathbb{N}$.*
(iii) *For every sequence (A_n) in $\mathcal{A}$ there exists $A \in \mathcal{A}$ such that $A \cap A_n \neq \emptyset$ for infinitely many $n \in \mathbb{N}$.*

The notion of an M-family is closely related to the notion of a selective co-ideal due to Mathias. We recall that a co-ideal $\mathcal{F}$ on $\mathbb{N}$ is said to be *selective*, or a *happy family* as it is called in [Ma], if for every decreasing sequence (D_n) in $\mathcal{F}$ there exists $D \in \mathcal{F}$ such that $D \setminus \{0, \dots, n\} \subseteq D_n$ for every $n \in D$. We have the following characterization of M-families which justifies our terminology.

Proposition 5. *Let $\mathcal{A}$ be a hereditary family on $\mathbb{N}$. Then $\mathcal{A}$ is an M-family if and only if the co-ideal $\mathrm{co}(\mathcal{A})$ generated by $\mathcal{A}$ is selective.*

Proof. First assume that the co-ideal $\mathrm{co}(\mathcal{A})$ is selective. Let (D_n) be a decreasing sequence in $\mathrm{co}(\mathcal{A})$. By the selectivity of $\mathrm{co}(\mathcal{A})$, there exists $D \in \mathrm{co}(\mathcal{A})$ with $D \setminus \{0, \dots, n\} \subseteq D_n$ for every $n \in D$. We select $A \in \mathcal{A}$ with $A \subseteq D$. Then $A \subseteq^* D_n$ for every $n \in \mathbb{N}$. By part (ii) of Fact 3, we see that $\mathcal{A}$ is an M-family.

Conversely, assume that $\mathcal{A}$ is an M-family. Let (D_n) be a decreasing sequence in $\mathrm{co}(\mathcal{A})$. By part (ii) of Fact 3, there exists $A \in \mathcal{A}$ with $A \subseteq^* D_n$ for every $n \in \mathbb{N}$. Recursively, we select a strictly increasing sequence (m_n) in $\mathbb{N}$ with $m_0 = \min(A)$ and $m_{n+1} \in A \cap D_{m_n}$ for every $n \in \mathbb{N}$. We set $D := \{m_n : n \in \mathbb{N}\}$. Then $D \subseteq A$ and $D \setminus \{0, \dots, n\} \subseteq D_n$ for every $n \in D$. Since $\mathcal{A}$ is hereditary, we obtain that $D \in \mathcal{A} \subseteq \mathrm{co}(\mathcal{A})$. Hence, $\mathrm{co}(\mathcal{A})$ is selective and the proof is completed. □

The following proposition shows that the notion of an M-family is, in a sense, the "dual" notion of bisequentiality.

Proposition 6. *Let X be a countable set.*

(i) *Let $\mathcal{A} \subseteq \mathcal{P}_\infty(X)$ be a hereditary family. If $\mathcal{A}^\perp$ is bisequential, then $\mathcal{A}$ is an M-family.*
(ii) *Let $\mathcal{I}$ be an ideal on X. If $\mathcal{I}$ is bisequential, then $\mathcal{I}^\perp$ is an M-family.*

Proof. (i) By part (ii) of Fact 3, it is enough to show that for every decreasing sequence (D_n) in $\mathrm{co}(\mathcal{A})$ there exists $A \in \mathcal{A}$ with $A \subseteq^* D_n$ for every $n \in \mathbb{N}$. So, let (D_n) be one. Since $\mathcal{A}^\perp$ is an ideal, the family $\{D_n \setminus L : n \in \mathbb{N} \text{ and } L \in \mathcal{A}^\perp\}$

has the finite intersection property. Hence, we may select $p \in \beta X$ with $\mathcal{A}^\perp \subseteq p^*$ and $D_n \in p$ for every $n \in \mathbb{N}$. Notice that p is non-principal. By the bisequentiality of $\mathcal{A}^\perp$, there exists a sequence (C_n) in p^* such that for every $B \in \mathcal{A}^\perp$ there exists $n \in \mathbb{N}$ with $B \subseteq^* C_n$. We may assume that the sequence (C_n) is increasing. Let $Q \in \mathcal{P}_\infty(X)$ be a diagonalization of the decreasing sequence $(D_n \setminus C_n)$. Then $Q \subseteq^* D_n$ and $Q \perp C_n$ for every $n \in \mathbb{N}$. By the properties of the sequence (C_n), we see that $Q \notin \mathcal{A}^\perp$. Since $\mathcal{A}$ is hereditary, there exists $A \subseteq Q$ with $A \in \mathcal{A}$. Hence $A \subseteq^* D_n$ for every $n \in \mathbb{N}$. Thus, $\mathcal{A}$ is an M-family.

(ii) By Fact 2, the ideal $\mathcal{I}$ has the Fréchet property. Thus, $\mathcal{I}^{\perp\perp}$ is bisequential and so the result follows by part (i). □

We notice that the converse of part (i) of Proposition 6 is also true, provided that the orthogonal $\mathcal{A}^\perp$ of $\mathcal{A}$ is analytic. Indeed, let $\mathcal{A}$ be an M-family such that $\mathcal{A}^\perp$ is $\mathbf{\Sigma}^1_1$. By Proposition 5, we see that the co-ideal $\mathrm{co}(\mathcal{A})$ generated by $\mathcal{A}$ is selective. It follows that $\mathcal{A}^\perp$ is an analytic ideal whose complement, $\mathrm{co}(\mathcal{A})$, is selective. By [To3, Exercise 12.3], we obtain that $\mathcal{A}^\perp$ is bisequential.

We proceed our discussion by presenting some examples of M-families.

Example 1. Let $\mathcal{I}_{\mathrm{c}}$ be the ideal on $\mathbb{N}^{<\mathbb{N}}$ generated by the infinite chains of $\mathbb{N}^{<\mathbb{N}}$. That is,

$$\mathcal{I}_{\mathrm{c}} := \Big\{ C \in \mathcal{P}_\infty(\mathbb{N}^{<\mathbb{N}}) : \exists \sigma_0, \dots, \sigma_k \in \mathbb{N}^{\mathbb{N}} \text{ with } C \subseteq \bigcup_{i=0}^{k} \{\sigma_i | n : n \in \mathbb{N}\} \Big\}. \tag{3}$$

Notice that $\mathcal{I}_{\mathrm{c}}$ has the Fréchet property. We set $\mathcal{A} = \mathcal{I}_{\mathrm{c}}^\perp$. Namely, $\mathcal{A}$ consists of all infinite subsets of $\mathbb{N}^{<\mathbb{N}}$ not containing an infinite chain. Then $\mathcal{A}$ is an ideal and it is easy to see that it is $\mathbf{\Pi}^1_1$-complete. The family $\mathcal{A}$ is an M-family. We will give a simple argument showing this. We will use part (ii) of Fact 3. So, let (D_n) be a decreasing sequence in $\mathrm{co}(\mathcal{A})$. For every $n \in \mathbb{N}$ there exists an infinite antichain A_n of $\mathbb{N}^{<\mathbb{N}}$ with $A_n \subseteq D_n$. Let $A_n = (t^n_m)$ be an enumeration of A_n. By an application of Ramsey's theorem, we may assume that $|t^n_m| \leqslant |t^k_l|$ for every $n < m < k < l$. Let

$$I := \big\{ (n < m < k < l) \in [\mathbb{N}]^4 : t^n_m \text{ is incomparable with } t^k_l \big\}.$$

By Ramsey's theorem again, there exists $L \in \mathcal{P}_\infty(\mathbb{N})$ such that either $[L]^4 \subseteq I$ or $[L]^4 \cap I = \emptyset$. Let $\{l_0 < l_1 < \cdots\}$ be the increasing enumeration of L. We claim that $[L]^4 \subseteq I$. If not, then $t^{l_0}_{l_1}$ is comparable with $t^{l_3}_{l_4}$ and since $|t^{l_0}_{l_1}| \leqslant |t^{l_3}_{l_4}|$, we obtain that $t^{l_0}_{l_1} \sqsubseteq t^{l_3}_{l_4}$. Similarly, we obtain that $t^{l_0}_{l_2} \sqsubseteq t^{l_3}_{l_4}$. But this implies that the nodes $t^{l_0}_{l_1}$ and $t^{l_0}_{l_2}$ are comparable, contradicting the fact that A_{l_0} is an antichain. Thus $[L]^4 \subseteq I$. Now set $A := \{t^{l_{2n}}_{l_{2n+1}} : n \in \mathbb{N}\}$. Then A is an infinite antichain, and so, $A \in \mathcal{A}$. Since $A \subseteq^* D_n$ for every $n \in \mathbb{N}$, this shows that $\mathcal{A}$ is an M-family.

Example 2. We notice that if an ideal $\mathcal{I}$ has the Fréchet property, then $\mathcal{I}^\perp$ is not necessarily an M-family. For instance, let $\mathcal{I}_{\mathrm{d}}$ be the ideal of all dominated subsets of $\mathbb{N}^{<\mathbb{N}}$, that is,

$$(4) \qquad \mathcal{I}_{\mathrm{d}} := \big\{ D \in \mathcal{P}_\infty(\mathbb{N}^{<\mathbb{N}}) : \exists \sigma \in \mathbb{N}^{\mathbb{N}} \text{ such that } \forall t \in D \ \forall i < |t| \ t(i) < \sigma(i) \big\}.$$

Also let

$$(5) \qquad \mathcal{I}_{\mathrm{wf}} := \big\{ W \in \mathcal{P}_\infty(\mathbb{N}^{<\mathbb{N}}) : \exists T \in \mathrm{WF} \text{ with } W \subseteq T \big\}$$

be the ideal on $\mathbb{N}^{<\mathbb{N}}$ generated by the set WF of all downwards closed, well-founded, infinite subtrees of $\mathbb{N}^{<\mathbb{N}}$. Clearly, $\mathcal{I}_{\mathrm{c}} \subseteq \mathcal{I}_{\mathrm{d}}$. It is easy to see that $\mathcal{I}_{\mathrm{d}}^{\perp} = \mathcal{I}_{\mathrm{wf}}$ and $\mathcal{I}_{\mathrm{wf}}^{\perp} = \mathcal{I}_{\mathrm{d}}$. Hence, the ideal $\mathcal{I}_{\mathrm{d}}$ has the Fréchet property. As in the above example, we set $\mathcal{A} = \mathcal{I}_{\mathrm{d}}^{\perp} = \mathcal{I}_{\mathrm{wf}}$. Again we see that $\mathcal{A}$ is a $\mathbf{\Pi}^1_1$-complete ideal. However, $\mathcal{A}$ is not an M-family. To see this, for every $n \in \mathbb{N}$ let $D_n = \{ t \in \mathbb{N}^{<\mathbb{N}} : 0^{n+1} \sqsubseteq t \}$. Then (D_n) is a decreasing sequence of sets in $\mathrm{co}(\mathcal{A})$. It is easy to check that if A is any infinite subset of $\mathbb{N}^{<\mathbb{N}}$ with $A \subseteq^* D_n$ for every $n \in \mathbb{N}$, then A must belong to $\mathcal{I}_{\mathrm{d}}$.

Example 3. Let E be a Polish space and let $\mathbf{f} = \{f_n\}$ be a pointwise bounded sequence of real-valued Baire-1 functions on E. Assume that the closure $\mathcal{K}$ of $\{f_n\}$ in $\mathbb{R}^E$ is a subset of the set of all Baire-1 functions on E, that is, $\mathcal{K}$ is a separable Rosenthal compact (see [Ro]). Let $f \in \mathcal{K}$ and set

$$(6) \qquad \mathcal{L}_f := \big\{ L \in \mathcal{P}_\infty(\mathbb{N}) : (f_n)_{n \in L} \text{ converges pointwise to } f \big\}.$$

The family $\mathcal{L}_f$ is a $\mathbf{\Pi}^1_1$ ideal. Also let

$$(7) \qquad \mathcal{I}_f := \big\{ L \in \mathcal{P}_\infty(\mathbb{N}) : f \notin \overline{\{f_n\}}^{p}_{n \in L} \big\}.$$

It is easy to see that $\mathcal{I}_f$ is a $\mathbf{\Sigma}^1_1$ ideal. Both $\mathcal{L}_f$ and $\mathcal{I}_f$ are well studied in the literature (see [ADK, Do, Kr, To3, To4]). By a result of Bourgain, Fremlin and Talagrand [BFT], we obtain that the orthogonal $\mathcal{L}_f^{\perp}$ of $\mathcal{L}_f$ is the family $\mathcal{I}_f$. An important fact concerning the structure of $\mathcal{I}_f$ is that it is bisequential. This is due to Pol [Po] and it can be also derived by the results of Debs in [De]. Hence, by part (i) of Proposition 6, we see that $\mathcal{L}_f$ is an M-family. Moreover, set

$$(8) \qquad \mathcal{F}_f := \big\{ L \in \mathcal{P}_\infty(\mathbb{N}) : f \in \overline{\{f_n\}}^{p}_{n \in L} \big\} = \mathcal{P}_\infty(\mathbb{N}) \setminus \mathcal{I}_f.$$

The equality $\mathcal{L}_f^{\perp} = \mathcal{I}_f$ yields that the co-ideal $\mathrm{co}(\mathcal{L}_f)$ generated by $\mathcal{L}_f$ is the family $\mathcal{F}_f$. By Proposition 5, it follows that $\mathcal{F}_f$ is a selective co-ideal, a fact discovered by Todorčević [To3].

4. Properties of M-families

This section is devoted to the study of the structural properties of M-families. We begin by noticing the following fact (the proof is left to the reader).

Fact 4. *Let X be a countable set.*

(i) *If $\mathcal{A} \subseteq \mathcal{P}_\infty(X)$ is a hereditary family and $\mathcal{B}$ is a hereditary subfamily of $\mathcal{A}$ cofinal in $\mathcal{A}$, then $\mathcal{A}$ is an M-family if and only if $\mathcal{B}$ is.*

(ii) *If $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}_\infty(X)$ are two M-families, then so is $\mathcal{A} \cup \mathcal{B}$.*

Most of the properties of M-families we will establish, are derived using an infinite-dimensional Ramsey-type argument. To state it, we need to introduce some pieces of notation. Let $\mathbf{C} = (C_n)$ be a sequence in $\mathcal{P}_\infty(\mathbb{N})$ such that $C_n \cap C_m = \emptyset$ for every $n \neq m$. For every $n \in \mathbb{N}$ let $\{x_0^n < x_1^n < \cdots\}$ be the increasing enumeration of the set C_n. We define $\Delta_{\mathbf{C}} \colon \mathcal{P}_\infty(\mathbb{N}) \to \mathcal{P}_\infty(\mathbb{N})$ as follows. If $L \in \mathcal{P}_\infty(\mathbb{N})$ with $L = \{l_0 < l_1 < \cdots\}$ its increasing enumeration, we set

$$\Delta_{\mathbf{C}}(L) = \big\{x_{l_{2n+1}}^{l_{2n}} : n \in \mathbb{N}\big\}. \tag{9}$$

Notice that the map $\Delta_{\mathbf{C}}$ is continuous.

Lemma 7. *Let $\mathcal{A} \subseteq \mathcal{P}_\infty(\mathbb{N})$ be an M-family and let $\mathbf{C} = (C_n)$ be a sequence in $\mathcal{A}$ such that $C_n \cap C_m = \emptyset$ for every $n \neq m$. Assume that $\mathcal{A}$ is C-measurable. Then for every $N \in \mathcal{P}_\infty(\mathbb{N})$ there exists $L \in \mathcal{P}_\infty(N)$ such that $\Delta_{\mathbf{C}}(M) \in \mathcal{A}$ for every $M \in \mathcal{P}_\infty(L)$.*

Proof. Let

$$\mathcal{C}_{\mathcal{A}} \coloneqq \{M \in \mathcal{P}_\infty(\mathbb{N}) : \Delta_{\mathbf{C}}(M) \in \mathcal{A}\}.$$

Then $\mathcal{C}_{\mathcal{A}}$ is C-measurable. By Ellentuck's theorem [El], there exists $L \in \mathcal{P}_\infty(N)$ such that either $\mathcal{P}_\infty(L) \subseteq \mathcal{C}_{\mathcal{A}}$ or $\mathcal{P}_\infty(L) \cap \mathcal{C}_{\mathcal{A}} = \emptyset$. It is enough to show that $\mathcal{P}_\infty(L) \cap \mathcal{C}_{\mathcal{A}} \neq \emptyset$. To this end we argue as follows. For every $n \in L$ we set

$$H_n \coloneqq \{x_i^n : i \in L \text{ and } i > n\}.$$

Then $H_n \subseteq C_n$ and so $H_n \in \mathcal{A}$ for every $n \in L$. By part (iii) of Fact 3, there exists $A \in \mathcal{A}$ such that $A \cap H_n \neq \emptyset$ for infinitely many $n \in L$. We may select $M = \{m_0 < m_1 < \cdots\} \in \mathcal{P}_\infty(L)$ such that $x_{m_{2n+1}}^{m_{2n}} \in A \cap H_{m_{2n}}$ for every $n \in \mathbb{N}$. Then $\Delta_{\mathbf{C}}(M) \subseteq A$. Since $\mathcal{A}$ is hereditary, we see that $\Delta_{\mathbf{C}}(M) \in \mathcal{A}$. Therefore, $\mathcal{P}_\infty(L) \cap \mathcal{C}_{\mathcal{A}} \neq \emptyset$ and the proof is completed. $\square$

The following proposition is the first application of Lemma 7.

Proposition 8. *Let X be a countable set and let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}_\infty(X)$ be two M-families. If $\mathcal{A}$ and $\mathcal{B}$ are C-measurable, then $\mathcal{A} \cap \mathcal{B}$ is an M-family.*

Proof. Clearly we may assume that $X = \mathbb{N}$. In order to show that $\mathcal{A} \cap \mathcal{B}$ is an M-family we will use part (ii) of Fact 3. So, let (D_n) be a decreasing sequence in $\mathrm{co}(\mathcal{A} \cap \mathcal{B})$. Since the family $\mathcal{A} \cap \mathcal{B}$ is hereditary, there exists a sequence $\mathbf{C} = (C_n)$ in $\mathcal{A} \cap \mathcal{B}$ with $C_n \subseteq D_n$ for every $n \in \mathbb{N}$. Refining if necessary, we may assume that $C_n \cap C_m = \emptyset$ for every $n \neq m$. Applying Lemma 7 successively two times, we find $L \in \mathcal{P}_\infty(\mathbb{N})$ such that $\Delta_{\mathbf{C}}(M) \in \mathcal{A}$ and $\Delta_{\mathbf{C}}(M) \in \mathcal{B}$ for every $M \in \mathcal{P}_\infty(L)$. Finally observe that $\Delta_{\mathbf{C}}(M) \subseteq^* D_n$ for every $n \in \mathbb{N}$ and every $M \in \mathcal{P}_\infty(L)$. The proof is completed. $\square$

Let $A, B \in \mathcal{P}_\infty(\mathbb{N})$ with $A = \{x_0 < x_1 < \cdots\}$ and $B = \{y_0 < y_1 < \cdots\}$ their increasing enumerations. We define the *diagonal product* $A \otimes B$ of A and B by

$$A \otimes B \coloneqq \{(x_n, y_n) : n \in \mathbb{N}\} \in \mathcal{P}_\infty(\mathbb{N} \times \mathbb{N}). \tag{10}$$

If $\mathcal{A},\mathcal{B}\subseteq\mathcal{P}_\infty(\mathbb{N})$ are two hereditary families, then we let

$$\mathcal{A}\otimes\mathcal{B}\coloneqq\{A\otimes B: A\in\mathcal{A} \text{ and } B\in\mathcal{B}\}. \tag{11}$$

Notice that $\mathcal{A}\otimes\mathcal{B}$ is a hereditary subfamily of $\mathcal{P}_\infty(\mathbb{N}\times\mathbb{N})$. We have the following proposition.

Proposition 9. *Let $\mathcal{A},\mathcal{B}\subseteq\mathcal{P}_\infty(\mathbb{N})$ be M-families. If $\mathcal{A}$ and $\mathcal{B}$ are C-measurable, then $\mathcal{A}\otimes\mathcal{B}$ is an M-family.*

Proof. Let (D_n) be a decreasing sequence in $\mathrm{co}(\mathcal{A}\otimes\mathcal{B})$. There exist sequences $\mathbf{A}=(A_n)$ and $\mathbf{B}=(B_n)$ in $\mathcal{A}$ and $\mathcal{B}$ respectively such that $A_n\otimes B_n\subseteq D_n$ for every $n\in\mathbb{N}$. Since the families $\mathcal{A}$ and $\mathcal{B}$ are hereditary, we may assume that $A_n\cap A_m=\emptyset$ and $B_n\cap B_m=\emptyset$ for every $n\neq m$. For every $n\in\mathbb{N}$ let $\{x^n_0<x^n_1<\cdots\}$ and $\{y^n_0<y^n_1<\cdots\}$ be the increasing enumerations of the sets A_n and B_n respectively. Applying Lemma 7 successively two times, we find $L\in\mathcal{P}_\infty(\mathbb{N})$ such that $\Delta_{\mathbf{A}}(M)\in\mathcal{A}$ and $\Delta_{\mathbf{B}}(M)\in\mathcal{B}$ for every $M\in\mathcal{P}_\infty(L)$. We select $I=\{i_0<i_1<\cdots\}\in\mathcal{P}_\infty(L)$ such that $x^{i_{2n}}_{i_{2n+1}}<x^{i_{2k}}_{i_{2k+1}}$ and $y^{i_{2n}}_{i_{2n+1}}<y^{i_{2k}}_{i_{2k+1}}$ for every $n<k$. It follows that

$$\Delta_{\mathbf{A}}(I)\otimes\Delta_{\mathbf{B}}(I)=\big\{(x^{i_{2n}}_{i_{2n+1}},y^{i_{2n}}_{i_{2n+1}}):n\in\mathbb{N}\big\}.$$

Therefore, $\Delta_{\mathbf{A}}(I)\otimes\Delta_{\mathbf{B}}(I)\subseteq^* D_n$ for every $n\in\mathbb{N}$ and $\Delta_{\mathbf{A}}(I)\otimes\Delta_{\mathbf{B}}(I)\in\mathcal{A}\otimes\mathcal{B}$. By part (ii) of Fact 3, we see that $\mathcal{A}\otimes\mathcal{B}$ is an M-family and the proof is completed. □

Proposition 9 has some topological implications which we are about to describe. Let us recall, first, some definitions. Let (Y,τ) be a (Hausdorff) topological space. A point $y\in Y$ is said to have the *weak diagonal sequence property* if for every doubly indexed sequence (y^n_k) in Y with $\lim_k y^n_k=y$ for every $n\in\mathbb{N}$, there exist $L\in\mathcal{P}_\infty(\mathbb{N})$ and a sequence $(k_n)_{n\in L}$ in $\mathbb{N}$ such that $\lim_{n\in L}y^n_{k_n}=y$. The space (Y,τ) has the weak diagonal sequence property if every point $y\in Y$ has it. Using part (iii) of Fact 3, it is easy to see that if X is a countable set, τ is a topology on X and $x\in X$, then the point x has the weak diagonal sequence property in the space (X,τ) if and only if the family $\mathcal{C}_x\coloneqq\{A\in\mathcal{P}_\infty(X):A\xrightarrow{\tau}x\}$ is an M-family. The following corollary of Proposition 9 yields a positive answer to Question 5.4 from [TU].

Corollary 10. *Let X,Y be two countable sets and τ_1,τ_2 two analytic topologies on X and Y respectively. Assume that both (X,τ_1) and (Y,τ_2) have the weak diagonal sequence property. Then $(X\times Y,\tau_1\times\tau_2)$ has the weak diagonal sequence property.*

Proof. Clearly we may assume that $X=Y=\mathbb{N}$. Let $x,y\in\mathbb{N}$ be arbitrary. As we have already remarked, it is enough to show that the family

$$\mathcal{C}_{(x,y)}\coloneqq\{C\in\mathcal{P}_\infty(\mathbb{N}\times\mathbb{N}):C\xrightarrow{\tau_1\times\tau_2}(x,y)\}$$

is an M-family. By our assumptions on τ_1 and τ_2, we see that the families

$$\mathcal{C}_x\coloneqq\{A\in\mathcal{P}_\infty(\mathbb{N}):A\xrightarrow{\tau_1}x\}\ \text{ and }\ \mathcal{C}_y\coloneqq\{B\in\mathcal{P}_\infty(\mathbb{N}):B\xrightarrow{\tau_2}y\}$$

are both co-analytic M-families on $\mathbb{N}$. By Proposition 9, it follows that the family $\mathcal{C}_x \otimes \mathcal{C}_y$ is an M-family. Notice that $\mathcal{C}_x \otimes \mathcal{C}_y \subseteq \mathcal{C}_{(x,y)}$. We set

$$\mathcal{C}^x_{(x,y)} \coloneqq \big\{C \in \mathcal{C}_{(x,y)} : C \subseteq \{x\} \times \mathbb{N}\big\} \text{ and } \mathcal{C}^y_{(x,y)} \coloneqq \big\{C \in \mathcal{C}_{(x,y)} : C \subseteq \mathbb{N} \times \{y\}\big\}.$$

Since $\mathcal{C}_y$ and $\mathcal{C}_x$ are M-families, it is easy to see that so are $\mathcal{C}^x_{(x,y)}$ and $\mathcal{C}^y_{(x,y)}$. By part (ii) of Fact 4, it follows that the family

$$\mathcal{B} \coloneqq \mathcal{C}^x_{(x,y)} \cup \mathcal{C}^y_{(x,y)} \cup (\mathcal{C}_x \otimes \mathcal{C}_y)$$

is an M-family. Now observe that $\mathcal{B}$ is a hereditary subfamily of $\mathcal{C}_{(x,y)}$ which is cofinal in $\mathcal{C}_{(x,y)}$. Hence, by part (i) of Fact 4, we conclude that $\mathcal{C}_{(x,y)}$ is an M-family and the proof is completed. □

We notice that, after a first draft of the paper, Todorčević informed us that he was also aware of the fact that the weak diagonal sequence property is productive within the class of countable analytic spaces.

We proceed by presenting another application of Lemma 7. To this end, let us notice that, by Fact 1, if $\mathcal{A}$ is a hereditary family, then $\mathcal{A}$ is cofinal in $\mathcal{A}^{\perp\perp}$. Hence, by part (i) of Fact 4, we see that if $\mathcal{A}$ is an M-family, then so is $\mathcal{A}^{\perp\perp}$. We have the following strengthening of part (iii) of Fact 3 for the family $\mathcal{A}^{\perp\perp}$, provided that $\mathcal{A}$ is reasonably definable.

Proposition 11. *Let X be a countable set and let $\mathcal{A} \subseteq \mathcal{P}_\infty(X)$ be an M-family and C-measurable. Then, for every sequence (A_n) in $\mathcal{A}^{\perp\perp}$ there exists $A \in \mathcal{A}^{\perp\perp}$ such that $A \cap A_n$ is infinite for infinitely many $n \in \mathbb{N}$.*

Proof. Clearly we may assume that $X = \mathbb{N}$. Let (A_n) be a sequence in $\mathcal{A}^{\perp\perp}$. By Fact 1, we may select a sequence $\mathbf{C} = (C_n)$ in $\mathcal{A}$ such that $C_n \subseteq A_n$ for every $n \in \mathbb{N}$ and $C_n \cap C_m = \emptyset$ for every $n \neq m$. By Lemma 7, there exists $L \in \mathcal{P}_\infty(\mathbb{N})$ such that $\Delta_{\mathbf{C}}(M) \in \mathcal{A}$ for every $M \in \mathcal{P}_\infty(L)$. For every $n \in \mathbb{N}$ let $\{x^n_0 < x^n_1 < \cdots\}$ be the increasing enumeration of the set C_n. We set

$$A \coloneqq \bigcup_{n \in L} \{x^n_i : i \in L \text{ and } i > n\}.$$

We claim that A is the desired set. First we notice that $A \cap C_n$ is infinite for every $n \in L$, and so, $A \cap A_n$ is infinite for infinitely many $n \in \mathbb{N}$. What remains is to show that $A \in \mathcal{A}^{\perp\perp}$. To this end, let $B \in \mathcal{P}_\infty(A)$ be arbitrary. It is easy to see that either there exists $n \in L$ such that $B \cap C_n$ is infinite, or there exists $M \in \mathcal{P}_\infty(L)$ such that $\Delta_{\mathbf{C}}(M) \subseteq B$. Since $\mathcal{A}$ is hereditary and $\Delta_{\mathbf{C}}(M) \in \mathcal{A}$ for every $M \in \mathcal{P}_\infty(L)$, we see that B contains an element of $\mathcal{A}$. Hence, by Fact 1, we conclude that $A \in \mathcal{A}^{\perp\perp}$ and the result follows. □

The following corollary is simply a restatement of Proposition 11 in the topological setting.

Corollary 12. *Let X be a countable set and τ an analytic topology on X. Assume that (X,τ) is Fréchet and has the weak diagonal sequence property. Let $x \in X$ and set $\mathcal{C}_x := \{A \in \mathcal{P}_\infty(X) : A \xrightarrow{\tau} x\}$. Then for every sequence (A_n) is $\mathcal{C}_x$ there exists $A \in \mathcal{C}_x$ such that $A \cap A_n$ is infinite for infinitely many $n \in \mathbb{N}$.*

Proof. As we have already seen in Corollary 10, the family $\mathcal{C}_x$ is a co-analytic M-family. Moreover, the assumption that (X,τ) is a Fréchet space simply reduces to the fact that $\mathcal{C}_x^{\perp\perp} = \mathcal{C}_x$. So the result follows by Proposition 11. □

We close this section with the following result concerning the effect of the notion of an M-family in the context of separation of families.

Proposition 13. *Let X be a countable set and let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}_\infty(X)$ be two hereditary families. Assume that $\mathcal{B}$ is an M-family. Then the following are equivalent.*

(i) *$\mathcal{A}$ and $\mathcal{B}$ are countably separated.*
(ii) *$\mathcal{A}$ is countably generated in $\mathcal{B}^\perp$.*

Proof. It is clear that (ii) implies (i). So we only have to show the other implication. Let us fix a sequence (C_n) in $\mathcal{P}_\infty(X)$ which separates $\mathcal{A}$ from $\mathcal{B}$. For every nonempty finite $F \subseteq \mathbb{N}$ we set $C_F := \bigcap_{n\in F} C_n$.

Claim. *For every $A \in \mathcal{A}$ there exists nonempty finite $F \subseteq \mathbb{N}$ such that $A \subseteq C_F$ and $C_F \in \mathcal{B}^\perp$.*

Proof of the claim. Assume not. Thus, there exists $A_0 \in \mathcal{A}$ such that for every nonempty finite $F \subseteq \mathbb{N}$ either $A_0 \nsubseteq C_F$ or $C_F \notin \mathcal{B}^\perp$. Set

$$L := \{n \in \mathbb{N} : A_0 \subseteq C_n\}$$

and note that L is nonempty. We claim that L is infinite. Assume not. Then $A_0 \subseteq C_L$ and so, by our assumptions, we obtain that $C_L \notin \mathcal{B}^\perp$. Hence, there exists $B_L \in \mathcal{B}$ with $B_L \subseteq C_L$. It follows that for every $n \in \mathbb{N}$ either $A_0 \nsubseteq C_n$ (that is, $n \notin L$) or $B_L \subseteq C_L \subseteq C_n$. This means that A_0 and B_L cannot be separated by the sequence (C_n), a contradiction.

Now let $\{l_0 < l_1 < \cdots\}$ be the increasing enumeration of L. For every $k \in \mathbb{N}$ set $D_k := C_{l_0} \cap \cdots \cap C_{l_k}$. Clearly (D_k) is a decreasing sequence. By our assumptions we see that $D_k \notin \mathcal{B}^\perp$, and so, $D_k \in \mathrm{co}(\mathcal{B})$ for every $k \in \mathbb{N}$. Since $\mathcal{B}$ is an M-family, by part (ii) of Fact 3, we see that there exists $B_0 \in \mathcal{B}$ such that $B_0 \subseteq^* D_k$ for every $k \in \mathbb{N}$. It follows that $B_0 \subseteq^* C_n$ for every $n \in L$. But then, for every $n \in \mathbb{N}$ we have that either $A_0 \nsubseteq C_n$ or $B_0 \subseteq^* C_n$. That it, the sets A_0 and B_0 cannot be separated by the sequence (C_n), a contradiction again. The claim is proved. □

By the above claim, for every $A \in \mathcal{A}$ there exists nonempty finite $F_A \subseteq \mathbb{N}$ with $C_{F_A} \in \mathcal{B}^\perp$ and $A \subseteq C_{F_A}$. The family $\{C_{F_A} : A \in \mathcal{A}\}$ is clearly countable, and so, $\mathcal{A}$ is countably generated in $\mathcal{B}^\perp$. The proof of Proposition 13 is completed. □

5. Proof of Theorem I

This section is devoted to the proof of Theorem I stated in the introduction. So, let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}_\infty(\mathbb{N})$ be a pair of hereditary orthogonal families such that $\mathcal{A}$ is $\boldsymbol{\Sigma}^1_1$ and $\mathcal{B}$ is C-measurable and an M-family. Assume that (i) does not hold true, that is, $\mathcal{A}$ is not countably generated in $\mathcal{B}^\perp$. We will find a perfect Lusin gap inside $(\mathcal{A}, \mathcal{B})$.

By Theorem 4, there exists a one-to-one map $\phi\colon \Sigma \to \mathbb{N}$ such that, setting

$$\mathcal{E} \coloneqq \{\phi^{-1}(A) : A \in \mathcal{A}\} \text{ and } \mathcal{H} \coloneqq \{\phi^{-1}(B) : B \in \mathcal{B}\},$$

properties (i) and (ii) of Theorem 4 are satisfied for $\mathcal{E}$ and $\mathcal{H}$. In what follows, we will work inside the tree Σ and with the families $\mathcal{E}$ and $\mathcal{H}$. Denote by $\mathcal{C}$ the family of all infinite chains of Σ. That is,

$$\mathcal{C} \coloneqq \big\{C \in \mathcal{P}_\infty(\Sigma) : \exists \sigma \in [\Sigma] \text{ with } C \subseteq \{\sigma|n : n \in \mathbb{N}\}\big\}.$$

Clearly $\mathcal{C}$ is a $\boldsymbol{\Pi}^0_2$ hereditary family. We notice the following properties of the families $\mathcal{E}$ and $\mathcal{H}$.

- (P1) $\mathcal{E}$ and $\mathcal{H}$ are hereditary and orthogonal.
- (P2) $\mathcal{E}$ is analytic and $\mathcal{C} \subseteq \mathcal{E}$.
- (P3) $\mathcal{H}$ is C-measurable and an M-family.
- (P4) For every $t \in \Sigma$ we have $\big\{t \cup \{n\} : n \in \mathbb{N} \text{ and } t < \{n\}\big\} \in \mathcal{H}$.

Properties (P1)–(P4) are rather straightforward consequences of the the way the families $\mathcal{E}$ and $\mathcal{H}$ are defined and of the fact that the map ϕ is one-to-one.

We are going to define a class of subsets of Σ which will play a decisive role in the proof of Theorem I.

Definition 14. *Let $\sigma \in [\Sigma]$ and $D \in \mathcal{P}_\infty(\Sigma)$. We say that D* descends *to σ, in symbols $D \downarrow \sigma$, if for every $k \in \mathbb{N}$ the set D is almost included in the set $\{t \in \Sigma : \sigma|k \sqsubseteq t\}$. We call such a set D a* descender.

We also need to introduce some pieces of notation. Let $\mathbf{B}$ be the set of all block sequences of Σ. For every $\mathbf{b} = (b_n) \in \mathbf{B}$ we set

$$\Sigma_{\mathbf{b}} \coloneqq \{t \in \Sigma : \exists b \in \langle \mathbf{b} \rangle \text{ with } t \sqsubseteq b\} \text{ and } \sigma_{\mathbf{b}} \coloneqq \bigcup_n b_n \tag{12}$$

where the set $\langle \mathbf{b} \rangle$ was defined in §2.2. Clearly $\Sigma_{\mathbf{b}}$ is a downwards closed subtree of Σ. Notice that $\sigma_{\mathbf{b}}$ is just the leftmost branch of the tree $\Sigma_{\mathbf{b}}$. We also observe the following properties.

- (O1) The set $[\Sigma_{\mathbf{b}}]$ of all branches of $\Sigma_{\mathbf{b}}$ is in one-to-one correspondence with the subsequences of $\mathbf{b} = (b_n)$. In particular, for every $\sigma \in [\Sigma_{\mathbf{b}}]$ there exists a unique subsequence (b_{l_n}) of (b_n), which we shall denote by $\mathbf{b}_\sigma$, such that $\sigma = \bigcup_n b_{l_n}$. Moreover, the map $[\Sigma_{\mathbf{b}}] \ni \sigma \mapsto \mathbf{b}_\sigma \in [\mathbf{b}]$ is continuous.
- (O2) If $\mathbf{c} \in [\mathbf{b}]$, then $\Sigma_{\mathbf{c}}$ is a downwards closed subtree of $\Sigma_{\mathbf{b}}$.

We define $\Delta\colon \mathbf{B} \to \mathcal{P}_\infty(\Sigma)$ by

$$\Delta\big((b_n)\big) = \Big\{ b_0 \cup \{\min(b_2)\}, \dots, \bigcup_{i=0}^{3n} b_i \cup \{\min(b_{3n+2})\}, \dots \Big\}. \tag{13}$$

We notice the following properties.

(O3) The map Δ is continuous.

(O4) For every block sequence $\mathbf{b} = (b_n)$ the set $\Delta(\mathbf{b})$ is a subset of the tree $\Sigma_{\mathbf{b}}$, is a descender and descends to the leftmost branch $\sigma_{\mathbf{b}} = \bigcup_n b_n$ of $\Sigma_{\mathbf{b}}$. Moreover, the sets $\{\sigma_{\mathbf{b}}|n : n \in \mathbb{N}\}$ and $\Delta(\mathbf{b})$ are disjoint.

The following lemma is a consequence of Theorem 2 and of the fact that $\mathcal{H}$ is an M-family. It can be considered as a parameterized version of Lemma 7. We notice that the arguments in its proof follow similar lines as in [ADK, Lemma 44].

Lemma 15. *There exists* $\mathbf{b} \in \mathbf{B}$ *such that* $\Delta(\mathbf{c}) \in \mathcal{H}$ *for every* $\mathbf{c} \in [\mathbf{b}]$.

Proof. We set

$$\mathcal{X} \coloneqq \{\mathbf{c} \in \mathbf{B} : \Delta(\mathbf{c}) \in \mathcal{H}\}.$$

Then $\mathcal{X}$ is a C-measurable subset of $[\mathbf{B}]$. By Theorem 2, there exists $\mathbf{b} = (b_n) \in \mathbf{B}$ such that $[\mathbf{b}]$ is monochromatic. We claim that $[\mathbf{b}] \subseteq \mathcal{X}$. To this end, we argue as follows. For every $n \in \mathbb{N}$ we set $t_n \coloneqq \bigcup_{k \leqslant n} b_k \in \Sigma$ and

$$A_n \coloneqq \big\{ t_n \cup \{\min(b_i)\} : i > n+1 \big\} \in \mathcal{P}_\infty(\Sigma).$$

The set A_n is a subset of the set $\big\{ t_n \cup \{m\} : m \in \mathbb{N} \text{ and } t_n < \{m\} \big\}$ which, by property (P4) above, belongs to $\mathcal{H}$. Since the family $\mathcal{H}$ is hereditary, we see that $A_n \in \mathcal{H}$ for every $n \in \mathbb{N}$. Invoking the fact that $\mathcal{H}$ is an M-family and part (iii) of Fact 3, there exists $A \in \mathcal{H}$ such that $A \cap A_n \neq \emptyset$ for infinitely many $n \in \mathbb{N}$. We may select $L = \{l_0 < l_1 < \cdots\}, M = \{i_0 < i_1 < \cdots\} \in \mathcal{P}_\infty(\mathbb{N})$ with $l_n + 1 < i_n < l_{n+1}$ and such that $t_{l_n} \cup \{\min(b_{i_n})\} \in A \cap A_{l_n}$ for every $n \in \mathbb{N}$. We set $s_n \coloneqq t_{l_n} \cup \{\min(b_{i_n})\}$ for every $n \in \mathbb{N}$. It follows that $\{s_n : n \in \mathbb{N}\} \in \mathcal{H}$ since $\{s_n : n \in \mathbb{N}\} \subseteq A \in \mathcal{H}$ and the family $\mathcal{H}$ is hereditary.

Now we define $\mathbf{c} = (c_n) \in [\mathbf{b}]$ as follows. We set $c_0 \coloneqq \bigcup_{k \leqslant l_0} b_n$ (that is, $c_0 = t_{l_0}$), $c_1 \coloneqq b_{l_0+1} \cup \cdots \cup b_{i_0-1}$ and $c_2 = b_{i_0}$. For every $n \geqslant 1$ let $I_n = [i_{n-1}+1, l_n]$ and $J_n = [l_n + 1, i_n - 1]$, and set

$$c_{3n} \coloneqq \bigcup_{k \in I_n} b_k, \ c_{3n+1} \coloneqq \bigcup_{k \in J_n} b_k \ \text{ and } \ c_{3n+2} \coloneqq b_{i_n}.$$

Clearly $\mathbf{c} \in [\mathbf{b}]$ and it is easy to see that $\Delta(\mathbf{c}) = \{s_n : n \in \mathbb{N}\}$. Thus, $\Delta(\mathbf{c}) \in \mathcal{H}$. It follows that $[\mathbf{b}] \cap \mathcal{X} \neq \emptyset$. Therefore, $[\mathbf{b}] \subseteq \mathcal{X}$ and the lemma is proved. $\square$

Let $\mathbf{b} = (b_n)$ be the block sequence obtained by Lemma 15. We set

$$\mathcal{F} \coloneqq \big\{ A \in \mathcal{P}_\infty(\Sigma) : \exists (b_{l_n}) \text{ subsequence of } (b_n) \text{ with } A \subseteq \Delta\big((b_{l_n})\big) \big\}. \tag{14}$$

By property (P1), the family $\mathcal{H}$ is hereditary. Hence, using the continuity of the map Δ and the fact that $\Delta(\mathbf{c}) \in \mathcal{H}$ for every $\mathbf{c} \in [\mathbf{b}]$, we see that

(P5) $\mathcal{F}$ is a hereditary analytic subfamily of $\mathcal{H}$.

Now consider the tree $\Sigma_{\mathbf{b}}$ corresponding to $\mathbf{b}$ as it was defined in (12) above and let $\sigma \in [\Sigma_{\mathbf{b}}]$ be arbitrary. By (O1), there exists a subsequence $\mathbf{b}_\sigma = (b_{l_n})$ of (b_n) such that $\sigma = \bigcup_n b_{l_n}$. By (O4) and (O2), we obtain that $\Delta\big((b_{l_n})\big) \subseteq \Sigma_{\mathbf{b}_\sigma} \subseteq \Sigma_{\mathbf{b}}$. Moreover, the set $\Delta\big((b_{l_n})\big)$ descends to σ and, by definition, belongs to the family $\mathcal{F}$. Summarizing, we arrive to the the following property of the family $\mathcal{F}$.

(P6) For every $\sigma \in [\Sigma_{\mathbf{b}}]$ there exists $D \in \mathcal{F}$ with $D \subseteq \Sigma_{\mathbf{b}}$ and $D \downarrow \sigma$.

We have the following lemma which is essentially a consequence of property (P6).

Lemma 16. *The families $\mathcal{C}$ and $\mathcal{F}$ are not countably separated.*

Proof. Assume, towards a contradiction, that there exists a sequence (C_k) in $\mathcal{P}_\infty(\Sigma)$ such that for every $C \in \mathcal{C}$ and every $B \in \mathcal{F}$ there exists $k \in \mathbb{N}$ with $C \subseteq C_k$ and $C_k \perp B$. For every k set

$$F_k \coloneqq \big\{\sigma \in [\Sigma_{\mathbf{b}}] : \{\sigma|n : n \in \mathbb{N}\} \subseteq C_k\big\}.$$

Then each F_k is a closed subset of $[\Sigma_{\mathbf{b}}]$. Moreover $[\Sigma_{\mathbf{b}}] = \bigcup_k F_k$.

For every $t \in \Sigma_{\mathbf{b}}$ and every $k \in \mathbb{N}$ there exists $s \in \Sigma_{\mathbf{b}}$ with $t \sqsubset s$ and such that either $V_s \cap F_k = \emptyset$ or $V_s \subseteq F_k$, where as usual by V_s we denote the clopen subset $\{\sigma \in [\Sigma_{\mathbf{b}}] : s \sqsubset \sigma\}$ of $[\Sigma_{\mathbf{b}}]$. Let us say that such a node s *decides* for (t,k). Observe that if s decides for (t,k) with $V_s \subseteq F_k$, then the set $\{w \in \Sigma_{\mathbf{b}} : s \sqsubseteq w\}$ is a subset of C_k.

Recursively, we select a sequence (s_k) in $\Sigma_{\mathbf{b}}$ such that s_0 decides for $(\emptyset, 0)$ and s_{k+1} decides for $(s_k, k+1)$ for every $k \in \mathbb{N}$. Notice that $s_k \sqsubset s_{k+1}$. Thus, setting $\tau = \bigcup_k s_k$, we see that $\tau \in [\Sigma_{\mathbf{b}}]$. By property (P6) above, there exists $B_0 \in \mathcal{F}$ with $B_0 \subseteq \Sigma_{\mathbf{b}}$ and $B_0 \downarrow \tau$. Now let $m \in \mathbb{N}$ with $\{\tau|n : n \in \mathbb{N}\} \subseteq C_m$. Then $\tau \in F_m$. Since $s_m \sqsubset \tau$, we see that $V_{s_m} \cap F_m \neq \emptyset$. The node s_m decides for every $m \in \mathbb{N}$, and so, $V_{s_m} \subseteq F_m$. As we have already remarked, this implies that $\{w \in \Sigma_{\mathbf{b}} : s_m \sqsubseteq w\} \subseteq C_m$. Since B_0 descends to τ, $B_0 \subseteq \Sigma_{\mathbf{b}}$ and $s_m \sqsubset \tau$, we obtain that

$$B_0 \subseteq^* \{w \in \Sigma_{\mathbf{b}} : s_m \sqsubseteq w\} \subseteq C_m.$$

Therefore, we see that for every $m \in \mathbb{N}$ either $\{\tau|n : n \in \mathbb{N}\} \nsubseteq C_m$ or $B_0 \subseteq^* C_m$. That is, the sequence (C_k) cannot separate the sets $\{\tau|n : n \in \mathbb{N}\}$ and B_0 although $\{\tau|n : n \in \mathbb{N}\} \in \mathcal{C}$ and $B_0 \in \mathcal{F}$, a contradiction. The lemma is proved. $\square$

The families $\mathcal{C}$ and $\mathcal{F}$ are hereditary, analytic and orthogonal. Thus, applying Theorem 3 to the pair $(\mathcal{C}, \mathcal{F})$ and invoking Lemma 16, we obtain that there exists a perfect Lusin gap inside $(\mathcal{C}, \mathcal{F})$. Since $\mathcal{C} \subseteq \mathcal{E}$ and $\mathcal{F} \subseteq \mathcal{H}$, we see that there exists a perfect Lusin gap $2^{\mathbb{N}} \ni x \mapsto (A_x, B_x)$ inside $(\mathcal{E}, \mathcal{H})$. Now recall that the map $\phi \colon \Sigma \to \mathbb{N}$ obtained by Theorem 4 is one-to-one. It follows that the map $2^{\mathbb{N}} \ni x \mapsto \big(\phi(A_x), \phi(B_x)\big)$ is a perfect Lusin gap inside $(\mathcal{A}, \mathcal{B})$. The proof of Theorem I is completed.

Remark 1. We would like to point out that one can construct the perfect Lusin gap inside $(\mathcal{E},\mathcal{H})$ without invoking Theorem 3. This can be done as follows. Let $\mathbf{b}=(b_n)$ be the block sequence obtained by Lemma 15. First we construct, recursively, a family $(t_s)_{s\in 2^{<\mathbb{N}}}$ in $\Sigma_{\mathbf{b}}$ such that the following are satisfied.

(C1) For every $s,s'\in 2^{<\mathbb{N}}$ we have $s\sqsubset s'$ if and only if $t_s\sqsubset t_{s'}$.

(C2) For every $s\in 2^{<\mathbb{N}}$ and every $\sigma\in[\Sigma_{\mathbf{b}}]$ with $t_{s^\frown 0}\sqsubset\sigma$ we have $t_{s^\frown 1}\in\Delta(\mathbf{b}_\sigma)$ where, as in (O1) above, by $\mathbf{b}_\sigma$ we denote the unique subsequence (b_{l_n}) of (b_n) such that $\sigma=\bigcup_n b_{l_n}$.

The construction proceeds as follows. We set $t_\emptyset:=\emptyset$. Assume that t_s has been defined for some $s\in 2^{<\mathbb{N}}$. We select $\tau\in\Sigma_{\mathbf{b}}$ with $t_s\sqsubset\tau$. Let $\mathbf{b}_\tau=(b_{l_n})$ be the unique subsequence of $\mathbf{b}$ with $\tau=\bigcup_n b_{l_n}$. By (O4) in the proof of Theorem I, the set $\Delta(\mathbf{b}_\tau)$ descends to τ. Since $t_s\sqsubset\tau$, there exists $t_{s^\frown 1}\in\Delta(\mathbf{b}_\tau)$ with $t_s\sqsubset t_{s^\frown 1}$. The map $[\Sigma_{\mathbf{b}}]\ni\sigma\mapsto\Delta(\mathbf{b}_\sigma)\in\mathcal{P}_\infty(\Sigma)$ is continuous. So, we may find a node $t_{s^\frown 0}$ incomparable to $t_{s^\frown 1}$ with $t_s\sqsubset t_{s^\frown 0}\sqsubset\tau$ and such that (C2) above is satisfied.

Having completed the construction, for every $x\in 2^{\mathbb{N}}$ set $\sigma_x:=\bigcup_n t_{x|n}\in[\Sigma_{\mathbf{b}}]$ and define

$$A_x:=\{\sigma_x|n:n\in\mathbb{N}\}\in\mathcal{E}\ \text{ and }\ B_x:=\Delta(\mathbf{b}_{\sigma_x})\in\mathcal{H}.$$

The perfect Lusin gap inside $(\mathcal{E},\mathcal{H})$ is the map $2^{\mathbb{N}}\ni x\mapsto(A_x,B_x)$. It is easy to check that it is one-to-one, continuous and $A_x\cap B_x=\emptyset$ for every $x\in 2^{\mathbb{N}}$. Finally, let $x,y\in 2^{\mathbb{N}}$ with $x\neq y$. We may assume that $x<y$ where $<$ stands for the lexicographical ordering of $2^{\mathbb{N}}$. There exists $s\in 2^{<\mathbb{N}}$ with $s^\frown 0\sqsubset x$ and $s^\frown 1\sqsubset y$. Then $t_{s^\frown 1}\in A_y$. Moreover, we have $t_{s^\frown 0}\sqsubset\sigma_x$. By (C2) above, we see that $t_{s^\frown 1}\in\Delta(\mathbf{b}_{\sigma_x})$. Thus $A_y\cap B_x\neq\emptyset$.

Remark 2. Let $\mathcal{A},\mathcal{B}\subseteq\mathcal{P}_\infty(\mathbb{N})$ be two hereditary, orthogonal, analytic families and assume that $\mathcal{B}$ is an M-family. We notice that, in this case, the dichotomy in Theorem I can be derived directly by Theorem 3. To see this, observe that if $\mathcal{A}$ is not countably generated in $\mathcal{B}^\perp$, then, by Proposition 13, the families $\mathcal{A}$ and $\mathcal{B}$ are not countably separated. Thus, part (ii) of Theorem 3 yields the existence of the gap inside $(\mathcal{A},\mathcal{B})$.

Remark 3. As in Example 3, let E be a Polish space and let $\mathbf{f}=\{f_n\}$ be a pointwise bounded sequence of real-valued Baire-1 functions on E such that the closure $\mathcal{K}$ of $\{f_n\}$ in $\mathbb{R}^E$ is a Rosenthal compact. We set

$$\mathcal{L}_{\mathbf{f}}:=\big\{L\in\mathcal{P}_\infty(\mathbb{N}):(f_n)_{n\in L}\text{ is pointwise convergent}\big\}. \tag{15}$$

For every $f\in\mathcal{K}$ let $\mathcal{L}_f$ be as in (6). In [To4, Lemmas G.9 and G.10], Todorčević proved that if f is any point of $\mathcal{K}$, then either

(A1) f is a G_δ point of $\mathcal{K}$, or

(A2) there exists a perfect Lusin gap in $(\mathcal{L}_{\mathbf{f}}\setminus\mathcal{L}_f,\mathcal{L}_f)$.

Let us see how Theorem I yields the above dichotomy. So, fix a point $f\in\mathcal{K}$. First we notice that, as it was explained in [Do, Remark 1(2)], by Debs' theorem [De] there exists a hereditary, Borel and *cofinal* subfamily $\mathcal{F}$ of $\mathcal{L}_{\mathbf{f}}$. We set $\mathcal{A} = \mathcal{F} \setminus \mathcal{L}_f$. Then $\mathcal{A}$ is an analytic, hereditary and cofinal subfamily of $\mathcal{L}_{\mathbf{f}} \setminus \mathcal{L}_f$. Moreover, as we mentioned in Example 3, the family $\mathcal{L}_f$ is a co-analytic M-family. Noticing that $\mathcal{A}$ and $\mathcal{L}_f$ are orthogonal, by Theorem I we obtain that either

(A3) $\mathcal{A}$ is countably generated in $\mathcal{L}_f^{\perp}$, or
(A4) there exists a perfect Lusin gap in $(\mathcal{A}, \mathcal{L}_f)$.

Clearly, we only have to check that (A3) implies (A1). Indeed, let (L_k) be a sequence in $\mathcal{L}_f^{\perp}$ that generates $\mathcal{A}$. Set $V_k := \mathcal{K} \setminus \overline{\{f_n\}}^{p}_{n \in L_k}$ and notice that $f \in V_k$ for every $k \in \mathbb{N}$. Taking into account that $\mathcal{A}$ is cofinal in $\mathcal{L}_{\mathbf{f}} \setminus \mathcal{L}_f$ and using the Bourgain–Fremlin–Talagrand theorem [BFT], we see that $\{f\} = \bigcap_k V_k$; that is, the point f is G_δ.

6. Proof of Theorem II

This section is devoted to the proof of Theorem II. Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}_\infty(\mathbb{N})$ be a pair of hereditary orthogonal families such that $\mathcal{A}$ is analytic and $\mathcal{B}$ is C-measurable and an M-family. Assume that $\mathcal{A}$ is not countably generated in $\mathcal{B}^{\perp}$. By Theorem 4, there exists a one-to-one map $\phi\colon \Sigma \to \mathbb{N}$ such that, setting $\mathcal{E} := \{\phi^{-1}(A) : A \in \mathcal{A}\}$ and $\mathcal{H} := \{\phi^{-1}(B) : B \in \mathcal{B}\}$, the following properties are satisfied for $\mathcal{E}$ and $\mathcal{H}$.

(P1) $\mathcal{E}$ and $\mathcal{H}$ are hereditary and orthogonal.
(P2) $\mathcal{E}$ is analytic and $\mathcal{C} \subseteq \mathcal{E}$.
(P3) $\mathcal{H}$ is C-measurable and an M-family.
(P4) For every $t \in \Sigma$ we have $\big\{t \cup \{n\} : n \in \mathbb{N} \text{ and } t < \{n\}\big\} \in \mathcal{H}$.

As in the proof of Theorem I, we shall work inside the tree Σ and with the families $\mathcal{E}$ and $\mathcal{H}$.

We introduce the following class of subsets of Σ. It will be used in a similar spirit as the class of descenders was used in the proof of Theorem I.

Definition 17. *An infinite subset F of Σ will be called a* fan *if F can be enumerated as $\{t_n : n \in \mathbb{N}\}$ and there exist $s \in \Sigma$ and a strictly increasing sequence (m_n) in $\mathbb{N}$ with $s < \{m_0\}$ and such that $s \cup \{m_n\} \sqsubseteq t_n$ for all $n \in \mathbb{N}$.*

The following fact is essentially well-known. We sketch a proof for completeness.

Fact 5. *Let $A \in \mathcal{P}_\infty(\Sigma)$. Then either A is dominated, or A contains a fan. In particular, if T is a downwards closed, well-founded, infinite subtree of Σ, then every infinite subset A of T contains a fan.*

Proof. Fix $A \in \mathcal{P}_\infty(\Sigma)$ and let $\hat{A} = \{t \in \Sigma : \exists s \in A \text{ with } t \sqsubseteq s\}$ be the downwards closure of A. It is easy to see that if $\hat{A}$ is finitely splitting, then A must be dominated while if $\hat{A}$ is not finitely splitting, then A must contain a fan.

For the second part, let T be a downwards closed, well-founded, infinite subtree of Σ and fix $A \in \mathcal{P}_\infty(T)$. If $\hat{A}$ is finitely splitting, then by an application of König's lemma we see that $[T] \neq \emptyset$, a contradiction. Thus $\hat{A}$ is not finitely splitting, and so, A contains a fan. $\square$

Notice that if $\mathbf{b} = (b_n)$ is a block sequence of Σ and $s \in \Sigma$ with $s < b_0$, then the set $\{s \cup b_n : n \in \mathbb{N}\}$ is a fan. A fan F of this form will be called a *block* fan. By $\mathcal{F}_{\mathrm{Block}}$ we denote the set of all block fans of Σ. We have the following elementary fact.

Fact 6. *Every fan contains a block fan.*

We define $\Phi\colon \mathbf{B} \to \mathcal{P}_\infty(\Sigma)$ by

$$\Phi\big((b_n)\big) = \big\{b_0 \cup b_1 \cup \{\min(b_2)\}, \dots, b_0 \cup b_{2n+1} \cup \{\min(b_{2n+2})\}, \dots\big\}. \tag{16}$$

We observe the following properties.

(O1) The map Φ is continuous.

(O2) For every $\mathbf{b} \in \mathbf{B}$ the set $\Phi(\mathbf{b})$ is a block fan.

We have the following analogue of Lemma 15.

Lemma 18. *There exists* $\mathbf{b} \in \mathbf{B}$ *such that* $\Phi(\mathbf{c}) \in \mathcal{H}$ *for every* $\mathbf{c} \in [\mathbf{b}]$.

Proof. We set

$$\mathcal{X} \coloneqq \{\mathbf{c} \in \mathbf{B} : \Phi(\mathbf{c}) \in \mathcal{H}\}.$$

Then $\mathcal{X}$ is a C-measurable. By Theorem 2, there exists $\mathbf{b} = (b_n) \in \mathbf{B}$ such that $[\mathbf{b}]$ is monochromatic. We claim that $[\mathbf{b}] \subseteq \mathcal{X}$. Indeed, for every $n \geqslant 1$ set

$$A_n \coloneqq \big\{b_0 \cup b_n \cup \{\min(b_k)\} : k > n\big\} \in \mathcal{P}_\infty(\Sigma).$$

The set A_n is a subset of the set $\big\{b_0 \cup b_n \cup \{m\} : m \in \mathbb{N} \text{ and } b_n < \{m\}\big\}$ which, by property (P4), belongs to $\mathcal{H}$. Therefore, by (P1), $A_n \in \mathcal{H}$ for every $n \in \mathbb{N}$. Since $\mathcal{H}$ is an M-family, by part (iii) of Fact 3, we may select $L = \{l_0 < l_1 < \cdots\}$, $M = \{m_0 < m_1 < \cdots\} \in \mathcal{P}_\infty(\mathbb{N})$ with $1 \leqslant l_n < m_n < l_{n+1}$ for all $n \in \mathbb{N}$ and such that

$$\big\{b_0 \cup b_{l_n} \cup \{\min(b_{m_n})\} : n \in \mathbb{N}\big\} \in \mathcal{H}.$$

We define $\mathbf{c} = (c_n)$ by $c_0 \coloneqq b_0$ and $c_{2n+1} \coloneqq b_{l_n}$, $c_{2n+2} \coloneqq b_{m_n}$ for every $n \in \mathbb{N}$. Then $\mathbf{c} \in [\mathbf{b}]$ and $\Phi(\mathbf{c}) = \{b_0 \cup b_{l_n} \cup \{\min(b_{m_n})\} : n \in \mathbb{N}\} \in \mathcal{H}$. Hence, $[\mathbf{b}] \cap \mathcal{X} \neq \emptyset$ and the result follows. $\square$

Let $\mathbf{b} = (b_n)$ be the block sequence obtained by Lemma 18. We are going to select a subset of Σ by defining an appropriate endomorphism of Σ (the desired subset will be the image of this endomorphism). In particular, we define $h\colon \Sigma \to \Sigma$ as follows.

(a) We set $h(\emptyset) = \emptyset$.

(b) If $t = (n)$ with $n \in \mathbb{N}$, then we set $h\big((n)\big) = b_0 \cup b_{2n+1} \cup \{\min(b_{2n+2})\}$.

(c) If $t = (n_0 < \cdots < n_k) \in \Sigma$ with $k \geqslant 1$, then we set

$$h(t) = b_0 \cup \Big(\bigcup_{i=0}^{k-1} (b_{2n_i+1} \cup b_{2n_i+2}) \Big) \cup b_{2n_k+1} \cup \{\min(b_{2n_k+2})\}.$$

It is easy to see that the map h is well-defined and one-to-one. We also observe the following property.

(O3) For every $s, t \in \Sigma$ we have $s \sqsubset t$ if and only if $h(s) \sqsubset h(t)$. Thus, if $C \in \mathcal{P}_\infty(\Sigma)$, then C is a chain of Σ if and only if $h(C)$ is.

The following fact shows the relation between the maps Φ and h.

Fact 7. *Let F be a block fan of Σ. Then there exists $\mathbf{c} \in [\mathbf{b}]$ such that $h(F) = \Phi(\mathbf{c})$.*

Proof. Let (u_n) be a block sequence of Σ and $s \in \Sigma$ with $s < u_0$ and such that $F = \{s \cup u_n : n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$ there exist $s_n \in \Sigma$ and $l_n \in \mathbb{N}$ with $s_n < \{l_n\}$ and $u_n = s_n \cup \{l_n\}$ (notice that s_n may be empty). We define $\mathbf{c} = (c_n) \in \mathbf{B}$ as follows. We set

$$c_0 \coloneqq b_0 \cup \bigcup_{k \in s} (b_{2k+1} \cup b_{2k+2})$$

with the convention that $\bigcup_{k\in s}(b_{2k+1} \cup b_{2k+2}) = \emptyset$ if $s = \emptyset$. For every $n \geqslant 1$ we set

$$c_{2n+1} \coloneqq \Big(\bigcup_{k \in s_n} (b_{2k+1} \cup b_{2k+2}) \Big) \cup b_{2l_n+1} \text{ and } c_{2n+2}v \coloneqq b_{2l_n+2}.$$

It is easy to see that $\mathbf{c} \in [\mathbf{b}]$ and $h(F) = \Phi(\mathbf{c})$, as desired. □

Finally, we define $\psi \colon \Sigma \to \mathbb{N}$ by $\psi(s) = \phi\big(h(s)\big)$ for every $s \in \Sigma$. Both ϕ and h are one-to-one, and so, the map ψ is one-to-one too. As in Example 2, let $\mathcal{I}_{\mathrm{wf}}$ be the ideal on Σ generated by the set WF of all downwards closed, well-founded, infinite subtrees of Σ, that is,

$$\mathcal{I}_{\mathrm{wf}} = \{W \in \mathcal{P}_\infty(\Sigma) : \exists T \in \mathrm{WF} \text{ with } W \subseteq T\}.$$

Lemma 19. *The following hold.*

(i) $\mathcal{C} \subseteq \{\psi^{-1}(A) : A \in \mathcal{A}\}$.
(ii) $\mathcal{F}_{\mathrm{Block}} \subseteq \{\psi^{-1}(B) : B \in \mathcal{B}\}$.
(iii) $\mathcal{I}_{\mathrm{wf}} \subseteq \{\psi^{-1}(B) : B \in \mathcal{B}^{\perp\perp}\}$.

Proof. Part (i) is an immediate consequence of property (P2) and observation (O3) above. Part (ii) follows by Lemma 18 and Fact 7. To see that part (iii) is satisfied, fix $W \in \mathcal{I}_{\mathrm{wf}}$. Let $A \in \mathcal{P}_\infty(W)$ be arbitrary. By Facts 5 and 6, there exists a block fan F with $F \subseteq A$. By part (ii), we see that $\psi(F) \in \mathcal{B}$. Hence, by Fact 1, we conclude that $\psi(W) \in \mathcal{B}^{\perp\perp}$, as desired. □

The trees Σ and $\mathbb{N}^{<\mathbb{N}}$ are isomorphic, that is, there exists a bijection $e \colon \mathbb{N}^{<\mathbb{N}} \to \Sigma$ with $|e(t)| = |t|$ for every $t \in \mathbb{N}^{<\mathbb{N}}$ and such that $t_1 \sqsubset t_2$ in $\mathbb{N}^{<\mathbb{N}}$ if and only if $e(t_1) \sqsubset e(t_2)$. Hence, by Lemma 19, the proof of Theorem II is completed.

Remark 4. In [Kr], Krawczyk proved that if $\mathcal{I}$ is a bisequential analytic ideal on $\mathbb{N}$, then either,

(A1) $\mathcal{I}$ is countably generated in $\mathcal{I}$, or

(A2) there exists a one-to-one map $\psi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ such that, setting

$$\mathcal{J} := \{\psi^{-1}(A) : A \in \mathcal{I}\},$$

we have $\mathcal{C} \subseteq \mathcal{J} \subseteq \mathcal{I}_{\mathrm{d}}$,

where $\mathcal{C}$ denotes the set of all infinite chains of $\mathbb{N}^{<\mathbb{N}}$ and $\mathcal{I}_{\mathrm{d}}$ denotes the ideal of all infinite dominated subsets of $\mathbb{N}^{<\mathbb{N}}$. Let us see how Theorem II yields the above result. So, fix a bisequential analytic ideal $\mathcal{I}$ on $\mathbb{N}$. We set $\mathcal{A} = \mathcal{I}$ and $\mathcal{B} = \mathcal{I}^{\perp}$. Clearly $\mathcal{A}$ and $\mathcal{B}$ are hereditary and orthogonal families. Moreover, $\mathcal{A}$ is $\mathbf{\Sigma}^1_1$ while $\mathcal{B}$ is $\mathbf{\Pi}^1_1$. By part (ii) of Proposition 6, we see that $\mathcal{B}$ is an M-family. By Fact 2, the ideal $\mathcal{I}$ has the Fréchet property, and so, $\mathcal{B}^{\perp} = \mathcal{I}$ and $\mathcal{B}^{\perp\perp} = \mathcal{I}^{\perp} = \mathcal{B}$. Thus, applying Theorem II, the result follows.

Remark 5. Let $\mathcal{A}$ and $\mathcal{B}$ be as in Theorem II and assume that $\mathcal{A}$ is not countably generated in $\mathcal{B}^{\perp}$. Let $\psi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ be the one-to-one map obtained by Theorem II. Notice that for every downwards closed, infinite subtree T of $\mathbb{N}^{<\mathbb{N}}$ we have $T \in \mathrm{WF}$ if and only if $\psi(T) \in \mathcal{B}^{\perp\perp}$, that is, the set WF is Wadge reducible to $\mathcal{B}^{\perp\perp}$. Thus, if $\mathcal{A}$ is not countably generated in $\mathcal{B}^{\perp}$, then the family $\mathcal{B}^{\perp\perp}$ is at least $\mathbf{\Pi}^1_1$-hard.

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National Technical University of Athens, Faculty of Applied Sciences, Department of Mathematics, Zografou Campus, 157 80, Athens, Greece.

E-mail address: pdodos@math.ntua.gr, bkanel@math.ntua.gr