# ON UNCONDITIONALLY SATURATED BANACH SPACES

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Abstract. We prove a structural property of the class of unconditionally saturated separable Banach spaces. We show, in particular, that for every analytic set $\mathcal{A}$, in the Effros–Borel space of subspaces of $C[0,1]$, of unconditionally saturated separable Banach spaces, there exists an unconditionally saturated Banach space $Y$, with a Schauder basis, that contains isomorphic copies of every space $X$ in the class $\mathcal{A}$.

## 1. Introduction

**(A)** An infinite-dimensional Banach space $X$ is said to be *unconditionally saturated* if every infinite-dimensional subspace $Y$ of $X$ contains an unconditional basic sequence. Although, by the discovery of Gowers and Maurey [GM], not every separable Banach space is unconditionally saturated, this class of spaces is quite extensive, includes the "classical" ones and has some desirable closure properties (it is closed, for instance, under taking subspaces and finite sums). Most important is the fact that within the class of unconditionally saturated spaces one can develop a strong structural theory. Among the numerous results found in the literature, there are two fundamental ones that deserve special attention. The first is due to James [Ja1] and asserts that any unconditionally saturated space contains either a reflexive subspace, or $\ell_1$, or $c_0$. The second result is due to Pełczyński [P] and provides a space $U$ with an unconditional basis $(u_n)$ with the property that any other unconditional basic sequence $(x_n)$, in some Banach space $X$, is equivalent to a subsequence of $(u_n)$.

**(B)** The main goal of this paper is to exhibit yet another structural property of the class of unconditionally saturated spaces which is of a global nature. To describe this property we need first to recall some standard facts. Quite often one needs a convenient way to treat separable Banach spaces as a unity. Such a way has been proposed by Bossard [Bos] and has been proved to be extremely useful. More precisely, let $F\big(C[0,1]\big)$ denote the set of all closed subspaces of the space $C[0,1]$ and consider the set

$$\mathrm{SB} := \big\{X \in F\big(C[0,1]\big) : X \text{ is a linear subspace}\big\}. \tag{1}$$

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It is easy to see that the set SB equipped with the relative Effros–Borel structure becomes a standard Borel space (see [Bos] for more details). As $C[0,1]$ is isometrically universal for all separable Banach spaces, we may identify any class of separable Banach spaces with a subset of SB. Under this point of view, we denote by US the subset of SB consisting of all $X \in \mathrm{SB}$ which are unconditionally saturated.

The above identification is ultimately related to universality problems in Banach space theory (see [AD, DF, D]). The connection is crystallized in the following definition, introduced in [AD].

**Definition 1.** *A class $\mathcal{C} \subseteq \mathrm{SB}$ is said to be* strongly bounded *if for every analytic subset $\mathcal{A}$ of $\mathcal{C}$ there exists $Y \in \mathcal{C}$ that contains isomorphic copies of every $X \in \mathcal{A}$.*

In [AD, Theorem 91(5)] it was shown that the class of unconditionally saturated Banach spaces with a Schauder basis is strongly bounded. We remove the assumption of the existence of a basis and we show the following theorem.

**Theorem 2.** *Let $\mathcal{A}$ be an analytic subset of* US. *Then there exists an unconditionally saturated Banach space $Y$, with a Schauder basis, that contains isomorphic copies of every $X \in \mathcal{A}$.*

*In particular, the class* US *is strongly bounded.*

We should point out that the above result is optimal. Indeed, it follows by a classical construction of Bourgain [Bou1] that there exists a co-analytic subset $\mathcal{B}$ of SB consisting of reflexive and unconditionally saturated separable Banach spaces with the following property. If $Y$ is a separable space that contains an isomorphic copy of every $X \in \mathcal{B}$, then $Y$ must contain every separable Banach space. In particular, there is no unconditionally saturated separable Banach space containing isomorphic copies of every $X \in \mathcal{B}$.

**(C)** By the results in [AD], the proof of Theorem 2 is essentially reduced to an embedding problem. Namely, given an unconditionally saturated separable Banach space $X$ one is looking for an unconditionally saturated space $Y(X)$, with a Schauder basis, that contains an isomorphic copy of $X$. In fact, for the proof of Theorem 2, one has to know additionally that this embedding is "uniform". This means, roughly, that the space $Y(X)$ is constructed from $X$ in a Borel way. In our case, the embedding problem has been already solved by Bourgain and Pisier [BP], while its uniform version has been recently obtained in [D]. These are the main ingredients of the proof of Theorem 2.

**(D)** At a more technical level, the paper also contains some results concerning the structure of a class of subspaces of a certain space constructed in [AD] and called as an $\ell_2$ Baire sum. Specifically, we study the class of $X$-singular subspaces of an $\ell_2$ Baire sum and we show the following (see §3.1 for the relevant definitions).

(1) Every $X$-singular subspace is unconditionally saturated (Theorem 11 in the main text).

(2) Every $X$-singular subspace contains an $X$-compact subspace (Corollary 16 in the main text). This answers a question from [AD] (see [AD, Remark 3]).

(3) Every normalized basic sequence in an $X$-singular subspace has a normalized block subsequence satisfying an upper $\ell_2$ estimate (Theorem 12 in the main text). Hence, an $X$-singular subspace can contain no $\ell_p$ for $1 \leqslant p < 2$. This generalizes the fact that the 2-stopping time Banach space (see [BO]) can contain no $\ell_p$ for $1 \leqslant p < 2$.

1.1. **General notation and terminology.** Let $\mathbb{N} = \{0, 1, 2, \dots\}$ denote the set of natural numbers. For every infinite subset $L$ of $\mathbb{N}$ by $[L]^\infty$ we denote the set of all infinite subsets of $L$. Our Banach space theoretic notation and terminology is standard and follows [LT], while our descriptive set theoretic terminology follows [Ke]. If $X$ and $Y$ are Banach spaces, then we shall denote the fact that $X$ and $Y$ are isomorphic by $X \cong Y$.

For the convenience of the reader, let us recall the following notions. A measurable space $(X, S)$ is said to be a *standard Borel space* if there exists a Polish topology[1] $\tau$ on $X$ such that the Borel $\sigma$-algebra of $(X, \tau)$ coincides with $S$. A subset $B$ of a standard Borel space $(X, S)$ is said to be *analytic* if there exists a Borel map $f\colon \mathbb{N}^\mathbb{N} \to X$ such that $f(\mathbb{N}^\mathbb{N}) = B$. Finally, a seminormalized sequence $(x_n)$ in a Banach space $X$ is said to be *unconditional* if there exists a constant $C > 0$ such that for every $k \in \mathbb{N}$, every $F \subseteq \{0, \dots, k\}$ and every $a_0, \dots, a_k \in \mathbb{R}$ we have

$$\Big\| \sum_{n\in F} a_n x_n \Big\| \leqslant C \Big\| \sum_{n=0}^{k} a_n x_n \Big\|. \tag{2}$$

1.2. **Trees.** The concept of a tree has been proved to be a very fruitful tool in the geometry of Banach spaces. It is also decisive throughout this paper. Below we gather all the conventions concerning trees that we need.

Let $\Lambda$ be a nonempty set. By $\Lambda^{<\mathbb{N}}$ we shall denote the set of all *nonempty*[2] finite sequences in $\Lambda$. By $\sqsubset$ we shall denote the (strict) partial order on $\Lambda^{<\mathbb{N}}$ of end-extension. For every $\sigma \in \Lambda^\mathbb{N}$ and every $n \in \mathbb{N}$ with $n \geqslant 1$ we set $\sigma|n = \big(\sigma(0), \dots, \sigma(n-1)\big) \in \Lambda^{<\mathbb{N}}$. Two nodes $s, t \in \Lambda^{<\mathbb{N}}$ are said to be *comparable* if either $s \sqsubseteq t$ or $t \sqsubseteq s$; otherwise, they are said to be *incomparable*. A subset of $\Lambda^{<\mathbb{N}}$ consisting of pairwise comparable nodes is said to be a *chain*, while a subset of $\Lambda^{<\mathbb{N}}$ consisting of pairwise incomparable nodes is said to be an *antichain*.

A *tree* $T$ on $\Lambda$ is a subset of $\Lambda^{<\mathbb{N}}$ satisfying

$$\forall s, t \in \Lambda^{<\mathbb{N}} \ (t \in T \text{ and } s \sqsubset t \Rightarrow s \in T). \tag{3}$$

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[1]A topology $\tau$ on a set $X$ is said to be Polish if the space $(X, \tau)$ is a separable and completely metrizable topological space.

[2]We should point out that in many standard textbooks, as for instance in [Ke], the empty sequence is included in $\Lambda^{<\mathbb{N}}$. We do not include the empty sequence for technical reasons that will become transparent in §3.

A tree $T$ is said to be *pruned* if for every $s \in T$ there exists $t \in T$ with $s \sqsubset t$. The *body* $[T]$ of a tree $T$ on $\Lambda$ is defined to be the set $\{\sigma \in \Lambda^{\mathbb{N}} : \sigma|n \in T \ \forall n \geqslant 1\}$. Notice that if $T$ is pruned, then $[T] \neq \emptyset$. A *segment* $\mathfrak{s}$ of a tree $T$ is a chain of $T$ satisfying

$$\forall s, t, w \in \Lambda^{<\mathbb{N}} \ (s \sqsubseteq w \sqsubseteq t \ \text{ and } s, t \in \mathfrak{s} \Rightarrow w \in \mathfrak{s}). \tag{4}$$

If $\mathfrak{s}$ is a segment of $T$, then by $\min(\mathfrak{s})$ we denote the $\sqsubseteq$-minimum node $t \in \mathfrak{s}$. We say that two segments $\mathfrak{s}$ and $\mathfrak{s}'$ of $T$ are *incomparable* if for every $t \in \mathfrak{s}$ and every $t' \in \mathfrak{s}'$ the nodes $t$ and $t'$ are incomparable (notice that this is equivalent to saying that $\min(\mathfrak{s})$ and $\min(\mathfrak{s}')$ are incomparable).

## 2. Embedding unconditionally saturated spaces into spaces with a basis

The aim of this section is to give the proof of the following result.

**Proposition 3.** *Let $\mathcal{A}$ be an analytic subset of* US*. Then there exists an analytic subset $\mathcal{A}'$ of* US *with the following properties.*

(i) *For every $Y \in \mathcal{A}'$ the space $Y$ has a Schauder basis*[3].
(ii) *For every $X \in \mathcal{A}$ there exists $Y \in \mathcal{A}'$ that contains an isometric copy of $X$.*

As we have already mention in the introduction, the proof of Proposition 3 is based on a construction of $\mathcal{L}_\infty$-spaces due to Bourgain and Pisier [BP], as well as, on its parameterized version which has been recently obtained in [D].

Let us recall, first, some definitions. If $X$ and $Y$ are two isomorphic Banach spaces (not necessarily infinite-dimensional), then their *Banach–Mazur distance* is defined by

$$d(X,Y) \coloneqq \inf\big\{\|T\| \cdot \|T^{-1}\| : T\colon X \to Y \text{ is an isomorphism}\big\}. \tag{5}$$

Now let $X$ be an infinite-dimensional Banach space and $\lambda \geqslant 1$. The space $X$ is said to be a $\mathcal{L}_{\infty,\lambda}$-space if for every finite-dimensional subspace $F$ of $X$ there exists a finite-dimensional subspace $G$ of $X$ with $F \subseteq G$ and $d(G, \ell_\infty^n) \leqslant \lambda$ where $n = \dim(G)$. The space $X$ is said to be a $\mathcal{L}_{\infty,\lambda+}$-space if it is a $\mathcal{L}_{\infty,\theta}$-space for every $\theta > \lambda$. Finally, $X$ is said to be a $\mathcal{L}_\infty$-space if it is $\mathcal{L}_{\infty,\lambda}$ for some $\lambda \geqslant 1$. The class of $\mathcal{L}_\infty$-spaces was defined by Lindenstrauss and Pełczyński [LP]. For a comprehensive account of the theory of $\mathcal{L}_\infty$-spaces as well as for a presentation of many remarkable examples we refer to the monograph of Bourgain [Bou2].

We also recall that a Banach space $X$ is said to have the *Schur property* if every weakly convergent sequence in $X$ is automatically norm convergent. It in an immediate consequence of Rosenthal's dichotomy [Ro] that every space $X$ with the Schur property is hereditarily $\ell_1$; that is, every subspace $Y$ of $X$ has a further subspace isomorphic to $\ell_1$ (hence, every space with the Schur property is unconditionally saturated).

[3]Throughout the paper, when we say that a Banach space $X$ has a Schauder basis, then we implicitly assume that $X$ is infinite-dimensional.

The following theorem summarizes some of the basic properties of the Bourgain–Pisier construction.

**Theorem 4** ([BP], Theorem 2.1)**.** *Let $\lambda > 1$ and $X$ be a separable Banach space. Then there exists a separable $\mathcal{L}_{\infty,\lambda+}$-space, denoted by $\mathcal{L}_\lambda[X]$, which contains $X$ isometrically and is such that the quotient $\mathcal{L}_\lambda[X]/X$ has the Radon–Nikodym and the Schur properties.*

The parameterized version of Theorem 4 is the content of the following theorem.

**Theorem 5** ([D], Theorem 16)**.** *For every $\lambda > 1$ the set $\mathcal{L}_\lambda \subseteq \mathrm{SB} \times \mathrm{SB}$ defined by*

$$(X, Y) \in \mathcal{L}_\lambda \Leftrightarrow Y \textit{ is isometric to } \mathcal{L}_\lambda[X]$$

*is analytic.*

We will also need the following Ramsey-type lemma. Although it is well-known, we sketch its proof for completeness.

**Lemma 6.** *Let $X$ be a Banach space and let $Y$ be a closed subspace of $X$. Then, for every subspace $Z$ of $X$ there exists a further subspace $Z'$ of $Z$ such that $Z'$ is either isomorphic to a subspace of $Y$, or isomorphic to a subspace of $X/Y$.*

*In particular, if $Y$ and $X/Y$ are both unconditionally saturated, then so is $X$.*

*Proof.* Let $Q\colon X \to X/Y$ be the natural quotient map. Consider the following (mutually exclusive) cases.

Case 1: *The operator $Q\colon Z \to X/Y$ is not strictly singular.* This case, by definition, yields the existence of a subspace $Z'$ of $Z$ such that $Q|_{Z'}$ is an isomorphic embedding.

Case 2: *The operator $Q\colon Z \to X/Y$ is strictly singular.* In this case our hypothesis implies that for every subspace $Z'$ of $Z$ and every $\varepsilon > 0$ we may find a normalized vector $z \in Z'$ such that $\|Q(z)\| \leqslant \varepsilon$. Hence, for every subspace $Z'$ of $Z$ and every $\varepsilon > 0$ there exist a normalized vector $z \in Z'$ and a vector $y \in Y$ such that $\|z - y\| < \varepsilon$. So, we may select a normalized basic sequence $(z_n)$ in $Z$ with basis constant 2 and a sequence $(y_n)$ in $Y$ such that $\|z_n - y_n\| < 1/8^n$ for every $n \in \mathbb{N}$. It follows that $(y_n)$ is equivalent to $(z_n)$ (see [LT]). Setting $Z' = \overline{\mathrm{span}}\{z_n : n \in \mathbb{N}\}$, we see that $Z'$ is isomorphic to a subspace of $Y$. The proof is completed. □

We are ready to proceed to the proof of Proposition 3.

*Proof of Proposition* 3. Let $\mathcal{A}$ be an analytic subset of US. Also let $\mathcal{L}_2$ be the subset of $\mathrm{SB} \times \mathrm{SB}$ obtained by applying Theorem 5 for $\lambda = 2$. We define $\mathcal{A}' \subseteq \mathrm{SB}$ by the rule

$$Y \in \mathcal{A}' \Leftrightarrow \exists X \ \big[X \in \mathcal{A} \text{ and } (X, Y) \in \mathcal{L}_2\big].$$

Since both $\mathcal{A}$ and $\mathcal{L}_2$ are analytic and the class of analytic sets is closed under projections, we see that $\mathcal{A}'$ is analytic. We claim that $\mathcal{A}'$ is the desired set. Indeed, notice that property (ii) is an immediate consequence of Theorem 4. To see (i), let $Y \in \mathcal{A}'$ be arbitrary. There exists $X \in \mathcal{A}$ such that $Y$ is isometric to $\mathcal{L}_2[X]$. By Theorem 4, we know that $\mathcal{L}_2[X]/X$ is unconditionally saturated. Recalling that $X$ is also unconditionally saturated, by Lemma 6, we see that $Y \in \mathrm{US}$. Finally, our claim that $Y$ has a Schauder basis is an immediate consequence of the fact that $Y$ is $\mathcal{L}_\infty$ and of a classical result due to Johnson, Rosenthal and Zippin [JRZ] asserting that every separable $\mathcal{L}_\infty$-space has a Schauder basis. The proof is completed. □

## 3. Schauder tree bases and $\ell_2$ Baire sums

3.1. **Definitions and statements of the main results.** We begin by recalling the following notion.

**Definition 7** ([AD], Definition 13)**.** *Let $X$ be a Banach space, $\Lambda$ a countable set and $T$ a pruned tree on $\Lambda$. Also let $(x_t)_{t\in T}$ be a normalized sequence in $X$ indexed by the tree $T$. We say that $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ is a* Schauder tree basis *if the following are satisfied.*

(a) *We have $X = \overline{\mathrm{span}}\{x_t : t \in T\}$.*
(b) *For every $\sigma \in [T]$ the sequence $(x_{\sigma|n})_{n\geqslant 1}$ is a (normalized) bimonotone basic sequence.*

Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis. For every $\sigma \in [T]$ we set

$$X_\sigma \coloneqq \overline{\mathrm{span}}\{x_{\sigma|n} : n \geqslant 1\}. \tag{6}$$

Notice that in Definition 7 we do not assume that the subspace $X_\sigma$ of $X$ is complemented. Also notice that if $\sigma, \tau \in [T]$ with $\sigma \neq \tau$, then this does not necessarily imply that $X_\sigma \neq X_\tau$.

**Example 1.** Let $X = c_0$ and $(e_n)$ be the standard unit vector basis of $c_0$. Also let $T = 2^{<\mathbb{N}}$ be the Cantor tree; that is, $T$ is the set of all nonempty finite sequences of 0's and 1's. For every $t \in T$, denoting by $|t|$ the length of the finite sequence $t$, we define $x_t \coloneqq e_{|t|-1}$. It is easy to see that the family $(X, 2, T, (x_t)_{t\in T})$ is a Schauder tree basis. Observe that for every $\sigma \in [T]$ the sequence $(x_{\sigma|n})_{n\geqslant 1}$ is the standard basis of $c_0$. Hence, this Schauder tree basis has been obtained by "spreading" along the branches of $2^{<\mathbb{N}}$ the standard basis of $c_0$.

The notion of a Schauder tree basis serves as a technical vehicle for the construction of a "tree-like" Banach space in the spirit of James [Ja2]. This is the content of the following definition.

**Definition 8** ([AD], §4.1)**.** *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis. The* $\ell_2$ Baire sum *of $\mathfrak{X}$, denoted by $T_2^{\mathfrak{X}}$, is defined to be the completion of $c_{00}(T)$ equipped with the norm*

$$\|z\|_{T_2^{\mathfrak{X}}} \coloneqq \sup\Big\{ \Big( \sum_{j=0}^{l} \Big\| \sum_{t\in \mathfrak{s}_j} z(t) x_t \Big\|_X^2 \Big)^{1/2} \Big\} \tag{7}$$

*where the above supremum is taken over all finite families* $(\mathfrak{s}_j)_{j=0}^l$ *of pairwise incomparable segments of* $T$*.*

**Example 2.** Let $\mathfrak{X}$ be the Schauder tree basis described in Example 1 and consider the corresponding $\ell_2$ Baire sum $T_2^{\mathfrak{X}}$. Notice that if $z \in T_2^{\mathfrak{X}}$, then its norm is given by the formula

$$\|z\|_{T_2^{\mathfrak{X}}} = \sup\Big\{ \Big( \sum_{j=0}^{l} z(t_j)^2 \Big)^{1/2} : (t_j)_{j=0}^l \text{ is an antichain of } 2^{<\mathbb{N}} \Big\}.$$

This space has been defined by Rosenthal and it is known in the literature as the 2-*stopping time* Banach space (see [BO]). It is usually denoted by $S_2$. A very interesting fact concerning the structure of $S_2$ is that it contains almost isometric copies of $\ell_p$ for every $2 \leqslant p < \infty$. This is due to Rosenthal and Schechtman (unpublished). On the other hand, the space $S_2$ contains no $\ell_p$ for $1 \leqslant p < 2$.

Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis and consider the corresponding $\ell_2$ Baire sum $T_2^{\mathfrak{X}}$ of $\mathfrak{X}$. Let $(e_t)_{t\in T}$ be the standard Hamel basis of $c_{00}(T)$. We fix a bijection $h\colon T \to \mathbb{N}$ such that for every pair $t, s \in T$ we have $h(t) < h(s)$ if $t \sqsubset s$. If $(e_{t_n})$ is the enumeration of $(e_t)_{t\in T}$ according to $h$, then it is easy to verify that the sequence $(e_{t_n})$ defines a normalized bimonotone Schauder basis of $T_2^{\mathfrak{X}}$.

For every $\sigma \in [T]$ consider the subspace $\mathcal{X}_\sigma$ of $T_2^{\mathfrak{X}}$ defined by

$$\mathcal{X}_\sigma := \overline{\mathrm{span}}\{e_{\sigma|n} : n \geqslant 1\}. \tag{8}$$

It is easily seen that the space $\mathcal{X}_\sigma$ is isometric to $X_\sigma$ and, moreover, it is 1-complemented in $T_2^{\mathfrak{X}}$ via the natural projection $P_\sigma\colon T_2^{\mathfrak{X}} \to \mathcal{X}_\sigma$. More generally, for every segment $\mathfrak{s}$ of $T$ we set $\mathcal{X}_{\mathfrak{s}} := \overline{\mathrm{span}}\{e_t : t \in \mathfrak{s}\}$. Again we see that $\mathcal{X}_{\mathfrak{s}}$ is isometric to the space $\overline{\mathrm{span}}\{x_t : t \in \mathfrak{s}\}$ and it is 1-complemented in $T_2^{\mathfrak{X}}$ via the natural projection $P_{\mathfrak{s}}\colon T_2^{\mathfrak{X}} \to \mathcal{X}_{\mathfrak{s}}$.

If $x$ is a vector in $T_2^{\mathfrak{X}}$, then by $\mathrm{supp}(x)$ we shall denote its *support*, that is, the set $\{t \in T : x(t) \neq 0\}$. The *range* of $x$, denoted by $\mathrm{range}(x)$, is defined to be the minimal interval $I$ of $\mathbb{N}$ satisfying $\mathrm{supp}(x) \subseteq \{t_n : n \in I\}$. We isolate, for future use, the following consequence of the enumeration $h$ of $T$.

**Fact 9.** *Let* $\mathfrak{s}$ *be a segment of* $T$ *and let* $I$ *be an interval of* $\mathbb{N}$*. Consider the set* $\mathfrak{s}' = \mathfrak{s} \cap \{t_n : n \in I\}$*. Then* $\mathfrak{s}'$ *is also a segment of* $T$*.*

Now let $Y$ be a subspace of $T_2^{\mathfrak{X}}$. Assume that there exist a subspace $Y'$ of $Y$ and a $\sigma \in [T]$ such that the operator $P_\sigma\colon Y' \to \mathcal{X}_\sigma$ is an isomorphic embedding. In such a case, the subspace $Y$ contains information about the Schauder tree basis $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$. On the other hand, there are subspaces of $T_2^{\mathfrak{X}}$ which are "orthogonal" to every $\mathcal{X}_\sigma$. These subspaces are naturally distinguished into two categories, as follows.

**Definition 10** ([AD], Definition 14)**.** *Let* $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ *be a Schauder tree basis and let* $Y$ *be a subspace of* $T_2^{\mathfrak{X}}$*.*

(a) *We say that* $Y$ *is* $X$-singular *if for every* $\sigma \in [T]$ *the operator* $P_\sigma \colon Y \to \mathcal{X}_\sigma$ *is strictly singular.*
(b) *We say that* $Y$ *is* $X$-compact *if for every* $\sigma \in [T]$ *the operator* $P_\sigma \colon Y \to \mathcal{X}_\sigma$ *is compact.*

In this section, we are focussed on the structure of the class of $X$-singular subspaces of an arbitrary $\ell_2$ Baire sum. Our main results are summarized below.

**Theorem 11.** *Let* $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ *be a Schauder tree basis and let* $Y$ *be an* $X$*-singular subspace of* $T_2^{\mathfrak{X}}$*. Then* $Y$ *is unconditionally saturated.*

**Theorem 12.** *Let* $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ *be a Schauder tree basis and let* $Y$ *be an* $X$*-singular subspace of* $T_2^{\mathfrak{X}}$*. Then for every normalized basic sequence* $(x_n)$ *in* $Y$ *there exists a normalized block sequence* $(y_n)$ *of* $(x_n)$ *satisfying an upper* $\ell_2$ *estimate. That is, there exists a constant* $C \geqslant 1$ *such that for every* $k \in \mathbb{N}$ *and every* $a_0, \dots, a_k \in \mathbb{R}$ *we have*

$$\Big\| \sum_{n=0}^{k} a_n y_n \Big\| \leqslant C \Big( \sum_{n=0}^{k} |a_n|^2 \Big)^{1/2}.$$

*In particular, every* $X$*-singular subspace* $Y$ *of* $T_2^{\mathfrak{X}}$ *contains no* $\ell_p$ *for* $1 \leqslant p < 2$.

We notice that in Theorem 12 one cannot expect to obtain a block sequence satisfying a lower $\ell_2$ estimate. Indeed, as it has been shown in [AD, Theorem 25], if $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ is a Schauder tree basis such that the tree $T$ is not small (precisely, if the tree $T$ contains a perfect[4] subtree), then one can find in $T_2^{\mathfrak{X}}$ a normalized block sequence $(x_n)$ which is equivalent to the standard basis of $c_0$ and which spans an $X$-singular subspace. Clearly, no block subsequence of $(x_n)$ can have a lower $\ell_2$ estimate.

The rest of this section is organized as follows. In §3.2 we provide a characterization of the class of $X$-singular subspaces of $T_2^{\mathfrak{X}}$. Using this characterization we show, for instance, that every $X$-singular subspace of $T_2^{\mathfrak{X}}$ contains an $X$-compact subspace. This can be seen as a "tree" version of the classical theorem of Kato asserting that for every strictly singular operator $T \colon X \to Y$ there is an infinite-dimensional subspace $Z$ of $X$ such that the operator $T \colon Z \to Y$ is compact. In §3.3 we present the proofs of Theorems 11 and 12.

3.2. **A characterization of $X$-singular subspaces.** We start with the following definition.

**Definition 13.** *Let* $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ *be a Schauder tree basis. The* $c_0$ Baire sum *of* $\mathfrak{X}$*, denoted by* $T_0^{\mathfrak{X}}$*, is defined to be the completion of* $c_{00}(T)$ *equipped with the norm*

$$\|z\|_{T_0^{\mathfrak{X}}} \coloneqq \sup\Big\{ \Big\| \sum_{t\in\mathfrak{s}} z(t) x_t \Big\|_X : \mathfrak{s} \text{ is a segment of } T \Big\}. \tag{9}$$

[4]A tree $T$ is perfect if every node $t \in T$ has at least two incomparable successors.

*By* $\mathrm{I}\colon T_2^{\mathfrak{X}} \to T_0^{\mathfrak{X}}$ *we shall denote the natural inclusion operator.*

Our characterization of $X$-singular subspaces of $T_2^{\mathfrak{X}}$ is achieved by considering the functional analytic properties of the inclusion operator $\mathrm{I}\colon T_2^{\mathfrak{X}} \to T_0^{\mathfrak{X}}$. Precisely, we have the following theorem.

**Proposition 14.** *Let* $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ *be a Schauder tree basis and let* $Y$ *be a subspace of* $T_2^{\mathfrak{X}}$*. Then the following are equivalent.*

(i) $Y$ *is an* $X$*-singular subspace of* $T_2^{\mathfrak{X}}$*.*
(ii) *The operator* $\mathrm{I}\colon Y \to T_0^{\mathfrak{X}}$ *is strictly singular.*

We isolate two consequences of Proposition 14. The first one is simply a restatement of Proposition 14.

**Corollary 15.** *Let* $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ *be a Schauder tree basis and let* $Y$ *be a block subspace of* $T_2^{\mathfrak{X}}$*. Assume that* $Y$ *is* $X$*-singular. Then for every* $\varepsilon > 0$ *we may select a finitely supported vector* $y \in Y$ *with* $\|y\| = 1$ *and such that* $\|P_{\mathfrak{s}}(y)\| \leqslant \varepsilon$ *for every segment* $\mathfrak{s}$ *of* $T$*.*

**Corollary 16.** *Let* $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ *be a Schauder tree basis and let* $Y$ *be an infinite-dimensional subspace of* $T_2^{\mathfrak{X}}$*. Assume that* $Y$ *is* $X$*-singular. Then there exists an infinite-dimensional subspace* $Y'$ *of* $Y$ *which is* $X$*-compact.*

*Proof.* By Proposition 14, the operator $\mathrm{I}\colon Y \to T_0^{\mathfrak{X}}$ is strictly singular. Hence, by [LT, Proposition 2.c.4], there exists an infinite-dimensional subspace $Y'$ of $Y$ such that the operator $\mathrm{I}\colon Y' \to T_0^{\mathfrak{X}}$ is compact. It is easy to see that $Y'$ must be an $X$-compact subspace of $T_2^{\mathfrak{X}}$ in the sense part (b) of Definition 10. □

For the proof of Proposition 14 we need a couple of results from [AD]. The first one is the following (see [AD, Lemma 17]).

**Lemma 17.** *Let* $(x_n)$ *be a bounded block sequence in* $T_2^{\mathfrak{X}}$ *and let* $\varepsilon > 0$ *be such that* $\limsup \|P_\sigma(x_n)\| < \varepsilon$ *for every* $\sigma \in [T]$*. Then there exists* $L \in [\mathbb{N}]^\infty$ *such that for every* $\sigma \in [T]$ *we have* $|\{n \in L : \|P_\sigma(x_n)\| \geqslant \varepsilon\}| \leqslant 1$*.*

The second result is the following special case of [AD, Proposition 33].

**Proposition 18.** *Let* $Y$ *be a block* $X$*-singular subspace of* $T_2^{\mathfrak{X}}$*. Then for every* $\varepsilon > 0$ *we may select a normalized block sequence* $(y_n)$ *in* $Y$ *such that for every* $\sigma \in [T]$ *we have* $\limsup \|P_\sigma(y_n)\| < \varepsilon$*.*

We are ready to proceed to the proof of Proposition 14.

*Proof of Proposition* 14. It is clear that (ii) implies (i). Hence we only need to show the converse implication. We argue by contradiction. So, assume that $Y$ is an $X$-singular subspace of $T_2^{\mathfrak{X}}$ such that the operator $\mathrm{I}\colon Y \to T_0^{\mathfrak{X}}$ is not strictly singular. By definition, there exists a further subspace $Y'$ of $Y$ such that $\mathrm{I}\colon Y' \to T_0^{\mathfrak{X}}$ is an isomorphic embedding. Using a sliding hump argument, we may recursively select a normalized basic sequence $(y_n)$ in $Y'$ and a normalized block sequence $(z_n)$ in $T_2^{\mathfrak{X}}$ such that, setting $Z \coloneqq \overline{\mathrm{span}}\{z_n : n \in \mathbb{N}\}$, the following are satisfied.

(a) The sequence $(z_n)$ is equivalent to $(y_n)$.
(b) The subspace $Z$ of $T_2^{\mathfrak{X}}$ is $X$-singular.
(c) The operator $\mathrm{I} : Z \to T_0^{\mathfrak{X}}$ is an isomorphic embedding.

The selection is fairly standard (we leave the details to the interested reader). By (c) above, there exists a constant $C > 0$ such that for every $z \in Z$ we have

$$C\|z\|_{T_2^{\mathfrak{X}}} \leqslant \|z\|_{T_0^{\mathfrak{X}}} \leqslant \|z\|_{T_2^{\mathfrak{X}}}. \tag{10}$$

We fix $k_0 \in \mathbb{N}$ and $\varepsilon > 0$ satisfying

$$k_0 > \frac{64}{C^4} \ \text{ and } \ \varepsilon < \min\Big\{\frac{C}{2}, \frac{1}{k_0}\Big\}. \tag{11}$$

By (b) above, we may apply Proposition 18 to the block subspace $Z$ of $T_2^{\mathfrak{X}}$ and the chosen $\varepsilon$. It follows that there exists a normalized block sequence $(x_n)$ in $Z$ such that $\limsup \|P_\sigma(x_n)\| < \varepsilon$ for every $\sigma \in [T]$. By Lemma 17 and by passing to a subsequence of $(x_n)$ if necessary, we may additionally assume that for every $\sigma \in [T]$ we have $|\{n \in \mathbb{N} : \|P_\sigma(x_n)\| \geqslant \varepsilon\}| \leqslant 1$. As the basis of $T_2^{\mathfrak{X}}$ is bimonotone, we may strengthen this property to the following one.

(d) For every segment $\mathfrak{s}$ of $T$ we have $|\{n \in \mathbb{N} : \|P_{\mathfrak{s}}(x_n)\| \geqslant \varepsilon\}| \leqslant 1$.

By Fact 9 and (10), for every $n \in \mathbb{N}$ we may select a segment $\mathfrak{s}_n$ of $T$ such that

(e) $\|P_{\mathfrak{s}_n}(x_n)\| \geqslant C$, and
(f) $\mathfrak{s}_n \subseteq \{t_k : k \in \mathrm{range}(x_n)\}$.

Since the sequence $(x_n)$ is block, we see that such a selection guarantees that

(g) $\|P_{\mathfrak{s}_n}(x_m)\| = 0$ for every $n, m \in \mathbb{N}$ with $n \neq m$.

We set $t_n = \min(\mathfrak{s}_n)$. Applying the classical Ramsey theorem, we find an infinite subset $L = \{l_0 < l_1 < \cdots\}$ of $\mathbb{N}$ such that one of the following (mutually exclusive) cases must occur.

Case 1: *The set $\{t_n : n \in L\}$ is an antichain.* Our hypothesis in this case implies that for every $n, m \in L$ with $n \neq m$ the segments $\mathfrak{s}_n$ and $\mathfrak{s}_m$ are incomparable. We define $z \coloneqq x_{l_0} + \cdots + x_{l_{k_0}}$. Since the family $(\mathfrak{s}_{l_i})_{i=0}^{k_0}$ consists of pairwise incomparable segments of $T$, we see that

$$\|z\| \geqslant \Big(\sum_{i=0}^{k_0} \|P_{\mathfrak{s}_{l_i}}(z)\|^2\Big)^{1/2} \overset{\text{(g)}}{=} \Big(\sum_{i=0}^{k_0} \|P_{\mathfrak{s}_{l_i}}(x_{l_i})\|^2\Big)^{1/2} \overset{\text{(e)}}{\geqslant} C\sqrt{k_0+1}. \tag{12}$$

We set $w \coloneqq z/\|z\| \in Z$. Invoking (d) above, inequality (12) and the choice of $k_0$ and $\varepsilon$ in (11), for every segment $\mathfrak{s}$ of $T$ we have

$$\|P_{\mathfrak{s}}(w)\| \leqslant \frac{1 + k_0\varepsilon}{C\sqrt{k_0+1}} < \frac{C}{2}.$$

It follows that

$$\|w\|_{T_0^{\mathfrak{X}}} \leqslant \frac{C}{2}$$

which contradicts inequality (10). Hence this case is impossible.

CASE 2: *The set* $\{t_n : n \in L\}$ *is a chain.* Let $\tau \in [T]$ be the branch of $T$ determined by the infinite chain $\{t_n : n \in L\}$. By (d) above and by passing to an infinite subset of $L$ if necessary, we may assume that $\|P_\tau(x_n)\| < \varepsilon$ for every $n \in L$. The basis of $T_2^{\mathfrak{X}}$ is bimonotone, and so, we have the following property.

(h) If $\mathfrak{s}$ is a segment of $T$ with $\mathfrak{s} \subseteq \tau$, then $\|P_{\mathfrak{s}}(x_n)\| < \varepsilon$ for every $n \in L$.

We set $\mathfrak{s}'_n = \mathfrak{s}_n \setminus \tau$. Observe that the set $\mathfrak{s}'_n$ is a sub-segment of $\mathfrak{s}_n$. Notice that $\mathfrak{s}_n$ is the disjoint union of the successive segments $\mathfrak{s}_n \cap \tau$ and $\mathfrak{s}'_n$. Hence, by properties (e) and (h) above and the choice of $\varepsilon$, we see that

$$\|P_{\mathfrak{s}'_n}(x_n)\| \geqslant C - \varepsilon \geqslant \frac{C}{2} \tag{13}$$

for every $n \in L$. Also notice that if $n, m \in L$ with $n \neq m$, then the segments $\mathfrak{s}'_n$ and $\mathfrak{s}'_m$ are incomparable. We set

$$z \coloneqq x_{l_0} + \cdots + x_{l_{k_0}} \quad \text{and} \quad w \coloneqq \frac{z}{\|z\|}.$$

Arguing precisely as in Case 1 and using the estimate in (13), we conclude that

$$\|w\|_{T_0^{\mathfrak{X}}} \leqslant \frac{C}{2}.$$

This is again a contradiction. The proof of Proposition 14 is completed. □

3.3. **Proof of Theorem 11 and of Theorem 12.** We start with the following lemma.

**Lemma 19.** *Let* $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t \in T})$ *be a Schauder tree basis. Let* $(w_n)$ *be a normalized block sequence in* $T_2^{\mathfrak{X}}$ *such that for every* $n \in \mathbb{N}$ *with* $n \geqslant 1$ *and every segment* $\mathfrak{s}$ *of* $T$ *we have*

$$\|P_{\mathfrak{s}}(w_n)\| \leqslant \frac{1}{\sum_{i=0}^{n-1} |\mathrm{supp}(w_i)|^{1/2}} \cdot \frac{1}{2^{n+2}}. \tag{14}$$

*Then the following are satisfied.*

(i) *The sequence* $(w_n)$ *is unconditional.*
(ii) *The sequence* $(w_n)$ *satisfies an upper* $\ell_2$ *estimate.*

*Proof.* We will only give the proof of part (i). For a proof of part (ii) we refer to [AD, Proposition 21].

So, let $k \in \mathbb{N}$ and let $a_0, \ldots, a_k \in \mathbb{R}$ be such that $\|\sum_{n=0}^{k} a_n w_n\| = 1$. Also let $F \subseteq \{0, \ldots, k\}$ and let $F = \{n_0 < \cdots < n_p\}$ denote its increasing enumeration. We will show that $\|\sum_{n \in F} a_n w_n\| \leqslant \sqrt{3}$. This will clearly finish the proof. For notational simplicity, we set

$$w \coloneqq \sum_{n=0}^{k} a_n w_n \quad \text{and} \quad z \coloneqq \sum_{n \in F} a_n w_n.$$

Let $(\mathfrak{s}_j)_{j=0}^l$ be an arbitrary collection of pairwise incomparable segments of $T$. We want to estimate the sum $\sum_{j=0}^l \|P_{\mathfrak{s}_j}(z)\|^2$. To this end we may assume that for every $j \in \{0,\dots,l\}$ there exists $i \in \{0,\dots,p\}$ with $\mathfrak{s}_j \cap \mathrm{supp}(w_{n_i}) \neq \emptyset$. We define, recursively, a partition $(\Delta_i)_{i=0}^p$ of $\{0,\dots,l\}$ by the rule

$$\begin{aligned} \Delta_0 &= \big\{ j \in \{0,\dots,l\} : \mathfrak{s}_j \cap \mathrm{supp}(w_{n_0}) \neq \emptyset \big\} \\ \Delta_1 &= \big\{ j \in \{0,\dots,l\} \setminus \Delta_0 : \mathfrak{s}_j \cap \mathrm{supp}(w_{n_1}) \neq \emptyset \big\} \\ &\ \vdots \\ \Delta_p &= \Big\{ j \in \{0,\dots,l\} \setminus \Big( \bigcup_{i=0}^{p-1} \Delta_i \Big) : \mathfrak{s}_j \cap \mathrm{supp}(w_{n_p}) \neq \emptyset \Big\}. \end{aligned}$$

The segments $(\mathfrak{s}_j)_{j=0}^l$ are pairwise incomparable and a fortiori disjoint. It follows that

$$|\Delta_i| \leqslant |\mathrm{supp}(w_{n_i})| \ \text{ for every } i \in \{0,\dots,p\}. \tag{15}$$

Also notice that for every $0 \leqslant i < q \leqslant p$ we have

$$\sum_{j \in \Delta_q} \|P_{\mathfrak{s}_j}(w_{n_i})\| = 0. \tag{16}$$

Let $j \in \{0,\dots,l\}$. There exists a unique $i \in \{0,\dots,p\}$ such that $j \in \Delta_i$. By Fact 9, we may select a segment $\mathfrak{s}'_j$ of $T$ such that

(a) $\mathfrak{s}'_j \subseteq \mathfrak{s}_j$,
(b) $\mathfrak{s}'_j \subseteq \{t_m : m \in \mathrm{range}(w_{n_i})\}$, and
(c) $\|P_{\mathfrak{s}_j}(a_{n_i} w_{n_i})\| = \|P_{\mathfrak{s}'_j}(a_{n_i} w_{n_i})\|$.

The above selection guarantees the following properties.

(d) The family $(\mathfrak{s}'_j)_{j=0}^l$ consists of pairwise incomparable segment of $T$. This is a straightforward consequence of (a) above and of our assumptions on the family $(\mathfrak{s}_j)_{j=0}^l$.
(e) We have $\|P_{\mathfrak{s}_j}(a_{n_i} w_{n_i})\| = \|P_{\mathfrak{s}'_j}(a_{n_i} w_{n_i})\| = \|P_{\mathfrak{s}'_j}(w)\|$. This is a consequence of (b) and (c) above and of the fact that the sequence $(w_n)$ is block.

We are ready for the last part of the argument. Let $i \in \{0,\dots,p\}$ and $j \in \Delta_i$. Our goal is to estimate the quantity $\|P_{\mathfrak{s}_j}(z)\|$. First we notice that

$$\begin{aligned} \|P_{\mathfrak{s}_j}(z)\| &\overset{(16)}{=} \|P_{\mathfrak{s}_j}(a_{n_i} w_{n_i} + \dots + a_{n_p} w_{n_p})\| \\ &\leqslant \|P_{\mathfrak{s}_j}(a_{n_i} w_{n_i})\| + \sum_{q=i+1}^{p} |a_{n_q}| \cdot \|P_{\mathfrak{s}_j}(w_{n_q})\|. \end{aligned}$$

Invoking the fact that the Schauder basis $(e_t)_{t \in T}$ of $T_2^{\mathfrak{X}}$ is bimonotone and (14), we see that for every $q \in \{i+1,\dots,p\}$ we have $\|P_{\mathfrak{s}_j}(w_{n_q})\| \leqslant |\mathrm{supp}(w_{n_i})|^{-1/2} \cdot 2^{-(q+2)}$

and $|a_{n_q}| \leqslant 1$. Hence, the previous estimate yields

$$\begin{aligned}
\|P_{\mathfrak{s}_j}(z)\| &\leqslant \|P_{\mathfrak{s}_j}(a_{n_i}w_{n_i})\| + \frac{1}{|\mathrm{supp}(w_{n_i})|^{1/2}} \cdot \sum_{q=i+1}^{p} \frac{1}{2^{q+2}} \\
&\overset{(15)}{\leqslant} \|P_{\mathfrak{s}_j}(a_{n_i}w_{n_i})\| + \frac{1}{|\Delta_i|^{1/2}} \cdot \frac{1}{2^{i+2}} \\
&\overset{\text{(e)}}{=} \|P_{\mathfrak{s}'_j}(w)\| + \frac{1}{|\Delta_i|^{1/2}} \cdot \frac{1}{2^{i+2}}.
\end{aligned}$$

The above inequality, in turn, implies that if $\Delta_i$ is nonempty, then

$$\begin{aligned}
\sum_{j\in\Delta_i} \|P_{\mathfrak{s}_j}(z)\|^2 &\leqslant 2\sum_{j\in\Delta_i} \|P_{\mathfrak{s}'_j}(w)\|^2 + 2\sum_{j\in\Delta_i} \frac{1}{|\Delta_i|} \cdot \frac{1}{2^{i+2}} \\
&\leqslant 2\sum_{j\in\Delta_i} \|P_{\mathfrak{s}'_j}(w)\|^2 + \frac{1}{2^{i+1}}. \qquad (17)
\end{aligned}$$

Summarizing, we see that

$$\sum_{j=0}^{l} \|P_{\mathfrak{s}_j}(z)\|^2 = \sum_{i=0}^{p}\sum_{j\in\Delta_i} \|P_{\mathfrak{s}_j}(z)\|^2 \overset{(17)}{\leqslant} 2\sum_{j=0}^{l} \|P_{\mathfrak{s}'_j}(w)\|^2 + 1 \overset{\text{(d)}}{\leqslant} 2\|w\|^2 + 1 \leqslant 3.$$

The family $(\mathfrak{s}_j)_{j=0}^{l}$ was arbitrary, and so, $\|z\| \leqslant \sqrt{3}$. The proof is completed. □

We continue with the proof of Theorem 11.

*Proof of Theorem* 11. Let $Y$ be an $X$-singular subspace of $T_2^{\mathfrak{X}}$. Clearly every subspace $Y'$ of $Y$ is also $X$-singular. Hence, it is enough to show that every $X$-singular subspace contains an unconditional basic sequence. So, let $Y$ be one. Using a sliding hump argument, we may additionally assume that $Y$ is a block subspace of $T_2^{\mathfrak{X}}$. Recursively and with the help of Corollary 15, we may select a normalized block sequence $(w_n)$ in $Y$ such that for every $n \in \mathbb{N}$ with $n \geqslant 1$ and every segment $\mathfrak{s}$ of $T$ we have

$$\|P_{\mathfrak{s}}(w_n)\| \leqslant \frac{1}{\sum_{i=0}^{n-1} |\mathrm{supp}(w_i)|^{1/2}} \cdot \frac{1}{2^{n+2}}.$$

By part (i) of Lemma 19, the sequence $(w_n)$ is unconditional. The proof is completed. □

We proceed to the proof of Theorem 12.

*Proof of Theorem* 12. Let $Y$ be an $X$-singular subspace of $T_2^{\mathfrak{X}}$. Also let $(x_n)$ be a normalized basic sequence in $Y$. A standard sliding hump argument allows us to construct a normalized block sequence $(v_n)$ of $(x_n)$ and a block sequence $(z_n)$ in $T_2^{\mathfrak{X}}$ such that, setting $Z := \overline{\mathrm{span}}\{z_n : n \in \mathbb{N}\}$, the following are satisfied.

(a) The sequences $(v_n)$ and $(z_n)$ are equivalent.
(b) The subspace $Z$ of $T_2^{\mathfrak{X}}$ is $X$-singular.

As in the proof of Theorem 11, using (b) above and Corollary 15, we select a normalized block sequence $(w_n)$ of $(z_n)$ such that for every $n \in \mathbb{N}$ with $n \geqslant 1$ and every segment $\mathfrak{s}$ of $T$ inequality (14) is satisfied for the sequence $(w_n)$. By part (ii) of Lemma 19, the sequence $(w_n)$ satisfies an upper $\ell_2$ estimate. Let $(b_n)$ be the block sequence of $(v_n)$ corresponding to $(w_n)$. Observe that, by (a) above, the sequence $(b_n)$ is seminormalized and satisfies an upper $\ell_2$ estimate. The property of being a block sequence is transitive, and so, $(b_n)$ is a normalized block sequence of $(x_n)$ as well. Hence, setting $y_n = b_n/\|b_n\|$ for every $n \in \mathbb{N}$, we see that the sequence $(y_n)$ is the desired one.

Finally, to see that every $X$-singular subspace of $T_2^{\mathfrak{X}}$ contains no $\ell_p$ for $1 \leqslant p < 2$, we argue by contradiction. So, assume that $Y$ is an $X$-singular subspace of $T_2^{\mathfrak{X}}$ containing an isomorphic copy of $\ell_{p_0}$ for some $1 \leqslant p_0 < 2$. There exists, in such a case, a normalized basic sequence $(x_n)$ in $Y$ which is equivalent to the standard unit vector basis $(e_n)$ of $\ell_{p_0}$. Let $(y_n)$ be a normalized block subsequence of $(x_n)$ satisfying an upper $\ell_2$ estimate. As any normalized block subsequence of $(e_n)$ is equivalent to $(e_n)$ (see [LT]), we see that there must exist constants $C \geqslant c > 0$ such that for every $k \in \mathbb{N}$ and every $a_0, \dots, a_k \in \mathbb{R}$ we have

$$c\Big(\sum_{n=0}^{k} |a_n|^{p_0}\Big)^{1/p_0} \leqslant \Big\|\sum_{n=0}^{k} a_n y_n\Big\| \leqslant C\Big(\sum_{n=0}^{k} |a_n|^2\Big)^{1/2}.$$

This is clearly a contradiction. The proof is completed. □

We close this section by recording the following consequence of Theorem 12.

**Corollary 20.** *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis. Let $1 \leqslant p < 2$. Then the following are equivalent.*

(i) *The space $T_2^{\mathfrak{X}}$ contains an isomorphic copy of $\ell_p$.*
(ii) *There exists $\sigma \in [T]$ such that $X_\sigma$ contains an isomorphic copy of $\ell_p$.*

*Proof.* It is clear that (ii) implies (i). Conversely, assume that $\ell_p$ embeds into $T_2^{\mathfrak{X}}$, and let $Y$ be a subspace of $T_2^{\mathfrak{X}}$ which is isomorphic to $\ell_p$. By Theorem 12, we see that $Y$ is not $X$-singular. Hence, there exist $\sigma \in [T]$ and an infinite-dimensional subspace $Y'$ of $Y$ such that $P_\sigma : Y' \to \mathcal{X}_\sigma$ is an isomorphic embedding. Recalling that every subspace of $\ell_p$ contains a copy of $\ell_p$ and that the spaces $\mathcal{X}_\sigma$ and $X_\sigma$ are isometric, the result follows. □

## 4. The main result

This section is devoted to the proof of Theorem 2 stated in the introduction. To this end, we will need the following correspondence principle between analytic classes of separable Banach spaces and Schauder tree bases (see [AD, Proposition 83] or [D, Lemma 32]).

**Lemma 21.** *Let $\mathcal{A}'$ be an analytic subset of* SB *such that every $Y \in \mathcal{A}'$ has a Schauder basis. Then there exist a separable Banach space $X$, a pruned tree $T$ on $\mathbb{N}\times\mathbb{N}$ and a normalized sequence $(x_t)_{t\in T}$ in $X$ such that the following are satisfied.*

(i) *The family $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ is a Schauder tree basis.*
(ii) *For every $Y \in \mathcal{A}'$ there exists $\sigma \in [T]$ with $Y \cong X_\sigma$.*
(iii) *For every $\sigma \in [T]$ there exists $Y \in \mathcal{A}'$ with $X_\sigma \cong Y$.*

We are now ready to proceed to the proof of Theorem 2. So, let $\mathcal{A}$ be an analytic subset of US. We apply Proposition 3 and we obtain a subset $\mathcal{A}'$ of SB with the following properties.

(a) The set $\mathcal{A}'$ is analytic.
(b) Every $Y \in \mathcal{A}'$ has a Schauder basis.
(c) Every $Y \in \mathcal{A}'$ is unconditionally saturated.
(d) For every $X \in \mathcal{A}$ there exists $Y \in \mathcal{A}'$ such that $Y$ contains an isometric copy of $X$.

By (a) and (b) above, we apply Lemma 21 to the set $\mathcal{A}'$ and we obtain a Schauder tree basis $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ satisfying the following properties.

(e) For every $Y \in \mathcal{A}'$ there exists $\sigma \in [T]$ with $Y \cong X_\sigma$.
(f) For every $\sigma \in [T]$ there exists $Y \in \mathcal{A}'$ such that $X_\sigma \cong Y$.

Consider the $\ell_2$ Baire sum $T_2^{\mathfrak{X}}$ of this Schauder tree basis $\mathfrak{X}$. We claim that the space $T_2^{\mathfrak{X}}$ is the desired one. Indeed, recall first that $T_2^{\mathfrak{X}}$ has a Schauder basis. Moreover, by (d) and (e) above, we see that $T_2^{\mathfrak{X}}$ contains an isomorphic copy of every $X \in \mathcal{A}$.

It remains is to check that $T_2^{\mathfrak{X}}$ is unconditionally saturated. To this end, let $Z$ be an arbitrary subspace of $T_2^{\mathfrak{X}}$. We have to show that the space $Z$ contains an unconditional basic sequence. We distinguish the following (mutually exclusive) cases.

Case 1: *The subspace $Z$ is not $\mathcal{X}$-singular.* In this case, by definition, there exist $\sigma \in [T]$ and a further subspace $Z'$ of $Z$ such that the operator $P_\sigma \colon Z' \to \mathcal{X}_\sigma$ is an isomorphic embedding. By (f) and (c) above, we see that $Z'$ must contain an unconditional basic sequence.

Case 2: *The subspace $Z$ is $\mathcal{X}$-singular.* By Theorem 11, we see that in this case the subspace $Z$ must also contain an unconditional basic sequence.

By the above, it follows that $T_2^{\mathfrak{X}}$ is unconditionally saturated. The proof of Theorem 2 is completed.

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