ON FILLING FAMILIES OF FINITE SUBSETS OF THE CANTOR SET

PANDELIS DODOS AND VASSILIS KANELLOPOULOS

ABSTRACT. Let $\varepsilon > 0$ and let $\mathcal{F}$ be a family of finite subsets of the Cantor set $\mathcal{C}$. Following Fremlin we say that $\mathcal{F}$ is *ε-filling* over $\mathcal{C}$ if $\mathcal{F}$ is hereditary and for every finite $F \subseteq \mathcal{C}$ there exists $G \subseteq F$ such that $G \in \mathcal{F}$ and $|G| \geqslant \varepsilon|F|$. We show that if $\mathcal{F}$ is ε-filling over $\mathcal{C}$ and C-measurable in $[\mathcal{C}]^{<\omega}$, then for every perfect $P \subseteq \mathcal{C}$ there exists perfect $Q \subseteq P$ such that $[Q]^{<\omega} \subseteq \mathcal{F}$. A similar result for weaker versions of density is also obtained.

1. Introduction

Let X be a set and let $\varepsilon > 0$. A family $\mathcal{F} \subseteq [X]^{<\omega}$ is said to be *ε-filling* over X if $\mathcal{F}$ is hereditary (that is, for every $F \in \mathcal{F}$ and every $G \subseteq F$ we have $G \in \mathcal{F}$) and for every $F \in [X]^{<\omega}$ there exists $G \subseteq F$ with $G \in \mathcal{F}$ and $|G| \geqslant \varepsilon|F|$. The notion of an ε-filling family is due to Fremlin [6] who posed the following problem. For which cardinals κ, λ we have that whenever $|X| = \kappa$ and $\mathcal{F} \subseteq [X]^{<\omega}$ is ε-filling, then there exists $A \subseteq X$ with $|A| = \lambda$ and such that $[A]^{<\omega} \subseteq \mathcal{F}$? It is well-known that if $\kappa = \omega$, then $\lambda < \omega$. A classical example is the Schreier family $\mathcal{S} := \{F \subseteq \omega : |F| \leqslant \min(F) + 1\}$. On the other hand, Fremlin has shown (see [6, Corollary 6D]) that large cardinal hypotheses imply the consistency of the statement that for every ε-filling family $\mathcal{F}$ over $\mathfrak{c}$, there exists an infinite subset A of $\mathfrak{c}$ such that $[A]^{<\omega} \subseteq \mathcal{F}$.

In this paper we look at the problem when X is the Cantor set $\mathcal{C} = 2^\omega$. Notice that $[\mathcal{C}]^{<\omega}$ has the structure of a Polish space since it is the direct sum of $[\mathcal{C}]^k$ $(k \geqslant 1)$. Argyros, Lopez-Abad and Todorčević asked whether the above mentioned result of Fremlin is valid without extra set-theoretic assumptions provided that $\mathcal{F}$ is reasonably definable. We prove the following theorem which answers this question positively.

Theorem A. *Let $\mathcal{F}$ be an ε-filling family over $\mathcal{C}$. If $\mathcal{F}$ is C-measurable in $[\mathcal{C}]^{<\omega}$, then for every perfect $P \subseteq \mathcal{C}$ there exists perfect $Q \subseteq P$ such that $[Q]^{<\omega} \subseteq \mathcal{F}$.*

Actually, we prove a more general result (Theorem 2 in the main text) which implies, for instance, that Theorem A is valid for an arbitrary ε-filling family in the Solovay model [9].

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Our second result concerns weaker versions of density. For every $\mathcal{F}\subseteq[\mathcal{C}]^{<\omega}$ and every $n\geqslant 1$ let $d_{\mathcal{F}}(n)$ be the *density* of $\mathcal{F}$ at n, that is,

$$d_{\mathcal{F}}(n) \coloneqq \min_{F\in[\mathcal{C}]^n} \max\big\{|G| : G\subseteq F \text{ and } G\in\mathcal{F}\big\}.$$

Notice that $\mathcal{F}$ is ε-filling if and only if $\mathcal{F}$ is hereditary and satisfies $\frac{d_{\mathcal{F}}(n)}{n}\geqslant\varepsilon$ for every $n\geqslant 1$. Although every C-measurable ε-filling family $\mathcal{F}$ over $\mathcal{C}$ is not compact, Fremlin has shown that for every $f\colon\omega\to\omega$ with $n\geqslant f(n)>0$ for every $n\geqslant 1$ and $\lim\frac{f(n)}{n}=0$, there exists a compact and hereditary family $\mathcal{F}$, closed in $[\mathcal{C}]^{<\omega}$ and such that $d_{\mathcal{F}}(n)\geqslant f(n)$ for every $n\geqslant 1$ (see [6, Proposition 4B]). The following theorem shows, however, that any such family $\mathcal{F}$ must still be large.

Theorem B. *Let $\mathcal{F}\subseteq[\mathcal{C}]^{<\omega}$ be hereditary. Assume that $\mathcal{F}$ has the Baire property in $[\mathcal{C}]^{<\omega}$ and satisfies*

$$(*)\qquad \limsup\frac{\log_2 d_{\mathcal{F}}(2^n)}{\log_2 n}=+\infty.$$

Then for every $k\geqslant 1$ there exists a perfect subset P of $\mathcal{C}$ such that $[P]^k\subseteq\mathcal{F}$.

The proof of Theorem B is based on Blass' theorem [4]. Theorem B has the following consequence which shows that we can increase the density of $\mathcal{F}$ by passing to a perfect subset. In particular, if $\mathcal{F}$ is C-measurable and satisfies equation $(*)$ above, then for every $f\colon\omega\to\omega$ with $n\geqslant f(n)>0$ for every $n\geqslant 1$ and $\lim\frac{f(n)}{n}=0$, and every perfect subset P of $\mathcal{C}$ there exists a perfect subset Q of P such that the density of $\mathcal{F}$ in Q is greater or equal to f. We also include some connections of the above results with Banach spaces.

2. Preliminaries

2.1. Let $\omega=\{0,1,\dots\}$. The cardinality of a set A is denoted by $|A|$. By $<$ we denote the (strict) lexicographical ordering on the Cantor set $\mathcal{C}=2^\omega$. If $A,B\subseteq\mathcal{C}$, then we write $A<B$ if for every $x\in A$ and every $y\in B$ we have $x<y$. For every $n\geqslant 1$ and every $P\subseteq\mathcal{C}$ by $[P]^n$ we denote the set of all $<$-increasing sequences of P of cardinality n, while by $[P]^{<\omega}$ the set of all finite $<$-increasing sequences of P.

2.2. By $2^{<\omega}$ we denote the Cantor tree, that is, the set of all finite sequences of 0's and 1's; we equipped $2^{<\omega}$ with the (strict) partial ordering $\sqsubset$ of initial segment. If $s,t\in 2^{<\omega}$, then by $s^\frown t$ we denote their concatenation. For every $s\in 2^{<\omega}$ the length $\ell(s)$ of s is defined to be the cardinality of the set $\{t\in 2^{<\omega}: t\sqsubset s\}$. For every $n\in\omega$ by 2^n we denote the set of all sequences in $2^{<\omega}$ of length n, while for every $n\geqslant 1$ by $2^{<n}$ we denote the set of all sequences of length less than n. For every $s,t\in 2^{<\omega}$ we denote by $s\wedge t$ the $\sqsubset$-maximal node w such that $w\sqsubseteq s$ and $w\sqsubseteq t$. Similarly, if $x,y\in\mathcal{C}$, then by $x\wedge y$ we denote the $\sqsubset$-maximal node t of $2^{<\omega}$ with $t\sqsubset x$ and $t\sqsubset y$. We write $s\prec t$ if $w^\frown 0\sqsubseteq s$ and $w^\frown 1\sqsubseteq t$, where $w=s\wedge t$.

2.3. We view every subset of $2^{<\omega}$ as a *subtree* of $2^{<\omega}$ equipped with the induced partial ordering. For every $m\in\omega$ and every subtree T of $2^{<\omega}$ by $T(m)$ we denote the *m-level* of T, that is, the set of all $t\in T$ such that $|\{s\in T: s\sqsubset t\}|=m$. A node $t\in T$ is said to be a *splitting node* of T if t has at least two immediate successors in T. By $\mathrm{Spl}(T)$ we denote the set of splitting nodes of T.

A subtree T of $2^{<\omega}$ is said to be *downwards closed* if for every $t\in T$ the set $\{s: s\sqsubseteq t\}$ is a subset of T. Notice that if T is a downwards closed subtree and $m\in\omega$, then $T(m)=\{t\in T: t\in 2^m\}$. The *body* $[T]$ of T is the set $\{x\in\mathcal{C}: x|n\in T\ \forall n\in\omega\}$, where $x|n=(x_0,\dots,x_{n-1})\in 2^{<\omega}$ if $n\geqslant 1$ and $x|0=\emptyset$ if $n=0$. If $t\in T$, then we set $[T]_t\coloneqq\{x\in[T]: t\sqsubset x\}$. In particular, for every $t\in 2^{<\omega}$ we have $\mathcal{C}_t=\{x\in\mathcal{C}: t\sqsubset x\}$.

If $A\subseteq 2^{<\omega}$, then the *downwards closure* $\hat{A}$ of A is the set $\{s\in 2^{<\omega}:\exists t\in A \text{ with } s\sqsubseteq t\}$. Moreover, for every $F\subseteq\mathcal{C}$ we set $T_F\coloneqq\{x|n: x\in F, n\in\omega\}$. Observe that F is closed if and only if $F=[T_F]$. It is easy to see that if F is a finite subset of $\mathcal{C}$, then $|\mathrm{Spl}(T_F)|=|F|-1$. Similarly, if A is a finite antichain of $2^{<\omega}$, then $|\mathrm{Spl}(\hat{A})|=|A|-1$.

A subtree T of $2^{<\omega}$ is said to be *pruned* if for every $t\in T$ there exists $s\in T$ with $t\sqsubset s$. It is said to be *skew* if for every $m\in\omega$ we have $|T(m)\cap\mathrm{Spl}(T)|\leqslant 1$.

2.4. We recall the notion of the *type* τ of a downwards closed, pruned, skew subtree T of $2^{<\omega}$ following the presentation of Louveau, Shelah and Veličković in [8]. We will only treat trees T with $[T]$ finite. So, let $k\geqslant 2$ and let T be a downwards closed, pruned, skew subtree of $2^{<\omega}$ such that $[T]$ has k elements. The type of T is a function $\tau\colon\{1,\dots,k-1\}\to\omega$ which is defined as follows. For every $n\in\{1,\dots,k-1\}$ let $m\in\omega$ be the least such that $T(m)$ has $n+1$ nodes, and write the set $T(m-1)$ in $\prec$-increasing order as $\{s_0\prec\dots\prec s_{n-1}\}$. Then we set $\tau(n)=d$ if s_d is the unique splitting node of $T(m-1)$. Every type of a tree T with $[T]=k$ will be called a k-type. It is easy to see that for every $k\geqslant 2$ there exist $(k-1)!$ k-types. We remark that the above definition is equivalent to the initial one, given by Blass [4]. If F is a finite subset of $\mathcal{C}$, then we say that F is of *type* τ if T_F is skew and of type τ. If $P\subseteq\mathcal{C}$ and τ is a k-type, then by $[P]^k_\tau$ we shall denote the set of all subsets of P of type τ.

2.5. We will also deal with the following class of subtrees of $2^{<\omega}$ which are not downwards closed. A subtree T of $2^{<\omega}$ is said to be *regular dyadic* if T can be written in the form $T=(t_s)_{s\in 2^{<\omega}}$ such that for every $s_1,s_2\in 2^{<\omega}$ we have

(a) $s_1\sqsubset s_2$ if and only if $t_{s_1}\sqsubset t_{s_2}$,
(b) $s_1\prec s_2$ if and only if $t_{s_1}\prec t_{s_2}$, and
(c) $\ell(s_1)=\ell(s_2)$ if and only if $\ell(t_{s_1})=\ell(t_{s_2})$.

It is easy to see that the representation of T as $(t_s)_{s\in 2^{<\omega}}$ is unique. In what follows, when we deal with a regular dyadic subtree T we will always use this unique representation. We also notice that if T is a regular dyadic tree, then $[\hat{T}]$ is a perfect subset of $\mathcal{C}$ homeomorphic to $\mathcal{C}$.

2.6. We recall that a subset A of an uncountable Polish space X is *C-measurable* if it belongs to the smallest σ-algebra which is closed under the Souslin operation and contains the open sets. We remark that the class of C-measurable sets is strictly bigger than the σ-algebra generated by the analytic sets (see, e.g., [7]).

3. Definable ε-filling families

We start with the following definition.

Definition 1. *Let $\mathcal{F} \subseteq [\mathcal{C}]^{<\omega}$. The family $\mathcal{F}$ is said to have the* Galvin property *if for every $n \in \omega$ and every tuple $P_0 < \cdots < P_n$ of perfect subsets of $\mathcal{C}$ there exist $Q_0, \ldots, Q_n$ such that the following hold.*

(a) *For every $i \in \{0, \ldots, n\}$ we have that Q_i is a perfect subset of P_i.*
(b) *Either $Q_0 \times \cdots \times Q_n \subseteq \mathcal{F}$ or $(Q_0 \times \cdots \times Q_n) \cap \mathcal{F} = \emptyset$.*

We notice that if for every $n \in \omega$ and every tuple $P_0 < \cdots < P_n$ of perfect subsets of $\mathcal{C}$ the set $\mathcal{F} \cap (P_0 \times \cdots \times P_n)$ has the Baire property in $P_0 \times \cdots \times P_n$, then the family $\mathcal{F}$ has the Galvin property. This is a consequence of a theorem of Galvin (see, e.g., [7, Theorem 19.6]). Under the above terminology, we have the following theorem.

Theorem 2. *Let $\varepsilon > 0$ and let $\mathcal{F}$ be an ε-filling family over $\mathcal{C}$. If $\mathcal{F}$ has the Galvin property, then for every perfect subset P of $\mathcal{C}$ there exists a perfect subset Q of P such that $[Q]^{<\omega} \subseteq \mathcal{F}$.*

For the proof of Theorem 2 we need to introduce the following definition.

Definition 3. *Let $\mathcal{F} \subseteq [\mathcal{C}]^{<\omega}$ and let $T = (t_s)_{s \in 2^{<\omega}}$ be a regular dyadic subtree of $2^{<\omega}$. We say that the tree T* decides *for $\mathcal{F}$ if for every $n \in \omega$, every $0 \leqslant d \leqslant 2^n - 1$ and every $F = \{s_0 \prec \cdots \prec s_d\} \subseteq 2^n$ we have that the product $[\hat{T}]_{t_{s_0}} \times \cdots \times [\hat{T}]_{t_{s_d}}$ either is included in or is disjoint from $\mathcal{F}$. In the case where $[\hat{T}]_{t_{s_0}} \times \cdots \times [\hat{T}]_{t_{s_d}}$ is included in $\mathcal{F}$, then we say that F is* trapped *in $\mathcal{F}$.*

The following lemma is the combinatorial part of the proof of Theorem 2.

Lemma 4. *Let $\mathcal{F} \subseteq [\mathcal{C}]^{<\omega}$ with the Galvin property and let P be a perfect subset of $\mathcal{C}$. Then there exists a regular dyadic tree $T = (t_s)_{s \in 2^{<\omega}}$ that decides for $\mathcal{F}$ and with $[\hat{T}] \subseteq P$.*

Proof. By recursion on the length of $s \in 2^{<\omega}$, we will select a regular dyadic tree $T = (t_s)_{s \in 2^{<\omega}}$ and a family $(P^s)_{s \in 2^{<\omega}}$ of subsets of $\mathcal{C}$ such that for every $n \in \omega$ the following are satisfied.

(C1) For every $s \in 2^n$ we have that P^s is a perfect subset of P.

(C2) If $n \geqslant 1$, then for every $s \in 2^{n-1}$ and every $i \in \{0,1\}$ we have that $P^{s^\frown i} \subseteq P^s \cap \mathcal{C}_{t_{s^\frown i}}$.

(C3) For every $0 \leqslant d \leqslant 2^n - 1$ and every $\{s_0 \prec \cdots \prec s_d\} \subseteq 2^n$ we have that the set $P^{s_0} \times \cdots \times P^{s_d}$ either is included in or is disjoint from $\mathcal{F}$.

We proceed to the selection. For $n = 0$ we set $t_\emptyset = \emptyset$. By the Galvin property of $\mathcal{F}$, there exists perfect $P^\emptyset \subseteq P$ such that either $[P^\emptyset]^1 \subseteq \mathcal{F}$ or $[P^\emptyset]^1 \cap \mathcal{F} = \emptyset$. Then (C1) and (C3) are satisfied. Next assume that for some $n \in \omega$, the nodes $(t_s)_{s \in 2^n}$ and the perfect sets $(P^s)_{s \in 2^n}$ have been selected. Since the family $\{P^s : s \in 2^n\}$ consists of perfect subsets of P and $P^s \subseteq \mathcal{C}_{t_s}$, we may select a sequence $(t_s)_{s \in 2^{n+1}}$ such that the following are satisfied.

(i) For every $s_1, s_2 \in 2^{n+1}$ we have $\ell(t_{s_1}) = \ell(t_{s_2})$.

(ii) For every $s \in 2^n$ the nodes $t_{s^\frown 0}$ and $t_{s^\frown 1}$ are successors of t_s and, moreover, $t_{s^\frown 0} \prec t_{s^\frown 1}$.

(iii) For every $s \in 2^n$ and every $i \in \{0,1\}$, setting $Q^{s^\frown i} := P^s \cap \mathcal{C}_{t_{s^\frown i}}$, we have that $Q^{s^\frown i}$ is a perfect subset of P^s.

Using the fact that the family $\mathcal{F}$ has the Galvin property, by an exhaustion argument over all subsets of 2^{n+1}, for every $s \in 2^{n+1}$ we select a perfect set $P^s \subseteq Q^s$ such that condition (C3) is satisfied. The recursive selection is thus completed.

We will check that $T = (t_s)_{s \in 2^{<\omega}}$ satisfies the desired properties. We first observe that, by (C1), (C2) and the fact that $P^\emptyset \subseteq P$, we have $[\hat{T}] \subseteq P$. On the other hand, by (C2), we have that $[\hat{T}]_{t_s} \subseteq P^s$ for every $s \in 2^{<\omega}$. Therefore, by (C3), we conclude that T decides for $\mathcal{F}$ as desired. □

Lemma 5. *Let $\mathcal{F} \subseteq [\mathcal{C}]^{<\omega}$ and let $T = (t_s)_{s \in 2^{<\omega}}$ be a regular dyadic tree that decides for $\mathcal{F}$. Assume that $\mathcal{F}$ is ε-filling for some $\varepsilon > 0$. Then the following hold.*

(a) *For every $n \in \omega$ there exists $F_n \subseteq 2^n$ with $|F_n| \geqslant \varepsilon \cdot 2^n$ and such that F_n is trapped in $\mathcal{F}$.*

(b) *Let $n, k \in \omega$ with $k \leqslant n$, $F \subseteq 2^n$ and $G \subseteq 2^k$ such that G is dominated by F (that is, for every $w \in G$ there exists $s \in F$ with $w \sqsubseteq s$). If F is trapped in $\mathcal{F}$, then so does G.*

Proof. (a) For every $s \in 2^n$ we select $x_s \in [\hat{T}]_{t_s}$. The family $\mathcal{F}$ is ε-filling, and so, there exists $F_n = \{s_0 \prec \cdots \prec s_{d-1}\} \subseteq 2^n$ with $d \geqslant \varepsilon \cdot 2^n$ and such that $\{x_s : s \in F_n\} \subseteq \mathcal{F}$. It follows that $([\hat{T}]_{t_{s_0}} \times \cdots \times [\hat{T}]_{t_{s_{d-1}}}) \cap \mathcal{F} \neq \emptyset$. Since the tree T decides for $\mathcal{F}$, we conclude that F_n is trapped in $\mathcal{F}$.

(b) First we notice that if F is trapped in $\mathcal{F}$, then every subset of F is also trapped in $\mathcal{F}$, as $\mathcal{F}$ is hereditary. Now let G be dominated by F. There exists F' subset of F with $|F'| = |G|$ and such that for every $w \in G$ there exists a unique $s \in F'$ with $w \sqsubseteq s$. Arguing as in (a) above, we see that G is trapped in $\mathcal{F}$. □

For every regular dyadic tree $T = (t_s)_{s \in 2^{<\omega}}$ we define a Borel probability measure μ_T on $[\hat{T}]$ by assigning to every $[\hat{T}]_{t_s}$, with $s \in 2^n$ and $n \in \omega$, measure equal to $\frac{1}{2^n}$. That is, μ_T is the image of the usual measure on $\mathcal{C}$ induced by the natural homeomorphism between $\mathcal{C}$ and $[\hat{T}]$. We remark that μ_T is continuous (that is, it vanishes on singletons) and regular. The final lemma is the analytic part of the argument.

Lemma 6. *Let $\mathcal{F} \subseteq [\mathcal{C}]^{<\omega}$ and let $T = (t_s)_{s\in 2^{<\omega}}$ a regular dyadic tree that decides for $\mathcal{F}$. Assume that $\mathcal{F}$ is ε-filling for some $\varepsilon > 0$. Then there exists $K \subseteq [\hat{T}]$ closed such that $\mu_T(K) \geqslant \varepsilon$ and $[K]^{<\omega} \subseteq \mathcal{F}$.*

Proof. By part (a) of Lemma 5, for every $n \in \omega$ there exists a subset F_n of 2^n with $|F_n| \geqslant \varepsilon \cdot 2^n$ and such that F_n is trapped in $\mathcal{F}$. Define

$$C_n := \bigcup_{s\in F_n} [\hat{T}]_{t_s}.$$

Then C_n is a clopen subset of $[\hat{T}]$ and, moreover, $\mu_T(C_n) \geqslant \varepsilon$ for every $n \in \omega$. Denote by $\mathcal{K}([\hat{T}])$ the hyperspace of all compact subsets of $[\hat{T}]$ equipped with the Vietoris topology. It is a compact metrizable space (see, e.g., [7]). Hence, there exist an infinite subset L of ω and $K \in \mathcal{K}([\hat{T}])$ such that the sequence $(C_n)_{n\in L}$ is convergent to K. Since the measure μ_T is regular, the map $\mathcal{K}([\hat{T}]) \ni K \mapsto \mu_T(K)$ is upper semicontinuous. It follows that

$$\mu_T(K) \geqslant \limsup_{n\in L} \mu_T(C_n) \geqslant \varepsilon.$$

It remains to show that $[K]^{<\omega} \subseteq \mathcal{F}$. Indeed, let $\{x_0 < \cdots < x_l\} \subseteq K$. Since $K \subseteq [\hat{T}]$, there exist $k \in \omega$ and $\{w_0 \prec \cdots \prec w_l\} \subseteq 2^k$ such that $t_{w_i} \sqsubset x_i$ for every $i \in \{0, \dots, l\}$; notice that $\ell(t_{w_0}) = \cdots = \ell(t_{w_l})$. The sequence $(C_n)_{n\in L}$ converges to K, and so, there exists $n_0 \in L$ such that for every $n \in L$ with $n \geqslant n_0$ the set $\{t_s : s \in F_n\}$ dominates the set $\{t_{w_0}, \dots, t_{w_l}\}$. The tree T is regular dyadic, and so, F_n dominates $\{w_0, \dots, w_l\}$. Since every F_n is trapped in $\mathcal{F}$, by part (b) of Lemma 5, we obtain that $\{w_0, \dots, w_l\}$ is trapped in $\mathcal{F}$ too. This clearly implies that $\{x_0, \dots, x_l\} \in \mathcal{F}$ and the proof is completed. □

We are ready to give the proof of Theorem 2.

Proof of Theorem 2. Let $P \subseteq \mathcal{C}$ be perfect. Since $\mathcal{F}$ has the Galvin property, by Lemma 4, there exists a regular dyadic tree T such that T decides for $\mathcal{F}$ and $[\hat{T}] \subseteq P$. Since $\mathcal{F}$ is ε-filling, by Lemma 6, there exists $K \subseteq [\hat{T}]$ closed with $\mu_T(K) \geqslant \varepsilon$ and such that $[K]^{<\omega} \subseteq \mathcal{F}$. As μ_T is continuous, K is an uncountable closed subset of P and the result follows. □

3.1. **Consequences.** We notice that for every Polish space X, every closed subset F of X and every C-measurable subset A of X the set $A \cap F$ is C-measurable in F. Invoking the classical fact that every C-measurable subset of a Polish space has the Baire property (hence, by the remarks at the beginning of the section, the Galvin property too), we obtain the following corollary of Theorem 2.

Corollary 7. *Let $\mathcal{F} \subseteq [\mathcal{C}]^{<\omega}$ be ε-filling. If $\mathcal{F}$ is C-measurable in $[\mathcal{C}]^{<\omega}$, then for every perfect $P \subseteq \mathcal{C}$ there exists perfect $Q \subseteq P$ such that $[Q]^{<\omega} \subseteq \mathcal{F}$.*

As projective determinacy (PD) implies that every projective set in a Polish space has the Baire property (see [7, Theorem 38.17]), under PD, Corollary 7 is also true for every projective set.

There are some natural limitations on the possibility of extending Corollary 7 for an arbitrary ε-filling family. Indeed, let B be a Bernstein set, that is, a subset of $\mathcal{C}$ such that neither B nor $\mathcal{C} \setminus B$ contain a perfect set. Setting $\mathcal{F} = [B]^{<\omega} \cup [\mathcal{C} \setminus B]^{<\omega}$, we see that $\mathcal{F}$ is $1/2$-filling, yet there does not exist a perfect set P with $[P]^{<\omega} \subseteq \mathcal{F}$. Notice, however, that the above counterexample depends on the axiom of choice. As a matter of fact, every counterexample known to us depends on the axiom of choice. This is not an accident. As in the proof of Theorem 2 we made no use of the axiom of choice, we have the following corollary.

Corollary 8. *Assume* $\mathrm{ZF} + \mathrm{DC}$ *and the statement that "every subset of a Polish space has the Baire property". Then for every ε-filling family $\mathcal{F}$ over $\mathcal{C}$ and every perfect $P \subseteq \mathcal{C}$ there exists perfect $Q \subseteq P$ such that $[Q]^{<\omega} \subseteq \mathcal{F}$.*

We notice that the hypotheses of Corollary 8 hold in the Solovay model [9]—see, also, [3, Section 5.3] for a discussion about this in a different but related context. A similar result has been also obtained by Apter and Džamonja [1].

Remark 1. By Corollary 7, it follows that if $\mathcal{F} \subseteq [\mathcal{C}]^{<\omega}$ is analytic and ε-filling, then $\mathcal{F}$ cannot be compact; that is, there exists an infinite subset A of $\mathcal{C}$ such that $[A]^{<\omega} \subseteq \mathcal{F}$. We should point out that this can also be derived by the results of Fremlin in [6]. To see this one argues by contradiction. So, assume that $\mathcal{F} \subseteq [\mathcal{C}]^{<\omega}$ is analytic, compact and ε-filling for some $\varepsilon > 0$. It was observed by Argyros, Lopez-Abad and Todorčević that the rank of $\mathcal{F}$ is a countable ordinal whenever $\mathcal{F}$ is analytic and compact. This follows by an application of the Kunen–Martin theorem (see, e.g., [7, Theorem 31.1]). On the other hand, by [6, Lemma 2C] applied to the ideal $\mathcal{I}$ of countable subsets of $\mathcal{C}$, we see that the rank of $\mathcal{F}$ must be greater or equal to ω_1, a contradiction.

Remark 2. By modifying the proof of Theorem 2, we obtain the following result for an arbitrary family $\mathcal{F}$.

Theorem 9. *Let $\varepsilon > 0$ and let $\mathcal{F} \subseteq [\mathcal{C}]^{<\omega}$ be an arbitrary ε-filling family. Then for every perfect $P \subseteq \mathcal{C}$ there exists perfect $Q \subseteq P$ such that for every perfect $R \subseteq Q$ and every $k \geqslant 1$ the set $\mathcal{F} \cap [R]^k$ is dense in $[R]^k$.*

The proof of Theorem 9 follows the arguments of the proof of Theorem 2. The only important change is that of the notion of a regular dyadic the decides for $\mathcal{F}$. Specifically, Definition 3 is modified as follows.

Definition 10. *Let $\mathcal{F} \subseteq [\mathcal{C}]^{<\omega}$ and let $T = (t_s)_{s\in 2^{<\omega}}$ be a regular dyadic subtree of $2^{<\omega}$. We say that the tree T* weakly decides *for $\mathcal{F}$ if for every $n \in \omega$, every $0 \leqslant d \leqslant 2^n - 1$ and every $F = \{s_0 \prec \cdots \prec s_d\} \subseteq 2^n$ we have that one of the following (mutually exclusive) alternatives holds.*

(A1) *Either $([\hat{T}]_{t_{s_0}} \times \cdots \times [\hat{T}]_{t_{s_d}}) \cap \mathcal{F} = \emptyset$, or*

(A2) *for every $i \in \{0, \ldots, d\}$ and every perfect subset Q_i of $[\hat{T}]_{t_{s_i}}$ we have that $(Q_0 \times \cdots \times Q_d) \cap \mathcal{F} \neq \emptyset$.*

If alternative (A2) *holds true, then we say that F is* weakly trapped *in $\mathcal{F}$.*

It is easily seen that the arguments of the proofs of Lemmas 4, 5 and 6 can be carried out using the above definition, yielding the proof of Theorem 9.

4. Families of weaker density

This section is devoted to the proof of Theorem B stated in the introduction. For the convenience of the reader, let us present the example of Fremlin which provides closed hereditary families over $\mathcal{C}$ (of weaker density) for which Theorem 2 is not valid.

Example 1. Let $f\colon \omega \to \omega$ be any function such that $n \geqslant f(n) > 0$ for every $n \geqslant 1$ and $\lim \frac{f(n)}{n} = 0$. Then there exists a family $\mathcal{F} \subseteq [\mathcal{C}]^{<\omega}$ with the following properties.

(P1) $\mathcal{F}$ is closed in $[\mathcal{C}]^{<\omega}$ and hereditary.

(P2) We have that $d_{\mathcal{F}}(n) \geqslant f(n)$ for every $n \geqslant 1$.

(P3) There does not exist $A \subseteq \mathcal{C}$ infinite with $[A]^{<\omega} \subseteq \mathcal{F}$.

Indeed, we select a strictly increasing sequence (n_k) in ω such that $n_0 = 1$ and $\sup_{i\geqslant n_k} \frac{f(i)}{i} \leqslant \frac{1}{2^k}$ for every $k \geqslant 1$, and we set

$$\mathcal{F} := \bigcup_{k\in\omega} \bigcup_{t\in 2^k} \Big\{ G : G \subseteq \mathcal{C}_t \text{ and } |G| \leqslant \lceil n_{k+1}/2^k \rceil \Big\}.$$

It is easy to see that (P1) and (P3) are satisfied. To verify (P2), let $F \subseteq \mathcal{C}$ with $|F| = n$. Let $k \in \omega$ such that $n_k \leqslant n < n_{k+1}$. Then F is partitioned into the sets $\{F \cap \mathcal{C}_t : t \in 2^k\}$. There exists $t_0 \in 2^k$ such that $|F \cap \mathcal{C}_{t_0}| \geqslant \lceil n/2^k \rceil$. Let G be any subset of $F \cap \mathcal{C}_{t_0}$ with $|G| = \lceil n/2^k \rceil$. By the definition of $\mathcal{F}$ and the fact that $n < n_{k+1}$, we see that $G \in \mathcal{F}$. Since $n \geqslant n_k$, we have $\frac{f(n)}{n} \leqslant \frac{1}{2^k}$, and so, $f(n) \leqslant \lceil n/2^k \rceil \leqslant d_{\mathcal{F}}(n)$.

We now proceed to the proof of the main result of this section (Theorem 16 below). Observe that for every perfect $P \subseteq \mathcal{C}$ and every k-type τ the set $[P]^k_\tau$ is nonempty. We will need a finite version of this fact. To this end, we introduce the following definitions.

Definition 11. *Let $n \geqslant 1$. A finite subtree T of the Cantor tree $2^{<\omega}$ is said to be n-*increasing *if T can be written in the form $T = (t_s)_{s\in 2^{<n}}$ such that for every $s_1, s_2 \in 2^{<n}$ we have*

(1) $t_{s_1} \sqsubset t_{s_2}$ *(respectively,* $t_{s_1} \prec t_{s_2}$*) if and only if* $s_1 \sqsubset s_2$ *(respectively,* $s_1 \prec s_2$*),*
(2) *if* $\ell(s_1) = \ell(s_2)$ *and* $s_1 \prec s_2$*, then* $\ell(t_{s_1}) < \ell(t_{s_2})$*, and*
(3) *if* $\ell(s_1) < \ell(s_2)$*, then* $\ell(t_{s_1}) < \ell(t_{s_2})$*.*

Definition 12. *A subset* F *of* $\mathcal{C}$ *with* $|F| = 2^n$ *is said to be* 2^n-increasing *if the set* $\mathrm{Spl}(T_F)$ *of splitting nodes of* T_F *is an* n*-increasing subtree of* $2^{<\omega}$*. The set of all* 2^n*-increasing subsets of* $\mathcal{C}$ *will be denoted by* $[\mathcal{C}]^{2^n}_{\vartriangle}$*.*

It is easy to see that if F is a 2^n-increasing subset of $\mathcal{C}$, then T_F is a skew subtree of $2^{<\omega}$. The class of increasing subsets of $\mathcal{C}$ has the following stability property.

Lemma 13. *Let* $n \geqslant 2$ *and* $k \geqslant 1$ *be such that* $2^n \geqslant n^k$*. Then for every* $F \in [\mathcal{C}]^{2^n}_{\vartriangle}$ *and every* $G \subseteq F$ *with* $|G| \geqslant n^k$ *there exists* $H \subseteq G$ *with* $H \in [\mathcal{C}]^{2^k}_{\vartriangle}$*.*

Proof. By our assumption, we have that $\mathrm{Spl}(T_F) = (t_s)_{s\in 2^{<n}}$ is n-increasing. For every $0 \leqslant j \leqslant n-1$ we set $L_F(j) := \{t_s \in \mathrm{Spl}(T_F) : s \in 2^j\}$. By the definition of n-increasing subtrees, the set $L_F(j)$ is the j-level of $\mathrm{Spl}(T_F)$, and so, it is an antichain of $2^{<\omega}$. Also let $\mathrm{Spl}(T_G)$ be the set of splitting nodes of the tree T_G. Clearly, $\mathrm{Spl}(T_G)$ is a subset of $\mathrm{Spl}(T_F)$.

Recursively, for every $0 \leqslant m \leqslant k-1$ we will select $j_m \in \omega$ and a subset A_m of $2^{<\omega}$ such that

(C1) $0 \leqslant j_m \leqslant n-1$, and if $m_1 < m_2$, then $j_{m_1} > j_{m_2}$,
(C2) $2^{j_m} \geqslant n^{k-m-1}$,
(C3) $A_m \subseteq L_F(j_m)$ and $|A_m| \geqslant n^{k-m-1}$,
(C4) if $0 \leqslant m_1 < m_2 \leqslant k-1$, then A_{m_2} is a subset of $\mathrm{Spl}(\hat{A}_{m_1})$, and
(C5) for every $0 \leqslant m \leqslant k-1$ we have $A_m \subseteq \mathrm{Spl}(T_G)$.

We begin the selection. Notice that the family $\{\mathrm{Spl}(T_G) \cap L_F(j) : 0 \leqslant j \leqslant n-1\}$ is a partition of $\mathrm{Spl}(T_G)$. Since $|\mathrm{Spl}(T_G)| = |G| - 1 \geqslant n^k - 1$, by the pigeonhole principle, there exists $l \in \{0, \dots, n-1\}$ such that $|\mathrm{Spl}(T_G) \cap L_F(l)| \geqslant n^{k-1}$. Notice that $|L_F(l)| = 2^l \geqslant n^{k-1}$. We set $j_0 = l$ and $A_0 = \mathrm{Spl}(T_G) \cap L_F(l)$. Then conditions (C2), (C3) and (C5) are satisfied. This completes the first step of the selection. Next, observe that A_0 is an antichain—it is a subset of $L_F(j_0)$—and so $|\mathrm{Spl}(\hat{A}_0)| = |A_0| - 1 \geqslant n^{k-1} - 1$. As in the first step, we observe that the family $\{\mathrm{Spl}(\hat{A}_0) \cap L_F(j) : 0 \leqslant j \leqslant j_0 - 1\}$ is a partition of $\mathrm{Spl}(\hat{A}_0)$. Hence, there exists $l' \in \{0, \dots, j_0 - 1\}$ such that $|\mathrm{Spl}(\hat{A}_0) \cap L_F(l')| \geqslant n^{k-2}$. We set $j_1 := l'$ and $A_1 := \mathrm{Spl}(\hat{A}_0) \cap L_F(l')$ and we proceed similarly.

We isolate the crucial properties established by the above selection.

(P1) For every $1 \leqslant m \leqslant k-1$ and every $w \in A_m$ the node w has at least two successors in A_{m-1}.
(P2) For every $0 \leqslant m \leqslant k-1$ if $w_1, w_2 \in A_m$ with $w_1 \prec w_2$, then $\ell(w_1) < \ell(w_2)$.
(P3) For every $0 \leqslant m_1 < m_2 \leqslant k-1$ if $w_1 \in A_{m_1}$ and $w_2 \in A_{m_2}$, then $\ell(w_1) > \ell(w_2)$.

Property (P1) follows by condition (C4) of the selection, while properties (P2) and (P3) follow by (C3) and (C1) above and the fact that $\mathrm{Spl}(T_F)$ is n-increasing.

Using (P1)–(P3) and starting from a node in A_{k-1} we may select a k-increasing subtree $T = (w_s)_{s\in 2^{<k}}$ which is, by condition (C5), a subset of $\mathrm{Spl}(T_G)$. This clearly implies the lemma. □

Lemma 14. *Let $k \geqslant 1$ and $H \in [\mathcal{C}]^{2^k}_{\Delta}$. Then for every $(k+1)$-type τ there exists a subset I of H of type τ.*

Proof. By induction on k. If $k = 1$, then we set $I \coloneqq H$. Suppose that the result holds for some $k \geqslant 1$. Let $H \in [\mathcal{C}]^{2^{k+1}}_{\Delta}$ and let $\tau\colon \{1,\dots,k+1\} \to \omega$ be a $(k+2)$-type. Write the set H in lexicographically increasing order as $H = \{y_0 < \cdots < y_{2^{k+1}-1}\}$, and set $E \coloneqq \{y_i : 0 \leqslant i < 2^{k+1} \text{ and } i \text{ even}\}$. Also write $\mathrm{Spl}(T_H) = (t_s)_{s\in 2^{<k+1}}$. It is easy to see that $\mathrm{Spl}(T_E) = (t_s)_{s\in 2^{<k}}$ and so $E \in [\mathcal{C}]^{2^k}_{\Delta}$. We set $\tau' \coloneqq \tau|_{\{1,\dots,k\}}$. Then τ' is a $(k+1)$-type. By our inductive assumption, there exists $I' \subseteq E$ of type τ'. There exists $\{i_0 < \cdots < i_k\} \subseteq \{0,\dots,2^k-1\}$ such that $I' = \{y_{2i_0} < \cdots < y_{2i_k}\}$. We set $I \coloneqq I' \cup \{y_{2i_{\tau(k+1)}+1}\}$. Then $I \subseteq G$ and is of type τ, as desired. □

Lemma 15. *Let $\mathcal{F} \subseteq [\mathcal{C}]^{<\omega}$ be hereditary and let $n \geqslant 2$ and $k \geqslant 2$ such that $d_{\mathcal{F}}(2^n) \geqslant n^{k-1}$. Then for every perfect $P \subseteq \mathcal{C}$ and every k-type τ there exists $I \in \mathcal{F} \cap [P]^k_{\tau}$.*

Proof. Since P is perfect, there exists a 2^n-increasing subset F of P. On the other hand, since $d_{\mathcal{F}}(2^n) \geqslant n^{k-1}$, there exists $G \subseteq F$ with $G \in \mathcal{F}$ and $|G| \geqslant n^{k-1}$. Notice that $2^n \geqslant d_{\mathcal{F}}(2^n) \geqslant n^{k-1}$. Therefore, by Lemma 13, there exists $H \subseteq G$ which is 2^{k-1}-increasing. By Lemma 14, there exists $I \subseteq H$ of type τ. Since $I \subseteq H \subseteq G \in \mathcal{F}$ and $\mathcal{F}$ is hereditary, the result follows. □

We are ready to state and prove the main result of this section. To this end, we recall that Blass' theorem [4] on partitions of $[\mathcal{C}]^k$ asserts that if U is open subset of $[\mathcal{C}]^k$ and τ is a k-type, then there exists a perfect subset P of $\mathcal{C}$ perfect (which is the body of a skew tree) such that either $[P]^k_{\tau} \subseteq U$ or $[P]^k_{\tau} \cap U = \emptyset$.

Theorem 16. *Let $\mathcal{F} \subseteq [\mathcal{C}]^{<\omega}$ be hereditary.*

(a) *Let $n \geqslant 2$ and $k \geqslant 1$ such that $d_{\mathcal{F}}(2^n) \geqslant n^{k-1}$. If $\mathcal{F} \cap [\mathcal{C}]^k$ has the Baire property, then there exists a perfect subset P of $\mathcal{C}$ such that $[P]^k \subseteq \mathcal{F}$.*

(b) *Assume that $\mathcal{F}$ has the Baire property in $[\mathcal{C}]^{<\omega}$ and satisfies*

$$(*) \qquad \limsup \frac{\log_2 d_{\mathcal{F}}(2^n)}{\log_2 n} = +\infty.$$

Then for every $k \geqslant 1$ there exists a perfect subset P of $\mathcal{C}$ such that $[P]^k \subseteq \mathcal{F}$.

Proof. First we argue for part (a). If $k = 1$ the result is trivial. So let $k \geqslant 2$ and assume that $\mathcal{F}$ has the Baire property in $[\mathcal{C}]^k$. By a classical result of Mycielski (see, e.g., [7]) and by passing to a perfect subset of $\mathcal{C}$, we may assume that $\mathcal{F} \cap [\mathcal{C}]^k$ is open. Fix a k-type τ. By Blass' theorem, there exists perfect $P \subseteq \mathcal{C}$ such that $[P]^k_{\tau}$

either is included in $\mathcal{F}$ or is disjoint from $\mathcal{F}$. The second alternative is impossible by Lemma 15. So, the result follows by a finite exhaustion argument over all possible k-types. Part (b) follows from part (a) by a direct computation. □

Remark 3. We do not know whether equation $(*)$ in part (b) of Theorem 16 is the optimal one. We notice, however, that the conclusion of part (b) of Theorem 16 is not valid if we merely assume that $\lim d_{\mathcal{F}}(n) = +\infty$. For instance, let $\mathcal{F}$ be the union of all strongly increasing and strongly decreasing finite subsets of $\mathcal{C}$. (Recall that a subset $\{x_0 < \cdots < x_k\}$ of $\mathcal{C}$ is said to be strongly increasing if $\ell(x_i \wedge x_{i+1}) < \ell(x_{i+1} \wedge x_{i+2})$ for every $i \in \{0, \dots, k-2\}$—a strongly decreasing subset of $\mathcal{C}$ is similarly defined.) Then $\mathcal{F}$ is closed in $[\mathcal{C}]^{<\omega}$ and it is easy to verify that $\lim d_{\mathcal{F}}(n) = +\infty$. However, for every $k \geqslant 4$ there does not exist a perfect subset P of $\mathcal{C}$ with $[P]^k \subseteq \mathcal{F}$.

4.1. **Consequences.** The following proposition shows that the families presented in Example 1 are essentially the only ones within the class of C-measurable hereditary families which satisfy $(*)$.

Proposition 17. *Let $\mathcal{F} \subseteq [\mathcal{C}]^{<\omega}$ be hereditary. Assume that $\mathcal{F}$ is C-measurable in $[\mathcal{C}]^{<\omega}$ and satisfies equation $(*)$ in Theorem 16. Then for every $g\colon \omega \to \omega$ and every perfect $P \subseteq \mathcal{C}$ there exists a regular dyadic subtree $T = (t_s)_{s \in 2^{<\omega}}$ with $[\hat{T}] \subseteq P$ and such that $\mathcal{G} \subseteq \mathcal{F}$ where $\mathcal{G} \coloneqq \bigcup_{k\in\omega} \bigcup_{s \in 2^k} \big\{G : G \subseteq [\hat{T}]_{t_s} \text{ and } |G| \leqslant g(k)\big\}$.*

Proof. By our assumptions and Theorem 16, for every perfect $P \subseteq \mathcal{C}$ and every $m \geqslant 1$ there exists perfect $Q \subseteq P$ such that $[Q]^m \subseteq \mathcal{F}$. Hence, arguing as in Lemma 4, we may select a regular dyadic subtree $T = (t_s)_{s\in 2^{<\omega}}$ with $t_\emptyset = \emptyset$ and a family $(P^s)_{s \in 2^{<\omega}}$ of perfect subsets of P such that the following are satisfied.

(i) For every $k \in \omega$, every $s \in 2^k$ and every $i \in \{0,1\}$ we have $P^{s^\frown i} \subseteq P^s \cap \mathcal{C}_{t_{s^\frown i}}$.
(ii) For every $k \in \omega$ and every $s \in 2^k$ we have $[P^s]^{g(k)} \subseteq \mathcal{F}$.

Clearly, T is as desired. □

We need to introduce some terminology. Let $f\colon \omega \to \omega$ such that $n \geqslant f(n) > 0$ for every $n \geqslant 1$. Also let $\mathcal{F} \subseteq [\mathcal{C}]^{<\omega}$ and $A \subseteq \mathcal{C}$. We say that $\mathcal{F}$ is *f-filling* over A if for every $n \geqslant 1$ and every $F \subseteq A$ with $|F| = n$ there exists $G \subseteq F$ with $G \in \mathcal{F}$ and $|G| \geqslant f(n)$. We notice that if $\mathcal{F} \subseteq [\mathcal{C}]^{<\omega}$ is an arbitrary hereditary family with $\lim d_{\mathcal{F}}(n) = +\infty$, then for every infinite subset A of $\mathcal{C}$ there exists a countable subset B of A such that $\mathcal{F}$ becomes 1/2-filling over B (this follows by an application of Ramsey's theorem). Although, by Theorem 2, this fact cannot be extended to perfect sets, it can be extended for weaker versions of density as follows.

Corollary 18. *Let $\mathcal{F} \subseteq [\mathcal{C}]^{<\omega}$ be as in Proposition 17 and let $f\colon \omega \to \omega$ such that $n \geqslant f(n) > 0$ for every $n \geqslant 1$ and $\lim \frac{f(n)}{n} = 0$. Then for every perfect $P \subseteq \mathcal{C}$ there exists perfect $Q \subseteq P$ such that $\mathcal{F}$ is f-filling over Q.*

Proof. We select a strictly increasing sequence (n_k) in ω such that $n_0 = 1$ and $\sup_{i\geqslant n_k} \frac{f(i)}{i} \leqslant \frac{1}{2^k}$ for every $k \in \omega$. Next, we define $g\colon \omega \to \omega$ by the rule $g(k) = \lceil n_{k+1}/2^k \rceil$. Let $T = (t_s)_{s\in 2^{<\omega}}$ be the regular dyadic subtree obtained by Proposition 17 for the function g and the given perfect set P. Setting $Q \coloneqq [\hat{T}]$ and arguing as in Example 1, we see that Q has the desired properties. □

5. Connections with Banach spaces

Theorem 2 has some Banach space theoretic implications which we are about to describe. Let $\mathcal{F} \subseteq [\mathcal{C}]^{<\omega}$ be hereditary and such that $[\mathcal{C}]^1 \subseteq \mathcal{F}$. We define a Banach space $X_{\mathcal{F}}$ associated with $\mathcal{F}$ as follows. Let $c_{00}(\mathcal{C})$ be the vector space of all real-valued functions on $\mathcal{C}$ with finite support, and denote by $(e_x)_{x\in\mathcal{C}}$ the standard Hamel basis of $c_{00}(\mathcal{C})$. The space $X_{\mathcal{F}}$ is the completion of $c_{00}(\mathcal{C})$ under the norm

$$\Big\| \sum_{i=0}^{n} a_i e_{x_i} \Big\|_{\mathcal{F}} \coloneqq \sup\Big\{ \sum_{i\in F} |a_i| : \{x_i : i \in F\} \in \mathcal{F} \Big\}.$$

We recall that a bounded sequence (e_n) in a Banach space E is called *Cesaro summable* if the sequence of averages $\frac{1}{n}(e_0 + \cdots + e_{n-1})$ converges in norm. Under the above terminology we have the following proposition.

Proposition 19. *Let $\mathcal{F} \subseteq [\mathcal{C}]^{<\omega}$ be hereditary, compact and such that $[\mathcal{C}]^1 \subseteq \mathcal{F}$. Assume that $\mathcal{F}$ is C-measurable and $\lim d_{\mathcal{F}}(n) = +\infty$. Then the following hold.*

(a) *For every sequence (x_i) in $\mathcal{C}$ there exists infinite $L \subseteq \omega$ such that for every infinite $N \subseteq L$ the sequence $(e_{x_i})_{i\in N}$ is not Cesaro summable in $X_{\mathcal{F}}$.*

But on the other hand,

(b) *for every perfect $P \subseteq \mathcal{C}$ there exists a sequence (x_i) in P such that the sequence (e_{x_i}) is Cesaro summable in $X_{\mathcal{F}}$.*

Proof. (a) Let (x_i) be a sequence in $\mathcal{C}$. As we have already remarked, by the fact that $\mathcal{F}$ is hereditary and $\lim d_{\mathcal{F}}(n) = +\infty$, there exists infinite $L \subseteq \omega$ such that $\mathcal{F}$ is 1/2-filling over $\{x_i : i \in L\}$. By the definition of the norm of $X_{\mathcal{F}}$, we see that for every finite $F \subseteq L$ we have $\| \sum_{i\in F} e_{x_i} \|_{\mathcal{F}} \geqslant \frac{|F|}{2}$. This clearly implies that for every infinite $N \subseteq L$ the sequence $(e_{x_i})_{i\in N}$ is not Cesaro summable in $X_{\mathcal{F}}$.

(b) Let $P \subseteq \mathcal{C}$ be perfect. By our assumptions, Lemma 4 can be applied. Hence, there exists a regular dyadic subtree $T = (t_s)_{s\in 2^{<\omega}}$ that decides for $\mathcal{F}$ and with $[\hat{T}] \subseteq P$. Let Z be the set of all eventually zero sequences in $\mathcal{C}$. We enumerate Z as (z_i) as follows. For every $i \in \omega$ let z_i be the unique element of Z satisfying $i = \sum_{k\in\omega} z_i(k)2^k$. By the uniqueness of the dyadic representation of every natural number, we see that if $i \neq j$, then $z_i \neq z_j$ and, moreover, if $n, i, j \in \omega$ are such that $i, j < 2^n$, then $z_i|n \neq z_j|n$.

For every $i \in \omega$ we set $x_i \coloneqq \bigcup_{k\in\omega} t_{z_i|k} \in [\hat{T}]$. We claim that the sequence (x_i) is as desired. Indeed, for every $n \in \omega$ let $s \in 2^n$ and set $l_n \coloneqq \ell(t_s)$ (since T is regular dyadic, l_n is well-defined and independent of the choice of s). By the above mentioned property of the sequence (z_i), for every $i, n \in \omega$ with $i < 2^n$ we have

$$|\{x_0|l_n, \dots, x_i|l_n\}| = |\{x_0, \dots, x_i\}| = i + 1. \tag{1}$$

For every $n \in \omega$ we set

$$M_n := \max\{|F| : F \subseteq 2^n \text{ and } F \text{ is trapped in } \mathcal{F}\}.$$

By (1) and the fact that the tree T decides for $\mathcal{F}$, for every $i < 2^n$ we have

$$\max\big\{|G| : G \subseteq \{x_0, \dots, x_i\} \text{ and } G \in \mathcal{F}\big\} \leqslant M_n.$$

Let $i, n \geqslant 1$ with $2^{n-1} \leqslant i < 2^n$. Then,

$$\Big\|\frac{1}{i+1}\sum_{k=0}^{i} e_{x_k}\Big\|_{\mathcal{F}} \leqslant \frac{M_n}{i+1} \leqslant \frac{M_n}{2^{n-1}} = 2\,\frac{M_n}{2^n}. \tag{2}$$

Finally, notice that

$$\lim \frac{M_n}{2^n} = 0.$$

For if not, arguing as in the proof of Lemma 6, we would obtain a perfect subset R of $[\hat{T}]$ such that $[R]^{<\omega} \subseteq \mathcal{F}$ which contradicts, of course, the fact that $\mathcal{F}$ is compact. Hence, by (2), we conclude that

$$\frac{1}{i+1}\sum_{k=0}^{i} e_{x_k} \to 0$$

and the proof is completed. □

Remark 4. (a) Part (b) of Proposition 19 can be also derived by [6, Theorem 3A] taking into account that every C-measurable, hereditary and compact family $\mathcal{F}$ is not ε-filling for every $\varepsilon > 0$. For completeness we have included a proof in the present setting.

(b) The fact that every subsequence of the sequence $(x_i)_{i\in L}$ obtained by part (a) of Proposition 19, is not Cesaro summable, is expected by the Erdős–Magidor theorem [5] (see, also, [2]).

(c) We notice that under the assumptions of Proposition 19 for every perfect $P \subseteq \mathcal{C}$ there exists perfect $Q \subseteq P$ with the following property. If (x_i) is a sequence in Q and the sequence (e_{x_i}) generates a spreading model (see, e.g., [2]), then this spreading model must be ℓ_1.

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National Technical University of Athens, Faculty of Applied Sciences, Department of Mathematics, Zografou Campus, 157 80, Athens, Greece.

E-mail address: `pdodos@math.ntua.gr, bkanel@math.ntua.gr`