A STRONG BOUNDEDNESS RESULT FOR SEPARABLE ROSENTHAL COMPACTA

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Abstract. It is proved that the class of separable Rosenthal compacta on the Cantor set having a uniformly bounded dense sequence of continuous functions, is strongly bounded.

1. Introduction

Our main result is a strong boundedness result for the class of separable Rosenthal compacta (that is, separable compact subsets of the first Baire class—see [ADK, Ro2]) on the Cantor set having a uniformly bounded dense sequence of continuous functions. We shall denote this class by SRC. The phenomenon of strong boundedness, which was first touched upon by Kechris and Woodin in [KW], is a strengthening of the classical property of boundedness of $\mathbf{\Pi}^1_1$-ranks. Abstractly, one has a $\mathbf{\Pi}^1_1$ set B, a natural notion of embedding between elements of B and a canonical $\mathbf{\Pi}^1_1$-rank ϕ on B which is coherent with the embedding in the sense that if $x, y \in B$ and x embeds into y, then $\phi(x) \leqslant \phi(y)$. The strong boundedness of B is the fact that for every analytic subset A of B there exists $y \in B$ such that x embeds into y for every $x \in A$. Basic examples of strongly bounded classes are the well-orderings WO and the well-founded trees WF (although, in these cases strong boundedness is easily seen to be equivalent to boundedness). Recently, it was shown (see [AD, DF]) that several classes of separable Banach spaces are strongly bounded, where the corresponding notion of embedding is that of (linear) isomorphic embedding. These results have, in turn, important consequences in the study of universality problems in Banach space theory.

We will add another example to the list of strongly bounded classes, namely the class SRC. We notice that every $\mathcal{K}$ in SRC can be naturally coded by its dense sequence of continuous functions. Hence, we identify SRC with the set

$$\big\{(f_n) \in B(2^{\mathbb{N}})^{\mathbb{N}} : \overline{\{f_n\}}^p \subseteq \mathcal{B}_1(2^{\mathbb{N}}) \text{ and } f_n \neq f_m \text{ if } n \neq m\big\}$$

where $B(2^{\mathbb{N}})$ stands for the closed unit ball of the separable Banach space $C(2^{\mathbb{N}})$. With this identification, the set SRC is $\mathbf{\Pi}^1_1$-true. A canonical $\mathbf{\Pi}^1_1$-rank on SRC comes from the work of Rosenthal. Specifically, for every $\mathbf{f} = (f_n)$ in SRC one is looking at the order of the ℓ_1-tree of the sequence (f_n). One also has a natural

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notion of topological embedding between elements of SRC. In particular, if $\mathbf{f} = (f_n)$ and $\mathbf{g} = (g_n)$ are in SRC, then we say that $\mathbf{g}$ *topologically embeds* into $\mathbf{f}$, if there exists a homeomorphic embedding of the compact space $\overline{\{g_n\}}^p$ into $\overline{\{f_n\}}^p$. However, this topological embedding is rather weak and is not coherent with the $\mathbf{\Pi}^1_1$-rank on SRC. Thus, we strengthen the notion of embedding by imposing extra metric conditions on the relation between $\mathbf{g}$ and $\mathbf{f}$. To motivate our definition, assume that $\mathbf{g} = (g_n)$ and $\mathbf{f} = (f_n)$ were in addition (Schauder) basic sequences. In this case the most natural thing to consider is equivalence of basic sequences, that is, $\mathbf{g}$ embeds into $\mathbf{f}$ if there exists $L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that (g_n) is equivalent to (f_{l_n}). In such a case, it is easily seen the order of the ℓ_1-tree of $\mathbf{g}$ is dominated by the one of $\mathbf{f}$.

Although not every sequence $\mathbf{f} \in$ SRC is basic, the following condition incorporates the above observation. So, we say that $\mathbf{g} = (g_n)$ *strongly embeds* into $\mathbf{f} = (f_n)$ if $\mathbf{g}$ topologically embeds into $\mathbf{f}$ and, moreover, for every $\varepsilon > 0$ there exists $L_\varepsilon = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that for every $k \in \mathbb{N}$ and $a_0, \ldots, a_k \in \mathbb{R}$ we have

$$\Big| \max_{0 \leqslant i \leqslant k} \big\| \sum_{n=0}^{i} a_n g_n \big\|_\infty - \big\| \sum_{n=0}^{k} a_n f_{l_n} \big\|_\infty \Big| \leqslant \varepsilon \sum_{n=0}^{k} \frac{|a_n|}{2^{n+1}}.$$

The notion of strong embedding is coherent with the $\mathbf{\Pi}^1_1$-rank on SRC and is consistent with our motivating observation in the sense that if $\mathbf{g} = (g_n)$ strongly embeds into $\mathbf{f} = (f_n)$ and (g_n) is basic, then there exists $L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that (f_{l_n}) is basic and equivalent to (g_n). Under the above terminology, we prove the following theorem.

Main Theorem. *Let A be an analytic subset of* SRC. *Then there exists* $\mathbf{f} \in$ SRC *such that for every* $\mathbf{g} \in A$ *the sequence* $\mathbf{g}$ *strongly embeds into* $\mathbf{f}$.

2. Background material

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers. By $[\mathbb{N}]^\infty$ we denote the set of all infinite subsets of $\mathbb{N}$, while for every $L \in [\mathbb{N}]^\infty$ by $[L]^\infty$ we denote the set of all infinite subsets of L. For every Polish space X by $\mathcal{B}_1(X)$ we denote the set of all real-valued, Baire-1 functions on X. If $\mathcal{F}$ is a subset of $\mathbb{R}^X$, then by $\overline{\mathcal{F}}^p$ we denote the closure of $\mathcal{F}$ in $\mathbb{R}^X$.

Our descriptive set theoretic notation and terminology follows [Ke]. If X, Y are Polish spaces, $A \subseteq X$ and $B \subseteq Y$, then we say that A is *Wadge* (respectively, *Borel*) *reducible* to B if there exists a continuous (respectively, Borel) map $f \colon X \to Y$ such that $f^{-1}(B) = A$. If A is $\mathbf{\Pi}^1_1$, then a map $\phi \colon A \to \omega_1$ is said to be a $\mathbf{\Pi}^1_1$-*rank* on A if there exist relations $\leqslant_\Sigma, \leqslant_\Pi$ in $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ respectively, such that for every $y \in A$ we have

$$x \in A \text{ and } \phi(x) \leqslant \phi(y) \Leftrightarrow x \leqslant_\Sigma y \Leftrightarrow x \leqslant_\Pi y.$$

We notice that if B is Borel reducible to a set A via a Borel map f and ϕ is a $\mathbf{\Pi}^1_1$-rank on A, then the map $\psi\colon B\to\omega_1$ defined by $\psi(y)=\phi\big(f(x)\big)$ for every $y\in B$, is a $\mathbf{\Pi}^1_1$-rank on B.

2.1. **Trees.** Let Λ be a nonempty set. By $\Lambda^{<\mathbb{N}}$ we denote the set of all finite sequences of Λ. We view $\Lambda^{<\mathbb{N}}$ as a tree equipped with the (strict) partial order $\sqsubset$ of end-extension. If $t\in\Lambda^{<\mathbb{N}}$, then the length $|t|$ of t is defined to be the cardinality of the set $\{s\in\Lambda^{<\mathbb{N}}: s\sqsubset t\}$. If $s,t\in\Lambda^{<\mathbb{N}}$, then by $s^\frown t$ we denote their concatenation. Two nodes $s,t\in\Lambda^{<\mathbb{N}}$ are said to be *comparable* if either $s\sqsubseteq t$ or $t\sqsubseteq s$; otherwise, they are said to be *incomparable*. A subset of $\Lambda^{<\mathbb{N}}$ consisting of pairwise comparable nodes is said to be a *chain*. If $L\in[\mathbb{N}]^\infty$, then by $\mathrm{FIN}(L)$ we denote the subset of $L^{<\mathbb{N}}$ consisting of all finite *strictly increasing* sequences in L. For every $x\in\Lambda^{\mathbb{N}}$ and every $n\geqslant 1$ we set $x|n=\big(x(0),\dots,x(n-1)\big)\in\Lambda^{<\mathbb{N}}$ while $x|0=\emptyset$.

A *tree* T on Λ is a downwards closed subset of $\Lambda^{<\mathbb{N}}$. By $\mathrm{Tr}(\Lambda)$ we denote the set of all trees on Λ. Hence,

$$T\in\mathrm{Tr}(\Lambda)\Leftrightarrow\forall s,t\in\Lambda^{<\mathbb{N}}\ (t\in T\ \wedge s\sqsubseteq t\Rightarrow s\in T).$$

A tree T on Λ is said to be *pruned* if for every $t\in T$ there exists $s\in T$ with $t\sqsubset s$. If $T\in\mathrm{Tr}(\Lambda)$, then the *body* $[T]$ of T is defined to be the set $\{x\in\Lambda^{\mathbb{N}}: x|n\in T\ \forall n\}$. A tree T is said to be *well-founded* if $[T]=\emptyset$. The subset of $\mathrm{Tr}(\Lambda)$ consisting of all well-founded trees on Λ will be denoted by $\mathrm{WF}(\Lambda)$. If $T\in\mathrm{WF}(\Lambda)$, then set $T':=\{t:\exists s\in T \text{ with } t\sqsubset s\}\in\mathrm{WF}(\Lambda)$. By transfinite recursion, we define the iterated derivatives $T^{(\xi)}$ of T. The *order* $o(T)$ of T is defined to be the least ordinal ξ such that $T^{(\xi)}=\emptyset$. If S,T are well-founded trees, then a map $\phi\colon S\to T$ is called *monotone* if $s_1\sqsubset s_2$ in S implies that $\phi(s_1)\sqsubset\phi(s_2)$ in T. Notice that in this case $o(S)\leqslant o(T)$. If Λ, M are nonempty sets, then we identify every tree T on $\Lambda\times M$ with the set of all pairs $(s,t)\in\Lambda^{<\mathbb{N}}\times M^{<\mathbb{N}}$ such that $|s|=|t|=k$ and $\big((s(0),t(0)),\dots,(s(k-1),t(k-1))\big)\in T$. If $\Lambda=\mathbb{N}$, then we shall simply denote by Tr and WF the sets of all trees and well-founded trees on $\mathbb{N}$ respectively. For every countable set Λ the set $\mathrm{WF}(\Lambda)$ is $\mathbf{\Pi}^1_1$-complete and the map $T\to o(T)$ is a $\mathbf{\Pi}^1_1$-rank on $\mathrm{WF}(\Lambda)$ (see [Ke]).

2.2. **Basic sequences.** A sequence (x_n) of non-zero vectors in a Banach space X is said to be a *basic sequence* if it is a Schauder basis of its closed linear span (see [LT]). This is equivalent to saying that there exists a constant $K\geqslant 1$ such that for every $m,k\in\mathbb{N}$ with $m<k$ and every $a_0,\dots,a_k\in\mathbb{R}$ we have

$$\Big\|\sum_{n=0}^{m}a_nx_n\Big\|\leqslant K\Big\|\sum_{n=0}^{k}a_nx_n\Big\|. \tag{1}$$

The least constant K for which inequality (1) holds is called the *basis constant* of (x_n). A basic sequence (x_n) is said to be *monotone* if $K=1$. It is said to be *seminormalized* (respectively, *normalized*) if there exists a constant $M>0$ such that $\frac{1}{M}\leqslant\|x_n\|\leqslant M$ (respectively, $\|x_n\|=1$) for every $n\in\mathbb{N}$.

Let X and Y be Banach spaces. If (x_n) and (y_n) are two sequences in X and Y respectively and $C \geqslant 1$, then we say that (x_n) is C-*equivalent* to (y_n) (or simply equivalent, if C is understood) if for every $k \in \mathbb{N}$ and every $a_0, \dots, a_k \in \mathbb{R}$ we have

$$\frac{1}{C}\Big\| \sum_{n=0}^{k} a_n y_n \Big\|_Y \leqslant \Big\| \sum_{n=0}^{k} a_n x_n \Big\|_X \leqslant C \Big\| \sum_{n=0}^{k} a_n y_n \Big\|_Y .$$

We denote by $(x_n) \overset{C}{\sim} (y_n)$ the fact that (x_n) is C-equivalent to (y_n).

3. Coding SRC

Let X be a compact metrizable space and let $\mathrm{SRC}(X)$ be the family of all separable Rosenthal compacta on X having a dense set of continuous functions which is uniformly bounded with respect to the supremum norm. We denote by $B(X)$ the closed unit ball of the separable Banach space $C(X)$. Notice that every $\mathcal{K} \in \mathrm{SRC}(X)$ is naturally coded by its dense sequence of continuous functions. Hence, we may identify $\mathrm{SRC}(X)$ with the set

$$\big\{ (f_n) \in B(X)^{\mathbb{N}} : \overline{\{f_n\}}^p \subseteq \mathcal{B}_1(X) \text{ and } f_n \neq f_m \text{ if } n \neq m \big\}.$$

We denote by $\mathbf{B}(X)$ the G_δ subset of $B(X)^{\mathbb{N}}$ consisting of all sequences $\mathbf{f} = (f_n)$ in $B(X)^{\mathbb{N}}$ such that $f_n \neq f_m$ if $n \neq m$. With the above identification the set $\mathrm{SRC}(X)$ becomes a subset of the Polish space $\mathbf{B}(X)$. Moreover, as for every compact metrizable space X the Banach space $C(X)$ embeds isometrically into $C(2^{\mathbb{N}})$, we shall denote by SRC the set $\mathrm{SRC}(2^{\mathbb{N}})$ and we view SRC as the set of all separable Rosenthal compacta having a uniformly bounded dense sequence of continuous functions and defined on a compact metrizable space (it is crucial that $C(X)$ embeds isometrically into $C(2^{\mathbb{N}})$—this will be clear later on). The following lemma provides an estimate for the complexity of the set $\mathrm{SRC}(X)$.

Lemma 1. *For every compact metrizable space X the set* $\mathrm{SRC}(X)$ *is* $\mathbf{\Pi}^1_1$. *Moreover, the set* SRC *is* $\mathbf{\Pi}^1_1$*-true.*

Proof. Instead of calculating the complexity of $\mathrm{SRC}(X)$ we will actually find a Borel map $\Phi\colon \mathbf{B}(X) \to \mathrm{Tr}$ such that $\Phi^{-1}(\mathrm{WF}) = \mathrm{SRC}(X)$. In other words, we will find a Borel reduction of $\mathrm{SRC}(X)$ to WF. This will not only show that $\mathrm{SRC}(X)$ is $\mathbf{\Pi}^1_1$, but also, it will provide a natural $\mathbf{\Pi}^1_1$-rank on $\mathrm{SRC}(X)$. This canonical reduction comes from the work of Rosenthal.

Specifically, let (e_i) be the standard basis of ℓ_1. With every $d \in \mathbb{N}$ with $d \geqslant 1$ and every $\mathbf{f} = (f_n)$ in $\mathbf{B}(X)$ we associate a tree $T^d_{\mathbf{f}}$ on $\mathbb{N}$ defined by

$$s \in T^d_{\mathbf{f}} \Leftrightarrow s = (n_0 < \dots < n_k) \in \mathrm{FIN}(\mathbb{N}) \text{ and } (e_i)_{i=0}^k \overset{d}{\sim} (f_{n_i})_{i=0}^k .$$

Notice that $(e_i)_{i=0}^k \overset{d}{\sim} (f_{n_i})_{i=0}^k$ if for every $a_0, \dots, a_k \in \mathbb{R}$ we have

$$\frac{1}{d}\sum_{i=0}^{k} |a_i| \leqslant \Big\| \sum_{i=0}^{k} a_i f_{n_i} \Big\|_\infty \leqslant d \sum_{i=0}^{k} |a_i| .$$

Observe that for every $t \in \mathbb{N}^{<\mathbb{N}}$ the set $\{\mathbf{f} : t \in T^d_{\mathbf{f}}\}$ is a closed subset of $\mathbf{B}(X)$. This yields that the map $\mathbf{B}(X) \ni \mathbf{f} \mapsto T^d_{\mathbf{f}} \in \mathrm{Tr}$ is Borel (actually it is Baire-1). Next, we glue the sequence of trees $\{T^d_{\mathbf{f}} : d \geqslant 1\}$ and we obtain a tree $T_{\mathbf{f}}$ on $\mathbb{N}$ defined by the rule

$$s \in T_{\mathbf{f}} \Leftrightarrow \exists d \geqslant 1\ \exists s' \text{ with } s = d^\frown s' \text{ and } s' \in T^d_{\mathbf{f}}.$$

The tree $T_{\mathbf{f}}$ is usually called the ℓ_1-tree of the sequence $\mathbf{f} = (f_n)$. Clearly the map $\Phi\colon \mathbf{B}(X) \to \mathrm{Tr}$ defined by $\Phi(\mathbf{f}) = T_{\mathbf{f}}$ is Borel.

We observe that

$$\mathbf{f} = (f_n) \in \mathrm{SRC}(X) \Leftrightarrow T_{\mathbf{f}} \in \mathrm{WF}.$$

This equivalence is essentially Rosenthal's dichotomy [Ro1] (see also [Ke, To]). Indeed, let $\mathbf{f} = (f_n)$ be such that $T_{\mathbf{f}}$ is well-founded. By Rosenthal's dichotomy, every subsequence of (f_n) has a further pointwise convergent subsequence. By the Main Theorem in [Ro2], the closure of $\{f_n\}$ in $\mathbb{R}^X$ is in $\mathcal{B}_1(X)$, and so, $\mathbf{f} \in \mathrm{SRC}(X)$. Conversely, assume that $T_{\mathbf{f}}$ is ill-founded. There exists $L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that the sequence (f_{l_n}) is equivalent to the standard unit vector basis of ℓ_1. By the fact that (f_n) is uniformly bounded and Lebesgue's dominated convergence theorem, we obtain that the sequence (f_{l_n}) has no pointwise convergent subsequence. This implies that the closure of $\{f_n\}$ in $\mathbb{R}^X$ contains a homeomorphic copy of $\beta\mathbb{N}$, and so, $\mathbf{f} \notin \mathrm{SRC}(X)$. It follows that the map Φ determines a Borel reduction of $\mathrm{SRC}(X)$ to WF. Hence, the set $\mathrm{SRC}(X)$ is $\mathbf{\Pi}^1_1$ and the map $\phi_X\colon \mathrm{SRC}(X) \to \omega_1$ defined by $\phi_X(\mathbf{f}) = o(T_{\mathbf{f}})$ is a $\mathbf{\Pi}^1_1$-rank on $\mathrm{SRC}(X)$.

We proceed to show that the set SRC is $\mathbf{\Pi}^1_1$-true. Denote by ϕ the canonical $\mathbf{\Pi}^1_1$-rank $\phi_{2^\mathbb{N}}$ on SRC defined above. In order to prove that SRC is $\mathbf{\Pi}^1_1$-true, by [Ke, Theorem 35.23], it is enough to show that $\sup\{\phi(\mathbf{f}) : \mathbf{f} \in \mathrm{SRC}\} = \omega_1$. In the argument below we shall use the following simple fact.

Fact 2. *Let X, Y be compact metrizable spaces and $e\colon X \to Y$ a continuous onto map. Let $\mathbf{f} = (f_n) \in \mathrm{SRC}(Y)$ and define $\mathbf{g} = (g_n) \in C(X)^\mathbb{N}$ by $g_n(x) = f_n(e(x))$ for every $x \in X$ and every $n \in \mathbb{N}$. Then $\mathbf{g} \in \mathrm{SRC}(X)$ and $\phi_Y(\mathbf{f}) = \phi_X(\mathbf{g})$.*

Now let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$ which is hereditary (that is, if $F \in \mathcal{F}$ and $G \subseteq F$, then $G \in \mathcal{F}$) and compact in the pointwise topology (that is, compact in $2^\mathbb{N}$). To every such family $\mathcal{F}$ one associates its *order* $o(\mathcal{F})$ which is simply the order of the downwards closed, well-founded tree $T_\mathcal{F}$ on $\mathbb{N}$ defined by

$$s \in T_\mathcal{F} \Leftrightarrow s = (n_0 < \cdots < n_k) \in \mathrm{FIN}(\mathbb{N}) \text{ and } \{n_0, \ldots, n_k\} \in \mathcal{F}.$$

Such families are well-studied in combinatorics and functional analysis and a detailed exposition can be found in [AT]. What we need is the simple fact that for every countable ordinal ξ one can find a compact hereditary family $\mathcal{F}$ with $o(\mathcal{F}) \geqslant \xi$.

So, fix a countable ordinal ξ and let $\mathcal{F}$ be a compact hereditary family with $o(\mathcal{F}) \geqslant \xi$. We will additionally assume that $\{n\} \in \mathcal{F}$ for every $n \in \mathbb{N}$. Define $\pi^\mathcal{F}_n\colon \mathcal{F} \to \mathbb{R}$ by $\pi^\mathcal{F}_n(F) = \mathbf{1}_F(n)$ for every $F \in \mathcal{F}$. Clearly for every $n \in \mathbb{N}$ we have

$\pi_n^{\mathcal{F}} \in C(\mathcal{F})$ and $\|\pi_n^{\mathcal{F}}\|_\infty = 1$. Moreover, as the family $\mathcal{F}$ contains all singletons, we see that $\pi_n^{\mathcal{F}} \neq \pi_m^{\mathcal{F}}$ if $n \neq m$. It is easy to see that the sequence $(\pi_n^{\mathcal{F}})$ converges pointwise to 0, and so, $(\pi_n^{\mathcal{F}}) \in \mathrm{SRC}(\mathcal{F})$.

Claim 3. *We have* $\phi_{\mathcal{F}}\big((\pi_n^{\mathcal{F}})\big) \geqslant o(\mathcal{F}) \geqslant \xi$.

Proof of Claim 3. The proof is essentially based on the fact that $\mathcal{F}$ is hereditary. Indeed, notice that if $F = \{n_0 < \cdots < n_k\} \in \mathcal{F}$, then $(e_i)_{i=0}^k \overset{2}{\sim} (\pi_{n_i}^{\mathcal{F}})_{i=0}^k$, or equivalently, $F \in T^2_{(\pi_n^{\mathcal{F}})}$. To see this, fix $F = \{n_0 < \cdots < n_k\} \in \mathcal{F}$ and let $a_0, \ldots, a_k \in \mathbb{R}$ be arbitrary. We set

$$I_+ = \big\{i \in \{0, \ldots, k\} : a_i \geqslant 0\big\} \ \text{ and } \ I_- = \{0, \ldots, k\} \setminus I_+.$$

Then, either $\sum_{i \in I_+} a_i \geqslant \frac{1}{2}\sum_{i=0}^k |a_i|$ or $-\sum_{i \in I_-} a_i \geqslant \frac{1}{2}\sum_{i=0}^k |a_i|$. Assume that the second case occurs (the argument is symmetric). Let $F_- = \{n_i : i \in I_-\} \subseteq F \in \mathcal{F}$. Then $F_- \in \mathcal{F}$ since $\mathcal{F}$ is hereditary. Now observe that

$$\frac{1}{2}\sum_{i=0}^k |a_i| \leqslant -\sum_{i \in I_-} a_i = \big|\sum_{i=0}^k a_i \pi_{n_i}^{\mathcal{F}}(F_-)\big| \leqslant \big\|\sum_{i=0}^k a_i \pi_{n_i}^{\mathcal{F}}\big\|_\infty \leqslant 2\sum_{i=0}^k |a_i|.$$

It follows by the above discussion that the identity map $\mathrm{Id}\colon T_{\mathcal{F}} \to T^2_{(\pi_n^{\mathcal{F}})}$ is a well-defined monotone map. The proof of Claim 3 is completed. □

By Fact 2 and Claim 3, we conclude that $\sup\{\phi(\mathbf{f}) : \mathbf{f} \in \mathrm{SRC}\} = \omega_1$, and so the entire proof is completed. □

4. Topological and strong embedding

Consider the classes $\mathrm{SRC}(X)$ and $\mathrm{SRC}(Y)$, where X and Y are compact metrizable spaces, as they were coded in the previous section. There is a canonical notion of embedding between elements of $\mathrm{SRC}(X)$ and $\mathrm{SRC}(Y)$ defined as follows.

Definition 4. *Let X, Y be compact metrizable spaces,* $\mathbf{f} = (f_n) \in \mathrm{SRC}(X)$ *and* $\mathbf{g} = (g_n) \in \mathrm{SRC}(Y)$. *We say that* $\mathbf{g}$ topologically embeds *into* $\mathbf{f}$ (*in symbols* $\mathbf{g} < \mathbf{f}$) *if there exists a homeomorphic embedding of* $\overline{\{g_n\}}^p$ *into* $\overline{\{f_n\}}^p$.

Clearly the notion of topological embedding is natural and meaningful, as $\mathbf{f}_1 < \mathbf{f}_2$ and $\mathbf{f}_2 < \mathbf{f}_3$ imply that $\mathbf{f}_1 < \mathbf{f}_3$. However, in this setting, one also has a canonical $\mathbf{\Pi}^1_1$-rank on SRC and any notion of embedding between elements of SRC should be coherent with this rank, in the sense that if $\mathbf{g} < \mathbf{f}$, then $\phi_Y(\mathbf{g}) \leqslant \phi_X(\mathbf{f})$. Unfortunately, the topological embedding is not strong enough in order to have this property.

Example 1. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two compact hereditary families of finite subsets of $\mathbb{N}$. As in the proof of Lemma 1, consider the sequences $(\pi_n^{\mathcal{F}_1}) \in \mathrm{SRC}(\mathcal{F}_1)$ and $(\pi_n^{\mathcal{F}_2}) \in \mathrm{SRC}(\mathcal{F}_2)$. Both of them are pointwise convergent to 0. Hence, they are topologically equivalent and clearly bi-embedable. However, it is easy to see that the corresponding ranks of the two sequences depend only on the order of the families $\mathcal{F}_1$ and $\mathcal{F}_2$, and so, they are totally unrelated.

We are going to strengthen the notion of topological embedding between the elements of SRC. To motivate our definition, let $\mathbf{f} = (f_n), \mathbf{g} = (g_n) \in \mathrm{SRC}$ and assume that both (f_n) and (g_n) are basic sequences. In this case, the most natural notion of embedding is that of equivalence, that is, $\mathbf{g}$ embeds into $\mathbf{f}$ if there exists $L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that the sequence (g_n) is equivalent to (f_{l_n}). It is easy to verify that, in this case, we do have that $\phi(\mathbf{g}) \leqslant \phi(\mathbf{f})$. Although not every $\mathbf{f} \in \mathrm{SRC}$ is a basic sequence, there is a metric relation we can impose on $\mathbf{f}$ and $\mathbf{g}$ which incorporates the above observation.

Definition 5. *Let X, Y be compact metrizable spaces, $\mathbf{f} = (f_n) \in \mathrm{SRC}(X)$ and $\mathbf{g} = (g_n) \in \mathrm{SRC}(Y)$. We say that $\mathbf{g}$* strongly embeds *into $\mathbf{f}$ (in symbols $\mathbf{g} \prec \mathbf{f}$) if $\mathbf{g}$ topologically embeds into $\mathbf{f}$ and, moreover, if for every $\varepsilon > 0$ there exists $L_\varepsilon = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that for every $k \in \mathbb{N}$ and every $a_0, \ldots, a_k \in \mathbb{R}$ we have*

$$\Big| \max_{0 \leqslant i \leqslant k} \Big\| \sum_{n=0}^{i} a_n g_n \Big\|_\infty - \Big\| \sum_{n=0}^{k} a_n f_{l_n} \Big\|_\infty \Big| \leqslant \varepsilon \sum_{n=0}^{k} \frac{|a_n|}{2^{n+1}}. \tag{2}$$

Below we gather the basic properties of the notion of strong embedding.

Proposition 6. *Let X, Y be compact metrizable spaces. Then the following hold.*

- (i) *If $\mathbf{f} \in \mathrm{SRC}(X)$ and $\mathbf{g} \in \mathrm{SRC}(Y)$ with $\mathbf{g} \prec \mathbf{f}$, then $\mathbf{g} < \mathbf{f}$.*
- (ii) *If $\mathbf{f} \in \mathrm{SRC}(X)$, $\mathbf{g} \in \mathrm{SRC}(Y)$ with $\mathbf{g} \prec \mathbf{f}$ and the sequence (g_n) is a normalized basic sequence, then there exists $L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that the sequence (f_{l_n}) is basic and equivalent to (g_n).*
- (iii) *If $\mathbf{f}_1 \prec \mathbf{f}_2$ and $\mathbf{f}_2 \prec \mathbf{f}_3$, then $\mathbf{f}_1 \prec \mathbf{f}_3$.*
- (iv) *If $\mathbf{f} \in \mathrm{SRC}(X)$ and $\mathbf{g} \in \mathrm{SRC}(Y)$ with $\mathbf{g} \prec \mathbf{f}$, then $\phi_Y(\mathbf{g}) \leqslant \phi_X(\mathbf{f})$.*
- (v) *Let Z be a compact metrizable space and $e\colon Z \to X$ a continuous onto map. Let $\mathbf{f} = (f_n) \in \mathrm{SRC}(X)$ and, as in Fact 2, define $\mathbf{h} = (h_n) \in \mathrm{SRC}(Z)$ by setting $h_n(z) = f_n(e(z))$ for every $n \in \mathbb{N}$ and every $z \in Z$. If $\mathbf{g} \in \mathrm{SRC}(Y)$ is such that $\mathbf{g} \prec \mathbf{f}$, then $\mathbf{g} \prec \mathbf{h}$.*

Proof. (i) It is straightforward.

(ii) Let $K \geqslant 1$ be the basis constant of (g_n). We are going to show that there exists $L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that (g_n) is $2K$-equivalent to (f_{l_n}). Indeed, let $0 < \varepsilon < \frac{1}{4K}$ and select $L_\varepsilon = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that inequality (2) is satisfied. Let $k \in \mathbb{N}$ and $a_0, \ldots, a_k \in \mathbb{R}$. Notice that

$$\Big\| \sum_{n=0}^{k} a_n g_n \Big\|_\infty \leqslant \max_{0 \leqslant i \leqslant k} \Big\| \sum_{n=0}^{i} a_n g_n \Big\|_\infty \leqslant K \Big\| \sum_{n=0}^{k} a_n g_n \Big\|_\infty. \tag{3}$$

Moreover, for every $m \in \{0,\dots,k\}$ we have

$$|a_m| \leqslant 2K\Big\| \sum_{n=0}^{k} a_n g_n \Big\|_\infty \tag{4}$$

as (g_n) is a normalized Schauder basic sequence (see [LT]). Plugging in inequalities (3) and (4) into (2) we obtain

$$\begin{aligned} \Big\|\sum_{n=0}^{k} a_n f_{l_n}\Big\|_\infty &\leqslant K\Big\|\sum_{n=0}^{k} a_n g_n\Big\|_\infty + 2K\varepsilon\Big\|\sum_{n=0}^{k} a_n g_n\Big\|_\infty \\ &\leqslant 2K\Big\|\sum_{n=0}^{k} a_n g_n\Big\|_\infty \end{aligned}$$

by the choice of ε. Arguing similarly, we see that

$$\frac{1}{2K}\Big\|\sum_{n=0}^{k} a_n g_n\Big\|_\infty \leqslant \Big\|\sum_{n=0}^{k} a_n f_{l_n}\Big\|_\infty.$$

Thus (g_n) is $2K$-equivalent to (f_{l_n}), as desired.

(iii) It is a simple calculation, similar to that of part (ii), and we prefer not to bother the reader with it.

(iv) Let $d \geqslant 1$. We fix $\varepsilon > 0$ with $\varepsilon < \frac{1}{2d}$ and we select $L_\varepsilon = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that inequality (2) is satisfied. For every $s = (m_0 < \cdots < m_k) \in T^d_{\mathbf{g}}$ we set $t_s = (l_{m_0} < \cdots < l_{m_k}) \in \mathrm{FIN}(\mathbb{N})$. Observe that for every $k \in \mathbb{N}$ and every $a_0,\dots,a_k \in \mathbb{R}$ we have

$$\begin{aligned} 2d\sum_{n=0}^{k}|a_n| \geqslant \Big\|\sum_{n=0}^{k} a_n f_{l_{m_n}}\Big\|_\infty &\geqslant \max_{0\leqslant i\leqslant k}\Big\|\sum_{n=0}^{i} a_n g_{m_n}\Big\|_\infty - \varepsilon\sum_{n=0}^{k}|a_n| \\ &\geqslant \Big\|\sum_{n=0}^{k} a_n g_{m_n}\Big\|_\infty - \varepsilon\sum_{n=0}^{k}|a_n| \\ &\geqslant \frac{1}{d}\sum_{n=0}^{k}|a_n| - \frac{1}{2d}\sum_{n=0}^{k}|a_n| \\ &= \frac{1}{2d}\sum_{n=0}^{k}|a_n|. \end{aligned}$$

This yields that $t_s \in T^{2d}_{\mathbf{f}}$. It follows that the map $s \mapsto t_s$ is a monotone map from $T^d_{\mathbf{g}}$ to $T^{2d}_{\mathbf{f}}$. Hence, $o(T^d_{\mathbf{g}}) \leqslant o(T^{2d}_{\mathbf{f}})$. Since d was arbitrary, this implies that $\phi_Y(\mathbf{g}) \leqslant \phi_X(\mathbf{f})$, as desired.

(v) It is also straightforward since the map e induces an isometric embedding of $C(X)$ into $C(Z)$. □

We are going to present another property of the notion of strong embedding which has a Banach space theoretic flavor. To this end, we give the following definition.

Definition 7. *Let E be a compact metrizable space and $\mathbf{g} = (g_n)$ a bounded sequence in $C(E)$. By $X_{\mathbf{g}}$ we shall denote the completion of $c_{00}(\mathbb{N})$ under the norm*

$$\|x\|_{\mathbf{g}} := \sup\Big\{\big\|\sum_{n=0}^{k} x(n)g_n\big\|_\infty : k \in \mathbb{N}\Big\}. \tag{5}$$

We shall denote by $(e_n^{\mathbf{g}})$ the standard Hamel basis of $c_{00}(\mathbb{N})$ regarded as a sequence in $X_{\mathbf{g}}$. We isolate some elementary properties of $(e_n^{\mathbf{g}})$.

- (P1) The sequence $(e_n^{\mathbf{g}})$ is a monotone basis of $X_{\mathbf{g}}$. Moreover, $(e_n^{\mathbf{g}})$ is normalized (respectively, seminormalized) if and only if (g_n) is.
- (P2) If (g_n) is basic with basis constant K, then $(e_n^{\mathbf{g}})$ is K-equivalent to (g_n).

Less trivial is the fact (which we will see in the next section) that $\mathbf{g} \in \mathrm{SRC}(E)$ if and only if $(e_n^{\mathbf{g}})$ is in $\mathrm{SRC}(K)$, where K is the closed unit ball of $X_{\mathbf{g}}^*$ with the weak* topology. In light of property (P2) above, the sequence $(e_n^{\mathbf{g}})$ may be regarded as an "approximation" of (g_n) by a basic sequence.

The following proposition relates the strong embedding of a sequence $\mathbf{g} = (g_n)$ into a sequence $\mathbf{f} = (f_n)$ with the existence of subsequences of (f_n) which are "almost isometric" to $(e_n^{\mathbf{g}})$. Its proof, which is left to the interested reader, is based on similar arguments as the proof of Proposition 6.

Proposition 8. *Let X and Y be compact metrizable spaces, $\mathbf{g} = (g_n) \in \mathrm{SRC}(X)$ and $\mathbf{f} = (f_n) \in \mathrm{SRC}(Y)$. If $\mathbf{g}$ strongly embeds into $\mathbf{f}$, then for every $\varepsilon > 0$ there exists $L_\varepsilon = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that $(e_n^{\mathbf{g}})$ is $(1+\varepsilon)$-equivalent to (f_{l_n}).*

5. The main result

We are ready to state and prove the strong boundedness result for the class SRC.

Theorem 9. *Let A be an analytic subset of* SRC. *Then there exists* $\mathbf{f} \in \mathrm{SRC}$ *such that for every $\mathbf{g} \in A$ we have $\mathbf{g} \prec \mathbf{f}$.*

We record the following consequence of Theorem 9 and Proposition 8.

Corollary 10. *Let X be a compact metrizable space and $\mathbf{g} = (g_n) \in \mathrm{SRC}(X)$. Then $(e_n^{\mathbf{g}})$ is in $\mathrm{SRC}(K)$ where K is the closed unit ball of $X_{\mathbf{g}}^*$ with the weak* topology.*

We proceed to the proof of Theorem 9.

Proof of Theorem 9. We fix a norm dense sequence (d_n) in the closed unit ball of $C(2^{\mathbb{N}})$ such that $d_n \neq d_m$ if $n \neq m$ and $d_n \neq 0$ for every $n \in \mathbb{N}$. We also fix a sequence (D_n) of infinite subsets of $\mathbb{N}$ such that $D_n \cap D_m = \emptyset$ if $n \neq m$ and $\mathbb{N} = \bigcup_n D_n$. Let A be an analytic subset of SRC and define $\tilde{A} \subseteq \mathbb{N}^{\mathbb{N}}$ by

$$\begin{aligned} \sigma \in \tilde{A} \quad \Leftrightarrow \quad & \exists \mathbf{g} = (g_n) \in A \ \exists \varepsilon > 0 \text{ such that} \\ & \big[\forall n\ \forall k\ \big(k \in D_n \Rightarrow \|g_n - d_{\sigma(k)}\|_\infty \leqslant \frac{\varepsilon}{2^{k+1}}\big)\big] \text{ and} \\ & \big[\forall n\ \forall m\ \big(n \neq m \Rightarrow \sigma(n) \neq \sigma(m)\big)\big]. \end{aligned}$$

Then $\tilde{A}$ is $\mathbf{\Sigma}^1_1$. Let T be the unique downwards closed, pruned tree on $\mathbb{N}\times\mathbb{N}$ such that $\tilde{A}=\mathrm{proj}[T]$. We define a sequence $(h_t)_{t\in T}$ in $C(2^{\mathbb{N}})$ as follows. If $t=(\emptyset,\emptyset)$, then we set $h_t\coloneqq 0$. If $t\in T$ with $t\neq(\emptyset,\emptyset)$, then $t=(s,w)$ with $s=(n_0,\dots,n_m)\in\mathbb{N}^{<\mathbb{N}}$. We set $h_t\coloneqq d_{n_m}$. Clearly $\|h_t\|_\infty\leqslant 1$ for every $t\in T$. We notice the following properties of the sequence $(h_t)_{t\in T}$.

(P1) For every $\sigma\in[T]$ there exist $\mathbf{g}=(g_n)\in A$ and $\varepsilon>0$ such that for every $n\in\mathbb{N}$ and every $k\geqslant 1$ with $k-1\in D_n$ we have $\|g_n-h_{\sigma|k}\|_\infty\leqslant\frac{\varepsilon}{2^k}$.

(P2) For every $\mathbf{g}=(g_n)\in A$ and every $\varepsilon>0$ there exists $\sigma\in[T]$ such that for every $n\in\mathbb{N}$ and every $k\geqslant 1$ with $k-1\in D_n$ we have $\|g_n-h_{\sigma|k}\|_\infty\leqslant\frac{\varepsilon}{2^k}$.

Let $\phi\colon T\to 2^{<\mathbb{N}}$ be an embedding such that for every $t,t'\in T$ we have $\phi(t)\sqsubset\phi(t')$ if and only if $t\sqsubset t'$. Also let $e\colon T\to\mathbb{N}$ be a bijection such that $e(t)<e(t')$ for every $t,t'\in T$ with $t\sqsubset t'$. We enumerate the nodes of T as (t_n) according to e. Now for every $n\in\mathbb{N}$ we define $f_n\colon 2^{\mathbb{N}}\times 2^{\mathbb{N}}\to\mathbb{R}$ by

$$f_n(\sigma_1,\sigma_2)=\mathbf{1}_{V_{\phi(t_n)}}(\sigma_1)\cdot h_{t_n}(\sigma_2) \tag{6}$$

where $V_{\phi(t_n)}\coloneqq\{\sigma\in 2^{\mathbb{N}}:\phi(t_n)\sqsubset\sigma\}$. Clearly, $f_n\in C(2^{\mathbb{N}}\times 2^{\mathbb{N}})$ and $\|f_n\|_\infty\leqslant 1$ for every $n\in\mathbb{N}$. Moreover, it is easy to check that $f_n\neq f_m$ if $n\neq m$.

It will be convenient to introduce the following notation. For every function $g\colon 2^{\mathbb{N}}\to\mathbb{R}$ and every $\tau\in 2^{\mathbb{N}}$ by $g*\tau\colon 2^{\mathbb{N}}\times 2^{\mathbb{N}}\to\mathbb{R}$ we denote the function defined by $g*\tau(\sigma_1,\sigma_2)=\delta_\tau(\sigma_1)\cdot g(\sigma_2)$ for every $(\sigma_1,\sigma_2)\in 2^{\mathbb{N}}\times 2^{\mathbb{N}}$. (Here, δ_τ stands for the Dirac function at τ.)

Claim 11. *We have* $(f_n)\in\mathrm{SRC}(2^{\mathbb{N}}\times 2^{\mathbb{N}})$.

Proof of Claim 11. By the Main Theorem in [Ro2], it is enough to show that every subsequence of (f_n) has a further pointwise convergent subsequence. So, let $N\in[\mathbb{N}]^\infty$ be arbitrary. By Ramsey's theorem, there exists $M\in[N]^\infty$ such that the family $\{\phi(t_n):n\in M\}$ either consists of pairwise incomparable nodes, or of pairwise comparable. In the first case we see that the sequence $(f_n)_{n\in M}$ is pointwise convergent to 0. In the second case we notice that, by the properties of ϕ and the enumeration of T, for every $n,m\in M$ with $n<m$ we have $t_n\sqsubset t_m$. It follows that there exists $\sigma\in[T]$ such that $t_n\sqsubset\sigma$ for every $n\in M$. We may also assume that $t_n\neq(\emptyset,\emptyset)$ for every $n\in M$. By property (P1) above, there exist $\mathbf{g}=(g_n)\in A$, $\varepsilon>0$ and a sequence $(k_n)_{n\in M}$ in $\mathbb{N}$ (with possible repetitions) such that $\|g_{k_n}-h_{t_n}\|_\infty\leqslant\frac{\varepsilon}{2^{|t_n|}}$ for every $n\in M$. Since $\mathbf{g}\in\mathrm{SRC}$, there exists $L\in[M]^\infty$ such that the sequence $(g_{k_n})_{n\in L}$ is pointwise convergent to a Baire-1 function g. By the fact that $\lim_{n\in L}|t_n|=\infty$, we see that the sequence $(h_{t_n})_{n\in L}$ is also pointwise convergent to g. Finally notice that the sequence $(\mathbf{1}_{V_{\phi(t_n)}})_{n\in L}$ converges pointwise to δ_τ where τ is the unique element of $2^{\mathbb{N}}$ determined by the infinite chain $\{\phi(t_n):n\in L\}$ of $2^{<\mathbb{N}}$. It follows that the sequence $(f_n)_{n\in L}$ is pointwise convergent to the function $g*\tau$. The proof of Claim 11 is completed. $\square$

Claim 12. *For every* $\mathbf{g}=(g_n)\in A$ *we have that* $\mathbf{g}$ *topologically embeds into* (f_n).

Proof of Claim 12. Let $\mathbf{g} = (g_n) \in A$. By property (P2), there exists $\sigma \in [T]$ such that for every $n \in \mathbb{N}$ and every $k \geqslant 1$ with $k-1 \in D_n$ we have $\|g_n - h_{\sigma|k}\|_\infty \leqslant \frac{1}{2^k}$. By the choice of ϕ, we see that there exists a unique $\tau \in 2^{\mathbb{N}}$ such that $\phi(\sigma|k) \sqsubset \tau$ for every $k \in \mathbb{N}$. Fix $n_0 \in \mathbb{N}$. By the fact that there exist infinitely many k with $\|g_{n_0} - h_{\sigma|k}\|_\infty \leqslant \frac{1}{2^k}$ and arguing as in Claim 11, we see that the function $g_{n_0} * \tau$ belongs to the closure of $\{f_n\}$ in $\mathbb{R}^{2^{\mathbb{N}} \times 2^{\mathbb{N}}}$. It follows that the map

$$\overline{\{g_n\}}^p \ni g \mapsto g * \tau \in \overline{\{f_n\}}^p$$

is a homeomorphic embedding. The proof of Claim 12 is completed. □

Claim 13. *For every* $\mathbf{g} = (g_n) \in A$ *we have that* $\mathbf{g}$ *strongly embeds into* (f_n).

Proof of Claim 13. Fix $\mathbf{g} = (g_n) \in A$. By Claim 12, it is enough to show that for every $\varepsilon > 0$ there exists $L_\varepsilon = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that inequality (2) is satisfied for (g_n) and (f_{l_n}). So, let $\varepsilon > 0$ be arbitrary. Invoking property (P2), we see that there exist $\sigma \in [T]$ such that for every $n \in \mathbb{N}$ and every $k \geqslant 1$ with $k-1 \in D_n$ we have $\|g_n - h_{\sigma|k}\|_\infty \leqslant \frac{\varepsilon}{2^k}$. There exists $D = \{m_0 < m_1 < \cdots\} \in [\mathbb{N}]^\infty$ with $m_0 \geqslant 1$ and such that $m_n - 1 \in D_n$ for every $n \in \mathbb{N}$. By the properties of the enumeration e of T, there exists $L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that $t_{l_n} = \sigma|m_n$ for every $n \in \mathbb{N}$. We isolate, for future use, the following facts.

- (F1) For every $n \in \mathbb{N}$ we have $\|g_n - h_{t_{l_n}}\|_\infty \leqslant \frac{\varepsilon}{2^{m_n}} \leqslant \frac{\varepsilon}{2^{n+1}}$.
- (F2) For every $n, m \in \mathbb{N}$ with $n < m$ we have $t_{l_n} \sqsubset t_{l_m}$.

We claim that the sequences (g_n) and (f_{l_n}) satisfy inequality (2) for the given $\varepsilon > 0$. Indeed, let $k \in \mathbb{N}$ and $a_0, \dots, a_k \in \mathbb{R}$. By (F1) above, for every $i \in \{0, \dots, k\}$ we have

$$\Big| \big\| \sum_{n=0}^{i} a_n g_n \big\|_\infty - \big\| \sum_{n=0}^{i} a_n h_{t_{l_n}} \big\|_\infty \Big| \leqslant \varepsilon \sum_{n=0}^{i} \frac{|a_n|}{2^{n+1}}.$$

This implies that

$$\Big| \max_{0 \leqslant i \leqslant k} \big\| \sum_{n=0}^{i} a_n g_n \big\|_\infty - \max_{0 \leqslant i \leqslant k} \big\| \sum_{n=0}^{i} a_n h_{t_{l_n}} \big\|_\infty \Big| \leqslant \varepsilon \sum_{n=0}^{k} \frac{|a_n|}{2^{n+1}}.$$

The above inequality is a consequence of the following elementary fact. If $(r_i)_{i=0}^k$, $(\theta_i)_{i=0}^k$ and $(\delta_i)_{i=0}^k$ are finite sequences of positive reals such that $|r_i - \theta_i| \leqslant \delta_i$ for every $i \in \{0, \dots, k\}$, then

$$\big| \max_{0 \leqslant i \leqslant k} r_i - \max_{0 \leqslant i \leqslant k} \theta_i \big| \leqslant \max_{0 \leqslant i \leqslant k} \delta_i.$$

So the claim will be proved once we show that

$$\max_{0 \leqslant i \leqslant k} \big\| \sum_{n=0}^{i} a_n h_{t_{l_n}} \big\|_\infty = \big\| \sum_{n=0}^{k} a_n f_{l_n} \big\|_\infty.$$

To this end we argue as follows. For every $t\in T$ the function h_t is continuous. So there exist $j\in\{0,\dots,k\}$ and $\sigma_2\in 2^{\mathbb{N}}$ such that

$$\max_{0\leqslant i\leqslant k}\Big\|\sum_{n=0}^{i}a_n h_{t_{l_n}}\Big\|_\infty=\Big|\sum_{n=0}^{j}a_n h_{t_{l_n}}(\sigma_2)\Big|.$$

By (F2), we have $t_{l_0}\sqsubset\cdots\sqsubset t_{l_k}$. Hence, by the properties of ϕ, we see that $\phi(t_{l_0})\sqsubset\cdots\sqsubset\phi(t_{l_k})$. It follows that there exists $\sigma_1\in 2^{\mathbb{N}}$ such that $\mathbf{1}_{V_{\phi(t_{l_n})}}(\sigma_1)=1$ if $n\in\{0,\dots,j\}$ while $\mathbf{1}_{V_{\phi(t_{l_n})}}(\sigma_1)=0$ otherwise. Therefore,

$$\Big\|\sum_{n=0}^{k}a_n f_{l_n}\Big\|_\infty\geqslant\Big|\sum_{n=0}^{k}a_n f_{l_n}(\sigma_1,\sigma_2)\Big|=\Big|\sum_{n=0}^{j}a_n h_{t_{l_n}}(\sigma_2)\Big|.$$

Conversely, let $(\sigma_3,\sigma_4)\in 2^{\mathbb{N}}\times 2^{\mathbb{N}}$ be such that

$$\Big\|\sum_{n=0}^{k}a_n f_{l_n}\Big\|_\infty=\Big|\sum_{n=0}^{k}a_n f_{l_n}(\sigma_3,\sigma_4)\Big|.$$

We notice that if $\mathbf{1}_{V_{\phi(t_{l_n})}}(\sigma_3)=1$ for some $n\in\mathbb{N}$, then for every $m\in\mathbb{N}$ with $m\leqslant n$ we also have that $\mathbf{1}_{V_{\phi(t_{l_m})}}(\sigma_3)=1$. Hence, there exists $p\in\{0,\dots,k\}$ such that $\mathbf{1}_{V_{\phi(t_{l_n})}}(\sigma_3)=1$ if $n\in\{0,\dots,p\}$ while $\mathbf{1}_{V_{\phi(t_{l_n})}}(\sigma_3)=0$ otherwise. This implies that

$$\begin{aligned}\Big\|\sum_{n=0}^{k}a_n f_{l_n}\Big\|_\infty &= \Big|\sum_{n=0}^{k}a_n f_{l_n}(\sigma_3,\sigma_4)\Big|=\Big|\sum_{n=0}^{p}a_n f_{l_n}(\sigma_3,\sigma_4)\Big|\\ &= \Big|\sum_{n=0}^{p}a_n h_{t_{l_n}}(\sigma_4)\Big|\leqslant\max_{0\leqslant i\leqslant k}\Big\|\sum_{n=0}^{i}a_n h_{t_{l_n}}\Big\|_\infty\end{aligned}$$

and the proof of Claim 13 is completed. □

Since $2^{\mathbb{N}}\times 2^{\mathbb{N}}$ is homeomorphic to $2^{\mathbb{N}}$, by Claims 11 and 13 and part (v) of Proposition 6, the proof of the theorem is completed. □

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