GENERICITY AND AMALGAMATION OF CLASSES OF BANACH SPACES

SPIROS A. ARGYROS AND PANDELIS DODOS

Abstract. We study universality problems in Banach space theory. We show that if A is an analytic class, in the Effros–Borel structure of subspaces of $C[0,1]$, of non-universal separable Banach spaces, then there exists a non-universal separable Banach space Y, with a Schauder basis, that contains isomorphic copies of each member of A with the bounded approximation property. The proof is based on the amalgamation technique of a class $\mathcal{C}$ of separable Banach spaces, introduced in the paper. We show, among others, that there exists a separable Banach space R not containing $L_1(0,1)$ such that the indices β and r_{ND} are unbounded on the set of Baire-1 elements of the ball of the double dual R^{**} of R. This answers two questions of Rosenthal.

We also introduce the concept of a strongly bounded class of separable Banach spaces. A class $\mathcal{C}$ of separable Banach spaces is strongly bounded if for every analytic subset A of $\mathcal{C}$ there exists $Y \in \mathcal{C}$ that contains all members of A up to isomorphism. We show that several natural classes of separable Banach spaces are strongly bounded, among them the class of non-universal spaces with a Schauder basis, the class of reflexive spaces with a Schauder basis, the class of spaces with a shrinking Schauder basis and the class of spaces with Schauder basis not containing a minimal Banach space X.

Contents

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1. Introduction

1.1. Problems concerning the structure of a separable Banach space X containing a class $\mathcal{C}$ of separable Banach spaces have attracted the attention of researchers for more than forty years. Indeed after the classical Mazur theorem that $C[0,1]$ is universal for all separable Banach spaces, Pelczynski [P] presented two universal spaces for the classes of spaces with a Schauder basis and an unconditional basis respectively. In 1968, Szlenk in his pioneering paper [Sz] showed that there does not exist a Banach space with separable dual that contains isomorphically every separable reflexive space. His proof was based on a transfinite analysis of every separable dual space, leading to the famous Szlenk index. In 1980, in two seminal papers [B1, B3], Bourgain proved that every separable Banach space containing either all separable reflexive Banach spaces or all $C(K)$ with K countable compact, is universal for all separable Banach spaces. For the case of reflexive spaces Bourgain's idea was to consider a representability tree of a given Banach space X into a Banach space Y. The complexity of this tree provides an index of the embedability of X into Y. The Kunen–Martin theorem and an appropriate transfinite sequence $\langle R_\xi : \xi < \omega_1 \rangle$ of separable reflexive Banach spaces yield the result. (Actually, the version of the Kunen–Martin theorem needed for Bourgain's application was known to the Russian and Polish set theorists.) Bourgain's approach is simple, efficient and it is essentially the unique method for showing that a given class $\mathcal{C}$ of separable Banach spaces is universal. In both results, Bourgain engaged results from descriptive set theory in his study. In the middle of 1990's Bossard [Bo1, Bo2, Bo3] considered universality problems in a pure descriptive set theoretic context. He showed that every analytic subset, in the Effros–Borel structure of subspaces of $C[0,1]$, that contains all separable reflexive spaces must also contain a universal space. To proceed with our discussion let us state the following definitions motivated by the corresponding results of Bourgain and Bossard.

Definition A. *Let $\mathcal{C}$ be an isomorphic invariant class of separable Banach spaces such that every $X \in \mathcal{C}$ is not universal.*

1. *We say that the class $\mathcal{C}$ is* Bourgain generic *if every separable Banach space Y that contains all members of $\mathcal{C}$, must be universal.*
2. *We say that the class $\mathcal{C}$ is* Bossard generic *if every analytic subset A that contains all members of $\mathcal{C}$ up to isomorphism, must also contain $Y \in A$ which is universal.*

We recall that a separable Banach space X is said to be universal *if it contains all separable Banach spaces up to isomorphism.*

It is clear that Bourgain genericity is what Banach space theory specialists are interested in. A glance at the definition of Bossard genericity gives the impression that it is related to descriptive set theory rather than Banach space theory. In the opposite, Godefroy [AGR, Go] has repeatedly stated that Bossard's approach provides the appropriate frame for studying several problems of Banach space theory. One of the goals of the present work is to support Godefroy's thesis for problems related to generic classes of separable Banach spaces. We believe that in the next few lines we will convince the reader for the importance of Bossard genericity. The following problem is central in our approach.

Problem B. *Is it true that a class $\mathcal{C}$ of separable Banach spaces is Bourgain generic if and only if it is Bossard generic?*

We have been informed that Kechris several years ago, motivated by the results of [KW2], had also posed a similar problem.

It is easy to see that Bossard genericity implies the Bourgain one. Therefore, the real problem concerns the converse implication. We conjecture that the above problem has an affirmative answer. Our optimism is based on the following theorem which is one of the main results of the paper.

Theorem C. *Let $\mathcal{C}$ be an analytic class of separable Banach spaces such that every $X \in \mathcal{C}$ is not universal. Then there exists a non-universal Banach space Y with a Schauder basis that contains isomorphic copies of each member of $\mathcal{C}$ with the bounded approximation property.*

The importance of a possible positive answer to Problem B arises from the fact that it provides an efficient tool in order to check the non-universality of certain classes of separable Banach spaces. Simply compute the complexity of the class in question. If it is analytic, then the class is not universal. For instance, let $\mathcal{C}_{\mathrm{uc}}$ be the class of all separable uniformly convex Banach spaces. Bourgain in [B1] had asked if there exists a reflexive Banach space universal for the class $\mathcal{C}_{\mathrm{uc}}$. Prus [Pr1, Pr2] answered affirmatively Bourgain's question for the subclass of uniformly convex spaces with the approximation property. We have been informed [OS1] that very recently Odell and Schlumprecht have succeeded to give a complete affirmative answer to the question [OS2]. Under our point of view the class $\mathcal{C}_{\mathrm{uc}}$ is Borel, and so a positive answer to Problem B would immediately imply that there exists a non-universal separable Banach space containing all members of $\mathcal{C}_{\mathrm{uc}}$. Other examples are the classes $\mathcal{C}_{\mathrm{type}}$ and $\mathcal{C}_{\mathrm{cotype}}$ of all separable spaces with non-trivial type and non-trivial cotype respectively. Both are Borel, and so Theorem C provides a non-universal separable Banach space Y containing all members of $\mathcal{C}_{\mathrm{type}}$ (respectively, $\mathcal{C}_{\mathrm{cotype}}$) with the bounded approximation property.

We proceed to discuss the proof of Theorem C. A basic ingredient in its proof is the *HI-amalgamation* (respectively, *p-amalgamation* for $1 < p < \infty$) of a family $\mathcal{C}$ of separable Banach spaces with a bimonotone Schauder basis. Roughly speaking the HI-amalgamation (respectively, p-amalgamation) of a class $\mathcal{C}$ is a Banach space $\mathcal{A}_{\mathrm{hi}}^{\mathcal{C}}$ (respectively, $\mathcal{A}_p^{\mathcal{C}}$) with a Schauder basis with the following properties.

($\mathcal{P}1$) Every $X \in \mathcal{C}$ is isomorphic to a complemented subspace of $\mathcal{A}_{\mathrm{hi}}^{\mathcal{C}}$ (respectively, $\mathcal{A}_p^{\mathcal{C}}$).

($\mathcal{P}2$) Every subspace Y of $\mathcal{A}_{\mathrm{hi}}^{\mathcal{C}}$ (respectively, $\mathcal{A}_p^{\mathcal{C}}$) either contains a HI subspace (respectively, a subspace isomorphic to ℓ_p), or there exists a finite sequence $(X_1, \dots, X_n)$ in $\mathcal{C}$ such that Y is isomorphic to a subspace of $\sum_{i=1}^n \oplus X_i$.

We prove the following theorem related to the above concept.

Theorem D. *Let $\mathcal{C}$ be an analytic class of separable Banach spaces, and set $\mathcal{C}_b = \{X \in \mathcal{C} : X$ has a Schauder basis$\}$. Then there exists a HI-amalgamation $\mathcal{A}_{\mathrm{hi}}^{\mathcal{C}_b}$ (respectively, p-amalgamation $\mathcal{A}_p^{\mathcal{C}_b}$ for $1 < p < \infty$) of the class $\mathcal{C}_b$. Moreover, the following hold.*

(a) *If every $X \in \mathcal{C}_b$ is reflexive, then $\mathcal{A}_{\mathrm{hi}}^{\mathcal{C}_b}$ (respectively, $\mathcal{A}_p^{\mathcal{C}_b}$) is reflexive.*

(b) *If $\mathcal{C}$ does not contain a universal member, then neither $\mathcal{A}_{\mathrm{hi}}^{\mathcal{C}_b}$ (respectively, $\mathcal{A}_p^{\mathcal{C}_b}$) does.*

To prove Theorem C from Theorem D we employ a result of Lusky [Lu] which asserts that for every Banach space X with the bounded approximation property, the space $X \oplus C_0$ has a Schauder basis. Here C_0 denotes the corresponding Johnson's space. Let us notice that results, similar to Theorem D, can also be obtained for the class $\mathcal{C}_{\mathrm{FDD}} = \{X \in \mathcal{C} : X$ has a Schauder FDD$\}$.

It does not seem easy to pass from Theorem C to a complete answer of Problem B, even for specific classes. A possible approach is the following. Starting from a class $\mathcal{C}$ as in Theorem C, to pass to a class $\mathcal{C}'$ which is also analytic, does not contain universal members and satisfies the following. For every $X \in \mathcal{C}$ there exists $X' \in \mathcal{C}'$ such that X' has a Schauder basis and X is isomorphic to a subspace of X'. In this direction the following is open for us.

Problem E. *Assume that X has non-trivial type (respectively, cotype). Does there exist a Banach space Y with a Schauder basis (or even Schauder FDD) with asymptotic non-trivial type (respectively, cotype) such that X is isomorphic to a subspace of Y?*

An affirmative answer to this problem would yield that the classes $\mathcal{C}_{\mathrm{type}}$ and $\mathcal{C}_{\mathrm{cotype}}$ are not universal. (The definitions of asymptotic non-trivial type and cotype are given in Section 9, Definition 9.11.)

A second approach is related to a deep result due to Zippin [Z]. A consequence of it and the interpolation theorem [DFJP], is that every separable reflexive Banach space is contained in a reflexive Banach space with a Schauder basis. However, in order to apply this result, one needs to know that such a selection is done in a uniform way. Zippin's approach does not appear to be able to provide this selection and it seems necessary to further understand the relation between the initial and the final space.

Next we extend the concepts of Bourgain and Bossard genericity for every separable Banach space X. A deep theorem of Rosenthal [Ro3] yields that when X is a universal space, then the X-genericities introduced below are equivalent to the ones defined above. There are examples showing that the additional assumptions are necessary. One of them was indicated to us by Rosendal and Schlumprecht.

Definition F. *Let X be a separable Banach space and let $\mathcal{C}$ be an isomorphic invariant class of separable Banach spaces such that X is not contained in any finite direct sum of members of $\mathcal{C}$.*

(1) *We say that the class $\mathcal{C}$ is* Bourgain X-generic *if for every separable Banach space Y that contains all members of $\mathcal{C}$, X is isomorphic to a subspace of a finite direct sum of Y.*
(2) *We say that the class $\mathcal{C}$ is* Bossard X-generic *if for every analytic subset A that contains all members of $\mathcal{C}$ up to isomorphism, X is isomorphic to a subspace of a finite direct sum of members of A.*

As we mentioned above for X universal these definitions coincide with the previous ones and if X is a minimal separable Banach space (e.g., an ℓ_p space), then the above definitions can be reduced to the corresponding analogue of Definition A. The following problem extends Problem B.

Problem G. *Let X be a separable Banach space and let $\mathcal{C}$ be an isomorphic invariant class of separable Banach spaces such that X is not contained in any finite direct sum of members of $\mathcal{C}$. Is it true $\mathcal{C}$ is Bourgain X-generic if and only if it is Bossard X-generic?*

It is open for us if the analogue of Theorem C is valid for an arbitrary separable Banach space X. However, there are several classes of Banach spaces (for instance, if X is unconditionally saturated, or HI saturated, or minimal) where the following analogue is proved.

Theorem H. *Let X be either an unconditional saturated, or HI saturated, or minimal separable Banach space. Also let A be an analytic class of separable Banach spaces such that X is not contained in any finite sum of members of A. Then there exists a separable Banach space Y that contains all members of A with a Schauder basis and X is not contained in any finite sum of Y.*

As a consequence we obtain that the subspaces with a Schauder basis of a Banach space X do not necessarily define a Bourgain X-generic class.

Corollary I. *Let X be a HI separable Banach space without a Schauder basis. Then the class $\mathcal{C}$ of all subspaces of X with a Schauder basis is not Bourgain X-generic and, consequently, not Bossard X-generic either.*

The existence of separable HI Banach spaces without a Schauder basis (even without a Schauder FDD) follows from a result of Allexandrov, Kutzarova and Plichko [AKP].

Now we shall present two other applications of amalgamations of classes of separable Banach spaces. The first one concerns two problems due to Rosenthal stated in [AGR, Problems 1 and 2, page 1043]. Following Rosenthal's notation, let us denote by $X^{**}_{\mathcal{B}_1}$ the set of all $x^{**}\in X^{**}$ which are the weak* limit of a sequence (x_n) of X. This is equivalent to saying that $x^{**}\colon (B_{X^*},w^*)\to\mathbb{R}$ is a Baire-1 function (see [OR]). As it is well known, for every real-valued Baire-1 function f on a compact metrizable space K several indices (scaled on countable ordinals) have been defined measuring the discontinuities of f. We refer to [AGR, KL2] for a detailed exposition. Rosenthal's problems concern the indices $\beta(x^{**}|_K)$ and $r_{\mathrm{ND}}(x^{**}|_K)$ where X is a separable Banach space, $x^{**}\in X^{**}_{\mathcal{B}_1}$ and $K=(B_{X^*},w^*)$. Before we state the problems let us mention two known results related to these indices. The first one is due to Bourgain [B2] and asserts that if X is separable and $\sup\{\beta(x^{**}|_K): x^{**}\in X^{**}_{\mathcal{B}_1}\}=\omega_1$, then ℓ_1 is isomorphic to a subspace of X. The second result is the c_0-index theorem [AK] asserting that if X is separable and $\sup\{r_{\mathrm{ND}}(x^{**}|_K): x^{**}\in X^{**}_{\mathcal{B}_1}\}=\omega_1$, then c_0 embeds into X. In his problems Rosenthal expresses the belief that if the two indices β and r_{ND} are unbounded, then the structure of X must be richer than the above two results indicate. Bossard [Bo4] has also pointed out that the only known examples of separable Banach spaces with unbounded β are the universal ones. Rosenthal's problems state the following.

Problem. (1) *Assume that for every countable ordinal ξ there exists $x^{**}\in X^{**}_{\mathcal{B}_1}$ such that $\xi\leqslant r_{\mathrm{ND}}(x^{**}|_K)<\omega_1$. Is the space X universal?*

(2) *Assume that β is unbounded on $X^{**}_{\mathcal{B}_1}$. Does $L_1(0,1)$ embed into X?*

In Section 10, we present a separable Banach space R which answers negatively both problems.

Theorem J. *There exists a separable Banach space R such both β and r_{ND} are unbounded on $R^{**}_{\mathcal{B}_1}$ and neither $L_1(0,1)$ nor $C\big(\omega^{\omega^2}\big)$ embeds into R.*

As both indices are unbounded on R, clearly ℓ_1 and c_0 embed into R. Actually, we show that every subspace Y of R either contains a reflexive subspace, or ℓ_1, or c_0. The space R is obtained either as the HI-amalgamation, or p-amalgamation for $2<p<\infty$, of the class $\mathcal{C}=\{J_X : X \text{ has an 1-unconditional basis}\}$ where J_X denotes the Bellenot–Haydon–Odell Jamesfication of X [BHO].

The second application of the amalgamations technique concerns a separable Banach space $\mathcal{A}^{\ell_1}_{\mathrm{hi}}$ which is the HI-amalgamation of ℓ_1, and where the HI and ℓ_1 structures co-exist in the following manner.

Theorem K. *There is a separable Banach space $\mathcal{A}^{\ell_1}_{\mathrm{hi}}$ with the following properties.*

1. *A subspace of $\mathcal{A}^{\ell_1}_{\mathrm{hi}}$ is reflexive if and only if it is HI.*
2. *The class $\mathcal{C}=\{X : X$ is a reflexive subspace of $\mathcal{A}^{\ell_1}_{\mathrm{hi}}\}$ is ℓ_1-Bossard generic.*
3. *Every non-reflexive subspace of $\mathcal{A}^{\ell_1}_{\mathrm{hi}}$ contains a complemented copy of ℓ_1.*
4. *If $\mathcal{A}^{\ell_1}_{\mathrm{hi}}=Z\oplus W$, then either Z or W is isomorphic to a subspace of ℓ_1.*

1.2. The paper is organized as follows. The second section contains preliminary notations and definitions for trees. Let us point out that trees are the central combinatorial tool for both the descriptive set theoretic as well as the Banach space theoretic part of the present work.

The third section is of descriptive set theoretic nature. We deal with the classes of hereditarily indecomposable, and indecomposable separable Banach spaces, as well as with the class of spaces not containing an unconditional sequence. It turns out that all these classes are co-analytic non-Borel (actually, they are co-analytic complete). Moreover, we provide some natural co-analytic ranks on these sets. The notion of a co-analytic rank is due to Moschovakis. It is an ordinal index on a co-analytic set A which satisfies some further definability assumptions. One of the important properties of co-analytic ranks is that they satisfy boundedness. This means that the rank is uniformly bounded below ω_1 for every analytic subset of the set A. Of particular importance is the embedability index introduced by Bourgain [B1] and further studied by Bossard [Bo3]. It is defined on a separable Banach space X and gives a quantitative estimate of how much a separable Banach space Z with a Schauder basis (e_n) embeds into X. The definition of the rank depends on the choice of the basis (e_n). Using the parameterized version of Lusin's classical theorem we show that there exists a co-analytic rank which dominates the embedability rank of X for every choice of the Schauder basis of Z. Similar results hold if the Banach space Z does not necessarily have a Schauder basis.

Sections 4 and 5 are devoted to the ℓ_2 Baire sum of a Schauder tree basis $(x_t)_{t\in T}$. By a Schauder tree basis we mean a bounded sequence $(x_t)_{t\in T}$ indexed by a countable tree of height ω and satisfying the property that for every branch σ of T the sequence $\{x_t : t \sqsubset \sigma\}$ is a bimonotone basic sequence. The ℓ_2 Baire sum of $(x_t)_{t\in T}$, denoted by $\mathcal{T}_2^X$, is a new norm defined on $(x_t)_{t\in T}$ similar to norms considered by Bourgain [B1] and Bossard [Bo3]. It follows easily from the definition that, for every branch σ of T the initial norm and the new one are isometric on the space $X_\sigma = \overline{\mathrm{span}}\{x_t : t \sqsubset \sigma\}$. Furthermore, for every σ the space X_σ is an 1-complemented subspace of $\mathcal{T}_2^X$ by a natural projection P_σ. Our investigation is focused on the X-singular subspaces on $\mathcal{T}_2^X$, that is, on the subspaces Y of $\mathcal{T}_2^X$ for which the operator $P_\sigma \colon Y \to \mathcal{T}_2^X$ is strictly singular for every $\sigma \in [T]$. It is shown that every X-singular subspace does not contain ℓ_1. On the other hand for every $(x_t)_{t\in T}$ with T perfect, the space c_0 is isomorphic to an X-singular subspace of $\mathcal{T}_2^X$. Next, we consider the set $W_X = \overline{\mathrm{conv}}\big\{\bigcup_{\sigma\in[T]} B_{X_\sigma}\big\}$ where B_{X_σ} denotes the unit ball of X_σ. On the pair $(\mathcal{T}_2^X, W_X)$ we apply the Davis–Figiel–Johnson–Pelczynski p-interpolation method (for $1 < p < \infty$), or its variant, the HI interpolation which is presented in this paper and it is a modification of the corresponding one in [AF]. The resulting spaces $\mathcal{A}_p^X$ and, respectively, $\mathcal{A}_{\mathrm{hi}}^X$ are the amalgamation spaces mentioned before. There is a key property of the set W_X, permitting us to establish the properties of the amalgamation of a class $\mathcal{C}$, which is related to the notion of thin sets (see [N1, AF]).

Theorem L. *For every X-singular subspace Y of $\mathcal{T}_2^X$ the set W_X is thin on Y.*

Theorem L requires several steps and uses some techniques from [AF].

Sections 6 and 7 are devoted to a brief presentation of the HI interpolation mentioned above. In Section 8, we establish the properties of $\mathcal{A}_{\mathrm{hi}}^X$ and we provide some applications. Sections 9 and 10 include the proofs of the results mentioned in the first part of the introduction. In an appendix, we have included the study of the structure of the dual of $\mathcal{T}_2^X$. More precisely, we show that $(\mathcal{T}_2^X)^* = \overline{\mathrm{span}}\{\bigcup_{\sigma\in[T]} X_\sigma^*\}$.

We also define the concept of a strongly bounded class of separable Banach spaces and we provide some examples of such classes. This notion is a strengthening of the classical property of boundedness of co-analytic ranks. Kechris had also asked for the existence of non-trivial strongly bounded classes of Banach spaces.

Definition M. *Let $\mathcal{C}$ be an isomorphic invariant class of separable Banach spaces. We say that $\mathcal{C}$ is* strongly bounded *if for every analytic subset A of $\mathcal{C}$ there exists $Y \in \mathcal{C}$ that contains all members of A up to isomorphism.*

Under the terminology of the above definition, Theorem C states that the class of non-universal separable Banach spaces with a Schauder basis is strongly bounded. The following theorem provides other examples of strongly bounded classes.

Theorem N. *Let $\mathcal{C}$ denote one of the following classes of Banach spaces.*

(1) *The reflexive spaces with a Schauder basis.*
(2) *The spaces with a shrinking Schauder basis.*
(3) *The ℓ_p-saturated for some $1 \leqslant p < \infty$, or c_0-saturated spaces with a Schauder basis.*
(4) *The unconditionally saturated spaces with a Schauder basis.*
(5) *The HI saturated spaces with a Schauder basis.*

Then $\mathcal{C}$ is strongly bounded.

We close this introduction by pointing out that all the results related to classes $\mathcal{C}$ of separable Banach spaces with a Schauder basis, remain valid for the wider class of spaces with a Schauder FDD.

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Addendum. Recently, Valentin Ferenczi and the second named author have shown that the classes of separable reflexive spaces and of spaces with separable dual are strongly bounded.

2. Trees

2.1. Let $\mathbb{N} = \{1, 2, \dots\}$ denote the set of positive integers. By $[\mathbb{N}]^\infty$ we denote the set of all infinite subsets on $\mathbb{N}$. If $L \in [\mathbb{N}]^\infty$, then by $[L]^\infty$ we denote the set of all infinite subsets of L. As it is common in Ramsey theory, for every $L \in [\mathbb{N}]^\infty$ by $[L]^2$ we denote the set of all pairs (i, j) such that $i, j \in L$ and $i < j$.

2.2. Let Λ be a countable set. By $\Lambda^{<\mathbb{N}}$ we denote the set of all *nonempty* finite sequences of Λ (we do not include the empty sequence for purely technical reasons). We view $\Lambda^{<\mathbb{N}}$ as a tree under the strict partial order $\sqsubset$ of extension. Notice that $\Lambda^{<\mathbb{N}}$ has infinitely many roots.

2.2. We use the letter t to denote the nodes of $\Lambda^{<\mathbb{N}}$.

2.3. If $t_1, t_2 \in \Lambda^{<\mathbb{N}}$ with $t_1 \sqsubseteq t_2$, then the set $\{t : t_1 \sqsubseteq t \sqsubseteq t_2\}$ is called a *segment* of $\Lambda^{<\mathbb{N}}$ (in particular, nodes are segments). The sets of the form $\{t' : t' \sqsubseteq t\}$ are called *initial segments* while the sets of the form $\{t' : t \sqsubseteq t'\}$ *final segments*. All segments will be denoted by s.

2.4. If $t \in \Lambda^{<\mathbb{N}}$, then the *length* of t is defined to be the cardinality of the set $\{t' : t' \sqsubseteq t\}$. It is denoted by $|t|$. Observe that if $t = (l_1, l_2, \dots, l_k)$, then $|t| = k$. If $n \in \mathbb{N}$, then the *n-level* of $\Lambda^{<\mathbb{N}}$ is defined to be the set $\{t : |t| = n\}$.

2.5. We identify the branches of $(\Lambda^{<\mathbb{N}}, \sqsubset)$ with the elements of the space $\Lambda^{\mathbb{N}}$. If we equip Λ with the discrete topology, then $\Lambda^{\mathbb{N}}$ is homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$, denoted by $\mathcal{N}$. For every $\sigma \in \Lambda^{\mathbb{N}}$ and every $n \in \mathbb{N}$ we set $\sigma|n = \big(\sigma(1), \dots, \sigma(n)\big)$. Notice that $\big|\sigma|n\big| = n$ for every $n \in \mathbb{N}$ and every $\sigma \in \Lambda^{\mathbb{N}}$.

2.6. Two nodes $t_1, t_2 \in \Lambda^{<\mathbb{N}}$ are called *comparable* if either $t_1 \sqsubseteq t_2$ or $t_2 \sqsubseteq t_1$. More generally, if $A_1, A_2 \subseteq \Lambda^{<\mathbb{N}}$, then A_1 and A_2 are called *comparable* if there exist $t_1 \in A_1$ and $t_2 \in A_2$ with t_1, t_2 comparable; otherwise, they are called *incomparable*. Notice that if A_1 and A_2 are incomparable, then they are disjoint.

2.7. If $t \in \Lambda^{<\mathbb{N}}$, then by $\mathcal{L}_t$ we denote the set of all segments s of $\Lambda^{<\mathbb{N}}$ for which there exists $t' \in s$ with $t \sqsubseteq t'$. Observe that the family $\{\mathcal{L}_t : t \in \Lambda^{<\mathbb{N}}\}$ restricted to the branches of $\Lambda^{<\mathbb{N}}$ forms the usual sub-basis of the topology of $\Lambda^{\mathbb{N}}$.

2.8. If s is a segment of $\Lambda^{<\mathbb{N}}$ and $A \subseteq \Lambda^{\mathbb{N}}$, then we write $s \cap A = \emptyset$ to denote the fact that the sets s and $\{t : \exists \sigma \in A \text{ with } t \sqsubset \sigma\}$ are disjoint. More generally, if s is a segment of $\Lambda^{<\mathbb{N}}$ and $A \subseteq \Lambda^{<\mathbb{N}}$, then we write $s \cap A = \emptyset$ to denote the fact that the sets s and $\{t : t \in A\}$ are disjoint.

2.9. Let A be a subset of $\Lambda^{<\mathbb{N}}$. We say that A is *segment complete* if for every $t_1, t_2, t_3 \in \Lambda^{<\mathbb{N}}$ with $t_1 \sqsubseteq t_2 \sqsubseteq t_3$ and $t_1, t_3 \in A$, we have that $t_2 \in A$.

2.10. By $\mathrm{Tr}(\Lambda)$ we denote the set of all *downward closed* subtrees of $\Lambda^{<\mathbb{N}}$, that is,

$$T \in \mathrm{Tr}(\Lambda) \Leftrightarrow \forall t, t' \in \Lambda^{<\mathbb{N}} \ (t' \sqsubseteq t \text{ and } t \in T \Rightarrow t' \in T).$$

(By convention, the empty set is a tree.) By identifying every $T \in \mathrm{Tr}(\Lambda)$ with its characteristic function, we see that $\mathrm{Tr}(\Lambda)$ is a closed subspace of $2^{\Lambda^{<\mathbb{N}}}$. A tree $T \in \mathrm{Tr}(\Lambda)$ is said to be *well-founded* if for every $\sigma \in \Lambda^{\mathbb{N}}$ there exists $n \in \mathbb{N}$ such that $\sigma|n \notin T$. The set of all well-founded trees is denoted by $\mathrm{WF}(\Lambda)$. A tree $T \in \mathrm{Tr}(\Lambda) \setminus \mathrm{WF}(\Lambda)$ is called *ill-founded*. The set of all ill-founded trees is denoted by $\mathrm{IF}(\Lambda)$. If $\Lambda = \mathbb{N}$, then the set of trees on $\mathbb{N}$ is simply denoted by Tr.

2.11. For every $T \in \mathrm{WF}(\Lambda)$ we set $T' := \{t \in T : \exists t' \in T \text{ with } t \sqsubset t'\}$. By transfinite recursion, for every $\xi < \omega_1$ we define $T^{(0)} = T$, $T^{(\xi+1)} = \big(T^{(\xi)}\big)'$ and $T^{(\zeta)} = \bigcap_{\xi<\zeta} T^{(\xi)}$ if ζ is a limit ordinal. The *order* of T is defined to be the least ordinal ξ such that $T^{(\xi)} = \emptyset$. It is denoted by $o(T)$.

2.12. A (downward closed) subtree T of $\Lambda^{<\mathbb{N}}$ is said to be *pruned* if for every $t \in T$ there exists $t' \in T$ such that $t \sqsubset t'$. Given a pruned tree T one defines the *body* $[T]$ of T to be the set

$$[T] := \{\sigma \in \Lambda^{\mathbb{N}} : \sigma|n \in T \text{ for every } n \in \mathbb{N}\}.$$

Notice that $[T]$ is a closed subset of $\Lambda^{\mathbb{N}}$. Actually the pruned subtrees of $\Lambda^{<\mathbb{N}}$ are in one-to-one correspondence with the closed subsets of $\Lambda^{\mathbb{N}}$ via the bijection $T \mapsto [T]$ (see [K, page 7]). There is a canonical way to assign to every tree T its pruned part T_{pr}. This is done using the derivative operation $T \mapsto T'$ defined above. Specifically, for every $T \in \mathrm{Tr}(\Lambda)$ set $T' := \{t \in T : \exists t' \in T \text{ with } t \sqsubset t'\}$, and notice that T is pruned if and only if $T' = T$. By transfinite recursion, we define the iterated derivatives $T^{(\xi)}$ of T for every $\xi < \omega_1$. Finally, we set $T_{\mathrm{pr}} = T^{(\infty)}$. Observe that $T \in \mathrm{WF}(\Lambda)$ if and only if $T_{\mathrm{pr}} = \emptyset$.

3. Complexity and ranks

We shall briefly review some basic concepts of descriptive set theory.

3.1. **Standard Borel spaces.** Let (X, Σ) be a measurable space. Then (X, Σ) is said to be a *standard Borel space* if there exists a Polish topology τ on X such that $\Sigma = B(X, \tau)$, that is, the Borel σ-algebra of (X, τ) coincides with Σ. Using the classical fact that for every Borel subset B of a Polish space X, there exists a finer Polish topology on X (with the same Borel sets) making B clopen (see [K, Theorem 13.1]) we see that if (X, Σ) is a standard Borel space and $B \in \Sigma$, then B equipped with the relative σ-algebra is a standard Borel space too.

An important example of a standard Borel space is the *Effros–Borel structure*. Let X be a Polish space and denote by $F(X)$ the set of all closed subsets of X. We endow $F(X)$ with the σ-algebra generated by the sets $\{F \in F(X) : F \cap U \neq \emptyset\}$ where U ranges over all open subsets of X. The space $F(X)$ equipped with this

σ-algebra is called the Effros-Borel space of $F(X)$. The basic fact is the following (see [K, Theorem 12.6]).

Theorem 3.1. *If X is Polish, then the Effros–Borel space of $F(X)$ is standard.*

The fact that a standard Borel space is just the Borel σ-algebra of a Polish space, allows us to speak about analytic, co-analytic and projective, in general, subsets of a standard Borel space. We will use the modern logical notation to denote these classes. Hence, $\mathbf{\Sigma}^1_1$ stands for the analytic sets while $\mathbf{\Pi}^1_1$ stands for the co-analytic ones. For more information we refer to [K].

3.2. **The standard Borel space of separable Banach spaces.** Let X be a separable Banach space, and set

$$\mathrm{Subs}(X) \coloneqq \{F \in F(X) : F \text{ is a closed linear subspace of } X\}.$$

Then $\mathrm{Subs}(X)$ is a Borel set in $F(X)$ (see [K, page 79]) and so a standard Borel space on its own right. If $X = C(2^{\mathbb{N}})$, then the space $\mathrm{Subs}\big(C(2^{\mathbb{N}})\big)$ is the standard Borel space of all separable Banach spaces and we denote it simply by SB. We recall some basic properties of SB.

3.2.1. If $X \in \mathrm{SB}$, then $\mathrm{Subs}(X)$ is a Borel subset of SB (see [K, page 76]).

3.2.2. The set of all infinite-dimensional separable Banach spaces is a Borel subset of SB. More generally, this holds for the infinite-dimensional subspaces of a fixed infinite-dimensional $X \in \mathrm{SB}$ (see [K, page 79]).

3.2.3. The relation $\{(Y, X) : Y \text{ is a closed subspace of } X\}$ is Borel in $\mathrm{SB} \times \mathrm{SB}$ (see [K, page 76]).

3.2.4. A simple application of the Kuratowski–Ryll-Nardzewski selection theorem (see [K, page 76]) yields that there exists a sequence $d_n \colon \mathrm{SB} \to C(2^{\mathbb{N}})$ $(n \in \mathbb{N})$ of Borel functions such that $\overline{\{d_n(X) : n \in \mathbb{N}\}} = X$ for every $X \in \mathrm{SB}$. As these functions can be chosen so that $d_n(X) \neq 0$ for every $n \in \mathbb{N}$ and every $X \in \mathrm{SB}$, this shows that there also exists a sequence $S_n \colon \mathrm{SB} \to C(2^{\mathbb{N}})$ $(n \in \mathbb{N})$ of Borel functions such that $\overline{\{S_n(X) : n \in \mathbb{N}\}} = S_X$ for every $X \in \mathrm{SB}$. Clearly all these facts can be relativized to $\mathrm{Subs}(X)$ for any $X \in \mathrm{SB}$.

3.2.5. The equivalence relation $\cong$ of isomorphism is analytic, that is, the set

$$\{(X, Y) : X \text{ is linearly isomorphic to } Y\}$$

is $\mathbf{\Sigma}^1_1$ in $\mathrm{SB} \times \mathrm{SB}$ (see [Bo3, page 127]).

As we are mainly interested in infinite-dimensional Banach spaces we will follow the convention that SB consists of all infinite-dimensional separable Banach spaces (the same also holds for $\mathrm{Subs}(X)$ of any infinite-dimensional $X \in \mathrm{SB}$). This causes no problems since, by 3.2.2, these are standard Borel spaces.

3.3. **The method of completeness.** Let X, Y be standard Borel spaces, $A \subseteq X$ and $B \subseteq Y$. We say that A is *reducible* to B, in symbols $A \leqslant B$, if there exists a Borel map $f\colon X \to Y$ such that

$$x \in A \Leftrightarrow f(x) \in B.$$

Notice that if $A \leqslant B$ and $B \leqslant C$, then $A \leqslant C$. Also observe that if $A \leqslant B$, then $X \setminus A \leqslant Y \setminus B$. Now let Γ be any class of sets in Polish spaces (such as $\mathbf{\Sigma}^1_1$, or $\mathbf{\Pi}^1_1$) and let $\check{\Gamma}$ be its dual class, that is, $\check{\Gamma} = \{A : X \setminus A \in \Gamma\}$.

Definition 3.2. *A subset B of a standard Borel space X is said to be* Γ-hard *if for any standard Borel space Y and any $A \subseteq Y$ which is in $\Gamma(Y)$ we have that A is reducible to B. If, in addition, B is in $\Gamma(X)$, then B is said to be* Γ-complete.

Notice that if Γ is closed under pre-images of Borel maps and not self-dual, that is, $\Gamma \neq \check{\Gamma}$, then no Γ-hard set is in $\check{\Gamma}$. In particular, any $\mathbf{\Pi}^1_1$-complete set is not analytic (whether the converse is true is one of the most fascinating questions in descriptive set theory). This gives us a method (which goes back to the beginnings of descriptive set theory) of proving that a subset B of a standard Borel space is not analytic. Select an already known $\mathbf{\Pi}^1_1$-complete set and show that it is reducible to B. A basic example of a $\mathbf{\Pi}^1_1$-complete set is the set WF of all well-founded trees on $\mathbb{N}$. In particular, we have the following theorem (see [K, page 243]).

Theorem 3.3. *The set* WF *is $\mathbf{\Pi}^1_1$-complete.*

Clearly, the above theorem yields that the set IF is $\mathbf{\Sigma}^1_1$-complete.

3.4. **Co-analytic ranks.** Let X be standard Borel space and let $A \subseteq X$ be a $\mathbf{\Pi}^1_1$ set. A map $\phi\colon A \to Ord$ is said to be a $\mathbf{\Pi}^1_1$-*rank* on A if there are relations $\leqslant_\Sigma, \leqslant_\Pi \subseteq X \times X$ in $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ respectively, such that for any $y \in A$ we have

$$\phi(x) \leqslant \phi(y) \Leftrightarrow x \leqslant_\Sigma y \Leftrightarrow x \leqslant_\Pi y.$$

The notion of a $\mathbf{\Pi}^1_1$-rank is due to Moschovakis (although its present form is due to Kechris). It is a fundamental fact of the structural theory of $\mathbf{\Pi}^1_1$ sets that every $\mathbf{\Pi}^1_1$ set admits a $\mathbf{\Pi}^1_1$-rank (see [Mo, K]). For our purposes the most important property of a $\mathbf{\Pi}^1_1$-rank ϕ is that it must satisfy *boundedness*. That is, if $\phi\colon A \to \omega_1$ is a $\mathbf{\Pi}^1_1$-rank on A and $B \subseteq A$ is $\mathbf{\Sigma}^1_1$, then we have (see [K, Theorem 35.23])

$$\sup\{\phi(x) : x \in B\} < \omega_1.$$

On the $\mathbf{\Pi}^1_1$-complete set WF, a canonical $\mathbf{\Pi}^1_1$-rank is the map which assigns to every well-founded tree T its order $o(T)$, which is of course a countable ordinal (see [K, page 269]).

Notice that if X and Y are standard Borel spaces, $A \subseteq X$ is reducible to $B \subseteq Y$ via a Borel map f and ϕ is a $\mathbf{\Pi}^1_1$-rank on B, then the map $\psi\colon A \to Ord$ defined by $\psi(x) = \phi\big(f(x)\big)$ for every $x \in A$, is a $\mathbf{\Pi}^1_1$-rank on A. This provides us a canonical way for producing natural $\mathbf{\Pi}^1_1$-ranks on $\mathbf{\Pi}^1_1$ sets. Simply find a natural reduction of the set in question to WF (which of course a priori exists, but may be artificial in some sense), and then assign to every point in our set the order of the well-founded tree to which the point is reduced. For more on $\mathbf{\Pi}^1_1$-ranks as well as applications of rank theory in analysis we refer to [K, KL1, KW1].

3.5. **Classes of separable Banach spaces.** In this subsection we will treat some classes of separable Banach spaces. We will give an upper bound for their complexity and provide natural ranks on them.

3.5.1. *Hereditarily indecomposable spaces.* Let HI be the set of all separable hereditarily indecomposable Banach spaces. Notice that

$$\begin{aligned} X \in \mathrm{HI} \quad &\Leftrightarrow \quad \forall Y, Z \hookrightarrow X \ d(S_Y, S_Z) = 0 \\ &\Leftrightarrow \quad \forall Y, Z \hookrightarrow X \ \forall k \ \exists n, m \text{ such that } \|S_n(Y) - S_m(Z)\| \leqslant \frac{1}{k}. \end{aligned}$$

(Here, (S_n) stands for the sequence of Borel functions defined in 3.2.4.) This shows that HI is $\mathbf{\Pi}^1_1$. For the convenience of the reader not familiar with descriptive set-theoretic calculations we will briefly describe a more detailed argument. Indeed, let $A_1 = \bigcup_{k\in\mathbb{N}} \bigcap_{n,m\in\mathbb{N}} \{(Y, Z) : \|S_n(Y) - S_m(Z)\| \geqslant \frac{1}{k}\}$. Since for every $n \in \mathbb{N}$ the function $S_n \colon \mathrm{SB} \to C(2^{\mathbb{N}})$ is Borel, we see that A_1 is Borel in $\mathrm{SB} \times \mathrm{SB}$. Now set $A = \{(X, Y, Z) : Y, Z \hookrightarrow X\} \cap (\mathrm{SB} \times A_1)$. The relation $\{(X, Y) : Y \hookrightarrow X\}$ is Borel in $\mathrm{SB} \times \mathrm{SB}$. Therefore, the set A is Borel in SB^3. Moreover,

$$X \notin \mathrm{HI} \Leftrightarrow X \in \mathrm{proj}_{\mathrm{SB}} A$$

where $\mathrm{proj}_{\mathrm{SB}}$ denotes the projection in the first coordinate. This implies that $\mathrm{SB} \setminus \mathrm{HI}$ is analytic, as desired.

We proceed to find a reduction of HI to the set of all well-founded trees on $\mathbb{N} \times \mathbb{N}$. To this end let $\mathrm{Tr}(\mathbb{N} \times \mathbb{N})$ be the set of all downward closed trees on $\mathbb{N} \times \mathbb{N}$. We identify every $T \in \mathrm{Tr}(\mathbb{N} \times \mathbb{N})$ with the set of all pairs $(t_1, t_2) \in \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{<\mathbb{N}}$ with $|t_1| = |t_2| = l$ and such that

$$\Big(\big(t_1(1), t_2(1)\big), \dots, \big(t_1(l), t_2(l)\big) \Big) \in T.$$

Before we describe the reduction let us introduce some terminology. For every $t \in \mathbb{N}^{<\mathbb{N}}$ with $t = (n_1, \dots, n_l)$ and every $X \in \mathrm{SB}$ set

$$X_t \coloneqq \overline{\mathrm{span}} \big\{ d_{n_1}(X), \dots, d_{n_l}(X) \big\}.$$

Observe that X_t is a finite-dimensional subspace of X. Moreover, notice that the vectors $d_{n_1}(X), \dots, d_{n_l}(X)$ are linearly independent if and only if $\dim X_t = l = |t|$. We say that t is X-*independent* if $\dim X_t = |t|$. Note that, for every $t \in \mathbb{N}^{<\mathbb{N}}$ the set $I_t = \{X \in \mathrm{SB} : t \text{ is } X\text{-independent}\}$ is Borel. To see this observe that

$$\begin{aligned} X \notin I_t \quad \Leftrightarrow \quad &\exists a_1, \dots, a_l \in \mathbb{R} \ \exists i \in \{1, \dots, l\} \text{ such that } a_i \neq 0 \text{ and} \\ &a_1 d_{n_1}(X) + \dots + a_l d_{n_l}(X) = 0. \end{aligned}$$

Now let $X \in \mathrm{SB}$ and $k \in \mathbb{N}$. We introduce a tree on $\mathbb{N}\times\mathbb{N}$, denoted by $T_{\mathrm{HI}}(X,k)$, as follows. We set

$$(t_1,t_2) \in T \Leftrightarrow |t_1| = |t_2|,\ t_1,t_2 \text{ are } X\text{-independent and } d(S_{X_{t_1}}, S_{X_{t_2}}) \geqslant \frac{1}{k}.$$

Next, we "glue" the trees $T_{\mathrm{HI}}(X,k)$ in a natural way to build a tree $T_{\mathrm{HI}}(X)$ on $\mathbb{N}\times\mathbb{N}$ defined by the rule

$$\begin{aligned} (t_1,t_2) \in T_{\mathrm{HI}}(X) \quad \Leftrightarrow \quad & \exists k \in \mathbb{N} \text{ such that } t_1(1) = t_2(1) = k \text{ and} \\ & \text{either } |t_1| = |t_2| = 1 \text{ or} \\ & t_1 = k^\frown t_1',\ t_2 = k^\frown t_2' \text{ and } (t_1',t_2') \in T_{\mathrm{HI}}(X,k). \end{aligned}$$

Clearly, the tree $T_{\mathrm{HI}}(X)$ describes all our attempts to produce a decomposable subspace of X. Moreover, we have the following lemma.

Lemma 3.4. *The map* $\mathrm{SB} \ni X \mapsto T_{\mathrm{HI}}(X) \in \mathrm{Tr}(\mathbb{N}\times\mathbb{N})$ *is a Borel reduction of* HI *to* $\mathrm{WF}(\mathbb{N}\times\mathbb{N})$.

Proof. First we check the Borel measurability of the map. Fix $(t_1,t_2) \in \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{<\mathbb{N}}$ with $|t_1| = |t_2|$. Using the Borel measurability of the functions (S_n), for every $k \in \mathbb{N}$ it is easy to see that the set $\{X : d(S_{X_{t_1}}, S_{X_{t_2}}) \geqslant \frac{1}{k}\}$ is Borel. Moreover, by the discussion before the lemma, the set $\{X : t_1 \text{ and } t_2 \text{ are } X\text{-independent}\}$ is Borel too. It follows that for any $(t_1,t_2) \in \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{<\mathbb{N}}$ with $|t_1| = |t_2|$ the pre-image of the set $\{T \in \mathrm{Tr}(\mathbb{N}\times\mathbb{N}) : (t_1,t_2) \in T\}$ is Borel in SB. As this family forms a sub-basis of the topology of $\mathrm{Tr}(\mathbb{N}\times\mathbb{N})$, the Borel measurability is clear.

Now we claim that

$$X \in \mathrm{HI} \Leftrightarrow \forall k\ T_{\mathrm{HI}}(X,k) \in \mathrm{WF}(\mathbb{N}\times\mathbb{N}) \Leftrightarrow T_{\mathrm{HI}}(X) \in \mathrm{WF}(\mathbb{N}\times\mathbb{N}).$$

To see this notice that if $X \notin \mathrm{HI}$, then a standard perturbation argument yields that there exists a $k \in \mathbb{N}$ such that $T_{\mathrm{HI}}(X,k)$ is not well-founded. Clearly, in this case $T_{\mathrm{HI}}(X)$ is not well-founded either. Conversely, if $T_{\mathrm{HI}}(X)$ is not well-founded, then there exists a $k \in \mathbb{N}$ such that $T_{\mathrm{HI}}(X,k)$ is not well-founded either (actually, this will also be the case for every $m \geqslant k$). The definition of $T_{\mathrm{HI}}(X,k)$ easily yields the existence of a decomposable subspace of X (here we made crucial use of the fact that the nodes of $T_{\mathrm{HI}}(X,k)$ correspond to linearly independent vectors). The proof is completed. □

By Lemma 3.4, we see that the Borel map $X \mapsto T_{\mathrm{HI}}(X)$ is a reduction of HI to $\mathrm{WF}(\mathbb{N}\times\mathbb{N})$. Since the map $T \mapsto o(T)$ is a $\mathbf{\Pi}^1_1$-rank on $\mathrm{WF}(\mathbb{N}\times\mathbb{N})$, it follows that the map $X \mapsto o\big(T_{\mathrm{HI}}(X)\big)$ is a $\mathbf{\Pi}^1_1$-rank on HI. We will see, later on, that HI is $\mathbf{\Pi}^1_1$-complete, and so this rank is unbounded on HI.

3.5.2. *Spaces with no unconditional sequence.* Let NUC be the set of all separable Banach spaces with no unconditional sequence. We will show that NUC is $\mathbf{\Pi}^1_1$. Actually instead of calculating the complexity of NUC we will find a reduction of

NUC to WF. Not only this will show that NUC is $\mathbf{\Pi}^1_1$, but also, as in the case of HI spaces, it will provide a natural $\mathbf{\Pi}^1_1$-rank on NUC.

For every $k \in \mathbb{N}$ let $T_{\mathrm{NUC}}(X,k)$ be the tree on $\mathbb{N}$ defined by the rule

$$t = (n_1, \dots, n_l) \in T_{\mathrm{NUC}}(X,k) \quad \Leftrightarrow \quad \begin{array}{l} \text{the sequence } d_{n_1}(X), \dots, d_{n_l}(X) \\ \text{is } k\text{-unconditional} \end{array}$$

where, as usual, a finite sequence $(x_i)_{i=1}^l$ is said to be *k-unconditional* if for every $a_1, \dots, a_l \in \mathbb{R}$ and $F \subseteq \{1, \dots, l\}$ we have $\|\sum_{i\in F} a_i x_i\| \leqslant k\|\sum_{i=1}^l a_i x_i\|$. This tree has been considered by Tomczak-Jaegermann in [TJ] (see also [BL, page 337]). As in the previous paragraph, we "glue" the trees $T_{\mathrm{NUC}}(X,k)$ and we produce a tree $T_{\mathrm{NUC}}(X)$. It is easy to check that the map $\mathrm{SB} \ni X \mapsto T_{\mathrm{NUC}}(X) \in \mathrm{Tr}$ is Borel and, moreover,

$$X \in \mathrm{NUC} \Leftrightarrow \forall k \; T_{\mathrm{NUC}}(X,k) \in \mathrm{WF} \Leftrightarrow T_{\mathrm{NUC}}(X) \in \mathrm{WF}.$$

This is the desired reduction.

3.5.3. *Indecomposable spaces.* Let I be the set of all separable indecomposable Banach spaces. We claim that it is $\mathbf{\Pi}^1_1$. Indeed,

$$\begin{aligned} X \in \mathrm{I} \quad &\Leftrightarrow \quad \forall Y, Z \hookrightarrow X \; (Y + Z \text{ is dense in } X \Rightarrow d(S_Y, S_Z) = 0) \\ &\Leftrightarrow \quad \forall Y, Z \hookrightarrow X \; \Big[\big(\forall n \; \forall i \; \exists m, k \text{ with } \|d_n(X) - d_m(Y) - d_k(Z)\| \leqslant \frac{1}{i}\big) \Rightarrow \\ &\qquad \big(\forall l \; \exists m', k' \text{ with } \|S_{m'}(Y) - S_{k'}(Z)\| \leqslant \frac{1}{l}\big)\Big]. \end{aligned}$$

Counting quantifiers and using the Borel measurability of the functions involved in the above expression we see that the class of indecomposable spaces is $\mathbf{\Pi}^1_1$. Using similar ideas as in the case of HI spaces, one may construct a $\mathbf{\Pi}^1_1$-rank on I (although in this case the construction is more involved). Instead of describing such a construction, we will take the opportunity to propose a natural $\mathbf{\Pi}^1_1$-rank on the set of all separable reflexive Banach spaces.

3.5.4. *Reflexive spaces.* Let REFL be the set of all separable reflexive Banach spaces. Bossard has shown in [Bo3] that the set REFL is $\mathbf{\Pi}^1_1$-complete. We will give a natural $\mathbf{\Pi}^1_1$-rank on REFL by finding a reduction of it to WF. To this end, for every $X \in \mathrm{SB}$ and every $k, n \in \mathbb{N}$ we define a tree $\mathbf{T}_{\mathrm{REFL}}(X,k,n)$ on X by

$$(x_i)_{i=1}^l \in \mathbf{T}_{\mathrm{REFL}}(X,k,n) \quad \Leftrightarrow \quad \begin{array}{l} \text{the finite sequence } (x_i)_{i=1}^l \text{ is } k\text{-Schauder, and} \\ \forall a_1, \dots, a_l \in \mathbb{R}^+ \text{ with } \displaystyle\sum_{i=1}^l a_i = 1 \;\text{ we have} \\ \displaystyle\|\sum_{i=1}^l a_i x_i\| \geqslant \frac{1}{n} \end{array}$$

where the finite sequence $(x_i)_{i=1}^l$ is said to be *k-Schauder* if for every $b_1, \dots, b_l \in \mathbb{R}$ and every $1 \leqslant m_1 \leqslant m_2 \leqslant l$ we have $\|\sum_{i=1}^{m_1} b_i x_i\| \leqslant k\|\sum_{i=1}^{m_2} b_i x_i\|$. Clearly, the tree $\mathbf{T}_{\mathrm{REFL}}(X,k,n)$ describes all our attempts to build a basic sequence in X with no weakly null subsequence. We have the following lemma.

Lemma 3.5. *Let $X \in \mathrm{SB}$. Then $X \in \mathrm{REFL}$ if and only if for every $k,n \in \mathbb{N}$ the tree $\mathbf{T}_{\mathrm{REFL}}(X,k,n)$ is well-founded.*

Proof. First assume that there exist $k,n \in \mathbb{N}$ such that the tree $\mathbf{T}_{\mathrm{REFL}}(X,k,n)$ is not well-founded. Let (x_i) be an infinite branch of $\mathbf{T}_{\mathrm{REFL}}(X,k,n)$. By Rosenthal's theorem [Ro1], there exists $L \in [\mathbb{N}]^\infty$ such that either the sequence $(x_i)_{i\in L}$ is equivalent to the ℓ_1 basis, or the sequence $(x_i)_{i\in L}$ is weakly Cauchy. In the first case, we immediately obtain that X is not reflexive. In the second case, we distinguish the following subcases. Either the sequence $(x_i)_{i\in L}$ is weakly convergent, or there exists $x^{**} \in X^{**} \setminus X$ such that $w^* - \lim_{i\in L} x_i = x^{**}$. Clearly, the second subcase implies that X is not reflexive. So we only have to deal with the case when $(x_i)_{i\in L}$ is weakly convergent. By the definition of the tree $\mathbf{T}_{\mathrm{REFL}}(X,k,n)$, we see that $(x_i)_{i\in L}$ is a basic sequence. Hence, $(x_i)_{i\in L}$ must be weakly null. By Mazur's theorem, there exist a finite convex combination z of $(x_i)_{i\in L}$ such that $\|z\| < \frac{1}{n}$. But this is clearly impossible by the definition of the tree. Hence, in any case we have that X is not reflexive.

Now assume that X is not reflexive. We must show that there exist $k,n \in \mathbb{N}$ such that $\mathbf{T}_{\mathrm{REFL}}(X,k,n)$ is not well-founded. If ℓ_1 embeds into X, then this is clearly possible. If not, then there exist $x^{**} \in X^{**} \setminus X$ with $\|x^{**}\| = 1$ and a sequence (x_i) such that $w^* - \lim x_i = x^{**}$. We select $x^* \in X^*$ with $\|x^*\| \leqslant 1$ such that $x^*(x_i) \geqslant \frac{1}{2}$ for every $i \in \mathbb{N}$. There exists $L \in [\mathbb{N}]^\infty$ such that the sequence $(x_i)_{i\in L}$ is basic with basis constant, say, $k \in \mathbb{N}$ (see [D, page 41]). Let $\{l_1 < l_2 < \cdots\}$ denote the increasing enumeration of L. Since $x^*(x_l) \geqslant \frac{1}{2}$ for every $l \in L$, we see that $(x_{l_i})_{i=1}^m \in \mathbf{T}_{\mathrm{REFL}}(X,k,2)$ for every $m \in \mathbb{N}$. The proof is completed. $\square$

Now we consider the following tree $T_{\mathrm{REFL}}(X,k,n)$ on $\mathbb{N}$ defined by

$$t \in T_{\mathrm{REFL}}(X,k,n) \Leftrightarrow t = (n_1,\dots,n_l) \text{ and } d_{n_1}(X),\dots,d_{n_l}(X) \in \mathbf{T}_{\mathrm{REFL}}(X,k,n).$$

The tree $T_{\mathrm{REFL}}(X,k,n)$ corresponds to a subtree of $\mathbf{T}_{\mathrm{REFL}}(X,k,n)$. Moreover, by Lemma 3.5, and a standard perturbation argument we obtain that

$$X \in \mathrm{REFL} \Leftrightarrow \forall k,n \ \mathbf{T}_{\mathrm{REFL}}(X,k,n) \text{ is well-founded} \Leftrightarrow \forall k,n \ T_{\mathrm{REFL}}(X,k,n) \in \mathrm{WF}.$$

By "gluing" the trees $T_{\mathrm{REFL}}(X,k,n)$ in the obvious way, we construct a Borel map $\mathrm{SB} \ni X \mapsto T_{\mathrm{REFL}}(X) \in \mathrm{Tr}$ such that

$$X \in \mathrm{REFL} \Leftrightarrow T_{\mathrm{REFL}}(X) \in \mathrm{WF}.$$

This is the desired reduction.

3.5.5. *Spaces with non-trivial type, or non-trivial cotype.* All the classes presented so far, as well as the classes presented in [Bo3], are actually $\mathbf{\Pi}^1_1$-complete. The following classes are important families of separable Banach spaces which are of low complexity.

Let $1 < p \leqslant 2$ and $2 \leqslant q < \infty$. Let Type(p) and Cotype(q) be the sets of all separable Banach spaces with type p and co-type q respectively. Both Type(p) and Cotype(q) are Borel in SB. To see this observe that

$$\begin{aligned} X \in \mathrm{Type}(p) \quad &\Leftrightarrow \quad \exists C > 0 \text{ such that } \forall (x_i)_{i=1}^k \text{ in } X \\ &\qquad \int_0^1 \big\| \sum_{i=1}^k r_i(t) x_i \big\| \, dt \leqslant C \Big(\sum_{i=1}^k \|x_i\|^p \Big)^{1/p} \\ &\Leftrightarrow \quad \exists C > 0 \text{ such that } \forall F = \{n_1, \dots, n_k\} \subseteq \mathbb{N} \text{ finite} \\ &\qquad \int_0^1 \big\| \sum_{i=1}^k r_i(t) d_{n_i}(X) \big\| \, dt \leqslant C \Big(\sum_{i=1}^k \|d_{n_i}(X)\|^p \Big)^{1/p}. \end{aligned}$$

(Here, (r_i) denotes the sequence of Rademacher functions—see [LT]). This shows that Type(p) is Borel. Similarly we verify that Cotype(q) is Borel. As a separable Banach space X has non-trivial type (respectively, non-trivial cotype) if and only if there exists $p \in \mathbb{Q}$ with $1 < p \leqslant 2$ (respectively, $q \in \mathbb{Q}$ with $2 \leqslant q < \infty$) such that $X \in \mathrm{Type}(p)$ (respectively, $X \in \mathrm{Cotype}(q)$), this also shows that the class of separable Banach space with non-trivial type (respectively, cotype) is a Borel subset of SB.

3.6. **Applications.** Our first application is the following theorem.

Theorem 3.6. *Let A be an analytic subset of* SB *that contains, up to isomorphism, all separable reflexive* HI *spaces. Then there exists $X \in A$ which is universal.*

We will see, later on, that a stronger version of Theorem 3.6 holds true. We will give a proof of this result which is based on the results in [Ar] and will illustrate the use of boundedness of $\mathbf{\Pi}_1^1$-ranks on this kind of results.

First we discuss some results presented by Bossard in [Bo3]. Let Z be a separable Banach space with a Schauder basis. Let (e_n) be a basis of Z, and let $C > 0$ be the basis constant of (e_n).

Let $X \in \mathrm{SB}$ and $k \in \mathbb{N}$. We construct a tree $\mathbf{T}(X, Z, (e_n), k)$ on X, sometimes called the *embeddability* tree of Z in X, as follows. Let

$$(x_i)_{i=1}^l \in \mathbf{T}(X, Z, (e_n), k) \Leftrightarrow (x_i)_{i=1}^l \text{ is } k\text{-equivalent to } (e_i)_{i=1}^l$$

where, as usual, $(x_i)_{i=1}^l$ is said to be *k-equivalent* to $(e_i)_{i=1}^l$ if for every $a_1, \dots, a_l \in \mathbb{R}$ we have

$$\frac{1}{k} \big\| \sum_{i=1}^l a_i e_i \big\|_Z \leqslant \big\| \sum_{i=1}^l a_i x_i \big\|_X \leqslant k \big\| \sum_{i=1}^l a_i e_i \big\|_Z.$$

The above defined tree was first consider by Bourgain (see [B1]). Notice that Z is isomorphic to a subspace of X if and only if there exists $k \in \mathbb{N}$ such that the tree

$\mathbf{T}(X,Z,(e_n),k)$ is not well-founded. We also construct a tree $T(X,Z,(e_n),k)$ on $\mathbb{N}$ as follows. We set

$$t \in T(X,Z,(e_n),k) \Leftrightarrow t=(n_1,\dots,n_l) \text{ and } (d_{n_i}(X))_{i=1}^l \in \mathbf{T}(X,Z,(e_n),k).$$

Then, $T(X,Z,(e_n),k)$ corresponds to a subtree of $\mathbf{T}(X,Z,(e_n),k)$. We need the following lemma (see also [Bo3]).

Lemma 3.7. *For every $X\in \mathrm{SB}$ we have*

$$\sup\Big\{o\big(\mathbf{T}(X,Z,(e_n),k)\big) : k\in\mathbb{N}\Big\} = \sup\Big\{o\big(T(X,Z,(e_n),k)\big) : k\in\mathbb{N}\Big\}.$$

Proof. It is clear that

$$\sup\Big\{o\big(\mathbf{T}(X,Z,(e_n),k)\big) : k\in\mathbb{N}\Big\} \geqslant \sup\Big\{o\big(T(X,Z,(e_n),k)\big) : k\in\mathbb{N}\Big\}.$$

Conversely, fix $X\in\mathrm{SB}$ and $k\in\mathbb{N}$. We have the following claim.

Claim. *We set $\mathbf{T}=\mathbf{T}(X,Z,(e_n),k)$ and $T=T(X,Z,(e_n),2k)$. Let $\xi<\omega_1$, $(x_i)_{i=1}^l\in\mathbf{T}^{(\xi)}$ and $t=(n_1,\dots,n_l)\in\mathbb{N}^{<\mathbb{N}}$ such that $\|x_i-d_{n_i}(X)\|\leqslant \frac{1}{2kC}\cdot\frac{1}{2^{i+1}}$ for every $i\in\{1,\dots,l\}$. Then we have $t\in T^{(\xi)}$.*

To prove the claim we use the classical fact (see, e.g., [LT]) that if $(x_i)_{i=1}^l$ is k-equivalent to $(e_i)_{i=1}^l$, and $(y_i)_{i=1}^l$ is such that $\|x_i-y_i\|\leqslant\frac{1}{2kC}\cdot\frac{1}{2^{i+1}}$ for every $i\in\{1,\dots,l\}$, then $(y_i)_{i=1}^l$ is $2k$-equivalent to $(e_i)_{i=1}^l$. Since the sequence $\big(d_n(X)\big)$ is dense in X, the claim follows easily by induction on countable ordinals. We conclude that $o\big(\mathbf{T}(X,Z,(e_n),k)\big)\leqslant o\big(T(X,Z,(e_n),2k)\big)$, and the proof is completed. □

We "glue" the trees $T(X,Z,(e_n),k)$ and we obtain a tree $T\big(X,Z,(e_n)\big)$ on $\mathbb{N}$ with the following properties.

(P1) Z is not isomorphic to a subspace of X if and only if $T\big(X,Z,(e_n)\big)$ is well-founded.

(P2) For every $k\in\mathbb{N}$ we have $o\big(T(X,Z,(e_n))\big)\geqslant o\big(\mathbf{T}(X,Z,(e_n),k)\big)$.

It is easy to see that the map $\mathrm{SB}\ni X\mapsto T\big(X,Z,(e_n)\big)\in\mathrm{Tr}$ is Borel and so, by property (P1), it is a reduction of the set NC_Z of all separable Banach spaces not containing Z to WF. It follows that the map $X\mapsto o\big(T(X,Z,(e_n))\big)$ is a $\mathbf{\Pi}^1_1$-rank on NC_Z. We are ready to give the proof of Theorem 3.6.

Proof of Theorem 3.6. Let A be as in the statement of the theorem. Let $A_{\cong}$ be the isomorphic saturation of A, that is, $A_{\cong}:=\{Y\in\mathrm{SB}:\exists X\in A \text{ such that } Y\cong X\}$. Notice that $A_{\cong}$ is analytic, since the equivalence relation of isomorphism is $\mathbf{\Sigma}^1_1$ in $\mathrm{SB}\times\mathrm{SB}$.

Let Z be an arbitrary separable Banach space with a Schauder basis (e_n). If there does not exist $X\in A_{\cong}$ with Z isomorphic to a subspace of X, then $A_{\cong}\subseteq\mathrm{NC}_Z$. Since the map $X\mapsto o\big(T(X,Z,(e_n))\big)$ is a $\mathbf{\Pi}^1_1$-rank on NC_Z and $A_{\cong}$ is $\mathbf{\Sigma}^1_1$, by boundedness, we obtain that

$$\sup\Big\{o\big(T(X,Z,(e_n))\big) : X\in A_{\cong}\Big\}<\omega_1. \tag{3.1}$$

However, as it has been shown in [Ar], for every separable Banach space Z with a Schauder basis (e_n) one can construct a transfinite sequence $\langle H_\xi(Z) : \xi < \omega_1\rangle$ of separable reflexive HI spaces such that for every $\xi < \omega_1$ we have that

$$\sup\Big\{o\big(\mathbf{T}(H_\xi(Z), Z, (e_n), k)\big) : k \in \mathbb{N}\Big\} \geqslant \xi.$$

Since the family $\langle H_\xi(Z) : \xi < \omega_1\rangle$ is clearly a subset of $A_\cong$ (recall that A contains all separable reflexive HI spaces up to isomorphism), by Lemma 3.7, we obtain that the rank must be unbounded on $A_\cong$, a contradiction by (3.1). Therefore, there exists $X \in A_\cong$ such that Z is isomorphic to a subspace of X. Applying the above for $Z = C[0,1]$ we obtain the result. □

Remark 1. (a) Using the results of Bourgain in [B1] instead of the results in [Ar], one can use the above argument to derive the following result of Bossard (see [Bo3, AGR]).

Theorem 3.8. *Let A be an analytic subset of* SB *that contains, up to isomorphism, all separable reflexive Banach spaces. Then there exists $X \in A$ which is universal.*

This is a typical use of techniques of rank theory in order to prove universality, and more generally existential, results (see [K, page 290]).

(b) Notice that, by Theorem 3.6, we have the following corollary.

Corollary 3.9. *If A is an analytic subset of* SB *with* HI $\subseteq A$*, then there exists $X \in A$ which is universal.*

Since no HI space (respectively, no indecomposable space, nor a space with no unconditional sequence) is universal, the above corollary implies that the class HI (respectively, I and NUC) is co-analytic non-Borel (the fact that no indecomposable separable Banach is universal follows from the classical fact that c_0 is separably injective [LT]). However, this does not show that HI is actually $\mathbf{\Pi}^1_1$-complete, the proof of which requires more elaborate techniques. This is a typical phenomenon in descriptive set theory.

Our second application concerns the embeddability rank of a separable Banach space Z with a Schauder basis. As we have seen the map $X \mapsto o\big(T(X, Z, (e_n))\big)$ is a $\mathbf{\Pi}^1_1$-rank on NC_Z for every Schauder basis (e_n) of Z. However, it appears that this rank depends on the choice of the Schauder basis. We will show that it is actually independent of such a choice in a very strong sense.

Theorem 3.10. *Let Z be a separable Banach space with Schauder basis. Then there exists a map $\phi_Z \colon \mathrm{SB} \to Ord$ such that*

$$X \in \mathrm{NC}_Z \Leftrightarrow \phi_Z(X) < \omega_1$$

and the map $\phi_Z \colon \mathrm{NC}_Z \to \omega_1$ is a $\mathbf{\Pi}^1_1$-rank on NC_Z*. Moreover, for every Schauder basis (e_n) of Z, every $k \in \mathbb{N}$ and every separable Banach space X we have*

$$o\big(\mathbf{T}(X, Z, (e_n), k)\big) \leqslant \phi_Z(X).$$

For the proof of Theorem 3.10 we need the following parameterized version of Lusin's classical theorem.

Theorem 3.11 (parameterized Lusin)**.** *Let X be a standard Borel space and let $A \subseteq X \times \mathrm{Tr}$ be analytic. Then there is a Borel map $f\colon X \to \mathrm{Tr}$ such that for every $x \in X$, if the section $A_x = \{T : (x,T) \in A\}$ is a subset of* WF*, then $f(x) \in \mathrm{WF}$ and $o\big(f(x)\big) \geqslant \sup\big\{o(T) : T \in A_x\big\}$, while if $A_x \cap \mathrm{IF} \neq \emptyset$, then $f(x) \in \mathrm{IF}$.*

Theorem 3.11 is certainly well-known among people working in descriptive set theory. However, we have not been able to find a reference (although it appears as a statement in [K, page 365]). For the sake of completeness we include a proof.

Proof of Theorem 3.11*.* Since any two uncountable standard Borel spaces are Borel isomorphic, we may assume that $X = \mathcal{N}$. In this case we will show that the map f is actually continuous. So let $A \subseteq \mathcal{N} \times \mathrm{Tr}$ be analytic and let $F \subseteq \mathcal{N} \times \mathrm{Tr} \times \mathcal{N}$ be closed such that $A = \mathrm{proj}_{\mathcal{N}\times\mathrm{Tr}} F$. For every $x \in \mathcal{N}$ define $T_x \in \mathrm{Tr}(\mathbb{N}\times\mathbb{N})$ by

$$\begin{aligned} T_x \coloneqq \big\{(t_1,t_2) \ : \ & \exists n \text{ with } |t_1| = |t_2| = n \text{ and } \exists (y,T,z) \in F \\ & \text{with } x|n = y|n, \ t_1 \in T \text{ and } t_2 = z|n\big\}. \end{aligned}$$

The map $h\colon \mathcal{N} \to \mathrm{Tr}(\mathbb{N}\times\mathbb{N})$ defined by $h(x) = T_x$ is clearly continuous.

Claim. *For every $x \in \mathcal{N}$ we have $T_x \in \mathrm{WF}(\mathbb{N}\times\mathbb{N})$ if and only if $A_x \subseteq \mathrm{WF}$.*

Proof of the claim. Fix $x \in \mathcal{N}$. Assume that T_x is well-founded. For every $T \in A_x$ we select $z \in \mathcal{N}$ such that $(x,T,z) \in F$. Define $\phi\colon T \to T_x$ by $\phi(t) = (t, z|n)$ where $n = |t|$. Then ϕ is a well-defined monotone map (that is, $t_1 \sqsubset t_2$ in T implies that $\phi(t_1) \sqsubset \phi(t_2)$ in T_x). Since $T_x \in \mathrm{WF}(\mathbb{N}\times\mathbb{N})$, we obtain that $T \in \mathrm{WF}$ and that $o(T) \leqslant o(T_x)$.

Conversely, assume that T_x is ill-founded. Let $\big((t^1_n, t^2_n)\big)$ be an infinite branch of T_x. For every $n \in \mathbb{N}$ we select $y_n \in \mathcal{N}$, $T_n \in \mathrm{Tr}$ and $z_n \in \mathcal{N}$ such that $(y_n, T_n, z_n) \in F$ and $y_n|n = x|n$, $t^1_n \in T_n$ and $z_n|n = t^2_n$. Then $y_n \to x$ and $z_n \to z$ where $z = \bigcup_n t^2_n$. Moreover, by passing to subsequences if necessary, we may assume that $T_n \to T$ in $\mathrm{Tr}(\mathbb{N}\times\mathbb{N})$ (the space $\mathrm{Tr}(\mathbb{N}\times\mathbb{N})$ is compact). By the fact that F is closed, we obtain that $(x,T,z) \in F$, and so $T \in A_x$. As the space $\mathrm{Tr}(\mathbb{N}\times\mathbb{N})$ consists of downward closed trees and (t^1_n) is a branch of $\mathbb{N}^{<\mathbb{N}}$, we see that $t^1_n \in T_i$ for every $n \leqslant i$, and so $t^1_n \in T$ for every $n \in \mathbb{N}$. Therefore, $T \in \mathrm{IF}$. □

Notice that, by the proof of the above claim, we also have that if $A_x \subseteq \mathrm{WF}$, then $\sup\big\{o(T) : T \in A_x\big\} \leqslant o(T_x)$. Now let $g\colon \mathrm{Tr}(\mathbb{N}\times\mathbb{N}) \to \mathrm{Tr}$ be any continuous map such that

(i) $T \in \mathrm{WF}(\mathbb{N}\times\mathbb{N})$ if and only if $g(T) \in \mathrm{WF}$, and
(ii) $o(T) \leqslant o\big(g(T)\big)$ for every $T \in \mathrm{Tr}(\mathbb{N}\times\mathbb{N})$ (with the usual convention that if T is ill-founded, then $o(T) = \omega_1$).

Finally, define $f\colon \mathcal{N} \to \mathrm{Tr}$ by setting $f(x) = g(T_x)$. Clearly f is as desired. □

We continue with the proof of Theorem 3.10.

Proof of Theorem 3.10. Let Z be a separable Banach space with a Schauder basis, and set

$$\mathcal{S} \coloneqq \big\{(e_n) \in Z^{\mathbb{N}} : (e_n) \text{ is a Schauder basis of } Z\big\}.$$

Then $\mathcal{S}$ is Borel in $Z^{\mathbb{N}}$. Indeed, the set $\mathcal{B}$ of all basic sequences of Z is F_σ in $Z^{\mathbb{N}}$, while the set $\mathcal{D}$ of all sequences (z_n) with dense linear span is Borel since

$$(z_n) \in \mathcal{D} \Leftrightarrow \forall k\ \forall m\ \exists l\ \exists a_1, \dots, a_l \in \mathbb{R} \text{ such that } \Big\| d_k - \sum_{n=1}^{l} a_n z_n \Big\| \leqslant \frac{1}{m}$$

where (d_k) is a fixed dense sequence in Z. Therefore, $\mathcal{D}$ is $\mathbf{\Pi}^0_3$ ($F_{\sigma\delta}$ in the classical notation). As $\mathcal{S} = \mathcal{B} \cap \mathcal{D}$, we conclude that $\mathcal{S}$ is Borel, and so a standard Borel space. Observe that the set

$$C \coloneqq \big\{((e_n), X, T) \in \mathcal{S} \times \mathrm{SB} \times \mathrm{Tr} : T = T\big(X, Z, (e_n)\big)\big\}$$

is Borel where $T\big(X, Z, (e_n)\big)$ denotes the tree on $\mathbb{N}$ defined in the beginning of this subsection by considering as Schauder basis of Z the sequence (e_n). It follows that the set

$$A \coloneqq \big\{(X, T) \in \mathrm{SB} \times \mathrm{Tr} : \exists (e_n) \in \mathcal{S} \text{ with } ((e_n), X, T) \in C\big\}$$

is analytic. Moreover, by Lemma 3.7, we have that

(1) $X \in \mathrm{NC}_Z$ if and only if $A_X = \{T \in \mathrm{Tr} : (X, T) \in A\} \subseteq \mathrm{WF}$, and
(2) for every $X \in \mathrm{NC}_Z$, every Schauder basis (e_n) of Z and every $k \in \mathbb{N}$ we have $o\big(\mathbf{T}(X, Z, (e_n), k)\big) \leqslant \sup\{o(T) : T \in A_X\}$.

We apply the parameterized Lusin theorem (Theorem 3.11) and we obtain a Borel function $f \colon \mathrm{SB} \to \mathrm{Tr}$ such that

(3) $X \in \mathrm{NC}_Z$ if and only if $f(X) \in \mathrm{WF}$ (that is, f is a reduction of NC_Z to WF), and
(4) for every $X \in \mathrm{NC}_Z$ we have $\sup\{o(T) : T \in A_X\} \leqslant o\big(f(X)\big)$.

We set $\phi_Z(X) = o\big(f(X)\big)$ with the standard convention that $o(T) = \omega_1$ if T is ill-founded. Clearly, ϕ_Z is as desired. $\square$

Remark 2. Although the $\mathbf{\Pi}^1_1$-rank ϕ_Z obtained by Theorem 3.10 may be considered as a universal embeddability rank for Z, we note that it is equivalent to the rank $X \mapsto o\big(T(X, Z, (e_n))\big)$ for every Schauder basis (e_n) of Z in the sense that for every $A \subseteq \mathrm{SB}$ we have $\sup\{o\big(T(X, Z, (e_n))\big) : X \in A\} = \omega_1$ if and only if $\sup\{\phi_Z(X) : X \in A\} = \omega_1$. To see this notice, first, that the "only if" part is an immediate consequence of Theorem 3.10. Conversely, observe that if

$$\sup\{o\big(T(X, Z, (e_n))\big) : X \in A\} = \xi < \omega_1,$$

then $A \subseteq B_\xi \coloneqq \{X \in \mathrm{SB} : o\big(T(X, Z, (e_n))\big) \leqslant \xi\}$. Since the map

$$X \mapsto o\big(T(X, Z, (e_n))\big)$$

is a $\mathbf{\Pi}^1_1$-rank on NC_Z, we see that B_ξ is Borel. Hence, as ϕ_Z is a $\mathbf{\Pi}^1_1$-rank on NC_Z, by boundedness, we obtain that $\sup\{\phi_Z(X) : X \in A\} \leqslant \sup\{\phi_Z(X) : X \in B_\xi\} < \omega_1$.

Bossard has extended the embeddability rank for the general case of a separable Banach space Z which does not necessarily have a Schauder basis (see [Bo2, Theorem 4.8] or [Bo3, Theorem 4.9]). We recall his definition taken from [Bo3]. Let Z be a separable Banach space and fix a sequence (z_n) of linearly independent vectors with dense linear span in Z. For every $X \in \mathrm{SB}$ and every $k \in \mathbb{N}$ we define a tree $\mathbf{T}(X, Z, (z_n), k)$ on $X^{<\mathbb{N}}$ as follows. A sequence $\big((x^1_1), (x^2_1, x^2_2), \dots, (x^n_1, \dots, x^n_n)\big)$ belongs to $\mathbf{T}(X, Z, (z_n), k)$ if the following are satisfied.

(i) For every $1 \leqslant i \leqslant j \leqslant l \leqslant n$ we have $\|x^j_i - x^l_i\| \leqslant \frac{k}{2j}$.
(ii) For every $1 \leqslant l \leqslant n$ the sequence $(x^l_i)^l_{i=1}$ is k-equivalent to $(z_i)^l_{i=1}$.

It is easy to see that Z is isomorphic to a subspace of X if and only if there exists $k \in \mathbb{N}$ such that the tree $\mathbf{T}(X, Z, (z_n), k)$ is not well-founded.

Now we consider a tree on $\mathbb{N}^{<\mathbb{N}}$, denoted by $T(X, Z, (z_n), k)$, which is defined as follows. A sequence $(t_1, t_2, \dots, t_n) \in (\mathbb{N}^{<\mathbb{N}})^{<\mathbb{N}}$ belongs to $T(X, Z, (z_n), k)$ if the following are satisfied.

(a) For every $i \in \{1, \dots, n\}$ we have $|t_i| = i$.
(b) For every $i \in \{1, \dots, n\}$ if $t_i = (l^i_1, \dots, l^i_i)$, then

$$\Big(\big(d_{l^1_1}(X)\big), \big(d_{l^2_1}(X), d_{l^2_2}(X)\big), \dots, \big(d_{l^n_1}(X), \dots, d_{l^n_n}(X)\big)\Big) \in \mathbf{T}(X, Z, (z_n), k).$$

Arguing as in Lemma 3.7, we see that

$$\sup\Big\{o\big(\mathbf{T}(X, Z, (z_n), k)\big) : k \in \mathbb{N}\Big\} = \sup\Big\{o\big(T(X, Z, (z_n), k)\big) : k \in \mathbb{N}\Big\}.$$

Since the set $\mathbb{N}^{<\mathbb{N}}$ is countable, the tree $T(X, Z, (z_n), k)$ may be considered as a tree on $\mathbb{N}$. By "gluing" the trees $\langle T(X, Z, (z_n), k) : k \in \mathbb{N}\rangle$ in a tree $T\big(X, Z, (z_n)\big)$ and using the parameterized Lusin theorem, we obtain the following analogue of Theorem 3.10.

Theorem 3.12. *Let Z be a separable Banach space. Then there exists a map $\psi_Z \colon \mathrm{SB} \to Ord$ such that*

$$X \in \mathrm{NC}_Z \Leftrightarrow \psi_Z(X) < \omega_1$$

and the map $\psi_Z \colon \mathrm{NC}_Z \to \omega_1$ is a $\mathbf{\Pi}^1_1$-rank on NC_Z. Moreover, for every sequence (z_n) of linearly independent vectors with dense linear span in Z, every $k \in \mathbb{N}$ and every separable Banach space X we have $o\big(\mathbf{T}(X, Z, (z_n), k)\big) \leqslant \psi_Z(X)$.

4. The ℓ_2 Baire sum of a Schauder tree basis

In this section we define the Schauder tree basis $(x_t)_{t\in T}$ of a separable Banach space X and the ℓ_2 Baire sum of a Schauder tree basis. Similar norms have been considered by Bourgain [B1] and Bossard [Bo3]. We study the structure of ℓ_2

Baire sums. The second and the third subsections are devoted to some preparatory lemmas. Most of these lemmas are of combinatorial nature and are based on applications of the classical Ramsey theorem. The central notion is that of an X-singular subspace of $\mathcal{T}_2^X$. We show that any such subspace does not contain ℓ_1. We also show that for every tree basis $(x_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ the corresponding ℓ_2 Baire sum contains c_0.

4.1. **Definitions.** Let $(X,\|\cdot\|_X)$ be a Banach space, let Λ be a countable set, let T be a (downwards closed) pruned subtree of $\Lambda^{<\mathbb{N}}$ and let $(x_t)_{t\in T}$ be a sequence (with possible repetitions) in X which is indexed by T. For every $\sigma\in[T]$ we set $X_\sigma := \overline{\mathrm{span}}\{x_t : t\sqsubset\sigma\}$.

Definition 4.1. *We say that a normalized sequence $(x_t)_{t\in T}$ is a* bimonotone Schauder tree basis *of X if the following are satisfied.*

(1) *We have $X=\overline{\mathrm{span}}\{x_t : t\in T\}$.*
(2) *For every $\sigma\in[T]$ the sequence $(x_{\sigma|n})$ is a bimonotone Schauder basis of X_σ.*

Let us present some examples of Schauder tree bases.

Example 1. Let X be a Banach space with a normalized bimonotone Schauder basis (x_n). For every $t\in\mathbb{N}^{<\mathbb{N}}$ set $x_t := x_{|t|}$. Then $(x_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ is a Schauder tree basis of X. Observe that, in this case, we have $X=X_\sigma$ for every $\sigma\in\mathcal{N}$.

Example 2. As above, let X be a Banach space with a normalized bimonotone Schauder basis (x_n). Fix a bijection $h\colon\mathbb{N}^{<\mathbb{N}}\to\mathbb{N}$ such that $t_1\sqsubset t_2$ implies that $h(t_1)<h(t_2)$. For every $t\in\mathbb{N}^{<\mathbb{N}}$ set $x_t := x_{h(t)}$. Then $(x_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ is a Schauder tree basis of X. In this case, notice that for every $\sigma\in\mathcal{N}$ the space X_σ is the space $\overline{\mathrm{span}}\{x_n : n\in L_\sigma\}$ where $L_\sigma := \{h(\sigma|n) : n\in\mathbb{N}\}\in[\mathbb{N}]^\infty$.

Example 3. It is a refinement of Example 2. In particular, notice that in Example 2 for every $\sigma\in\mathcal{N}$ the space X_σ is a subspace of X spanned by a subsequence of the basis. In this example we show that, by a more careful enumeration, the converse may also be true. That is, for every $L\in[\mathbb{N}]^\infty$ there exists $\sigma_L\in\mathcal{N}$ such that $X_{\sigma_L}=\overline{\mathrm{span}}\{x_n : n\in L\}$. To define this enumeration let $[\mathbb{N}]^{<\mathbb{N}}$ be the downward closed subtree of $\mathbb{N}^{<\mathbb{N}}$ consisting of all nonempty, increasing finite sequences. Notice that every $t\in[\mathbb{N}]^{<\mathbb{N}}$ has infinitely many immediate successors in $[\mathbb{N}]^{<\mathbb{N}}$. Hence, there exists a bijection $g\colon\mathbb{N}^{<\mathbb{N}}\to[\mathbb{N}]^{<\mathbb{N}}$ such that

(1) $|g(t)|=|t|$ for every $t\in\mathbb{N}^{<\mathbb{N}}$, and
(2) if $t_1,t_2\in\mathbb{N}^{<\mathbb{N}}$, then $t_1\sqsubset t_2$ if and only if $g(t_1)\sqsubset g(t_2)$.

Next, let $\pi\colon[\mathbb{N}]^{<\mathbb{N}}\to\mathbb{N}$ be defined by setting $\pi(t)=n_k$ if $t=(n_1,\dots,n_k)$. (Notice that, since $t\in[\mathbb{N}]^{<\mathbb{N}}$, we have $n_1<\dots<n_k$.) Finally, define $f\colon\mathbb{N}^{<\mathbb{N}}\to\mathbb{N}$ by the rule $f(t)=\pi\big(g(t)\big)$, and set $x_t := x_{f(t)}$. It is clear that $(x_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ is a Schauder tree basis of X and, moreover, for every $\sigma\in\mathcal{N}$ the space X_σ is the space spanned by the sequence $(x_n)_{n\in L_\sigma}$ where $L_\sigma := \{f(\sigma|n) : n\in\mathbb{N}\}$. Conversely, let $L\in[\mathbb{N}]^\infty$

and let $\{l_1 < l_2 < \cdots\}$ denote its increasing enumeration. For every $n \in \mathbb{N}$ set $t_n := g^{-1}\big((l_1, \dots, l_n)\big)$ and $\sigma_L := \bigcup_{n\in L} t_n \in \mathcal{N}$. Then the space X_{σ_L} is the space spanned by the subsequence $(x_n)_{n\in L}$. The above construction is motivated by a construction of Schechtman [Sch] (see also [LT, page 93]).

We proceed with the main definition in this section. Let $(X, \|\cdot\|_X)$ be a Banach space, let Λ be a countable set, let T be a pruned subtree of $\Lambda^{<\mathbb{N}}$ and let $(x_t)_{t\in T}$ be a normalized bimonotone Schauder tree basis of X. We define the ℓ_2 *Baire sum* of $(x_t)_{t\in T}$, denoted by $\mathcal{T}_2^X$, to be the completion of $c_{00}(T)$ with the norm

$$\|z\|_{\mathcal{T}_2^X} := \sup\Big\{ \Big(\sum_{i=1}^{l} \big\| \sum_{t\in s_i} z(t)x_t \big\|_X^2 \Big)^{\frac{1}{2}} : (s_i)_{i=1}^{l} \text{ are incomparable segments of } T \Big\}.$$

If $T = \mathbb{N}^{<\mathbb{N}}$ and the Schauder tree basis $(x_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ is as in Example 1, then we call this space the *Schauder tree space associated with* X and we denote it by $\mathcal{N}_2^X$.

We denote by $(e_t)_{t\in T}$ the standard Hamel basis of $c_{00}(T)$. We fix a bijection $h\colon T \to \mathbb{N}$ such that $t_1 \sqsubset t_2$ implies that $h(t_1) < h(t_2)$, and we enumerate T as (t_n) using h. The sequence (e_{t_n}) is a normalized Schauder basis of $\mathcal{T}_2^X$. We notice the following important property. If (x_n) is block sequence in $\mathcal{T}_2^X$ and s is a segment of T, then for every $n_1 < n_2$, every $t_1 \in \mathrm{supp}(x_{n_1}) \cap s$ and every $t_2 \in \mathrm{supp}(x_{n_2}) \cap s$ we have that $t_1 \sqsubset t_2$.

For every $\sigma \in [T]$ we set $\mathcal{X}_\sigma := \overline{\mathrm{span}}\{e_t : t \sqsubset \sigma\}$. Note that $\mathcal{X}_\sigma$ is isometric to X_σ (thus, if we deal with $\mathcal{N}_2^X$, the space $\mathcal{X}_\sigma$ is isometric to X). Let $P_\sigma\colon \mathcal{T}_2^X \to \mathcal{X}_\sigma$ be defined by $P_\sigma(x) = \sum_{t\sqsubset\sigma} x(t)e_t$ and observe that P_σ is a norm-one projection. Also notice that if (x_n) is a block sequence in $\mathcal{T}_2^X$ such that $\sigma \cap \mathrm{supp}(x_n) \neq \emptyset$ for every $n \in \mathbb{N}$, then $\big(P_\sigma(x_n)\big)$ is also a block sequence in $\mathcal{X}_\sigma$ (this is a consequence of the enumeration of T). More generally, if $A \subseteq T$ is segment complete, then it is easy to see that the operator $P_A\colon \mathcal{T}_2^X \to \mathcal{X}_A$ defined by $P_A(x) = \sum_{t\in A} x(t)e_t$ is a norm-one projection onto the subspace $\mathcal{X}_A := \overline{\mathrm{span}}\{e_t : t \in A\}$.

Definition 4.2. *Let Y be a closed infinite-dimensional subspace of $\mathcal{T}_2^X$.*

1. *Y is said to be* X-singular *if for every $\sigma \in [T]$ the operator $P_\sigma\colon Y \to \mathcal{X}_\sigma$ is strictly singular.*
2. *Y is said to be* X-compact *if for every $\sigma \in [T]$ the operator $P_\sigma\colon Y \to \mathcal{X}_\sigma$ is compact.*

Remark 3. It is well-known (see [LT]) that for every strictly singular operator $T\colon Y \to Z$ there exists an infinite-dimensional subspace W of Y such that $T|_W\colon W \to Z$ is compact. It is open if for every X-singular subspace Y of $\mathcal{T}_2^X$ there exists an infinite-dimensional subspace W of Y which is X-compact.

4.2. **General lemmas.** In what follows let X be a Banach space, let Λ be a countable set, let T be a pruned subtree of $\Lambda^{<\mathbb{N}}$ and let $(x_t)_{t\in T}$ be a normalized bimonotone Schauder tree basis of X. We start with the following lemma.

Lemma 4.3. *Let (x_n) be a bounded block sequence in $\mathcal{T}_2^X$. Also let $\varepsilon > 0$ and $L \in [\mathbb{N}]^\infty$. Then there exist finite $A \subseteq [T]$ and $M \in [L]^\infty$ such that for every $\sigma \in [T] \setminus A$ we have $\limsup_{n\in M} \|P_\sigma(x_n)\| < \varepsilon$.*

Proof. Assume not. Then we may select, recursively, a decreasing sequence (L_i) of infinite subsets of L and a sequence (σ_i) in $[T]$ such that

$$\|P_{\sigma_i}(x_n)\| > \frac{\varepsilon}{3} \ \text{ for every } n \in L_i \text{ and every } i \in \mathbb{N}. \tag{4.1}$$

Set $C \coloneqq \sup\{\|x_n\| : n \in \mathbb{N}\} < \infty$. We select $k_0 \in \mathbb{N}$ such that $k_0 > \frac{9C^2}{\varepsilon^2}$. Since $\sigma_1, \dots, \sigma_{k_0}$ are different elements of $[T]$, we may select $n_0 \in \mathbb{N}$ such that the σ_i's restricted after the n_0-level become pairwise incomparable.

Now let $l_0 \in L_{k_0}$ such that for every $l \geqslant l_0$ with $l \in L_{k_0}$ and every $t \in \mathrm{supp}(x_l)$ we have that if $t \in \sigma_i$ for some $i \in \{1, \dots, k_0\}$, then $|t| > n_0$; this is possible since the sequence (x_n) is block. As the sequence (L_i) is decreasing, we have that $l_0 \in L_i$ for every $1 \leqslant i \leqslant k_0$. Notice that, by (4.1), for every $i \in \{1, \dots, k_0\}$ there exists a segment $s_i \subseteq \sigma_i$ such that $\|\sum_{t\in s_i} x_{l_0}(t)x_t\| \geqslant \frac{\varepsilon}{3}$. By the choice of l_0, we see that the s_i's can be selected to be pairwise incomparable. Hence,

$$C \geqslant \|x_{l_0}\| \geqslant \Big(\sum_{i=1}^{k_0} \big\| \sum_{t\in s_i} x_{l_0}(t) x_t \big\|_X^2 \Big)^{1/2} \geqslant \sqrt{k_0 \frac{\varepsilon^2}{9}} > C$$

a contradiction. The lemma is proved. $\square$

We will need the following slight variant of Lemma 4.3.

Lemma 4.4. *Let (x_n) be a bounded block sequence in $\mathcal{T}_2^X$. Also let $\varepsilon > 0$ and $L \in [\mathbb{N}]^\infty$. Then there exist finite $A \subseteq [T]$ and $M \in [L]^\infty$ such that for every segment s of T with $s \cap A = \emptyset$ we have $\limsup_{n\in M} \|P_s(x_n)\| < \varepsilon$.*

The proof of Lemma 4.4 is identical to that of Lemma 4.3 and so we omit it. We proceed with the following lemma.

Lemma 4.5. *Let (x_n) be a bounded block sequence in $\mathcal{T}_2^X$. Also let $\varepsilon > 0$ such that for every $\sigma \in [T]$ we have $\limsup \|P_\sigma(x_n)\| < \varepsilon$. Then there exists $L \in [\mathbb{N}]^\infty$ such that for every $\sigma \in [T]$ we have $|\{n \in L : \|P_\sigma(x_n)\| \geqslant \varepsilon\}| \leqslant 1$.*

Proof. Assume not. Then for every $L \in [\mathbb{N}]^\infty$ there exist $(n_1, n_2) \in [L]^2$ and $\sigma \in [T]$ with $\|P_\sigma(x_{n_i})\| \geqslant \varepsilon$ for $i \in \{1, 2\}$. By Ramsey's theorem [R], there exists $L \in [\mathbb{N}]^\infty$ such that for every $(n_1, n_2) \in [L]^2$ there exists $\sigma \in [T]$ with $\|P_\sigma(x_{n_i})\| \geqslant \varepsilon$ for $i \in \{1, 2\}$. Hence, by passing to a subsequence, we may assume that for every $n < k$ there exists $\sigma_{n,k} \in [T]$ such that $\|P_{\sigma_{n,k}}(x_n)\| \geqslant \varepsilon$ and $\|P_{\sigma_{n,k}}(x_k)\| \geqslant \varepsilon$.

Let $k \in \mathbb{N}$ be arbitrary. For every $n < k$ set $o_n \coloneqq \min\{|t| : t \in \mathrm{supp}(x_k) \cap \sigma_{n,k}\}$ and $s_{n,k} \coloneqq \{t \in \sigma_{n,k} : 1 \leqslant |t| < o_n\}$. Then $s_{n,k}$ is an initial segment of T and $s_{n,k} \subseteq \sigma_{n,k}$. Also notice that $\mathrm{supp}(x_n) \cap \sigma_{n,k} \subseteq s_{n,k}$ as the sequence (x_n) is block. Hence, $\|P_{s_{n,k}}(x_n)\| \geqslant \varepsilon$. Moreover, $s_{n,k} \cap \big(\mathrm{supp}(x_k) \cap \sigma_{n,k}\big) = \emptyset$; actually,

$s_{n,k}$ is the maximal initial segment of $\sigma_{n,k}$ which does not intersect $\mathrm{supp}(x_k)$. Set $C \coloneqq \sup\{\|x_n\| : n \in \mathbb{N}\} < \infty$.

Claim. *For every $k \geqslant 2$ we have $|\{s_{n,k} : n < k\}| \leqslant \lceil C^2/\varepsilon^2 \rceil$.*

Proof of the claim. Let $\{s_1, \dots, s_l\}$ be an enumeration of the set $\{s_{n,k} : n < k\}$; thus, for every $i \in \{1, \dots, l\}$ there exists $n_i < k$ such that $s_i = s_{n_i,k}$. Set $s'_i \coloneqq \{t \in \sigma_{n_i,k} : |t| \geqslant o_{n_i}\}$, and notice that s'_i is a final segment of T and, moreover, $\sigma_{n_i,k} \cap \mathrm{supp}(x_k) \subseteq s'_i$. By our assumptions, this implies that $\|P_{s'_i}(x_k)\| \geqslant \varepsilon$. Therefore, for every $i \in \{1, \dots, l\}$ we have $\| \sum_{t \in s'_i} x_k(t)\, x_t\| \geqslant \varepsilon$. Next notice that, since the segments $(s_i)_{i=1}^l$ are mutually *different*, the segments $(s'_i)_{i=1}^l$ are pairwise *incomparable*. Indeed, for every $i \in \{1, \dots, l\}$ let t_i be the $\sqsubset$-least element of s'_i. Observe that $t_i \in \mathrm{supp}(x_k)$. Also notice that if $i \neq j$, then neither $t_i \sqsubset t_j$ nor $t_j \sqsubset t_i$ holds true. Suppose, on the contrary, that $t_i \sqsubset t_j$ (the argument is symmetric). Then $t_i \in \sigma_{n_j,k}$ and so $o_{n_j} \leqslant |t_i| < |t_j| = o_{n_j}$ which is a contradiction. Finally, note that $t_i \neq t_j$. Indeed, if $t_i = t_j$, then, by the definition of the segments $(s_i)_{i=1}^l$, we would have that $s_i = s_j$ and again we derive a contradiction. It follows that the segments $(s'_i)_{i=1}^l$ are pairwise incomparable and, consequently,

$$C \geqslant \|x_k\| \geqslant \Big(\sum_{i=1}^{l} \big\| \sum_{t \in s'_i} x_k(t)\, x_t \big\|^2 \Big)^{1/2} \geqslant \varepsilon \sqrt{l}$$

which yields the desired estimate. The proof of the claim is completed. □

Set $M \coloneqq \lceil C^2/\varepsilon^2 \rceil$. By the above claim, for every $n < k$ there exists a family $\big\{s_{i,k} : i \in \{1, \dots, M\}\big\}$ of initial segments of T such that for every $n \in \{1, \dots, k-1\}$ there exists $i \in \{1, \dots, M\}$ with $\|P_{s_{i,k}}(x_n)\| \geqslant \varepsilon$. By passing to subsequences, we may assume that $s_{i,k} \to s_i$ in $2^{\Lambda^{<\mathbb{N}}}$ for every $i \in \{1, \dots, M\}$. Notice that each s_i, if nonempty, is an initial segment of T (but it might be finite, of course).

For every $n \in \mathbb{N}$ and every $i \in \{1, \dots, M\}$ let us say that a positive integer k is i-good for n if $k > n$ and $\|P_{s_{i,k}}(x_n)\| \geqslant \varepsilon$. Notice that for every $n \in \mathbb{N}$ there exists $i \in \{1, \dots, M\}$ such that the set $H^i_n \coloneqq \{k : k > n \text{ and } k \text{ is } i\text{-good for } n\}$ is an infinite subset of $\mathbb{N}$. Hence, there exists $i_0 \in \{1, \dots, M\}$ and $L \in [\mathbb{N}]^\infty$ such that for every $n \in L$ the set $H^{i_0}_n$ is infinite. It follows that for every $n \in L$ we have $\|P_{s_{i_0,k}}(x_n)\| \geqslant \varepsilon$ for infinitely many k. Since $s_{i_0,k} \to s_{i_0}$, this yields that $\|P_{s_{i_0}}(x_n)\| \geqslant \varepsilon$ for every $n \in L$. Next recall that the sequence (x_n) is block, and that s_{i_0} is an initial segment of T. Therefore, we have $s_{i_0} \in [T]$. But then $\limsup \|P_{s_{i_0}}(x_n)\| \geqslant \limsup_{n \in L} \|P_{s_{i_0}}(x_n)\| \geqslant \varepsilon$ which is a contradiction. □

Using Lemma 4.5, we obtain the following lemma.

Lemma 4.6. *Let (x_n) be a bounded block sequence in $\mathcal{T}^X_2$. Also let $\varepsilon > 0$ such that for every $\sigma \in [T]$ we have $\limsup \|P_\sigma(x_n)\| < \varepsilon$. Then for every $L \in [\mathbb{N}]^\infty$ there exists a finite convex combination w of $(x_n)_{n \in L}$ such that for every $\sigma \in [T]$ we have $\|P_\sigma(w)\| \leqslant 2\varepsilon$.*

Proof. We Lemma 4.5 for the sequence $(x_n)_{n\in L}$ and obtain $M \in [L]^\infty$ such that for every $\sigma \in [T]$ we have $|\{n \in M : \|P_\sigma(x_n)\| \geqslant \varepsilon\}| \leqslant 1$. Let $\{m_1 < m_2 < \cdots\}$ be the increasing enumeration of M. We set $C \coloneqq \sup\{\|x_n\| : n \in \mathbb{N}\} < \infty$ and we select $k_0 \in \mathbb{N}$ such that $\big(C+\varepsilon(k_0-1)\big)/k_0 \leqslant 2\varepsilon$. We define $w = \frac{1}{k_0}(x_{m_1}+x_{m_2}+\cdots+x_{m_{k_0}})$. Then for every $\sigma \in [T]$ we have

$$\|P_\sigma(w)\| \leqslant \frac{\sum_{i=1}^{k_0} \|P_\sigma(x_{m_i})\|}{k_0} \leqslant \frac{C+\varepsilon(k_0-1)}{k_0} \leqslant 2\varepsilon$$

as desired. □

We will also need the following two variants of Lemmas 4.5 and 4.6.

Lemma 4.7. *Let (x_n) be a bounded block sequence in $\mathcal{T}_2^X$. Also let $\varepsilon > 0$ and finite $A \subseteq [T]$ such that for every s segment of T with $s \cap A = \emptyset$ we have $\limsup \|P_s(x_n)\| < \varepsilon$. Then there exists $L \in [\mathbb{N}]^\infty$ such that for every s segment of T with $s \cap A = \emptyset$ we have $|\{n \in L : \|P_s(x_n)\| \geqslant \varepsilon\}| \leqslant 1$.*

Proof. It is very similar to the proof of Lemma 4.5, and so we shall only indicate the necessary changes. Again, arguing by contradiction and using Ramsey's theorem, we may assume that for every $n < k$ there exist a segment $s_{n,k}$ of T which is disjoint from A and is such that $\|P_{s_{n,k}}(x_n)\|, \|P_{s_{n,k}}(x_k)\| \geqslant \varepsilon$. Define the quantity o_n as in the proof of Lemma 4.5, and for every $n < k$ let $i_{n,k}$ be the maximal segment of T which contains $s_{n,k} \cap \mathrm{supp}(x_n)$ and does not intersect neither $\mathrm{supp}(x_k)$ nor A. It is easy to see that the estimate obtained in the claim in the proof of Lemma 4.5 is also valid for the family $\{i_{n,k} : n < k\}$. Next observe that if (s_n) is a sequence of segments of T each of which is disjoint from A, and $s_n \to s$ in $2^{\Lambda^{<\mathbb{N}}}$, then s is a segment of T which is also disjoint from A. Using this observation, the rest of proof is identical to that in Lemma 4.5. □

Lemma 4.8. *Let (x_n), ε and A be as in Lemma 4.7. Then for every $L \in [\mathbb{N}]^\infty$ there exists a finite convex combination w of $(x_n)_{n\in L}$ such that for every s segment of T with $s \cap A = \emptyset$ we have $\|P_s(w)\| \leqslant 2\varepsilon$.*

Proof. It is identical to the proof of Lemma 4.6, using Lemma 4.7 instead of Lemma 4.5. □

4.3. **Sequences satisfying an upper ℓ_2 estimate.** Our first goal is to prove the following proposition.

Proposition 4.9. *Let (x_n) be a bounded block sequence in $\mathcal{T}_2^X$ such that for every $\sigma \in [T]$ we have that $\lim \|P_\sigma(x_n)\| = 0$. Then there exists a block sequence (w_n) of finite convex combinations of (x_n) satisfying an upper ℓ_2 estimate. That is, there exists $C > 0$ such that for every $k \in \mathbb{N}$ and every $a_1, \dots, a_k \in \mathbb{R}$ we have $\|\sum_{n=1}^k a_n w_n\| \leqslant C\big(\sum_{n=1}^k a_n^2\big)^{1/2}$.*

Proof. Recursively and using Lemma 4.6, we select a block sequence (w_n) of finite convex combinations of (x_n) such that for every $n \geqslant 2$ and every $\sigma \in [T]$ we have

$$\|P_\sigma(w_n)\| \leqslant \frac{1}{\sum_{i=1}^{n-1} |\mathrm{supp}(w_i)|^{1/2}} \ \frac{1}{2^{2n}}.$$

We will show that (w_n) is as desired. Set $M \coloneqq \sup\{\|x_n\| : n \in \mathbb{N}\} < \infty$ and notice that $\|w_n\| \leqslant M$ for every n. Let $k \in \mathbb{N}$ and $a_1, \dots, a_k \in \mathbb{R}$ with $\sum_{i=1}^k a_i^2 = 1$. We will show that $\| \sum_{i=1}^k a_i w_i\| \leqslant \sqrt{2M^2+2}$. This will finish the proof.

Let $(s_j)_{j=1}^l$ be an arbitrary family of pairwise incomparable segments of T. We define a partition of $\{1, \dots, l\}$ by setting

$$\begin{aligned}
I_1 &= \big\{ j \in \{1,\dots,l\} : s_j \cap \mathrm{supp}(w_1) \neq \emptyset \big\} \\
I_2 &= \big\{ j \in \{1,\dots,l\} \setminus I_1 : s_j \cap \mathrm{supp}(w_2) \neq \emptyset \big\} \\
&\ \ \vdots \\
I_k &= \big\{ j \in \{1,\dots,l\} \setminus \big(\bigcup_{i=1}^{k-1} I_i \big) : s_j \cap \mathrm{supp}(w_k) \neq \emptyset \big\}.
\end{aligned}$$

Since the segments $(s_j)_{j=1}^l$ are pairwise incomparable, we see that

$$|I_i| \leqslant |\mathrm{supp}(w_i)| \ \text{ for every } i \in \{1,\dots,k\}. \tag{4.2}$$

Also we observe that for every $1 \leqslant m < i \leqslant k$ we have

$$\sum_{j \in I_i} \|P_{s_j}(w_m)\| = 0. \tag{4.3}$$

Let $i \in \{1, \dots, k\}$ and $j \in I_i$. We will estimate the quantity

$$\begin{aligned}
\|P_{s_j}(a_1 w_1 + \dots + a_n w_n)\| &\overset{(4.3)}{=} \|P_{s_j}(a_i w_i + \dots + a_n w_n)\| \\
&\leqslant a_i \|P_{s_j}(w_i)\| + \sum_{k=i+1}^{n} a_k \|P_{s_j}(w_k)\|.
\end{aligned}$$

Since the Schauder tree basis $(x_t)_{t\in T}$ of X is bimonotone, by the choice of the sequence (w_n), we see that for every $k \in \{i+1, \dots, n\}$ we have

$$\|P_{s_j}(w_k)\| \leqslant \frac{1}{|\mathrm{supp}(w_i)|^{1/2}} \ \frac{1}{2^{2k}}.$$

Therefore,

$$\begin{aligned}
\|P_{s_j}(a_1 w_1 + a_2 w_2 + \dots + a_n w_n)\| &\leqslant a_i \|P_{s_j}(w_i)\| + \frac{1}{|\mathrm{supp}(w_i)|^{1/2}} \sum_{k=i+1}^{n} \frac{1}{2^{2k}} \\
&\overset{(4.2)}{\leqslant} a_i \|P_{s_j}(w_i)\| + \frac{1}{|I_i|^{1/2}} \ \frac{1}{2^i}.
\end{aligned}$$

Notice that for every $i \in \{1,\dots,k\}$ we have $\sum_{j\in I_i} \|P_{s_j}(w_i)\|^2 \leqslant \|w_i\|^2 \leqslant M^2$ as the segments $(s_j)_{j\in I_i}$ are pairwise incomparable. Hence,

$$\begin{aligned}\sum_{j\in I_i} \|P_{s_j}(a_1w_1+\cdots+a_nw_n)\|^2 &\leqslant \sum_{j\in I_i} \Big(a_i\|P_{s_j}(w_i)\| + \frac{1}{|I_i|^{1/2}}\,\frac{1}{2^i}\Big)^2 \\ &\leqslant 2a_i^2 \sum_{j\in I_i} \|P_{s_j}(w_i)\|^2 + 2\sum_{j\in I_i} \frac{1}{|I_i|}\,\frac{1}{2^i} \\ &\leqslant 2a_i^2M^2 + \frac{2}{2^i}.\end{aligned}$$

By the above, we obtain that

$$\sum_{i=1}^k \sum_{j\in I_i} \|P_{s_j}(a_1w_1+\cdots+a_nw_n)\|^2 \leqslant \sum_{i=1}^k 2a_i^2M^2 + \sum_{i=1}^k \frac{2}{2^i} \leqslant 2M^2+2.$$

The segments $(s_j)_{j=1}^l$ were arbitrary, and so $\|\sum_{i=1}^k a_iw_i\|^2 \leqslant 2M^2+2$. The proof is completed. □

By Proposition 4.9, we obtain the following criterion for checking that a block sequence (x_n) is weakly null.

Proposition 4.10. *Let (x_n) be a bounded block sequence in $\mathcal{T}_2^X$. Assume that for every $\sigma\in[T]$ we have $w-\lim P_\sigma(x_n)=0$ in $\mathcal{X}_\sigma$. Then (x_n) is weakly null.*

Proof. Assume not. Then there exist $L\in[\mathbb{N}]^\infty$, $\varepsilon>0$ and $x^*\in(\mathcal{T}_2^X)^*$ with $\|x^*\|=1$ such that $x^*(x_n)>\varepsilon$ for every $n\in L$. By repeated applications of Lemma 4.3, we obtain a decreasing sequence (M_k) of infinite subsets of L and an increasing sequence (A_k) of finite subsets of $[T]$ such that for every $k\in\mathbb{N}$ we have that $\limsup_{n\in M_k}\|P_\sigma(x_n)\|<\frac{1}{k}$ for every $\sigma\in[T]\setminus A_k$. Thus, if M_∞ denotes the infinite diagonal set of (M_k) and $A=\bigcup_{k\in\mathbb{N}}A_k$, then for every $\sigma\in[T]\setminus A$ we have $\limsup_{n\in M_\infty}\|P_\sigma(x_n)\|=0$. Notice that A is countable. Moreover, observe that for every convex block sequence (y_n) of $(x_n)_{n\in M_\infty}$ and every $\sigma\in[T]\setminus A$ we also have that $\limsup\|P_\sigma(y_n)\|=0$. Since convex combinations of convex combinations are convex combinations, using our assumption, a diagonal argument and Mazur's theorem, we obtain a convex block sequence (y_n) of $(x_n)_{n\in M_\infty}$ such that for every $\sigma\in[T]$ we have

$$\lim\|P_\sigma(y_n)\|=0 \tag{4.4}$$

and, moreover, $x^*(y_n)>\varepsilon$ for every $n\in\mathbb{N}$. The sequence (y_n) is bounded and block and so, by (4.4), we may apply Proposition 4.9 and we obtain a further convex block sequence (z_n) of (y_n) (and, consequently, of $(x_n)_{n\in L}$) which satisfies an upper ℓ_2 estimate. Since for the sequence (z_n) we still have that $x^*(z_n)>\varepsilon$ for every $n\in\mathbb{N}$, this yields to a contradiction. □

4.4. **Subspaces of $\mathcal{T}_2^X$.** Our first goal in this subsection is to show that every X-singular subspace of $\mathcal{T}_2^X$ does not contain ℓ_1. To this end we need the following lemma.

Lemma 4.11. *Let (x_n) be a normalized block sequence in $\mathcal{T}_2^X$ such that for every $\sigma\in[T]$, the sequence $(P_\sigma(x_n))$ is weak Cauchy saturated, that is, for every $L\in[\mathbb{N}]^\infty$ there exists $M\in[L]^\infty$ such that the sequence $(P_\sigma(x_n))_{n\in M}$ is weakly Cauchy. Then (x_n) has a weakly Cauchy subsequence.*

Proof. Arguing as in the proof of Proposition 4.10, by repeated applications of Lemma 4.3, we obtain $M\in[\mathbb{N}]^\infty$ and a countable subset A of $[T]$ such that for every $\sigma\in[T]\setminus A$ we have $\lim_{n\in M}\|P_\sigma(x_n)\|=0$. Using another diagonal argument and our assumptions, we obtain $N\in[M]^\infty$ such that for every $\sigma\in A$ the sequence $(P_\sigma(x_n))_{n\in N}$ is weak Cauchy. Let $\{n_1<n_2<\cdots\}$ be the increasing enumeration of N, and set $y_k:=x_{n_{2k}}-x_{n_{2k-1}}$ for every $k\in\mathbb{N}$. Then for every $\sigma\in[T]$ the sequence $(P_\sigma(y_k))$ is weakly null. By Proposition 4.10, we obtain that the sequence (y_k) is also weakly null. Therefore, the $(x_n)_{n\in N}$ is weakly Cauchy, as desired. □

Theorem 4.12. *Let Y be an X-singular subspace of $\mathcal{T}_2^X$. Then Y does not contain a copy of ℓ_1.*

Proof. Let (x_n) be a normalized block sequence in Y. By our assumptions and Rosenthal's theorem, we see that for every $\sigma\in[T]$ the sequence $(P_\sigma(x_n))$ is weak Cauchy saturated. Lemma 4.11 yields that (x_n) contains a weakly Cauchy subsequence. By Rosenthal's theorem, we conclude that Y does not contain ℓ_1. □

We proceed with the following theorem.

Theorem 4.13. *Let X be a Banach space, Λ a countable set and $(x_t)_{t\in\Lambda^{<\mathbb{N}}}$ a normalized bimonotone Schauder tree basis of X. Then the space $\mathcal{T}_2^X$ contains c_0.*

Proof. Let T be a downward closed, uniquely rooted, subtree of $\Lambda^{<\mathbb{N}}$ such that every node of T has four immediate successors in T. Therefore, for every $n\in\mathbb{N}$ the n-level of T has 4^{n-1} nodes; let T_n denote the n-level of T, and set $y_n:=\sum_{t\in T_n}\frac{1}{2^{n-1}}e_t$. The set T_n consists of pairwise incomparable nodes, and this is easily seen to imply that $\|y_n\|=1$ for every $n\in\mathbb{N}$. It is also easy to see that (y_n) is a basic sequence.

We will show that the subspace $Y:=\overline{\mathrm{span}}\{y_n:n\in\mathbb{N}\}$ contains a copy of c_0. By a result of Johnson (see, e.g., [D, page 245]), it is enough to show that $\sup\{\|\sum_{i\in F}y_i\|:\emptyset\neq F\subseteq\mathbb{N}\text{ is finite}\}<\infty$. To this end, we start with the following observation. Let s be a segment of $\Lambda^{<\mathbb{N}}$ and set $o_s:=\min\{|t|:t\in s\}$. Then for every nonempty finite $F\subseteq\mathbb{N}$ we have

$$\big\|P_s\Big(\sum_{i\in F}y_i\Big)\big\|\leqslant\sum_{i=o_s}^{\infty}\|P_s(y_i)\|\leqslant\sum_{i=o_s}^{\infty}\frac{1}{2^{i-1}}=4\,\frac{1}{2^{o_s}}.$$

Now let $(s_j)_{j=1}^l$ be an arbitrary family of mutually incomparable segments. For every $j\in\{1,\dots,l\}$ set $o_j:=\min\{|t|:t\in s_j\}$, and write all these $(o_j)_{j=1}^l$ in increasing order as $o_1 < \cdots < o_m$ (notice that $m \leqslant l$). For every $n \in \{1,\ldots,m\}$ set $I_n \coloneqq \big\{j \in \{1,\ldots,l\} : o_j = o_n\big\}$. The family $(I_n)_{n=1}^m$ is a partition of $\{1,\ldots,l\}$. We claim that

$$\frac{|I_1|}{4^{o_1}} + \frac{|I_2|}{4^{o_2}} + \cdots + \frac{|I_m|}{4^{o_m}} \leqslant \frac{1}{4}. \tag{4.5}$$

Indeed, notice that every node t in T_{o_n} has precisely $4^{o_m - o_n}$ successors in T_{o_m}. Since the family $(s_j)_{j=1}^l$ consists of pairwise incomparable segments, we see that

$$4^{o_m-o_1}|I_1| + 4^{o_m-o_2}|I_2| + \cdots + 4^{o_m-o_{m-1}}|I_{m-1}| + |I_m| \leqslant 4^{o_m-1}$$

which gives the desired estimate.

Thus, if F is a nonempty finite subset of $\mathbb{N}$ finite, then we have

$$\Big(\sum_{j=1}^{l} \big\|P_{s_j}\big(\sum_{i\in F} y_i\big)\big\|^2\Big)^{1/2} \leqslant 4\Big(\sum_{j=1}^{l}\frac{1}{4^{o_j}}\Big)^{1/2} = 4\Big(\sum_{n=1}^{m}\frac{|I_n|}{4^{o_n}}\Big)^{1/2} \leqslant 2$$

where the last inequality follows by (4.5). The proof is completed. □

Remark 4. It is easy to see that for every $k \in \mathbb{N}$ and every $a_1,\ldots,a_k \in \mathbb{R}$ we have $\sup\big\{|a_i| : i \in \{1,\ldots,k\}\big\} \leqslant \|\sum_{i=1}^k a_i y_i\|_{\mathcal{T}_2^X} \leqslant 2\sup\big\{|a_i| : i \in \{1,\ldots,k\}\big\}$ where (y_n) is the sequence constructed in proof of Theorem 4.13. Thus, the sequence (y_n) is actually 2-equivalent to the standard unit vector basis of c_0.

We close this subsection with the following (essentially known) result concerning the subspaces of $\mathcal{T}_2^X$ generated by well-founded trees (see, e.g., [Ar, Bo3, B1]).

Proposition 4.14. *Let $(x_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ be a Schauder tree basis of X. Then for every well-founded tree T with infinitely many nodes, the space $\mathcal{X}_T \coloneqq \overline{\mathrm{span}}\{e_t : t \in T\}$ is reflexive and ℓ_2-saturated.*

Proof. Both properties are proved by induction on the order of the tree T. We shall only sketch the argument that the space $\mathcal{X}_T$ is ℓ_2-saturated. So let $T \in \mathrm{WF}$ with $o(T) = \xi$. Assume that the result has been proved for every $T' \in \mathrm{WF}$ with $o(T') < \xi$. (If $o(T) = 1$, then the result is straightforward as in this case $\mathcal{X}_T$ is isometric to ℓ_2.) For every $n \in \mathbb{N}$ set $T_n \coloneqq \{t : n^\frown t \in T\} \in \mathrm{Tr}$. Let Y be an arbitrary subspace of $\mathcal{X}_T$. Then, either for every $n \in \mathbb{N}$ the operator $P_{T_n} \colon Y \to \mathcal{X}_{T_n}$ is strictly singular, or there exist $n \in \mathbb{N}$ and a subspace Y' of Y such that the operator $P_{T_n} \colon Y' \to \mathcal{X}_{T_n}$ is an isomorphic embedding. In the first case, we see that ℓ_2 is contained in Y. In the second case, since $o(T_n) < o(T)$, the inductive assumption yields that ℓ_2 is contained in Y', as desired. □

We isolate, for future use, the following corollary of Proposition 4.14.

Corollary 4.15. *Let Z be a separable Banach space with a Schauder basis (e_n). Then for every countable ordinal ξ there exists a reflexive and ℓ_2-saturated separable Banach space X such that $o\big(T(X,Z,(e_n))\big) \geqslant \xi$.*

5. Thin sets

The main notion in this section (and, actually, of the whole paper) is that of a thin set. For every Schauder tree basis $(x_t)_{t\in T}$ we consider the set W_X defined in Definition 5.3 below. Our goal is to show that the set W_X is thin on every X-singular subspace of the ℓ_2 Baire sum $\mathcal{T}_2^X$ of $(x_t)_{t\in T}$. The proof of this fact requires several steps. The key ingredient is Proposition 5.10. Although the conclusion is similar to the corresponding results in [AF], the proof requires a new approach which is based on the definition of the norm of the ℓ_2 Baire sum. Theorems 5.15 and 5.16 have a central role in establishing the properties of the amalgamations. As we have mentioned in the introduction, the amalgamation spaces will be the interpolation spaces $\Delta^{\mathrm{hi}}_{(X,W_X)}$ or $\Delta^{p}_{(X,W_X)}$. The thinness of W_X will permit us to understand the structure of the subspaces of the interpolation spaces by studying the geometric relation between their natural image in $\mathcal{T}_2^X$ and W_X.

5.1. Definitions and preliminary results.

First we recall some definitions.

Definition 5.1. *Let X be a Banach space, let $A,B\subseteq X$ and let $\varepsilon>0$.*

(a) *We say that A* ε-absorbs *B if $B\subseteq \lambda A+\varepsilon B_X$ for some $\lambda>0$.*
(b) *we say that A* almost absorbs *B if A ε-absorbs B for every $\varepsilon>0$.*

We proceed to introduce the following slight variant of the notion of a thin set defined by Neidinger [N1].

Definition 5.2. *Let X be a Banach space and let W be a closed, bounded, convex and symmetric subset of X. Also let Y be a closed infinite-dimensional subspace of X. We say that W is* thin on *Y if W does not almost absorb the ball B_Z of any infinite-dimensional subspace Z of Y, that is, for every subspace Z of Y there exists $\varepsilon>0$ such that for every $\lambda>0$ we have $B_Z\nsubseteq \lambda W+\varepsilon B_Z$. The set W is said to be* thin *if it is thin on X.*

Definition 5.3. *Let X be a Banach space, let Λ be a countable set, let T be a pruned subtree of $\Lambda^{<\mathbb{N}}$ and let $(x_t)_{t\in T}$ be a normalized bimonotone Schauder tree basis of X. We set*

$$W_X^0 := \mathrm{conv}\Big\{\bigcup_{\sigma\in[T]} B_{\mathcal{X}_\sigma}\Big\} \ \ \text{and} \ \ W_X := \overline{\mathrm{conv}}\Big\{\bigcup_{\sigma\in[T]} B_{\mathcal{X}_\sigma}\Big\}.$$

Notice that W_X is a closed, bounded, convex and symmetric subset of $\mathcal{T}_2^X$.

Next we introduce the following definition.

Definition 5.4. *We say that a sequence (x_n) in $\mathcal{T}_2^X$ is* pointwise-null *provided that $\lim e_t^*(x_n)=0$ for every $t\in T$.*

Remark 5. Related to Definition 5.4 the following hold.

(a) Every block sequence in $\mathcal{T}_2^X$ is pointwise-null.

(b) Every infinite-dimensional subspace Y of $\mathcal{T}_2^X$ contains a pointwise-null sequence.
(c) If (x_n) is a pointwise-null sequence in $\mathcal{T}_2^X$, then for every $\varepsilon > 0$ there exist $L \in [\mathbb{N}]^\infty$ and a block sequence $(y_n)_{n\in L}$ such that $\sum_{n\in L} \|x_n - y_n\| < \varepsilon$.

Parts (a) and (b) are straightforward. Part (c) follows by a sliding-hump argument.

We also need the following weaker version of the notion of X-singularity.

Definition 5.5. *A subspace Y of $\mathcal{T}_2^X$ is said to be* weakly X-singular *if for every finite-codimensional subspace Z of Y and every finite $A \subseteq [T]$ the operator $P_A\colon Z \to \mathcal{X}_A$ is not an isomorphism.*

Remark 6. The following hold.

(a) If Y is X-singular, then Y is weakly X-singular.
(b) If Y is weakly X-singular and Z is a finite-codimensional subspace of Y, then Z is also weakly X-singular.
(c) For every finite $A \subseteq [T]$ there exists a normalized, pointwise-null sequence (y_n) in Y such that $\lim \|P_\sigma(y_n)\| = 0$ for every $\sigma \in A$. If, in addition, Y is a block subspace, then the sequence (y_n) can be selected to be block.

We have the following proposition.

Proposition 5.6. *Let Y be a weakly X-singular subspace of $\mathcal{T}_2^X$. Then for every $\varepsilon > 0$ there exists a normalized pointwise-null sequence (y_n) in Y such that for every $\sigma \in [T]$ we have $\limsup \|P_\sigma(y_n)\| < \varepsilon$.*

Proof. We will give the proof under the additional assumption that Y is a block subspace. The proof for the general case is identical and follows by part (c) of Remark 5 and a standard sliding-hump argument.

So, let Y be a block weakly X-singular subspace of $\mathcal{T}_2^X$ and assume, towards a contradiction, that there exists $\varepsilon > 0$ such that for every normalized block sequence (y_n) in Y there exists $\sigma \in [T]$ such that $\limsup \|P_\sigma(y_n)\| \geqslant \varepsilon$. We select $k_0 \in \mathbb{N}$ and $r > 0$ to be determined later. We start with a normalized block sequence (y_n^1) in Y. Our assumption yields that there exists (at least one) $\sigma_1 \in [T]$ such that $\limsup \|P_{\sigma_1}(y_n^1)\| \geqslant \varepsilon$. Hence, there exists $L_1 \in [\mathbb{N}]^\infty$ such that $\|P_{\sigma_1}(y_n^1)\| \geqslant \varepsilon/2$ for every $n \in L_1$. Next, we apply Lemma 4.4 and we obtain $M_1 \in [L_1]^\infty$ and finite $A_1 \subseteq [T]$ such that for every segment s of T with $s \cap A_1 = \emptyset$ we have $\limsup_{n\in M_1} \|P_s(y_n^1)\| < r$. By Lemma 4.7, there exists $N_1 \in [M_1]^\infty$ such that for every segment s of T with $s \cap A_1 = \emptyset$ we have $|\{n \in N_1 : \|P_s(y_n^1)\| \geqslant r\}| \leqslant 1$. Summing up, we obtain $\sigma_1 \in [T]$, finite $A_1 \subseteq [T]$ and $N_1 \in [\mathbb{N}]^\infty$ such that

(P1) $\|P_{\sigma_1}(y_n^1)\| \geqslant \varepsilon/2$ for every $n \in N_1$, and
(P2) $|\{n \in N_1 : \|P_s(y_n^1)\| \geqslant r\}| \leqslant 1$ for every segment s of T with $s \cap A_1 = \emptyset$.

Since Y is weakly X-singular, by part (c) of Remark 6, we select a normalized block sequence (y^2_n) in Y such that for every $\sigma\in A_1\cup\{\sigma_1\}$ we have

$$\lim \|P_\sigma(y^2_n)\|=0. \tag{5.1}$$

As before, we select $\sigma_2\in[T]$ and $L_2\in[\mathbb{N}]^\infty$ such that $\|P_{\sigma_2}(y^2_n)\|\geqslant\frac{\varepsilon}{2}$ for every $n\in L_2$. Notice that, by (5.1), we have that $\sigma_2\notin(A_1\cup\{\sigma_1\})$. Next, we select finite $A_2\subseteq[T]$ and $N_2\in[L_2]^\infty$ such that for every segment s of T with $s\cap A_2=\emptyset$ we have $|\{n\in N_2:\|P_s(y^2_n)\|\geqslant r\}|\leqslant 1$. Once again, we remark that, by (5.1), the set A_2 can be selected so that $A_2\cap(A_1\cup\{\sigma_1\})=\emptyset$. We proceed inductively up to k_0.

For every $i\in\{1,\dots,k_0\}$ we enumerate the sequence $(y^i_n)_{n\in N_i}$ as (z^i_n), and we set $G_i\coloneqq A_i\cup\{\sigma_i\}$. By the above selection, the sets $G_1,\dots,G_{k_0}$ are finite and mutually disjoint. Therefore, there exists $l_0\in\mathbb{N}$ such that if we restrict every $\sigma\in G_1\cup\dots\cup G_{k_0}$ after the l_0-level of T, then these final segments of T are mutually *incomparable*. Also set $T_i\coloneqq\{t:\exists\sigma\in G_i \text{ with } t\sqsubset\sigma \text{ and } |t|=l_0\}$ and notice that, by the choice of l_0, for every $i,j\in\{1,\dots,k_0\}$ with $i\neq j$ and every $t_1\in T_i$ and $t_2\in T_j$ the nodes t_1 and t_2 are incomparable.

As the sequences (z^i_n) $(1\leqslant i\leqslant k_0)$ are block, we may assume that for every $n\in\mathbb{N}$ and every $i\in\{1,\dots,k_0\}$ we have that

$$\text{if } t\in\operatorname{supp}(z^i_n)\cap\sigma \text{ for some } \sigma\in G_i, \text{ then } |t|>l_0. \tag{5.2}$$

For every $i\in\{1,\dots,k_0\}$ we set $s_i\coloneqq\{t\in\sigma_i:|t|\geqslant l_0\}$, that is, s_i is the final segment of T obtained by restricting σ_i after the l_0-level of T. By (5.2) and (P1), we see that

$$\|P_{s_i}(z^i_n)\|\geqslant\frac{\varepsilon}{2} \ \text{ for every } n\in\mathbb{N} \text{ and every } i\in\{1,\dots,k_0\}. \tag{5.3}$$

Also notice that the segments $(s_i)_{i=1}^{k_0}$ are mutually incomparable.

We set $w_n\coloneqq z^1_n+\dots+z^{k_0}_n$. Then we have

$$\|w_n\|\geqslant\Big(\sum_{i=1}^{k_0}\|P_{s_i}(z^i_n)\|^2\Big)^{1/2}\geqslant\frac{\varepsilon}{2}\sqrt{k_0}. \tag{5.4}$$

Moreover, by passing to a common subsequence of each (z^i_n) if necessary, we may assume that the sequence (w_n) is block. Finally, we define $y_n\coloneqq\frac{w_n}{\|w_n\|}$ for every $n\in\mathbb{N}$. Clearly, (y_n) is a normalized block sequence in Y. We will show that for appropriate choices of k_0 and r we have that $\limsup\|P_\sigma(y_n)\|\leqslant\frac{\varepsilon}{2}$ for every $\sigma\in[T]$. This, clearly, leads to a contradiction.

To this end, let $\sigma\in[T]$ be arbitrary. Notice that there exists at most one $j\in\{1,\dots,k_0\}$ with the following property. There exists $t\in T_j$ with $t\sqsubset\sigma$. For this j we have the trivial estimate $\|P_\sigma(z^j_n)\|\leqslant 1$ for every $n\in\mathbb{N}$. Next, fix $i\in\{1,\dots,k_0\}$ with $i\neq j$. Then for every $t\in T_i$ the node t is not an initial segment of σ. We set $s^i\coloneqq\{t:t\sqsubset\sigma \text{ and } t\notin\sigma' \text{ for every } \sigma'\in G_i\}$, that is, s^i is the unique maximal final segment of σ which is disjoint from each G_i. Note that, by the choice of l_0, we have $\min\{|t|:t\in s^i\}\leqslant l_0$ and so, by (5.2), we have $\|P_\sigma(z^i_n)\|=\|P_{s^i}(z^i_n)\|$

for every $n \in \mathbb{N}$. On the other hand, the definition of s^i yields that $s^i \cap A_i = \emptyset$ and so, by (P2), there exists $n_i \in \mathbb{N}$ (clearly depending on σ) such that $\|P_{s^i}(z_n^i)\| < r$ for every $n \geqslant n_i$. Setting $n_\sigma \coloneqq \max\big\{n_i : i \in \{1, \dots, k_0\} \text{ with } i \neq j\big\}$, we see that $\|P_\sigma(z_n^i)\| < r$ for every $n \geqslant n_\sigma$ and every $i \in \{1, \dots, k_0\}$ with $i \neq j$. If follows that for every $n \geqslant n_\sigma$ we have

$$\|P_\sigma(w_n)\| = \|P_\sigma(z_n^1 + \dots + z_n^{k_0})\| \leqslant 1 + (k_0 - 1)r. \tag{5.5}$$

Combining inequalities (5.4) and (5.5), we obtain that

$$\|P_\sigma(y_n)\| = \Big\|P_\sigma\Big(\frac{w_n}{\|w_n\|}\Big)\Big\| \leqslant 2\ \frac{1 + (k_0 - 1)r}{\varepsilon\sqrt{k_0}}$$

for every $n \geqslant n_\sigma$. Hence, for every $\sigma \in [T]$ we have

$$\limsup \|P_\sigma(y_n)\| \leqslant 2\ \frac{1 + (k_0 - 1)r}{\varepsilon\sqrt{k_0}}.$$

Thus, it $k_0 > \frac{36}{\varepsilon^4}$ and $r < \frac{1}{2(k_0-1)}$, then we have $\limsup \|P_\sigma(y_n)\| \leqslant \frac{\varepsilon}{2}$ for every $\sigma \in [T]$ which is a contradiction. □

Lemma 5.7. *Let (x_n) be a bounded block sequence in $\mathcal{T}_2^X$ and let (ε_n) be a sequence of positive real numbers with $\lim \varepsilon_n = 0$. Assume that for every $n \in \mathbb{N}$ and every $\sigma \in [T]$ we have $\|P_\sigma(x_n)\| \leqslant \varepsilon_n$. Then (x_n) has a subsequence satisfying an upper ℓ_2 estimate.*

Proof. Since $\lim \varepsilon_n = 0$, recursively we select a subsequence (w_n) of (x_n) such that for every $n \geqslant 2$ and every $\sigma \in [T]$ we have $\|P_\sigma(w_n)\| \leqslant \frac{1}{\sum_{i=1}^{n-1} |\mathrm{supp}(w_i)|}\,\frac{1}{2^{2n}}$. The rest of the proof is identical to that of Proposition 4.9. □

We introduce the following definition.

Definition 5.8. *Let Z be a subspace of $\mathcal{T}_2^X$. We say that Z satisfies property $(*)$ if there exist $\varepsilon > 0$ and $\delta > 0$ such that for every normalized pointwise-null sequence (z_n) in Z with*

$$\limsup \|P_\sigma(z_n)\| < \delta$$

for every $\sigma \in [T]$, there exists $L \in [\mathbb{N}]^\infty$ such that the sequence $(z_n)_{n \in L}$ satisfies an ε-lower ℓ_2 estimate, that is, if $\{l_1 < l_2 < \cdots\}$ denotes the increasing enumeration of the set L, then for every $k \in \mathbb{N}$ and every $a_1, \dots, a_k \in \mathbb{R}$ we have that $\varepsilon\big(\sum_{i=1}^k a_i^2\big)^{1/2} \leqslant \|\sum_{i=1}^k a_i z_{l_i}\|$.

The importance of property $(*)$ is illustrated in the following proposition.

Proposition 5.9. *Let Y be a weakly X-singular subspace of $\mathcal{T}_2^X$ which satisfies property $(*)$. Then there exists a normalized pointwise-null sequence (y_n) in Y with the following properties.*

(i) *For every $\sigma \in [T]$ we have $\lim \|P_\sigma(y_n)\| = 0$.*
(ii) *The sequence (y_n) is equivalent to the standard unit vector basis of ℓ_2.*

Proof. As in Proposition 5.6, we will present the proof for block subspaces; the general case follows by identical arguments. So, let Y be a block weakly X-singular subspace of $\mathcal{T}_2^X$. First we remark that if Z is a block, finite-codimensional subspace of Y, then Z is also weakly X-singular and satisfies property $(*)$ with the same constants ε and δ. We continue with the following claim.

Claim. *For every block, finite-codimensional subspace Z of Y and every $r > 0$ there exists $z \in Z$ with $\|z\| = 1$ such that $\|P_\sigma(z)\| < r$ for every $\sigma \in [T]$.*

Proof of the claim. Let $r' > 0$ with $r' < \min\{r, \delta\}$ whose exact value will be determined later. Since Z is weakly X-singular, by Proposition 5.6, there exists a normalized block sequence (w_n) in Z such that for every $\sigma \in [T]$ we have

$$\limsup \|P_\sigma(w_n)\| \leqslant r'. \tag{5.6}$$

By Lemma 4.5, there exists $L \in [\mathbb{N}]^\infty$ such that

$$|\{n \in L : \|P_\sigma(w_n)\| \geqslant r'\}| \leqslant 1 \tag{5.7}$$

for every $\sigma \in [T]$. In particular, by (5.6) and the choice of r', for every $\sigma \in [T]$ we have $\limsup_{n\in L} \|P_\sigma(w_n)\| \leqslant \limsup \|P_\sigma(w_n)\| \leqslant r' < \delta$. Applying property $(*)$ for the sequence $(w_n)_{n\in L}$, we select $M \in [L]^\infty$ such that the sequence $(w_n)_{n\in M}$ satisfies an ε-lower ℓ_2 estimate. Let $\{m_1 < m_2 < \cdots\}$ be the increasing enumeration of M.

Let $k \in \mathbb{N}$ be arbitrary. Since $(w_n)_{n\in M}$ satisfies an ε-lower ℓ_2 estimate, we see that $\|\sum_{i=1}^k w_{m_i}\| \geqslant \varepsilon\sqrt{k}$. Next, let $\sigma \in [T]$ be arbitrary. By (5.7), we have that $\|P_\sigma\big(\sum_{i=1}^k w_{m_i}\big)\| \leqslant 1 + r'(k-1)$. Hence, setting

$$z_k := \frac{w_{m_1} + \cdots + w_{m_k}}{\|w_{m_1} + \cdots + w_{m_k}\|},$$

we conclude that $\|z_k\| = 1$ and $\|P_\sigma(z_k)\| \leqslant \frac{1+r'(k-1)}{\varepsilon\sqrt{k}}$ for every $k \in \mathbb{N}$ and every $\sigma \in [T]$. Thus, if k_0 and r' satisfy $k_0 \geqslant \frac{4}{\varepsilon^2 r^2}$ and $r' < \frac{1}{k_0 - 1}$, then the vector z_{k_0} is as desired. The claim is proved. □

By the above claim, there exists a normalized block sequence (y_n) in Y such that for every $n \in \mathbb{N}$ and every $\sigma \in [T]$ we have $\|P_\sigma(y_n)\| \leqslant \frac{1}{n}$. By Lemma 5.7, there exists $L \in [\mathbb{N}]^\infty$ such that the sequence $(y_n)_{n\in L}$ satisfies an upper ℓ_2 estimate. Invoking property $(*)$ once again, we select $M \in [L]^\infty$ such that the sequence $(y_n)_{n\in M}$ satisfies a lower ℓ_2 estimate. The sequence $(y_n)_{n\in M}$ is as desired. □

5.2. **Finding incomparable sets of nodes.** For every finitely supported vector z of $\mathcal{T}_2^X$ we denote by $\mathrm{range}(z)$ the minimal interval of $\mathbb{N}$ that contains $\mathrm{supp}(z)$. It is an immediate consequence of the enumeration of the basis of $\mathcal{T}_2^X$ that $\mathrm{range}(z)$, considered as a subset of T, is segment complete. (Recall that we enumerate T using a fixed bijection $h\colon T \to \mathbb{N}$ which satisfies that $h(t_1) < h(t_2)$ for every $t_1, t_2 \in T$ with $t_1 \sqsubset t_2$.) Our next goal is to prove the following proposition.

Proposition 5.10. *Let (z_n) be a normalized block sequence in $\mathcal{T}_2^X$ and let $\lambda > 0$ such that*
$$\{z_n : n \in \mathbb{N}\} \subseteq \lambda W_X + \frac{1}{200} B_{\mathcal{T}_2^X}.$$
Also let $r \leqslant \frac{1}{100^3 \cdot \lambda}$ and assume that
$$\limsup \|P_\sigma(z_n)\| < r$$
for every $\sigma \in [T]$. Then there exist $L \in [\mathbb{N}]^\infty$ and for every $n \in L$ a segment complete subset A_n of T such that the following are satisfied.

(I) *For every $n \in L$ we have $A_n \subseteq \mathrm{range}(z_n)$.*
(II) *If $n, m \in L$ with $n \neq m$, then A_n is incomparable with A_m.*
(III) *For every $n \in L$ we have $\|P_{A_n}(z_n)\| \geqslant 2/3$.*

Proof. Let $n \in \mathbb{N}$. By our assumptions, there exist $w_n \in W_X^0$ and $x_n \in \mathcal{T}_2^X$ such that $\|x_n\| \leqslant 1/100$ and $z_n = \lambda w_n + x_n$. Hence, $\|z_n - \lambda w_n\| \leqslant 1/100$. Set $R_n \coloneqq \mathrm{range}(z_n)$, and notice that we may assume that $\mathrm{supp}(w_n) \subseteq R_n$ for every $n \in \mathbb{N}$. Indeed, since R_n is segment complete, P_{R_n} is a norm-one projection, and so $P_{R_n}(W_X) \subseteq W_X$ and $P_{R_n}(B_{\mathcal{T}_2^X}) \subseteq B_{\mathcal{T}_2^X}$ for every $n \in \mathbb{N}$. Thus, in what follows, we will assume that $\mathrm{supp}(w_n) \subseteq R_n$; this implies, in particular, that the sequence (w_n) is block.

For every $n \in \mathbb{N}$ let $\{s_{1,n}, \dots, s_{d_n,n}\}$ be a collection of pairwise incomparable segments of T such that $s_{i,n} \subseteq \mathrm{range}(w_n) \subseteq R_n$ and
$$\|\lambda w_n\| = \Big(\sum_{i=1}^{d_n} \|P_{s_{i,n}}(\lambda w_n)\|^2 \Big)^{1/2}.$$
Since $\|z_n\| = 1$ and $\|z_n - \lambda w_n\| \leqslant 1/100$, we have $99/100 \leqslant \|\lambda w_n\| \leqslant 101/100$. Next, set $\theta \coloneqq 8^2/(100^2\lambda^2)$ and notice that $\lambda\sqrt{\theta} = 8/100$. We define
$$G_n \coloneqq \big\{ i \in \{1, \dots, d_n\} : \|P_{s_{i,n}}(w_n)\| \geqslant \theta \big\}.$$

Claim 1. *For every $n \in \mathbb{N}$ the following hold.*

(a) *We have $|G_n| \leqslant 4/(\lambda^2\theta^2)$.*
(b) *We have $\big(\sum_{i \in G_n} \|P_{s_{i,n}}(z_n)\|^2\big)^{1/2} \geqslant 9/10$.*

Proof of the claim. (a) Notice that
$$2 \geqslant \|\lambda w_n\| \geqslant \Big(\sum_{i \in G_n} \|P_{s_{i,n}}(\lambda w_n)\|^2 \Big)^{1/2} \geqslant \lambda \Big(\sum_{i \in G_n} \theta^2 \Big)^{1/2} = \lambda\, \theta \sqrt{|G_n|}$$
which implies that $|G_n| \leqslant 4/(\lambda^2\theta^2)$.

(b) Since $w_n \in W_X^0$, we have that $w_n = \sum_{j=1}^{k_n} a_j^n x_j^n$ where $\sum_{j=1}^{k_n} a_j^n = 1$, $a_j^n > 0$ and $x_j^n \in B_{\mathcal{X}_{\sigma_j^n}}$ for some $\sigma_j^n \in [T]$. For every $i \in \{1, \dots, d_n\}$ set $\beta_{i,n} \coloneqq \|P_{s_{i,n}}(w_n)\|$. We claim that $\sum_{i=1}^{d_n} \beta_{i,n} \leqslant 1$. Indeed, for every $i \in \{1, \dots, d_n\}$ set
$$H_i \coloneqq \big\{ j \in \{1, \dots, k_n\} : \mathrm{supp}(x_j^n) \cap s_{i,n} \neq \emptyset \big\}.$$

Since $\mathrm{supp}(x_j^n)$ is a chain and the family $\{s_{1,n},\dots,s_{d_n,n}\}$ consists of pairwise incomparable segments, we see that $H_{i_1}\cap H_{i_2}=\emptyset$ if $i_1\neq i_2$. Moreover,

$$\beta_{i,n}=\|P_{s_{i,n}}(w_n)\|=\Big\|P_{s_{i,n}}\Big(\sum_{j=1}^{k_n}a_j^n x_j^n\Big)\Big\|=\Big\|P_{s_{i,n}}\Big(\sum_{j\in H_i}a_j^n x_j^n\Big)\Big\|\leqslant\sum_{j\in H_i}a_j^n$$

and so

$$\sum_{i=1}^{d_n}\beta_{i,n}\leqslant\sum_{i=1}^{d_n}\sum_{j\in H_i}a_j^n\leqslant\sum_{j=1}^{k_n}a_j^n=1$$

which yields the desired estimate. By the definition of G_n, we see that if $i\notin G_n$, then $\beta_{i,n}<\theta$. By the choice of θ, it follows that

$$\begin{aligned}\Big(\sum_{i\notin G_n}\|P_{s_{i,n}}(\lambda w_n)\|^2\Big)^{1/2} &= \lambda\Big(\sum_{i\notin G_n}\beta_{i,n}^2\Big)^{1/2}<\lambda\Big(\sum_{i\notin G_n}\beta_{i,n}\theta\Big)^{1/2}\\ &= \lambda\sqrt{\theta}\Big(\sum_{i\notin G_n}\beta_{i,n}\Big)^{1/2}\leqslant\lambda\sqrt{\theta}=\frac{8}{100}.\end{aligned}$$

Also notice that

$$\begin{aligned}\frac{99}{100}\leqslant\|\lambda w_n\| &= \Big(\sum_{i\in G_n}\|P_{s_{i,n}}(\lambda w_n)\|^2+\sum_{i\notin G_n}\|P_{s_{i,n}}(\lambda w_n)\|^2\Big)^{1/2}\\ &\leqslant \Big(\sum_{i\in G_n}\|P_{s_{i,n}}(\lambda w_n)\|^2\Big)^{1/2}+\Big(\sum_{i\notin G_n}\|P_{s_{i,n}}(\lambda w_n)\|^2\Big)^{1/2}.\end{aligned}$$

Therefore, $\big(\sum_{i\in G_n}\|P_{s_{i,n}}(\lambda w_n)\|^2\big)^{1/2}\geqslant 91/100$. Finally, observe that

$$\begin{aligned}\frac{91}{100} &\leqslant \Big(\sum_{i\in G_n}\|P_{s_{i,n}}(\lambda w_n)\|^2\Big)^{1/2}\\ &\leqslant \Big(\sum_{i\in G_n}\|P_{s_{i,n}}(z_n)\|^2\Big)^{1/2}+\Big(\sum_{i\in G_n}\|P_{s_{i,n}}(\lambda w_n-z_n)\|^2\Big)^{1/2}\\ &\leqslant \Big(\sum_{i\in G_n}\|P_{s_{i,n}}(z_n)\|^2\Big)^{1/2}+\|z_n-\lambda w_n\|\\ &\leqslant \Big(\sum_{i\in G_n}\|P_{s_{i,n}}(z_n)\|^2\Big)^{1/2}+\frac{1}{100}\end{aligned}$$

which yields the desired estimate. The claim is proved. □

By part (a) of Claim 1, the choice of θ and by passing to a subsequence of (z_n) if necessary, we may assume that $|G_n|=k$ for every $n\in\mathbb{N}$ where $k\leqslant\big(\frac{4\cdot 100^4}{8^4}\big)\lambda^2$. For every $n\in\mathbb{N}$ we re-enumerate the family $\{s_{i,n}:i\in G_n\}$ of incomparable segments of T as $\{s_{1,n},\dots,s_{k,n}\}$.

Claim 2. *Let $i\in\{1,\dots,k\}$ and let $M_i\in[\mathbb{N}]^\infty$. Then there exists $N_i\in[M_i]^\infty$ and for every $n\in N_i$ disjoint segments $g_{i,n}$ and $b_{i,n}$ such that the following are satisfied.*

(i) *For every* $n \in N_i$ *we have* $s_{i,n} = g_{i,n} \cup b_{i,n}$ (*that is, the segments* $g_{i,n}$ *and* $b_{i,n}$ *form a partition of* $s_{i,n}$) *and, moreover, if* $t \in b_{i,n}$ *and* $t' \in g_{i,n}$, *then we have* $t \sqsubset t'$.

(ii) *For every* $n \in N_i$ *we have* $\|P_{b_{i,n}}(z_n)\| < r$.

(iii) *For every* $n, m \in N_i$ *with* $n \neq m$, *if* $g_{i,n}$ *and* $g_{i,m}$ *are nonempty, then* $g_{i,n}$ *is incomparable with* $g_{i,m}$.

Proof of the claim. For every $n \in M_i$ let t_n denote the $\sqsubset$-minimum of $s_{i,n}$. By Ramsey's theorem, there exists $I \in [M_i]^\infty$ such that either the sequence $(t_n)_{n\in I}$ consists of pairwise incomparable nodes, or the nodes $(t_n)_{n\in I}$ are mutually comparable. In the first case, we set $N_i := I$, and $g_{i,n} := s_{i,n}$ and $b_{i,n} := \emptyset$ for every $n \in N_i$. So, assume that the nodes $(t_n)_{n\in I}$ are pairwise comparable. Since $t_n \in s_{i,n} \subseteq \mathrm{range}(w_n) \subseteq \mathrm{range}(z_n)$ and the sequence (z_n) is block, we see that if $n, m \in I$ with $n < m$, then $t_n \sqsubset t_m$. Set $\sigma_i := \bigcup_{n\in I}\{t \in T : t \sqsubseteq t_n\} \in [T]$. By our assumptions for the sequence (z_n), we have

$$\limsup_{n\in I} \|P_{\sigma_i}(z_n)\| \leqslant \limsup \|P_{\sigma_i}(z_n)\| < r.$$

Hence, there exists $N_i \in [I]^\infty$ such that $\|P_{\sigma_i}(z_n)\| < r$ for every $n \in N_i$. For every $n \in N_i$ set $b_{i,n} := s_{i,n} \cap \sigma_i$ and $g_{i,n} := s_{i,n} \setminus b_{i,n}$. Since $s_{i,n}$ is a segment and σ_i is a branch, we see that both $b_{i,n}$ and $g_{i,n}$ are segments; consequently, part (i) of the claim is satisfied. The Schauder tree basis $(x_t)_{t\in T}$ of X is bimonotone, and so for every $n \in N_i$ we have $\|P_{b_{i,n}}(z_n)\| \leqslant \|P_{\sigma_i}(z_n)\| < r$; that is, part (ii) is satisfied. We will verify part (iii). To this end let $n, m \in N_i$ with $n < m$ and assume, towards a contradiction, that $g_{i,n}$ and $g_{i,m}$ are nonempty and comparable. The sequence (z_n) is block and $n < m$. Therefore, there exists $t \in g_{i,n}$ with $t \sqsubset t_m$. (Recall that t_m is the $\sqsubset$-minimum node of $s_{i,m}$.) It follows that $t \sqsubset \sigma_i$ which contradicts the definition of $g_{i,n}$. The claim is proved. □

Applying Claim 2 recursively for every $i \in \{i, \dots, k\}$, we obtain $N \in [\mathbb{N}]^\infty$ and for every $n \in N$ and every $i \in \{1, \dots, k\}$ disjoint segments $g_{i,n}$ and $b_{i,n}$ such that the following are satisfied.

(P1) For every $n \in N$ and every $i \in \{1, \dots, k\}$ we have $s_{i,n} = g_{i,n} \cup b_{i,n}$ (that is, the segments $g_{i,n}$ and $b_{i,n}$ form a partition of $s_{i,n}$), and if $t \in b_{i,n}$ and $t' \in g_{i,n}$, then $t \sqsubset t'$.

(P2) For every $n \in N$ and every $i \in \{1, \dots, k\}$ we have $\|P_{b_{i,n}}(z_n)\| < r$.

(P3) For every $i \in \{1, \dots, k\}$ and every $n, m \in N$ with $n \neq m$ if $g_{i,n}$ and $g_{i,m}$ are nonempty, then $g_{i,n}$ is incomparable with $g_{i,m}$.

For every $i, j \in \{1, \dots, k\}$ we set

$$C_{i,j} := \big\{(n, m) \in [N]^2 : g_{i,n} \text{ and } g_{j,m} \text{ are nonempty and comparable}\big\}.$$

and

$$B := [N]^2 \setminus \Big(\bigcup_{i,j\in\{1,\dots,k\}} C_{i,j} \Big).$$

By Ramsey's theorem, there exists $L \in [N]^\infty$ which is monochromatic. We claim that $[L]^2 \subseteq B$. Assume not. Then there exist $i, j \in \{1, \dots, k\}$ such that $[L]^2 \subseteq C_{i,j}$. Let $\{l_1 < l_2 < l_3 < \cdots\}$ denote the increasing enumeration of L. Notice that both g_{i,l_1} and g_{i,l_2} are comparable with g_{j,l_3}. Let t_1, t_2 and t_3 be the $\sqsubset$-minimum nodes of g_{i,l_1}, g_{i,l_2} and g_{j,l_3} respectively. Since $l_1 < l_2 < l_3$ and the sequence (z_n) is block, we see that $t_1 \sqsubset t_3$ and $t_2 \sqsubset t_3$. But then t_1 must be comparable with t_2, which implies that g_{i,l_1} is comparable with g_{i,l_2}. This contradicts property (P3), since $l_1, l_2 \in L$ and $L \in [N]^\infty$. Therefore, $[L]^2 \subseteq B$.

For every $n \in L$ we set $A_n := \bigcup_{i \in \{1,\dots,k\}} g_{i,n}$. The fact that $[L]^2 \subseteq B$ implies that if $n, m \in L$ with $n \neq m$, then A_n is incomparable with A_m. Also notice that for every $n \in L$ we have $A_n \subseteq \bigcup_{i \in \{1,\dots,k\}} s_{i,n} \subseteq \mathrm{range}(w_n) \subseteq \mathrm{range}(z_n)$. It remains to estimate the quantity $\|P_{A_n}(z_n)\|$ for every $n \in L$. Fix $n \in L$. By our hypotheses on r and the estimate on k, we have

$$r\sqrt{k} \leqslant \Big(\frac{1}{100^3 \cdot \lambda}\Big) \cdot \Big(\frac{2 \cdot 100^2 \cdot \lambda}{8^2}\Big) \leqslant \frac{1}{10}.$$

By property (P1), we have $\|P_{s_{i,n}}(z_n)\| \leqslant \|P_{g_{i,n}}(z_n)\| + \|P_{b_{i,n}}(z_n)\|$. Therefore, by property (P2), we obtain that

$$\begin{aligned}
\Big(\sum_{i=1}^k \|P_{s_{i,n}}(z_n)\|^2\Big)^{1/2} &\leqslant \Big(\sum_{i=1}^k \|P_{g_{i,n}}(z_n)\|^2\Big)^{1/2} + \Big(\sum_{i=1}^k \|P_{b_{i,n}}(z_n)\|^2\Big)^{1/2} \\
&\leqslant \Big(\sum_{i=1}^k \|P_{g_{i,n}}(z_n)\|^2\Big)^{1/2} + r\sqrt{k} \\
&\leqslant \Big(\sum_{i=1}^k \|P_{g_{i,n}}(z_n)\|^2\Big)^{1/2} + \frac{1}{10}.
\end{aligned}$$

By part (b) of Claim 1, we conclude that

$$\begin{aligned}
\|P_{A_n}(z_n)\| &\geqslant \Big(\sum_{i=1}^k \|P_{g_{i,n}}(z_n)\|^2\Big)^{1/2} \geqslant \Big(\sum_{i=1}^k \|P_{s_{i,n}}(z_n)\|^2\Big)^{1/2} - \frac{1}{10} \\
&\geqslant \frac{9}{10} - \frac{1}{10} \geqslant \frac{2}{3}
\end{aligned}$$

and the proof is completed. □

5.3. **Singularity and thinness.** We start with the following lemma.

Lemma 5.11. *Let Z be a subspace of $\mathcal{T}_2^X$ and let $\lambda > 0$ such that*

$$B_Z \subseteq \lambda W_X + \frac{1}{200} B_{\mathcal{T}_2^X}.$$

Then Z satisfies property $(*)$ for $\delta = \frac{1}{100^3 \cdot \lambda}$ and $\varepsilon = \frac{1}{2}$.

Proof. As in Proposition 5.6, we will give the proof under the assumption that Z is a block subspace, and we will work with block sequences instead of pointwise-null sequences; the general case follows using identical arguments. By Definition 5.8, in order to verify that Z has property $(*)$ for $\delta = \frac{1}{100^3\cdot\lambda}$ and $\varepsilon = \frac{1}{2}$, let (z_n) be a normalized block sequence in Z such that for every $\sigma \in [T]$ we have

$$\limsup \|P_\sigma(z_n)\| < \frac{1}{100^3\cdot\lambda}.$$

By our assumptions on Z, we may apply Proposition 5.10 for the sequence (z_n) and $r = \frac{1}{100^3\lambda}$, and we obtain $L \in [\mathbb{N}]^\infty$ and for every $n \in L$ a segment complete set A_n such that the following are satisfied.

(I) For every $n \in L$ we have $A_n \subseteq \mathrm{range}(z_n)$.
(II) If $n, m \in L$ with $n \neq m$, then A_n is incomparable with A_m.
(III) For every $n \in L$ we have $\|P_{A_n}(z_n)\| \geqslant 2/3$.

We select a sequence $(z_n^*)_{n\in L}$ in $(\mathcal{T}_2^X)^*$ with the following properties.

(a) For every $n \in L$ we have $\|z_n^*\| \leqslant 1$ and $z_n^*(z_n) \geqslant 1/2$.
(b) For every $n \in L$ we have that $\mathrm{supp}(z_n^*) \subseteq A_n$.

Let $\{l_1 < l_2 < \cdots\}$ be the increasing enumeration of L, and observe that the following hold.

(i) For every $k \in \mathbb{N}$, if $a_1, \dots, a_k \in \mathbb{R}$ with $\sum_{i=1}^k a_i^2 = 1$, then the functional $\sum_{i=1}^k a_i z_{l_i}^*$ has norm at most one.
(ii) By (I) and (b), for every $i, n \in L$ with $i \neq n$ we have $z_n^*(z_i) = 0$.

Using (i) and (ii), it is easy to verify that the sequence $(z_n)_{n\in L}$ satisfies an $\frac{1}{2}$-lower ℓ_2 estimate. The proof is completed. □

Lemma 5.12. *Let Z be a weakly X-singular subspace of $\mathcal{T}_2^X$. Assume that W_X almost absorbs B_Z. Then there exist a sequence (z_n) in Z, a sequence (A_n) of subsets of T and a sequence (z_n^*) in $(\mathcal{T}_2^X)^*$ such that the following are satisfied.*

(1) *(z_n) is normalized, pointwise-null and equivalent to the ℓ_2 basis.*
(2) *For every $n \in \mathbb{N}$ we have that A_n is segment complete, and if $n \neq m$, then A_n is incomparable with A_m. Moreover, if $n < m$, then $h(A_n) < h(A_m)$ where $h\colon T \to \mathbb{N}$ is the fixed enumeration of T.*
(3) *For every $n \in \mathbb{N}$ we have* $\mathrm{supp}(z_n^*) \subseteq A_n$, $\|z_n^*\| \leqslant 1$ *and* $z_n^*(z_n) \geqslant 1/2$.

Proof. Again, we will assume that Z is a block subspace and we will work with block sequences. Since W_X almost absorbs B_Z, there exists $\lambda > 0$ such that $B_Z \subseteq \lambda W_X + \frac{1}{200} B_{\mathcal{T}_2^X}$. By Lemma 5.11, we see that Z has property $(*)$ for $\delta = \frac{1}{100^3\cdot\lambda}$ and $\varepsilon = \frac{1}{2}$. By Proposition 5.9 (and its proof), there exists a normalized block sequence (z_n) in Z such that the following are satisfied.

(I) There exists $C > 0$ such that for every $k \in \mathbb{N}$ and $a_1, \dots, a_k \in \mathbb{R}$ we have

$$\frac{1}{2}\Big(\sum_{i=1}^k a_i^2\Big)^{1/2} \leqslant \Big\|\sum_{i=1}^k a_i z_i\Big\| \leqslant C\Big(\sum_{i=1}^k a_i^2\Big)^{1/2}.$$

(II) For every $\sigma \in [T]$ we have $\lim \|P_\sigma(z_n)\| = 0$.

Using (II), the fact that $B_Z \subseteq \lambda W_X + \frac{1}{200} B_{\mathcal{T}_2^X}$ and arguing as in the proof of Lemma 5.11 for the sequence (z_n), we obtain $L \in [\mathbb{N}]^\infty$ and for every $n \in L$ a segment complete set $A_n \subseteq \mathrm{range}(z_n)$ and $z_n^* \in (\mathcal{T}_2^X)^*$ such that (2) and (3) in the statement of the lemma are satisfied. The proof is completed. □

Proposition 5.13. *Let Z be a weakly X-singular subspace of $\mathcal{T}_2^X$. Then the set W_X does not almost absorb B_Z.*

Proof. As in the previous two lemmas, we will assume that Z is a block subspace. Assume, towards a contradiction, that W_X almost absorbs B_Z. Since Z is weakly X-singular, we obtain sequences (z_n), (A_n) and (z_n^*) as described in Lemma 5.12. We set $\Omega := \overline{\mathrm{span}}\{z_n : n \in \mathbb{N}\}$ and we define $P \colon \mathcal{T}_2^X \to \Omega$ by $P(x) = \sum_{n\in\mathbb{N}} \frac{z_n^*(x)}{z_n^*(z_n)} z_n$. Using the fact that the sequence (z_n) is equivalent to the ℓ_2 basis and that the vectors (z_n^*) are supported in incomparable sets of nodes, we see that P is a bounded projection. We set $C := \|P\| < \infty$. (Actually, it is easy to see that $C \leqslant 2$.)

Claim. *We have $P(W_X) \subseteq \overline{\mathrm{conv}}\{\pm 2z_n : n \in \mathbb{N}\}$.*

Proof of the claim. Let $w \in W_X^0$ be arbitrary. Then w is of the form $w = \sum_{i=1}^l a_i x_i$ where $\sum_{i=1}^l a_i = 1$ with $a_i > 0$ and $x_i \in B_{\mathcal{X}_{\sigma_i}}$ for some $\sigma_i \in [T]$. For every $i \in \{1, \dots, l\}$ let s_i be the unique minimal segment of T that contains $\mathrm{supp}(x_i)$. For every $n \in \mathbb{N}$ we set $F_n := \big\{ i \in \{1, \dots, l\} : s_i \cap A_n \neq \emptyset \big\}$. The sets (A_n) are pairwise incomparable, and so $F_n \cap F_m = \emptyset$ if $n \neq m$. Moreover,

$$z_n^*(w) = z_n^*\big(\sum_{i\in F_n} a_i x_i \big) = \sum_{i \in F_n} a_i z_n^*(x_i) \leqslant \sum_{i\in F_n} a_i.$$

Thus, setting $\theta_n := \frac{z_n^*(w)}{z_n^*(z_n)}$, we see that $\sum_{n\in\mathbb{N}} |\theta_n| \leqslant 2$. By the definition of P, this yields that $P(W_X^0) \subseteq \mathrm{conv}\{\pm 2z_n : n \in \mathbb{N}\}$ and the proof is completed. □

Since Ω is a subspace of Z and W_X almost absorbs B_Z, we see that W_X must also almost absorb B_Ω. Thus, there exists $r > 0$ such that $B_\Omega \subseteq rW_X + \frac{1}{2C} B_{\mathcal{T}_2^X}$. Consequently, we have $B_\Omega \subseteq rP(W_X) + \frac{1}{2} B_\Omega$, since P is a projection with $\|P\| = C$. By standard arguments (see, e.g., [AF, Lemma 4.8]), we obtain that $B_\Omega \subseteq 2r\overline{P(W_X)}$ and so, by the previous claim, we conclude that

$$B_\Omega \subseteq 2r\, \overline{\mathrm{conv}}\{\pm 2z_n : n \in \mathbb{N}\}. \tag{5.8}$$

This is a contradiction, since (5.8) implies that the ℓ_2 norm is equivalent to the ℓ_1 norm. The proof is completed. □

The following proposition is the analogue of Proposition 5.13 for sequences.

Proposition 5.14. *Let (v_k) be a bounded block sequence in $\mathcal{T}_2^X$ and let $\varepsilon > 0$ such that the following hold.*

- (I) *For every $k \in \mathbb{N}$ we have $\|v_k\| > \varepsilon$.*
- (II) *For every $\sigma \in [T]$ we have $\lim \|P_\sigma(v_k)\| = 0$.*
- (III) *The set W_X almost absorbs the set $\{v_k : k \in \mathbb{N}\}$.*

Then there exists $L \in [\mathbb{N}]^\infty$ and for every $k \in L$ a segment complete subset A_k of T and a vector $z_k^* \in (\mathcal{T}_2^X)^*$ such that the following hold.

(a) *The sets $(A_k)_{k\in L}$ are pairwise incomparable.*
(b) *For every $k \in L$ we have $\|z_k^*\| \leqslant 1$ and $\mathrm{supp}(z_k^*) \subseteq A_k \subseteq \mathrm{range}(v_k)$.*
(c) *For every $k \in L$ we have $z_k^*(v_k) \geqslant \frac{\varepsilon}{2}$.*

Proof. We set $C \coloneqq \sup\{\|v_k\| : k \in \mathbb{N}\} < \infty$ and $z_k \coloneqq \frac{v_k}{\|v_k\|}$ for every $k \in \mathbb{N}$. Then (z_k) is a normalized block sequence and, moreover, $\lim \|P_\sigma(z_k)\| = 0$ for every $\sigma \in [T]$. The set W_X almost absorbs the set $\{v_k : k \in \mathbb{N}\}$, and so there exists $\lambda' > 0$ such that $\{v_k : k \in \mathbb{N}\} \subseteq \lambda' W_X + \frac{\varepsilon}{200} B_{\mathcal{T}_2^X}$. Set $\lambda \coloneqq \lambda'/\varepsilon$ and notice that

$$z_k = \frac{v_k}{\|v_k\|} \in \frac{\lambda'}{\|v_k\|} W_X + \frac{\varepsilon}{200\,\|v_k\|} B_{\mathcal{T}_2^X} \subseteq \lambda W_X + \frac{1}{200} B_{\mathcal{T}_2^X}$$

for every $k \in \mathbb{N}$. By Proposition 5.10 applied for (z_k) and $r = \frac{1}{100^3\cdot\lambda}$, we obtain $L \in [\mathbb{N}]^\infty$ and for every $k \in L$ a segment complete subset A_k of T such that the following are satisfied.

(I) For every $k \in L$ we have $A_k \subseteq \mathrm{range}(z_k)$.
(II) If $n, m \in L$ with $n \neq m$, then A_n is incomparable with A_m.
(III) For every $k \in L$ we have $\|P_{A_k}(z_k)\| \geqslant 2/3$.

Next, as in the proof of Lemma 5.11, we select a sequence $(z_k^*)_{k\in L}$ in $(\mathcal{T}_2^X)^*$ such that the following are satisfied.

(a) For every $k \in L$ we have $\|z_k^*\| \leqslant 1$ and $\mathrm{supp}(z_k^*) \subseteq A_k \subseteq \mathrm{range}(v_k)$.
(b) For every $k \in L$ we have $z_k^*(z_k) \geqslant \frac{1}{2}$ and, consequently, $z_k^*(v_k) \geqslant \frac{\|v_k\|}{2} \geqslant \frac{\varepsilon}{2}$.

The proof is completed. □

We are ready to state the main results in this subsection.

Theorem 5.15. *Let Y be an X-singular subspace of $\mathcal{T}_2^X$. Then W_X is thin on Y.*

Proof. Assume, towards a contradiction, that W_X is not thin on Y. Thus, there exists a subspace Z of Y such that W_X almost absorbs B_Z. Clearly Z is X-singular and so, by part (a) of Remark 6, Z is weakly X-singular. By Proposition 5.13, we derive a contradiction. □

We also need the following slightly stronger version of Theorem 5.15.

Theorem 5.16. *Let Y be a subspace of $\mathcal{T}_2^X$. If W_X almost absorbs B_Y, then there exists finite $A \subseteq [T]$ such that the operator $P_A\colon Y \to \mathcal{X}_A$ is an isomorphic embedding.*

Proof. Assume not, that is, for every finite $A \subseteq [T]$ the operator $P_A\colon Y \to \mathcal{X}_A$ is not an isomorphic embedding. According to our terminology, this is equivalent to saying that Y is weakly X-singular. By Proposition 5.13, we see that W_X does not almost absorb B_Y and we derive a contradiction. □

6. HI Schauder sums

In Sections 6 and 7 we present, briefly, the definition and the main properties of HI interpolations. Our definition is similar to the one introduced in [AF]. In the present setting the interpolation space has a Schauder basis under some mild assumptions on the set W. Furthermore, the HI Schauder sums are defined with the use of the $\big(\mathcal{A}_{n_l}, \frac{1}{m_l}\big)$-saturation families. We note that the reader who is interested exclusively in p-amalgamations can skip Sections 6 and 7, and proceed directly to Section 8.

6.1. We start by introducing some pieces of notation.

Notation. We define $j, \pi\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ by setting $j\big((n,k)\big)=n$ and $\pi\big((n,k)\big)=k$. Moreover, for every $x\in c_{00}(\mathbb{N}\times\mathbb{N})$ by range(x) we denote the rectangle $I\times J$ with I,J intervals of $\mathbb{N}$, and which is the minimal rectangle of this form that contains the support supp(x) of the vector x.

Notation. Let $A,B\subseteq\mathbb{N}\times\mathbb{N}$. We write $A\prec_\pi B$ provided that $\pi(A)<\pi(B)$ (that is, $\max\{k : k\in\pi(A)\}<\min\{k:\pi(B)\}$); respectively, we write $A\prec_j B$ if $j(A)<j(B)$. Finally, we write $A\prec_{(j,\pi)} B$ if $A\prec_j B$ and $A\prec_\pi B$.

More generally, given $x,y\in c_{00}(\mathbb{N}\times\mathbb{N})$ we write $x\prec_j y$ (respectively, $x\prec_\pi y$ and $x\prec_{(j,\pi)} y$) provided that $\mathrm{supp}(x)\prec_j \mathrm{supp}(y)$ (respectively, $\mathrm{supp}(x)\prec_\pi \mathrm{supp}(y)$ and $\mathrm{supp}(x)\prec_{(j,\pi)}\mathrm{supp}(y)$).

Definition 6.1. *We say that a sequence (x_n) in $c_{00}(\mathbb{N}\times\mathbb{N})$ is* j-block *(respectively,* π-block*) if $x_n\prec_j x_{n+1}$ (respectively, $x_n\prec_\pi x_{n+1}$) for every $n\in\mathbb{N}$. We say that (x_n) is* diagonally block *if $x_n\prec_{(j,\pi)} x_{n+1}$ for every $n\in\mathbb{N}$.*

Definition 6.2. *Let (X_n) be a sequence of separable Banach spaces. An* HI Schauder sum *of (X_n) is a Banach space $\mathfrak{X}=(\sum_{n\in\mathbb{N}}\oplus X_n)_{\mathrm{hi}}$ with the following properties.*

(i) *The sequence (X_n) defines a Schauder decomposition of $\mathfrak{X}$ (that is, every $x\in\mathfrak{X}$ has a unique representation of the form $x=\sum_n x_n$ with $x_n\in X_n$ for every $n\in\mathbb{N}$).*
(ii) *Every subspace Y of $\mathfrak{X}$ either contains a HI subspace, or there exists $n\in\mathbb{N}$ such that the natural projection $j_n\colon Y\to X_n$ is not strictly singular.*

Remark 7. In [AF], it was shown that for every sequence (X_n) of separable Banach spaces there exists a HI Schauder sum of (X_n). The purpose of this section is to provide a variant of the construction presented in [AF] in the special case where each X_n has a bimonotone Schauder basis $(x_{n,k})_{k\in\mathbb{N}}$. This variant satisfies the additional property that the HI Schauder sum $\mathfrak{X}$ admits an alternative Schauder decomposition $\mathfrak{X}=(\sum_{k\in\mathbb{N}}\oplus Z_k)_\#$ where $Z_k=\overline{\mathrm{span}}\{x_{n,k}:n\in\mathbb{N}\}$ for every $k\in\mathbb{N}$. This property (together with some additional hypotheses) will be used that the interpolation space Δ_X has a Schauder basis.

Definition 6.3. *Let X be a Banach space with a bimonotone basis (x_n). We define the subset G_X of $c_{00}(\mathbb{N})$ by the rule*

$$G_X \coloneqq \Big\{ \sum_{i=1}^{n} a_i x_i^* : n \in \mathbb{N},\, a_1, \dots, a_n \in \mathbb{Q} \text{ and } \Big\| \sum_{i=1}^{n} a_i x_i^* \Big\| \leqslant 1 \Big\}.$$

Observe that the following hold.

(i) The Banach space X is isometric (in the natural way) with the completion of $c_{00}(\mathbb{N})$ with the norm $\|\cdot\|_{G_X}$ where, as usual, for every $x \in c_{00}(\mathbb{N})$ we set $\|x\|_{G_X} \coloneqq \sup\{\phi(x) : \phi \in G_X\}$.

(ii) The set G_X is countable, symmetric, closed in the restrictions on intervals (since the basis (x_n) is bimonotone) and contains the sequence (x_n^*).

Definition 6.4. *Let (X_n) be a sequence of Banach spaces each of which has a bimonotone Schauder basis $(x_{n,k})_{k\in\mathbb{N}}$. For every $n \in \mathbb{N}$ we denote by G_n the set G_{X_n} described in Definition 6.3, and we view G_n as a subset of $c_{00}(\{n\} \times \mathbb{N})$ in the natural way. In particular, $\bigcup_n G_n$ is a (well-defined) subset of $c_{00}(\mathbb{N} \times \mathbb{N})$.*

6.2. For the rest of this section let (X_n) denote a sequence of Banach spaces each of which has a bimonotone Schauder basis $(x_{n,k})_{k\in\mathbb{N}}$, and let G_n denote the subsets of $c_{00}(\mathbb{N} \times \mathbb{N})$ described in Definition 6.4. We fix two sequence (m_l) and (n_l) defined recursively by setting $m_1 \coloneqq 2, m_{l+1} \coloneqq m_l^5$, and $n_1 \coloneqq 4$, $n_{l+1} \coloneqq (5n_l)^{s_l}$ where $s_l \coloneqq \log_2 m_{l+1}$. We define the set G to be the minimal subset of $c_{00}(\mathbb{N} \times \mathbb{N})$ with the following properties.

(I) We have $\bigcup_n G_n \subseteq G$; moreover, G is closed under the projection on rectangles of the form $I \times J$ where I, J are intervals of $\mathbb{N}$ (that is, if $f \in G$ and I, J are intervals of $\mathbb{N}$, then $(I \times J) \cdot f \coloneqq \mathbf{1}_{I\times J} \cdot f \in G$).

(II) For every $l \in \mathbb{N}$ the set G is closed in the $\big(\mathcal{A}_{n_{2l}}, \frac{1}{m_{2l}}\big)$-operation on j-block sequences; that is, if $f_1 \prec_j \cdots \prec_j f_{n_{2l}}$, then $\frac{1}{m_{2l}} \sum_{i=1}^{n_{2l}} f_i \in G$.

(III) For every $l \in \mathbb{N}$ the set G is closed in the $\big(\mathcal{A}_{n_{2l-1}}, \frac{1}{m_{2l-1}}\big)$-operation on (n_{2l-1})-special sequences.

(IV) The set G is rationally convex.

Of course, we need to determine the (n_{2l-1})-special sequences; they are defined using a coding function σ. Before we proceed to the details, we introduce some terminology. For every $l \in \mathbb{N}$ if $f \in G$ is the result of the $\big(\mathcal{A}_{n_l}, \frac{1}{m_l}\big)$-operation, then we denote the positive integer m_l by $w(f)$ and we call it the *weight* $w(f)$ of f. Note that $w(f)$ is not uniquely defined.

6.3. **The coding function σ.** First we consider a set $\mathcal{S}$ of finite sequences in $c_{00}(\mathbb{N} \times \mathbb{N})$ defined by the rule

$$\mathcal{S} \coloneqq \big\{ (\phi_1, \dots, \phi_d) \,:\, \phi_1 \prec_j \cdots \prec_j \phi_d, \text{ and } \phi_i(n,k) \in \mathbb{Q} \text{ for every } (n,k) \in \mathbb{N} \times \mathbb{N} \text{ and every } i \in \{1, \dots, d\} \big\}.$$

We fix a pair Ω_1, Ω_2 of disjoint infinite subsets of $\mathbb{N}$. Since $\mathcal{S}$ is countable, we may also fix an injection $\sigma\colon \mathcal{S} \to \{2l : l \in \Omega_2\}$ such that

$$m_{\sigma(\phi_1,\dots,\phi_d)} > \max\Big\{\frac{1}{|\phi_i(n,k)|} : (n,k) \in \mathrm{supp}(\phi_i) \text{ and } i \in \{1,\dots,d\}\Big\} \times \\ \times \max\{k : (n,k) \in \mathrm{supp}(\phi_d)\}.$$

We say that a finite sequence $(f_i)_{i=1}^{n_{2l-1}}$ is (n_{2l-1})-*special* provided that

(a) $(f_1,\dots,f_{n_{2l-1}}) \in \mathcal{S}$ and $f_i \in G$ for every $i \in \{1,\dots,n_{2l-1}\}$, and
(b) $w(f_1) = m_{2k}$ with $k \in \Omega_1$, $m_{2k}^{1/2} > n_{2l-1}$, and $w(f_i) = m_{\sigma(f_1,\dots,f_{i-1})}$ for every $i \in \{2,\dots,n_{2l-1}\}$.

Remark 8. As we have already pointed out, the weight $w(f)$ of a functional f is not unique. However, if $(f_1,\dots,f_{n_{2l-1}})$ is a (n_{2l-1})-special sequence, then for every $i \in \{2,\dots,n_{2l-1}\}$ we set $w(f_i) \coloneqq m_{\sigma(f_1,\dots,f_{i-1})}$.

6.4. We define

$$\mathfrak{X}_G \coloneqq T\Big[(G_n), \big(\mathcal{A}_{n_l}, \frac{1}{m_l}\big), \sigma\Big]$$

to be the completion of $c_{00}(\mathbb{N}\times\mathbb{N})$ with the norm $\|\cdot\|_G$ where G is the set defined in Subsection 6.2.

Remark 9. The following hold.

(1) For every $n \in \mathbb{N}$ the space X_n is isometric to $\overline{\mathrm{span}}\{x_{n,k} : k \in \mathbb{N}\} \hookrightarrow \mathfrak{X}_G$.
(2) For every pair I, J of (finite or infinite) intervals of $\mathbb{N}$ the projection

$$P_{I\times J}\colon \mathfrak{X}_G \to \mathfrak{X}_{I\times J} \coloneqq \overline{\mathrm{span}}\{x_{n,k} : n \in I, k \in J\}$$

has norm one. Consequently, the following are satisfied.
   (a) The sequence (X_n) defines a Schauder decomposition of $\mathfrak{X}_G$.
   (b) Setting $Z_k = \overline{\mathrm{span}}\{x_{n,k} : n \in \mathbb{N}\}$ for every $k \in \mathbb{N}$, the sequence (Z_k) also defines a Schauder decomposition of $\mathfrak{X}_G$.
(3) Every j-block, and every π-block, sequence is a bimonotone basic sequence. In particular, every diagonally block sequence is bimonotone basic sequence.

6.5. We proceed to present the basic ingredients which are needed for the proof of the fact that certain block sequences in $\mathfrak{X}_G$ generate HI spaces. We start with the following definition.

Definition 6.5. *Let $x \in c_{00}(\mathbb{N}\times\mathbb{N})$ and $C > 1$. We say that x is a $C - \ell_k^1$* average *if there exists a j-block sequence $x_1 \prec_j \cdots \prec_j x_k$ such that $x = \frac{1}{k}(x_1 + \cdots + x_k)$, $\|x_i\|_G \leqslant C$ for every $i \in \{1,\dots,k\}$, and $\|x\|_G = 1$.*

We have the following lemma; see, e.g., [ATo, Lemma 2.22] or [AM, Lemma 4.6] for a proof.

Lemma 6.6. *For every j-block sequence (y_n) and every $k \in \mathbb{N}$ there exists a $2 - \ell_k^1$ average in* $\mathrm{span}\{y_n : n \in \mathbb{N}\}$.

The following result has its roots in Schlumprecht's paper [Schl].

Lemma 6.7. *Let (x_q) be a j-block sequence such that each x_q is a $C-\ell^1_{k_q}$ average, where $C>1$ and the sequence (k_q) is increasing to infinity. Then for every $l\in\mathbb{N}$ there exists $q_1<\cdots<q_{n_{2l}}$ such that*

$$\Big\|\frac{1}{n_{2l}}(x_{q_1}+\cdots+x_{q_{n_{2l}}})\Big\|\leqslant\frac{3C}{m_{2l}}.$$

The proof of Lemma 6.7 (which is based on the concept of an R.I.S. sequence and the basic inequality) is identical to the proof of [ATo, Subsection 2.2].

Definition 6.8 (exact pair)**.** *Let $x\in c_{00}(\mathbb{N}\times\mathbb{N})$ and let $\phi\in G$. Also let $C>0$ and $l\in\mathbb{N}$. We say that (x,ϕ) is* (C,l)-exact pair *if the following are satisfied.*

(1) *We have $1\leqslant\|x\|_G\leqslant C$; moreover, for every $f\in G$ with $w(f)=m_q$ and $q\neq l$ we have $|f(x)|\leqslant 3C/m_q$ if $q<l$, while $|f(x)|\leqslant C/m_l^2$ if $q>l$.*
(2) *The functional ϕ is the result of the $\big(\mathcal{A}_{n_l},\frac{1}{m_l}\big)$-operation; thus, $w(\phi)=m_l$.*
(3) *We have that $\phi(x)=1$ and* $\mathrm{range}(x)=\mathrm{range}(\phi)$. (*Recall that the range of a vector in $c_{00}(\mathbb{N}\times\mathbb{N})$ is the minimal rectangle generated by intervals which contains its support.*)

The following proposition is a consequence of Lemmas 6.6 and 6.7.

Proposition 6.9. *If (x_q) is a j-block sequence, then for every $l\in\mathbb{N}$ there exists an $(6,2l)$-exact pair (x,ϕ) where $x\in\mathrm{span}\{x_q:q\in\mathbb{N}\}$ and $\phi\in G$.*

We need to introduce some terminology. We say that a (possibly finite) j-block sequence in $c_{00}(\mathbb{N}\times\mathbb{N})$ is *special j-block* if either it is diagonally block, or there exists $k\in\mathbb{N}$ such that its members are all supported in $\mathbb{N}\times\{k\}$.

Definition 6.10 (dependent sequences)**.** *Let $C>0$ and $l\in\mathbb{N}$. Let $(x_k)_{k=1}^{n_{2l-1}}$ be a special j-block sequence, and for every $k\in\{1,\dots,n_{2l-1}\}$ let $\phi_k\in G$. We say that $(x_k,\phi_k)_{k=1}^{n_{2l-1}}$ is* $(C,2l-1)$-dependent sequence *if there exists a sequence $(2l_k)_{k=1}^{n_{2l-1}}$ of even integers such that the following hold.*

(i) *We have that $(\phi_k)_{k=1}^{n_{2l-1}}$ is a (n_{2l-1})-special sequence with $w(\phi_k)=m_{2l_k}$ for every $k\in\{1,\dots,n_{2l-1}\}$.*
(ii) *Each (x_k,ϕ_k) is a $(C,2l_k)$-exact par.*

We have the following proposition.

Proposition 6.11. *Let $(x_k,\phi_k)_{k=1}^{n_{2l-1}}$ be a $(C,2l-1)$-dependent sequence. Then*

$$\Big\|\frac{1}{n_{2l-1}}\sum_{k=1}^{n_{2l-1}}x_k\Big\|\geqslant\frac{1}{m_{2l-1}}\tag{6.1}$$

and

$$\Big\|\frac{1}{n_{2l-1}}\sum_{k=1}^{n_{2l-1}}(-1)^k x_k\Big\|\leqslant\frac{8C}{m_{2l-1}^2}.\tag{6.2}$$

We notice that inequality (6.1) is straightforward, since the special functional $\frac{1}{m_{2l-1}}\sum_{k=1}^{n_{2l-1}}\phi_k$ belongs to G. However, the estimate (6.2) is not easy. It follows arguing precisely as in [ATo, Proposition 3.6].

Remark 10. Proposition 6.11 is the main tool for showing the HI property of certain subspaces of $\mathfrak{X}_G$; we present the precise statement below. At this point we want to comment on the role of special j-block sequences in the definition of dependent sequences. A key ingredient needed for the proof of (6.2) is a "tree-like" property satisfied by all (n_{2l-1})-special sequences—see, e.g., [ATo, Proposition 3.3]. When we deal with norms on $c_{00}(\mathbb{N})$, then this "tree-like" property is also satisfied by all restrictions of the special sequences on intervals of $\mathbb{N}$. On the other hand, when we deal with $c_{00}(\mathbb{N}\times\mathbb{N})$ and we consider restrictions on rectangles generated by intervals of $\mathbb{N}$, then this is no longer valid. However, this problem can be resolved if we work with special j-block sequences. This is the reason why we introduced this concept.

The following proposition is an easy consequence of the previous results.

Proposition 6.12. *Let (x_n) and (y_n) be diagonally block sequences. Then for every $n\in\mathbb{N}$ there exists a $(6,2l-1)$-dependent sequence $(z_k,\phi_k)_{k=1}^{n_{2l-1}}$ such that $z_{2k-1}\in\mathrm{span}\{x_n : n\in\mathbb{N}\}$ and $z_{2k}\in\mathrm{span}\{y_n : n\in\mathbb{N}\}$. The same result also holds true provided that (x_n) and (y_n) are both j-block sequences in Z_k for some $k\in\mathbb{N}$.*

We proceed with the following proposition.

Proposition 6.13. *If Y is a subspace of $\mathfrak{X}_G$, then one of the following is satisfied.*

(a) *There exists $n\in\mathbb{N}$ such that $j_n\colon Y\to X_n$ is not strictly singular.*
(b) *There exists $k\in\mathbb{N}$ such that $\pi_k\colon Y\to Z_k$ is not strictly singular.*
(c) *For every $r>0$ there exists a normalized sequence (y_n) in Y and a diagonally block sequence (w_n) such that $\sum_{n\in\mathbb{N}}\|y_n-w_n\|<r$.*

Proof. Assume that neither (a) nor (b) is satisfied. Then for every $n\in\mathbb{N}$ and every subspace Y' of Y there exists a subspace Y'' of Y' such that the operator $j_{\{1,\dots,n\}}\colon Y''\to\sum_{i=1}^n\oplus X_n$ is strictly singular (see, e.g., [Ar, Lemma 3.7]). Note that the same also holds for the projections $\pi_{\{1,\dots,m\}}$ $(m\in\mathbb{N})$. Hence, for every $\varepsilon>0$ and every $n,m\in\mathbb{N}$ there exists a subspace Y' of Y such that $\|j_{\{1,\dots,n\}}|_{Y'}\|<\varepsilon$ and $\|\pi_{\{1,\dots,m\}}|_{Y'}\|<\varepsilon$. Using this fact and a standard sliding hump argument, we easily verify that (c) is satisfied. □

By Propositions 6.12 and 6.13, we obtain the following corollary.

Corollary 6.14. *The following hold.*

(a) *For every $k\in\mathbb{N}$ the space Z_k is HI.*
(b) *If (y_n) is diagonally block, then the space $\overline{\mathrm{span}}\{y_n : n\in\mathbb{N}\}$ is HI.*
(c) *If Y is a subspace of $\mathfrak{X}_G$ such that $j_n\colon Y\to X_n$ and $\pi_k\colon Y\to Z_k$ are strictly singular for every $n,k\in\mathbb{N}$, then Y is HI.*

Proof. Parts (a) and (b) follow by Proposition 6.12. For part (c) let Y be an arbitrary subspace of $\mathfrak{X}_G$ such that the operators $j_n\colon Y \to X_n$ and $\pi_k\colon Y \to Z_k$ are strictly singular for every $n,k \in \mathbb{N}$. Let Y_1, Y_2 be subspaces of Y, and let $\varepsilon > 0$. By Proposition 6.13, there exist two normalized block sequences (y_n^1), (y_n^2) and a diagonally block sequence (w_n) such that the following are satisfied.

(1) For every $n \in \mathbb{N}$ we have $y_n^1 \in Y_1$ and $y_n^2 \in Y_2$.
(2) We have $\sum_{n\in\mathbb{N}} \|w_{2n-1} - y_n^1\| < \varepsilon$ and $\sum_{n\in\mathbb{N}} \|w_{2n} - y_n^2\| < \varepsilon$.

By part (b), the space $W := \overline{\mathrm{span}}\{w_n : n \in \mathbb{N}\}$ is HI. Since ε was arbitrary, it follows that $d(S_{Y_1}, S_{Y_2}) = 0$; but the subspaces Y_1, Y_2 of Y were also arbitrary, and so Y is HI. $\square$

6.6. We close this section with the following two properties of $\mathfrak{X}_G$ (see also [AM]).

Proposition 6.15. *Every j-block sequence (x_n) in $\mathfrak{X}_G$ is boundedly complete.*

Proof. If not, then there exist a sequence (a_n) in $\mathbb{R}$ and $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ we have $\|\sum_{k=1}^n a_k x_k\| \leqslant 1$ and $\|\sum_{k=n+1}^\infty a_k x_k\| > \varepsilon$. Thus, there exists a sequence (I_n) of successive intervals of $\mathbb{N}$ such that for every $d \in \mathbb{N}$, setting $w_d = \sum_{k\in I_d} a_k x_k$, we have $\|w_d\| > \varepsilon$. We select $\phi_d \in G$ with $\mathrm{range}(\phi_d) = \mathrm{range}(w_d)$ and $\phi_d(w_d) > \varepsilon$. Notice that $n_l/m_l \to \infty$ as $l \to \infty$. Hence, for appropriate $l, n \in \mathbb{N}$, we obtain that $\big(\frac{1}{m_{2l}} \sum_{d=1}^{n_{2l}} \phi_d\big)\big(\sum_{k=1}^n a_k x_k\big) > 1$ which yields a contradiction. $\square$

Proposition 6.16. *We have $\mathfrak{X}_G^* = \overline{\mathrm{span}}\big\{\bigcup_{n\in\mathbb{N}} X_n^*\big\}$.*

Proof. Assume not. Then there exist $x^{**} \in \mathfrak{X}_G^{**}$ and $x^* \in B_{\mathfrak{X}_G^*}$ such that $\|x^{**}\| = 1$, $x^{**}(x^*) > 1/2$ and $\bigcup_n X_n^* \subseteq \mathrm{Ker}(x^{**})$. We select a net $(x_i)_{i\in I}$ in $B_{\mathfrak{X}_G}$ with $w^* - \lim_{i\in I} x_i = x^{**}$. Clearly, we may assume that

$$x^*(x_i) > \frac{1}{2} \ \text{ for every } i \in I. \tag{6.3}$$

Observe that $w - \lim_{i\in I} j_{\{1,\dots,n\}}(x_i) = 0$. Hence, by Mazur's theorem and a sliding hump argument, we may select two sequences (y_n) and (z_n) such that the following are satisfied.

(i) For every $n \in \mathbb{N}$ we have $y_n \in \mathrm{conv}\{x_i : i \in I\}$.
(ii) We have that (z_n) is a j-block sequence.
(iii) We have $\sum_n \|y_n - z_n\| < 1/8$.

Notice that for every $k \in \mathbb{N}$ and every $n_1 < \dots < n_k$ we have

$$\Big\|\frac{z_{n_1} + \dots + z_{n_k}}{k}\Big\| \geqslant \frac{1}{4}. \tag{6.4}$$

Indeed, by (i) and (6.3), we have that $x^*(y_n) > 1/2$ for every $n \in \mathbb{N}$. Hence, by (iii), we obtain that $x^*(z_n) > 1/4$ for every $n \in \mathbb{N}$ which clearly implies (6.4). Thus, we may select a j-block sequence (w_k) with $w_k = \frac{1}{k}\sum_{n\in F_k} z_n$ where (F_k) is a sequence of successive intervals of $\mathbb{N}$. Since (w_k) is a j-block sequence of $4-\ell_k^1$ averages, by Lemma 6.7, we have that for every $l\in\mathbb{N}$ there exist $k_1<\cdots<k_{n_{2l}}$ such that

$$\|\frac{1}{n_{2l}}\sum_{i=1}^{n_{2l}} w_{k_i}\|\leqslant \frac{12}{m_{2l}}. \tag{6.5}$$

Set $v_l\coloneqq \frac{1}{n_{2l}}\sum_{i=1}^{n_{2l}} w_{k_i}$ and notice that v_l is a convex combination of z_k's. Let v_l' be the corresponding convex combination of y_n's. By (i) and (6.3), we have $\|v_l'\|>1/2$. On the other hand, by (iii), we see that $\|v_l-v_l'\|<1/8$. Since $m_l\to\infty$ as $l\to\infty$, by (6.5), we obtain that $\|v_l\|\to 0$ which is clearly a contradiction. $\square$

7. HI interpolations

Let X be a Banach space with a bimonotone Schauder basis (x_k). Also let $W\subseteq X$ be closed, bounded, convex and symmetric. For every $n\in\mathbb{N}$ let $\|\cdot\|_n$ denote the equivalent norm on X defined by the Minkowski gauge of the set $2^nW+\frac{1}{2^n}B_X$. We will assume that (x_k) remains a bimonotone Schauder basis of $X_n\coloneqq (X,\|\cdot\|_n)$. By $\mathfrak{X}_{(X,W)}$ we denote the HI Schauder sum of the sequence (X_n) as described in the previous section.

Definition 7.1. *The* HI interpolation space $\Delta_{(X,W)}$ *is the* (*closed*) *subspace of* $\mathfrak{X}_{(X,W)}$ *consisting of the vectors* $(x,x,\dots)\in\mathfrak{X}_{(X,W)}$ *with* $x\in X$.

Remark 11. This definition is a variant of the corresponding definition in [AF], which in turn follows the general scheme of the classical Davis–Figiel–Johnson–Pelczynski interpolation method [DFJP]. As we have already pointed out in the previous section, the present variant will allow us to obtain HI amalgamations with a Schauder basis.

We proceed to present some general results concerning the structure of $\Delta_{(X,W)}$. We start with the following lemma which provides a general condition for the existence of HI interpolations.

Lemma 7.2. *Let* (x_k) *be a bimonotone Schauder basis of* X*, and let* $W\subseteq X$ *be closed, bounded, convex and symmetric. Assume that* $P_I(W)\subseteq W$ *for every interval* I *of* $\mathbb{N}$ *where* $P_I\colon X\to\overline{\mathrm{span}}\{x_k: k\in I\}$ *is the natural projection. Then for every* $n\in\mathbb{N}$ *the basis* (x_k) *remains bimonotone in* X_n.

Proof. Let $n\in\mathbb{N}$ and $x\in X_n$. Let $\lambda>0$ such that $x\in\lambda(2^nW+\frac{1}{2^n}B_X)$. By our assumptions, for every interval I of $\mathbb{N}$ we have

$$P_I(x)\in\lambda\big(2^nP_I(W)+\frac{1}{2^n}P_I(B_X)\big)\subseteq\lambda\big(2^nW+\frac{1}{2^n}B_X\big).$$

This implies that $\|P_I(x)\|_n\leqslant\|x\|_n$, as desired. $\square$

Remark 12. Note that the ℓ_2 Baire sum $\mathcal{T}_2^X$ of a normalized bimonotone Schauder tree basis $(x_t)_{t\in T}$ and the set W_X defined in Definition 5.3 satisfy the assumptions of Lemma 7.2. Consequently, the space $\Delta_{(\mathcal{T}_2^X, W_X)}$ is well defined. We will need this observation later on.

Proposition 7.3. *Let $X, (x_k)$ and W be as in Lemma 7.2, and assume that $x_k \in W$ for every $k \in \mathbb{N}$. Then $\bar{x}_k \coloneqq (x_k, x_k, \dots) \in \Delta_{(X,W)}$ for every $k \in \mathbb{N}$ and, moreover, the sequence $(\bar{x}_k)$ defines a bimonotone Schauder basis of $\Delta_{(X,W)}$.*

Proof. Notice, first, that $\|x_k\|_n \leqslant \frac{1}{2^n}$ for every $n, k \in \mathbb{N}$. Therefore, $\bar{x}_k \in \Delta_{(X,W)}$ for every $k \in \mathbb{N}$. Next, let $\bar{x} = (x, x, \dots) \in \Delta_{(X,W)}$ with $x = \sum_k a_k x_k$. Let $k \in \mathbb{N}$ and consider the projection $\pi_k \colon \Delta_{(X,W)} \to Z_k$. We claim that $\pi_k(\bar{x}) = a_k \bar{x}_k$. Indeed, observe that the sequence $(x_{n,k})_{n\in\mathbb{N}}$ is a Schauder basis of Z_k (not normalized) and $x^*_{n,k}\big(\pi_k(\bar{x})\big) = x^*_{n,k}(x) = a_k$ for every $n \in \mathbb{N}$. Hence, $\pi_k(\bar{x}) = \sum_{n\in\mathbb{N}} a_k x_{n,k} = a_k \bar{x}_k$. This is easily seen to imply that for every (nonempty) finite interval I of $\mathbb{N}$ we have $\pi_I(\Delta_{(X,W)}) = \mathrm{span}\{\bar{x}_k : k \in I\}$, and so $\pi_I(\bar{x}) = \sum_{k\in I} a_k \bar{x}_k$ where, as before, $\bar{x} = (x, x, \dots)$ and $x = \sum_k a_k x_k$.

The above argument and the fact that $\|\pi_I\| = 1$ for every finite interval $I \subseteq \mathbb{N}$, yield that the sequence $(\bar{x}_k)$ is a bimonotone Schauder basis of $\overline{\mathrm{span}}\{\bar{x}_k : k \in \mathbb{N}\}$. Thus, it suffices to show that the space $\overline{\mathrm{span}}\{\bar{x}_k : k \in \mathbb{N}\}$ coincides with $\Delta_{(X,W)}$. To this end let $\bar{x} = (x, x, \dots)$ with $x = \sum_k a_k x_k$. We will show that the partial sums $\sum_{k=1}^d a_k \bar{x}_k$ converge weakly to $\bar{x}$; clearly, this is enough to complete the proof. First observe that $\sum_{k=1}^d a_k \bar{x}_k = \pi_{\{1,\dots,d\}}(\bar{x})$, and so $\|\sum_{k=1}^d a_k \bar{x}_k\| \leqslant \|\bar{x}\|$. Moreover, for every $x^* \in \bigcup_{n\in\mathbb{N}} B_{X_n^*}$ we have that $x^*\big(\sum_{k=1}^d a_k \bar{x}_k\big) \to x^*(\bar{x})$. On the other hand, by Proposition 6.16, the vector space $\mathrm{span}\big\{\bigcup_{n\in\mathbb{N}} B_{X_n^*}\big\}$ is norm dense in $\mathfrak{X}^*_{(X,W)}$. Therefore, the partial sums $\sum_{k=1}^d a_k \bar{x}_k$ must converge weakly to $\bar{x}$, and the proof is completed. □

Proposition 7.3 justifies the following definition.

Definition 7.4. *Let X be a Banach space and $W \subseteq X$. We say that the pair (X, W)* admits a HI interpolation *if X has a bimonotone Schauder basis (x_k), W is closed, bounded, convex and symmetric, $x_k \in W$ for every $k \in \mathbb{N}$, and for every interval I of $\mathbb{N}$ we have $P_I(W) \subseteq W$.*

Notation. In what follows $J \colon \Delta_{(X,W)} \to X$ we denote the one-to-one linear operator defined by $J(\bar{x}) = x$ for every $\bar{x} = (x, x, \dots) \in \Delta_{(X,W)}$.

Proposition 7.5. *Assume that the pair (X, W) admits a HI interpolation.*

(a) *If Y is a subspace of $\Delta_{(X,W)}$ such that $J \colon Y \to X$ is strictly singular, then Y is HI.*
(b) *If Y, Z are subspaces of $\Delta_{(X,W)}$ such that both $J|_Y$ and $J|_Z$ are strictly singular, then $d(S_Y, S_Z) = 0$.*

Proof. (a) Note that for every $k \in \mathbb{N}$ the image of the operator $\pi_k \colon \Delta_{(X,W)} \to Z_k$ has dimension 1; therefore, this operator is strictly singular. Also observe that for every $\bar{x} \in \Delta_{(X,W)}$ and every $n \in \mathbb{N}$ we have $j_n(\bar{x}) = J(\bar{x})$. Since every X_n is isomorphic to X, we conclude that $j_n|_Y$ is strictly singular. By part (c) of Corollary 6.14, the result follows.

(b) As in part (a), we first observe that for every $k \in \mathbb{N}$ the operators $\pi_k|_Y$ and $\pi_k|_Z$ are strictly singular. Moreover, by our assumptions, for every $n \in \mathbb{N}$ the operators $j_n|_Y$ and $j_n|_Z$ are also strictly singular. Let $\varepsilon > 0$ be arbitrary. Arguing as in the proof of part (c) of Corollary 6.14, we may select two normalized sequences (y_n) and (z_n), and a diagonally block sequence (w_n) such that the following hold.

(i) For every $n \in \mathbb{N}$ we have $y_n \in Y$ and $z_n \in Z$.
(ii) We have $\sum_n \|w_{2n-1} - y_n\| < \varepsilon$ and $\sum_n \|w_{2n} - z_n\| < \varepsilon$.

By part (c) of Corollary 6.14, the space $W := \overline{\mathrm{span}}\{w_n : n \in \mathbb{N}\}$ is HI. Hence, setting $W_1 := \overline{\mathrm{span}}\{w_{2n-1} : n \in \mathbb{N}\}$ and $W_2 := \overline{\mathrm{span}}\{w_{2n} : n \in \mathbb{N}\}$, we see that $d(S_{W_1}, S_{W_2}) = 0$. Since ε was arbitrary, by (ii) above, we conclude that $d(S_Y, S_Z) = 0$, as desired. $\square$

Remark 13. Although Y and Z in part (b) of Proposition 7.5 are HI and satisfy $d(S_Y, S_Z) = 0$, the space $\overline{Y+Z}$ may not be HI subspace. Actually, there are examples of such pairs Y, Z with $\overline{Y+Z} = \Delta_{(X,W)}$ and $\Delta_{(X,W)}$ not HI.

Theorem 7.6. *Assume that the pair (X, W) admits a HI interpolation. Let Y be a (closed) subspace of X and assume that W is thin on Y. Then $J^{-1}(Y)$ is either HI, or finite-dimensional.*

Proof. Set $Z := J^{-1}(Y)$, and assume that Z is infinite-dimensional. We will show that the operator $J\colon Z \to X$ is strictly singular. Indeed, if not, then there exists $Z_1 \hookrightarrow Z$ such that $J\colon Z_1 \to X$ is an isomorphic embedding. Then $J(Z_1)$ is a closed subspace of Y and there exists $C > 0$ such that for every $x \in J(Z_1)$ with $\|x\| \leqslant 1$ we have that $\|\bar{x}\| \leqslant C$. It follows that $\|x\|_n \leqslant C$ and, consequently, $B_{J(Z_1)} \subseteq C2^n W + \frac{C}{2^n} B_X$ for every $n \in \mathbb{N}$; that is, the set W almost absorbs $B_{J(Z_1)}$. This is a contradiction since W is thin on Y. Therefore, $J\colon Z \to X$ is strictly singular, and the result follows by part (a) of Proposition 7.5. $\square$

We proceed with the following (essentially known) proposition which establishes an important property of the operator J.

Proposition 7.7. *The operator $J\colon \Delta_{(X,W)} \to X$ is Tauberian; that is, for every $x^{**} \in \Delta^{**}_{(X,W)} \setminus \Delta_{(X,W)}$ we have $J^{**}(x^{**}) \in X^{**} \setminus X$.*

Proof. We have the following claim.

Claim. *For every $\bar{x}^{**} \in \Delta^{**}_{(X,W)}$ there exists $y^{**} \in X^{**}$ such that, setting*

$$\bar{y}^{**}_n := (\underbrace{y^{**}, y^{**}, \dots, y^{**}}_{n\text{-times}}, 0, \dots) \tag{7.1}$$

for every $n \in \mathbb{N}$, we have $\bar{x}^{**} = w^* - \lim \bar{y}^{**}_n$.

Proof of the claim. By Proposition 6.16, we have $(\sum_n \oplus X_n)^*_{\mathrm{hi}} = \overline{\mathrm{span}}\{\bigcup_{n\in\mathbb{N}} X_n^*\}$ and, consequently,

$$\big(\sum_n \oplus X_n\big)^{**}_{\mathrm{hi}} = \Big\{w^* - \sum_{n=1}^{\infty} x_n^{**} \ :\ x_n^{**} \in X_n^{**} \text{ for every } n\in\mathbb{N}, \text{ and}$$
$$\exists C>0 \text{ with } \big\|\sum_{n=1}^{k} x_n^{**}\big\| \leqslant C \text{ for every } k\in\mathbb{N}\Big\}.$$

Let $\bar{x}^{**}\in\Delta^{**}_{(X,W)}$ and select a net $(\bar{x}_i)_{i\in I}$ in $\Delta_{(X,W)}$ such that $w^*-\lim_{i\in I}\bar{x}_i=\bar{x}^{**}$. By the previous discussion, it follows that $\bar{x}^{**}=w^*-\lim\bar{y}_n^{**}$ where each $\bar{y}_n^{**}$ is as in (7.1) for the vector $y^{**}:=w^*-\lim_{i\in I}J^{**}(\bar{x}_i)\in X^{**}$. The claim is proved. □

Now let $\bar{x}^{**}\in\Delta^{**}_{(X,W)}\setminus\Delta_{(X,W)}$ be arbitrary, and let $y^{**}\in X^{**}$ be such that $\bar{x}^{**}=w^*-\lim\bar{y}_n^{**}$ where the sequence $(\bar{y}_n^{**})$ is as in (7.1). It is enough to show that $y^{**}\in X^{**}\setminus X$. Suppose, towards a contradiction, that $y^{**}\in X$ and note that the sequence $(\bar{y}_n^{**})$ is norm bounded by $\|x^{**}\|$. By Proposition 6.15, the space $(\sum_n\oplus X_n)_{\mathrm{hi}}$ is j-block boundedly complete. Therefore, the sequence $(\bar{y}_n^{**})$ is norm convergent to x^{**} which in turn implies that $\bar{x}^{**}\in\Delta_{(X,W)}$, a contradiction. The proof is completed. □

Corollary 7.8. *If X is reflexive and the pair (X,W) admits a HI interpolation, then $\Delta_{(X,W)}$ is reflexive.*

Our last result in this section is the following strengthening of Corollary 7.8 (see also [DFJP]).

Proposition 7.9. *If the pair (X,W) admits a HI interpolation and $W\subseteq X$ is weakly compact, then $\Delta_{(X,W)}$ is reflexive.*

Proof. Recall that if $T\colon X\to Y$ is a Tauberian operator and $W\subseteq X$, then W is relatively weakly compact if and only if $T(W)$ is relatively weakly compact (see, e.g., [N2]). Also recall that, by a classical result of Grothendieck [Gr], a set $K\subseteq X$ is relatively weakly compact if for every $\varepsilon>0$ there exists a weakly compact set $K_\varepsilon\subseteq X$ such that $K\subseteq K_\varepsilon+\varepsilon B_X$.

Now assume that W is weakly compact. It is easy to see that the set W almost absorbs the set $J(B_{\Delta_{(X,W)}})$, that is, for every $\varepsilon>0$ there exists $\lambda>0$ such that $J(B_{\Delta_{(X,W)}})\subseteq\lambda W+\varepsilon B_X$. By Grothendieck's criterion, it follows that $J(B_{\Delta_{(X,W)}})$ is relatively weakly compact. By Proposition 7.7, J is a Tauberian operator. Hence, $B_{\Delta_{(X,W)}}$ is also relatively weakly compact, which is equivalent to saying that $\Delta_{(X,W)}$ is reflexive. □

8. Amalgamations of Schauder tree bases

8.1. **Existence of HI-amalgamations and p-amalgamations.** Let X be a Banach space, let Λ be a countable set, let T be a pruned subtree of $\Lambda^{<\mathbb{N}}$ and let $(x_t)_{t\in T}$ be a normalized bimonotone Schauder tree basis of X.

Definition 8.1. *A Banach space $\mathcal{A}^X_{\mathrm{hi}}$ is said to be a* HI-amalgamation *of $(x_t)_{t\in T}$ if the following are satisfied.*

(1) *The space $\mathcal{A}^X_{\mathrm{hi}}$ has a Schauder basis (e_n) which can be written as $(e_t)_{t\in T}$ where $e_t = e_{h(t)}$ for every $t \in T$ and $h\colon T \to \mathbb{N}$ denotes the fixed enumeration of T described in Section 4.*
(2) *Setting $\tilde{\mathcal{X}}_\sigma := \overline{\mathrm{span}}\{e_t : t \sqsubset \sigma\}$ for every $\sigma \in [T]$ and letting $\tilde{P}_\sigma\colon \mathcal{A}^X_{\mathrm{hi}} \to \tilde{\mathcal{X}}_\sigma$ denote the natural projection, we have that $\{\|\tilde{P}_\sigma\| : \sigma \in [T]\}$ is bounded.*
(3) *For every $\sigma \in [T]$ the space $\tilde{\mathcal{X}}_\sigma$ is isomorphic to $X_\sigma = \overline{\mathrm{span}}\{x_t : t \sqsubset \sigma\}$ with constant independent of σ.*
(4) *The following hold.*
   (a) *Every X-singular subspace Y of $\mathcal{A}^X_{\mathrm{hi}}$ (that is, for every $\sigma \in [T]$ the operator $\tilde{P}_\sigma\colon Y \to \tilde{\mathcal{X}}_\sigma$ is strictly singular) is HI.*
   (b) *If Y and Z are X-singular subspaces of $\mathcal{A}^X_{\mathrm{hi}}$, then $d(S_Y, S_Z) = 0$.*
(5) *Every X-compact subspace Y of $\mathcal{A}^X_{\mathrm{hi}}$ (that is, for every $\sigma \in [T]$ the operator $\tilde{P}_\sigma\colon Y \to \tilde{\mathcal{X}}_\sigma$ is compact) is reflexive and HI.*
(6) *If Y is a subspace of $\mathcal{A}^X_{\mathrm{hi}}$ not containing an X-singular subspace, then there exists finite $A \subseteq [T]$ such that the operator*
$$\tilde{P}_A\colon Y \to \tilde{\mathcal{X}}_A := \overline{\mathrm{span}}\{e_t : t \in A\}$$
is an isomorphic embedding.

Definition 8.2. *A Banach space $\mathcal{A}^X_p$ is said to be a* p-amalgamation *of $(x_t)_{t\in T}$, where $1 < p < \infty$, if (1), (2), (3) and (6) in Definition 8.1 are satisfied, and (4) and (5) are replaced with the following.*

(4)$'$ *Every X-singular subspace Y of $\mathcal{A}^X_p$ contains a copy of ℓ_p.*
(5)$'$ *Every X-compact subspace Y of $\mathcal{A}^X_p$ is reflexive and contains a copy of ℓ_p.*

Our goal in this section is to prove the following theorem.

Theorem 8.3. *For every normalized bimonotone Schauder tree basis $(x_t)_{t\in T}$ there exists a HI-amalgamation space $\mathcal{A}^X_{\mathrm{hi}}$ of $(x_t)_{t\in T}$. Respectively, for any $1 < p < \infty$ there exists a p-amalgamation space $\mathcal{A}^X_p$ of $(x_t)_{t\in T}$.*

The proof of the existence of HI-amalgamations is almost identical to the proof of the existence of p-amalgamations; the first proof uses the HI interpolation, while the second proof uses the classical Davis–Figiel–Johnson–Pelczynski interpolation scheme [DFJP]. We will present the proof simultaneously for both cases indicating the differences in the arguments whenever it is necessary.

First, we consider the ℓ_2 Baire sum $\mathcal{T}^X_2$ of $(x_t)_{t\in T}$ as constructed in Section 3. Also let W_X be as in Definition 5.3 in Section 5. The HI-amalgamation space $\mathcal{A}^X_{\mathrm{hi}}$ of $(x_t)_{t\in T}$ is the HI interpolation space $\Delta^{\mathrm{hi}}_{(\mathcal{T}^X_2, W_X)}$. (Note that, by Remark 12, this space is well-defined.) Respectively, the p-amalgamation space $\mathcal{A}^X_p$ is the p interpolation space $\Delta^p_{(\mathcal{T}^X_2, W_X)}$ in the sense of [DFJP]. It remains to show that these spaces satisfy the properties in Definitions 8.1 and 8.2 respectively.

Proposition 7.3 and Remark 12 yield that $(\bar{e}_t)_{t\in T}$ defines (after an appropriate enumeration) a bimonotone Schauder basis of $\mathcal{A}^X_{\mathrm{hi}}$. Also observe that $\|\bar{y}\| \leqslant 1$ for every $y \in B_{\mathcal{X}_\sigma}$. Hence, setting $\tilde{\mathcal{X}}_\sigma \coloneqq \overline{\mathrm{span}}\{\bar{e}_t : t \sqsubset \sigma\}$, we see that the operator $J\colon \tilde{\mathcal{X}}_\sigma \to \mathcal{X}_\sigma$ is an onto isomorphism and, moreover, $\|(J|_{\tilde{\mathcal{X}}_\sigma})^{-1}\| \leqslant 2$. Thus, the operator $\tilde{P}_\sigma = (J|_{\tilde{\mathcal{X}}_\sigma})^{-1} \circ P_\sigma \circ J$ is a projection from $\mathcal{A}^X_{\mathrm{hi}}$ onto $\tilde{\mathcal{X}}_\sigma$ which satisfies $\tilde{P}_\sigma(\bar{e}_t) = 0$ for every $t \notin \sigma$. Since the space $\mathcal{X}_\sigma$ is isometric to X_σ for every $\sigma \in [T]$, we see that properties (1), (2) and (3) are satisfied. We argue similarly for $\mathcal{A}^X_p$.

We proceed to verify property (4) for the HI-amalgamation space $\mathcal{A}^X_{\mathrm{hi}}$. For part (4.a), first we observe that, by Proposition 7.5, if $J\colon Y \to \mathcal{T}^X_2$ is strictly singular, then property (4.a) is satisfied. So assume, towards a contradiction, that the operator $J\colon Y \to \mathcal{T}^X_2$ is not strictly singular, or equivalently, that there exists a subspace Z_1 of Y such that the operator $J\colon Z_1 \to \mathcal{T}^X_2$ is an isomorphic embedding. It follows that the set W_X almost absorbs $B_{J(Z_1)}$. On the other hand, we have that $J(Z_1)$ is an X-singular subspace of $\mathcal{T}^X_2$. Theorem 5.15 yields a contradiction. Using similar arguments, we verify property (4.b). For the corresponding property $(4)'$ of $\mathcal{A}^X_p$ notice that if Y is any X-singular subspace of $\mathcal{A}^X_p$, then (as before) the operator $J\colon Y \to \mathcal{T}^X_2$ is strictly singular. Hence, by standard arguments, we see that Y contains a copy of ℓ_p.

Next, we show that property (6) is satisfied. Let Y be a subspace of $\mathcal{A}^X_{\mathrm{hi}}$ not containing an X-singular subspace. We claim that the operator $J\colon Y \to \mathcal{T}^X_2$ is an isomorphic embedding. Indeed, if not, then there exists a subspace Z of Y such that the operator $J\colon Z \to \mathcal{T}^X_2$ is compact. It follows that for every $\sigma \in [T]$ the operator $\tilde{P}_\sigma\colon Z \to \tilde{\mathcal{X}}_\sigma$ is also compact, a contradiction. Thus, $J\colon Y \to \mathcal{T}^X_2$ is an isomorphic embedding and, consequently, the set W_X almost absorbs $B_{J(Y)}$. By Theorem 5.16, there exists finite $A \subseteq [T]$ such that $P_A\colon J(Y) \to \mathcal{X}_A$ is an isomorphic embedding. This is easily seen to imply that the operator $\tilde{P}_A\colon Y \to \tilde{\mathcal{X}}_A$ is also an isomorphic embedding, as required. The proof of the corresponding property for the p-amalgamation space is identical.

Finally, properties (5) and $(5)'$ follow from the following theorem.

Theorem 8.4. *Let Y be an X-compact subspace of $\mathcal{A}^X_{\mathrm{hi}}$ (respectively, of $\mathcal{A}^X_p$). Then Y is reflexive.*

Proof. As we have already proved, if Y is an X-compact subspace of $\mathcal{A}^X_{\mathrm{hi}}$, then Y is HI; consequently, ℓ_1 does not embed into Y. On the other hand, in the case of p-amalgamations, we have that Y is ℓ_p-saturated, and so ℓ_1 also does not embed into Y. From this point on the arguments for both spaces are identical.

Assume that Y is not reflexive. Since ℓ_1 does not embed into Y, there exist a normalized sequence $(\bar{y}_n)$ in Y and $\bar{y}^{**} \in Y^{**} \setminus Y$ such that $w^* - \lim \bar{y}_n = \bar{y}^{**}$. We set $y^{**} \coloneqq J(\bar{y}^{**})$ and $y_n \coloneqq J(\bar{y}_n)$ for every $n \in \mathbb{N}$. By Proposition 7.7, J is a Tauberian operator; therefore, $w^* - \lim y_n = y^{**} \in (\mathcal{T}^X_2)^{**} \setminus \mathcal{T}^X_2$. Notice that there exist $\varepsilon > 0$ and $y^* \in (\mathcal{T}^X_2)^*$ with $\|y^*\| \leqslant 1$ such that $y^*(y_n) > \varepsilon$ for every

$n \in \mathbb{N}$. Moreover, by passing to subsequences if necessary, we may assume that $a_t := \lim e_t^*(y_n)$ exists for every $t \in T$. Recall that we enumerate the basis of $\mathcal{T}_2^X$ as (e_{t_n}). For every $d \in \mathbb{N}$ set $z_d := \sum_{n=1}^d a_{t_n} e_{t_n}$ and notice that $\|z_d\| \leqslant \|y^{**}\|$. We consider the following cases.

CASE 1: *We have* $\sum_{t\in T} a_t e_t \in \mathcal{T}_2^X$. We set $z := \sum_{t\in T} a_t e_t$ and $w_n := y_n - z$ for every $n \in \mathbb{N}$. Observe that $\lim e_t^*(w_n) = 0$ for every $t \in T$. By passing to a subsequence of (w_n) if necessary, for every $r > 0$ we may select a block sequence (b_n) in $\mathcal{T}_2^X$ such that $\sum_{n\in\mathbb{N}} \|b_n - w_n\| < r$. Let $\sigma \in [T]$ be arbitrary. Since Y is a X-compact subspace, the set $\{P_\sigma(y_n) : n \in \mathbb{N}\} \subseteq \mathcal{X}_\sigma$ is relatively compact (indeed, observe that $P_\sigma\big(J(\bar{y}_n)\big) = P_\sigma(y_n)$ for every $n \in \mathbb{N}$.) This property and the fact that the sequence (b_n) is block yield that $\lim \|P_\sigma(b_n)\| = 0$ for every $\sigma \in [T]$. By Proposition 4.10, we see that the sequence (b_n) is weakly null. This, in turn, implies that the sequence (w_n) is also weakly null, and so $w - \lim y_n = z$. This is clearly a contradiction.

CASE 2: *We have* $\sum_{t\in T} a_t e_t \notin \mathcal{T}_2^X$. In this case, there exist $\varepsilon > 0$ and a sequence (I_k) of successive intervals of $\mathbb{N}$ such that, setting $v_k := \sum_{n\in I_k} a_{t_n} e_{t_n}$ for all $k \in \mathbb{N}$, the sequence (v_k) is bounded and block, and satisfies $\|v_k\| > \varepsilon$ for every $k \in \mathbb{N}$.

Claim 1. *For every* $\delta > 0$ *there exists* $\lambda_\delta > 0$ *such that* $v_k \in \lambda_\delta W_X + \delta B_{\mathcal{T}_2^X}$ *for every* $k \in \mathbb{N}$.

Proof of the claim. First observe that, since the sequence $(\bar{y}_n)$ is normalized, for every $\delta > 0$ there exists $\lambda_\delta > 0$ such that

$$\{y_n : n \in \mathbb{N}\} \subseteq \lambda_\delta W_X + \delta B_{\mathcal{T}_2^X}. \tag{8.1}$$

On the other hand, by the definition of v_k, for every $k \in \mathbb{N}$ and every $\delta > 0$ there exists n_0 such that $\|P_{I_k}(y_n) - v_k\| = \|P_{I_k}(y_n) - \sum_{n\in I_k} a_{t_n} e_{t_n}\| < \delta$ for every $n \geqslant n_0$. By (8.1) and the fact that $P_{I_k}(W_X) \subseteq W_X$ and $P_{I_k}(B_{\mathcal{T}_2^X}) \subseteq B_{\mathcal{T}_2^X}$, we see that

$$P_{I_k}(y_n) \subseteq \lambda_\delta W_X + \delta B_{\mathcal{T}_2^X}. \tag{8.2}$$

Hence, for every $\delta > 0$ there exists $\lambda_\delta > 0$ such that $\{v_k : k \in \mathbb{N}\} \subseteq \lambda_\delta W_X + 2\delta B_{\mathcal{T}_2^X}$ as desired. □

Claim 2. *For every* $\sigma \in [T]$ *we have* $\lim \|P_\sigma(v_k)\| = 0$.

Proof of the claim. As we have already pointed out, the fact that Y is X-compact implies that for every $\sigma \in [T]$ the set $\{P_\sigma(y_n) : n \in \mathbb{N}\} \subseteq \mathcal{X}_\sigma$ is relatively compact. Also note that if $L \in [\mathbb{N}]^\infty$ is such that the sequence $(P_\sigma(y_n))_{n\in L}$ is convergent, then $\lim_{n\in L} P_\sigma(y_n) = \sum_{t\sqsubset\sigma} a_t e_t$. It follows that the sequence $\big(P_\sigma(y_n)\big)$ is convergent for every $\sigma \in [T]$.

Fix $\sigma \in [T]$ and let $\delta > 0$ be arbitrary. There exists $n_0 \in \mathbb{N}$ such that for every $n > m \geqslant n_0$ we have

$$\|P_\sigma(y_n - y_m)\| < \frac{\delta}{3}. \tag{8.3}$$

The sequence (e_{t_n}) is a Schauder basis of $\mathcal{T}_2^X$ and so there exists $k_0 \in \mathbb{N}$ such that for every $k \geqslant k_0$,

$$\|P_{I_k}(y_{n_0})\| < \frac{\delta}{3}. \tag{8.4}$$

By the definition of v_k, for every $k \geqslant k_0$ there exists $m_k > n_0$ such that

$$\|P_{I_k}(y_{m_k}) - v_k\| < \frac{\delta}{3}. \tag{8.5}$$

Moreover,

$$\begin{aligned} \|P_\sigma(v_k)\| \;\; &\leqslant \;\; \|P_\sigma\big(P_{I_k}(y_{m_k}) - v_k\big)\| + \|P_\sigma\big(P_{I_k}(y_{m_k}) - P_{I_k}(y_{n_0})\big)\| + \\ &\quad + \|P_\sigma\big(P_{I_k}(y_{n_0})\big)\|. \end{aligned} \tag{8.6}$$

Since $P_\sigma \circ P_{I_k} = P_{I_k} \circ P_\sigma$, $\|P_{I_k}\| = 1$ and $m_k > n_0$, by (8.3), we have

$$\|P_\sigma\big(P_{I_k}(y_{m_k}) - P_{I_k}(y_{n_0})\big)\| < \frac{\delta}{3}. \tag{8.7}$$

Finally, notice that $\|P_\sigma\| = 1$, thus combining inequalities (8.4)–(8.7) we obtain that $\|P_\sigma(v_k)\| \leqslant \delta$ for every $k \geqslant k_0$. The claim is proved. □

Summarizing, we see that (v_k) is a bounded block sequence in $\mathcal{T}_2^X$ with the following properties.

(I) For every $k \in \mathbb{N}$ we have $\|v_k\| > \varepsilon$.
(II) For every $\sigma \in [T]$ we have $\lim \|P_\sigma(v_k)\| = 0$.
(III) The set W_X almost absorbs the set $\{v_k : k \in \mathbb{N}\}$.

We apply Proposition 5.14 and we obtain $L \in [\mathbb{N}]^\infty$ and for every $k \in L$ a segment complete set $A_k \subseteq T$ and a vector $z_k^* \in (\mathcal{T}_2^X)^*$ such that the following are satisfied.

(a) The sets $(A_k)_{k\in L}$ are pairwise incomparable.
(b) For every $k \in L$ we have $\|z_k^*\| \leqslant 1$ and $\mathrm{supp}(z_k^*) \subseteq A_k \subseteq \mathrm{range}(v_k)$.
(c) For every $k \in L$ we have $z_k^*(v_k) \geqslant \frac{\varepsilon}{2}$.

Let $\{k_1 < k_2 < \dots\}$ denote the increasing enumeration of L. By (a) and (b), for every $\ell \in \mathbb{N}$ we have $\|\frac{1}{\sqrt{\ell}} \sum_{i=1}^{\ell} z_{k_i}^*\| \leqslant 1$. Therefore, by (b) and (c), we see that

$$\Big\|\sum_{i=1}^{\ell} v_{k_i}\Big\| \geqslant \Big(\frac{1}{\sqrt{\ell}} \sum_{i=1}^{\ell} z_{k_i}^*\Big)\Big(\sum_{i=1}^{\ell} v_{k_i}\Big) \geqslant \sqrt{\ell}\,\frac{\varepsilon}{2}. \tag{8.8}$$

The functional $\frac{1}{\sqrt{\ell}} \sum_{i=1}^{\ell} z_{k_i}^*$ is supported in $\bigcup_{i=1}^{\ell} I_{k_i}$ and so, by (8.8),

$$\lim_{d\to\infty} \Big\|\sum_{n=1}^{d} a_{t_n} e_{t_n}\Big\| = \infty.$$

But as we have indicated in the beginning of the proof, for every $d \in \mathbb{N}$ we have

$$\Big\|\sum_{n=1}^{d} a_{t_n} e_{t_n}\Big\| \leqslant \|y^{**}\|$$

which yields a contradiction. The proof is completed. □

We proceed to present another important property of the amalgamation spaces.

Proposition 8.5. *Let $(x_t)_{t\in T}$ be a normalized bimonotone Schauder tree basis such that X_σ is reflexive for every $\sigma\in[T]$. Then there exists a reflexive HI-amalgamation space (respectively, p-amalgamation space for any $1<p<\infty$) of $(x_t)_{t\in T}$.*

Proof. Note that it is enough to show that the set W_X is weakly compact; in the case of HI-amalgamations this is a consequence of Proposition 7.9, while in the case of p-amalgamations this follows from the results in [DFJP]. Set $C\coloneqq\bigcup_{\sigma\in[T]}B_{\mathcal{X}_\sigma}$.

Claim. *The set C is relatively weakly compact.*

Proof of the claim. Let (x_n) be an arbitrary sequence in C. Clearly, we may assume that every x_n is finitely supported. Let s_n be the unique initial segment of T which contains $\mathrm{supp}(x_n)$, and let t_n denote the $\sqsubset$-maximal node of s_n. By Ramsey's theorem, there exists $L\in[\mathbb{N}]^\infty$ such that the nodes $(t_n)_{n\in L}$ are either pairwise comparable, or pairwise incomparable. In the first case, there exists $\sigma\in[T]$ such that $\mathrm{supp}(x_n)\subseteq\sigma$ for every $n\in L$. By our assumptions, we obtain $M\in[L]^\infty$ such that the sequence $(x_n)_{n\in M}$ is weakly convergent.

So assume that the nodes $(t_n)_{n\in L}$ are pairwise incomparable. By passing to a further subsequence, we may additionally assume that $\lim_{n\in L}x_n(t)=x(t)$ for every $t\in T$. Observe that there exists $\tau\in[T]$ such that $\{t\in T: x(t)\neq 0\}\subseteq\tau$ and, moreover, the sequence $(P_\tau(x_n))_{n\in L}$ converges weakly to the vector $x\coloneqq\sum_{t\sqsubset\sigma}x(t)$; in particular, this yields that $x\in\mathcal{T}_2^X$. For every $n\in L$ we set $y_n\coloneqq x_n-x$. Then $\lim_{n\in L}y_n(t)=0$ for every $t\in T$ and so, by a standard sliding hump argument, we may assume that the sequence $(y_n)_{n\in L}$ is block. Notice that $\lim_{n\in L}P_\sigma(y_n)=0$ for every $\sigma\in[T]$. By Proposition 4.10, we see that $(y_n)_{n\in L}$ is weakly null which implies that the sequence $(x_n)_{n\in L}$ is weakly convergent. The claim is proved. □

Since $W_X=\overline{\mathrm{conv}}(C)$, by the Krein–Smulian theorem and the above claim, we conclude that the set W_X is weakly compact. □

We have the following refinement of Proposition 8.5.

Theorem 8.6. *Let $(x_t)_{t\in T}$ be a normalized bimonotone Schauder tree basis, and let $\mathcal{A}^X_{\mathrm{hi}}$ (respectively, $\mathcal{A}^X_p$) denote the HI-amalgamation (respectively, p-amalgamation for $1<p<\infty$) of $(x_t)_{t\in T}$ constructed in the proof of Theorem 8.3.*

1. *If for every $\sigma\in[T]$ the basic sequence $(x_{\sigma|n})$ is boundedly complete, then the basis of $\mathcal{A}^X_{\mathrm{hi}}$ (respectively, $\mathcal{A}^X_p$) is boundedly complete.*
2. *If for every $\sigma\in[T]$ the basic sequence $(x_{\sigma|n})$ is shrinking, then the basis of $\mathcal{A}^X_{\mathrm{hi}}$ (respectively, $\mathcal{A}^X_p$) is shrinking.*

Proof. (1) Let $(\bar{e}_{t_n})$ be the basis of $\mathcal{A}^X_{\mathrm{hi}}$, and assume that it is not boundedly complete. Then there exist a sequence (a_n) in $\mathbb{R}$, a sequence (I_k) of successive intervals of $\mathbb{N}$ and $r>0$ such that, setting $\bar{y}_d\coloneqq\sum_{n=1}^d a_n\bar{e}_{t_n}$ for every $d\in\mathbb{N}$, we have $\|\bar{y}_d\| \leqslant 1$ and $\|\sum_{n\in I_k} a_n \bar{e}_{t_n}\| \geqslant r$ for every $n \in \mathbb{N}$. We set $y_d := J(\bar{y}_d)$. By Proposition 7.7, the operator J is Tauberian. This implies that the sequence (y_d) is not Cauchy. Indeed, notice that there exist a subnet $(\bar{y}_i)_{i\in I}$ of $(\bar{y}_d)$ and $\bar{y}^{**} \in (\mathcal{A}_{\mathrm{hi}}^X)^{**} \setminus \mathcal{A}_{\mathrm{hi}}^X$ such that $w^*-\lim_{i\in I} \bar{y}_i = \bar{y}^{**}$. It follows that the corresponding subnet $(y_i)_{i\in I}$ of (y_d) must be weak* convergent to a vector $y^{**} \in (\mathcal{T}_2^X)^{**} \setminus \mathcal{T}_2^X$ which yields that the sequence (y_d) is not Cauchy. Thus, there exist $\varepsilon > 0$ and a subsequence (y_{d_k}) of (y_d) such that, setting $v_k := y_{d_{k+1}} - y_{d_k}$ for every $k \in \mathbb{N}$, the following hold.

(a) The sequence (v_k) is bounded and block, and, $\|v_k\| \geqslant \varepsilon$ for every $k \in \mathbb{N}$.
(b) Since for every $\sigma \in [T]$ the sequence $(x_{\sigma|n})$ is boundedly complete, we have $\lim \|P_\sigma(v_k)\| = 0$ for every $\sigma \in [T]$.
(c) The set W_X almost absorbs the set $\{v_k : k \in \mathbb{N}\}$. Indeed, first observe that the set W_X almost absorbs the set $\{y_d : d \in \mathbb{N}\}$. Next, notice that for every $k \in \mathbb{N}$ there exists an interval J_k of $\mathbb{N}$ such that $v_k = \pi_{J_k}(y_{d_{k+1}})$. Since W_X is closed under projections on intervals, it follows that W_X also almost absorbs the set $\{v_k : k \in \mathbb{N}\}$.

By Proposition 5.14 and arguing as in the proof of Theorem 8.4, we see that $\lim \|y_d\| = 0$ which is clearly a contradiction. The proof for the case of p-amalgamations is identical.

(2) Assume that for every $\sigma \in [T]$ the basic sequence $(x_{\sigma|n})$ is shrinking. By Theorem A.5, we have that $(\mathcal{T}_2^X)^* = \overline{\mathrm{span}}\{\bigcup_{\sigma\in[T]} B_{\mathcal{X}_\sigma^*}\}$. It follows that the basis (e_{t_n}) of $\mathcal{T}_2^X$ is also shrinking. First we will deal with the case of HI-amalgamations. For every $n \in \mathbb{N}$ let X_n be the space $\mathcal{T}_2^X$ equipped with the norm defined by the Minkowski gauge of the set $2^n W_x + \frac{1}{2^n} B_{\mathcal{T}_2^X}$, and let Z denote the HI Schauder sum of (X_n). Also let $\mathrm{Id}\colon \mathcal{A}_{\mathrm{hi}}^X \to Z$ be the identity operator, and let $\mathrm{Id}^* : Z^* \to (\mathcal{A}_{\mathrm{hi}}^X)^*$ denote the dual onto map. It is easy to verify that for every $n \in \mathbb{N}$ we have $\mathrm{Id}^*(e_{n,t}^*) = \lambda_n \bar{e}_t^*$ for some $\lambda_n \in \mathbb{R}$. On the other hand, by Proposition 6.16, we see that $Z^* = \overline{\mathrm{span}}\{\bigcup_n X_n^*\}$. Since $X_n^* = \overline{\mathrm{span}}\{e_{n,t}^* : t \in T\}$, we obtain that $Z^* = \overline{\mathrm{span}}\{e_{n,t}^* : n \in \mathbb{N}, t \in T\}$. It follows that $(\mathcal{A}_{\mathrm{hi}}^X)^* = \overline{\mathrm{span}}\{\bar{e}_t^* : t \in T\}$, and the proof for the case of HI-amalgamations is completed. The proof for the case of p-amalgamations is identical (actually it is simpler, since in this case Proposition 6.16 is straightforward). □

We proceed with the following proposition.

Proposition 8.7. *Let $\mathcal{A}_{\mathrm{hi}}^X$ be the HI-amalgamation of $(x_t)_{t\in T}$, and assume that $\mathcal{A}_{\mathrm{hi}}^X \cong Y \oplus W$. Then there exists finite $A \subseteq [T]$ such that either $\tilde{P}_A\colon Y \to \tilde{\mathcal{X}}_A$ or $\tilde{P}_A\colon W \to \tilde{\mathcal{X}}_A$ is an isomorphic embedding.*

Proof. First we claim that either Y or W does not contain an X-singular subspace. Indeed, suppose that there exist a subspace Y' of Y and a subspace W' of W such that Y' and W' are both X-singular. By property (4.b) in Definition 8.1, we see that $d(S_Y, S_W) = 0$ which is clearly a contradiction. The result then follows by property (6) in Definition 8.1. $\square$

We introduce the following definition.

Definition 8.8. (1) *Let $1 \leqslant p < \infty$ and let (x_n) denote the standard unit vector basis of ℓ_p. We enumerate the sequence (x_n) as $(x_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ as in Example 1; that is, for every $t \in \mathbb{N}^{<\mathbb{N}}$ we set $x_t \coloneqq x_{|t|}$. We denote by $\mathcal{A}_{\mathrm{hi}}^{\ell_p}$ the HI-amalgamation space of $(x_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$.*

(2) *Let U and V denote Pelczynski's universal spaces for basic sequences and unconditional basic sequences respectively (see [P, LT]). Recall that U has a Schauder basis (u_n) and for every basic sequence (x_k) there exists $L \in [\mathbb{N}]^\infty$ such that $(u_n)_{n\in L}$ is equivalent to (x_k); respectively, V has an unconditional Schauder basis (v_n) and for every unconditional basic sequence (y_k) there exists $L \in [\mathbb{N}]^\infty$ such that $(u_n)_{n\in L}$ is equivalent to (y_k). Let $(u_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ and $(v_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ be the enumerations of (u_n) and (v_n) as described in Example 3. By $\mathcal{A}_{\mathrm{hi}}^{U}$ and $\mathcal{A}_{\mathrm{hi}}^{V}$ we denote the HI-amalgamations of the Schauder trees bases $(u_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ and $(v_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ respectively.*

Remark 14. A remarkable feature of the space U is that $\mathcal{A}_{\mathrm{hi}}^{U}$, $\mathcal{T}_2^{U}$ and U are all mutually isomorphic.

We have the following theorem.

Theorem 8.9. *There exists a separable Banach space X which satisfies the following properties.*

- (i) *If Z is a subspace of X, then Z is reflexive if and only if it is HI.*
- (ii) *Every separable Banach space Y which contains all reflexive subspaces of X must also contain ℓ_1; that is, the class $\mathcal{C}$ of reflexive subspaces of X is Bourgain ℓ_1-generic.*
- (iii) *Every non-reflexive subspace Z of X contains a complemented copy of ℓ_1.*
- (iv) *If $X \cong Y \oplus W$, then either Y or W is contained in ℓ_1.*

Proof. The desired space X is the space $\mathcal{A}_{\mathrm{hi}}^{\ell_1}$. Indeed, let Z be a subspace of X. If Z is reflexive, then, by the lifting property of ℓ_1, Z must be ℓ_1-singular. By property (4.a) in Definition 8.1, we obtain that Z is HI. Conversely, assume that Z is HI. Then, clearly, Z is ℓ_1-singular. Invoking the lifting property of ℓ_1 once again, we see that Z must be ℓ_1-compact. By property (5) in Definition 8.1, we conclude that Z is reflexive. Thus, property (i) is satisfied.

We proceed to show that property (ii) is satisfied. Let Y be a separable Banach space that contains (up to isomorphism) all reflexive subspaces of X. For every $T \in \mathrm{Tr}$ set $\tilde{\mathcal{X}}_T \coloneqq \overline{\mathrm{span}}\{e_t : t \in T\}$. Since the sequence $(e_t)_{t\in T}$ defines (after a re-enumeration) a Schauder basis of $\tilde{X}_T$, it is easy to verify that the map $\Phi\colon \mathrm{Tr} \to \mathrm{SB}$ defined by $\Phi(T) = \tilde{\mathcal{X}}_T$ is Borel (see, e.g., [Bo3, Lemma 2.4]). Moreover, notice that if $T \in \tilde{\mathrm{WF}}$, then for every $\sigma \in \mathcal{N}$ the operator $\tilde{P}_\sigma\colon \tilde{\mathcal{X}}_T \to \ell_1$ is compact. Hence, by properties (4.a) and (5) in Definition 8.1, we see that $\tilde{\mathcal{X}}_T$ is either reflexive HI, or finite-dimensional. Let $B \subseteq \mathrm{SB}$ denote the isomorphic saturation of $\mathrm{Subs}(Y)$; it is analytic. Thus, the set $A \coloneqq \Phi^{-1}(B) \subseteq \mathrm{Tr}$ is analytic and, by our assumption, we have that $A \supseteq \mathrm{WF}$. But the set WF is $\mathbf{\Pi}^1_1$-complete, and so there exists $T \in \mathrm{IF}$ such that $\tilde{\mathcal{X}}_T$ is isomorphic to a subspace of Y. Since for every ill-founded tree T the space $\tilde{\mathcal{X}}_T$ contains ℓ_1, we conclude that property (ii) is satisfied.

Next, let Z be a non-reflexive subspace of $\mathcal{A}^{\ell_1}_{\mathrm{hi}}$. As we have already mentioned in the proof of property (i), the non-reflexivity of Z implies that Z is not ℓ_1-singular. It follows that there exists $\sigma \in \mathcal{N}$ such that the operator $\tilde{P}_\sigma \colon Z \to \ell_1$ is not strictly singular. This implies[1] that Z must contain a complemented copy of ℓ_1, and so property (iii) is satisfied.

Finally, let Y and W be subspaces of $\mathcal{A}^{\ell_1}_{\mathrm{hi}}$ such that $\mathcal{A}^{\ell_1}_{\mathrm{hi}} \cong Y \oplus W$. By Proposition 8.7, there exists finite $A \subseteq \mathcal{N}$ such that either Y, or W is isomorphic to a subspace of $\tilde{\mathcal{X}}_A$. Noticing that for every finite $A \subseteq \mathcal{N}$ the space $\tilde{\mathcal{X}}_A$ is isomorphic to ℓ_1, the result follows. □

8.2. **Applications.** We start by determining the descriptive set theoretic complexity of the classes HI, I and NUC presented in Section 3.

Theorem 8.10. *The classes* HI*,* I *and* NUC *are all* $\mathbf{\Pi}^1_1$*-complete.*

Proof. As we have shown in Section 3, all these classes are co-analytic non-Borel. It remains to prove that they are actually complete. Let $\tilde{\mathrm{Tr}}$ denote the set of all trees on $\mathbb{N}$ which have infinitely many nodes (we need to work with this class of trees since we are dealing with infinite-dimensional separable Banach spaces). Also let $\tilde{\mathrm{WF}}$ denote the set of all well-founded trees in $\tilde{\mathrm{Tr}}$. It is easy to see that the set $\tilde{\mathrm{Tr}}$ is Borel in $2^{\mathbb{N}^{<\mathbb{N}}}$ (thus, a standard Borel space); moreover, the set $\tilde{\mathrm{WF}}$ is $\mathbf{\Pi}^1_1$-complete. We will present a reduction of $\tilde{\mathrm{WF}}$ to HI which is also a reduction to I and NUC.

To this end, let $X = C[0,1]$ and let (x_n) be a normalized bimonotone Schauder basis of X. We enumerate the sequence (x_n) as $(x_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ as in Example 1, that is, for every $t \in \mathbb{N}^{<\mathbb{N}}$ we set $x_t \coloneqq x_{|t|}$. Then $(x_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ is a normalized bimonotone Schauder tree basis of X. Let $\mathcal{A}^X_{\mathrm{hi}}$ be the HI-amalgamation of $(x_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$, and let $(e_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ be the Schauder tree basis of $\mathcal{A}^X_{\mathrm{hi}}$. As in the proof of Theorem 8.9, for every $T \in \tilde{\mathrm{Tr}}$ we set $\tilde{\mathcal{X}}_T \coloneqq \overline{\mathrm{span}}\{e_t : t \in T\}$. The map $\tilde{\mathrm{Tr}} \ni T \mapsto \tilde{\mathcal{X}}_T \in \mathrm{SB}$ is Borel and, moreover, for every $T \in \tilde{\mathrm{WF}}$ the space $\tilde{\mathcal{X}}_T$ is reflexive and HI. On the other hand, if $T \notin \tilde{\mathrm{WF}}$, then there exists $\sigma \in \mathcal{N}$ such that $\tilde{\mathcal{X}}_\sigma$ is a subspace of $\tilde{\mathcal{X}}_T$ and so, by property (3) in Definition 8.1 and the choice of X, we see that $C[0,1]$ is isomorphic to a subspace of $\tilde{\mathcal{X}}_T$. Therefore,

$$T \in \tilde{\mathrm{WF}} \Leftrightarrow \tilde{\mathcal{X}}_T \in \mathrm{HI} \Leftrightarrow \tilde{\mathcal{X}}_T \in \mathrm{I} \Leftrightarrow \tilde{\mathcal{X}}_T \in \mathrm{NUC}.$$

[1]Recall the well-known fact that if X is a Banach space and there exists a non-strictly singular operator $T \colon X \to \ell_p$ (for $1 \leqslant p < \infty$), then X contains a complemented copy of ℓ_p.

The proof is completed. □

Remark 15. Note that the above reduction also shows the fact, first proved by Bossard, that the class REFL of separable reflexive spaces, the class SD of spaces with separable dual, the class Nℓ_1 of separable Banach spaces not containing ℓ_1 and the class NU of all non-universal separable Banach spaces are $\mathbf{\Pi}^1_1$-complete. Moreover, using the result of Tomczak-Jaegermann [TJ] that every HI space is arbitrarily distortable, we see that the class AD of all separable arbitrarily distortable Banach spaces is $\mathbf{\Pi}^1_1$-hard.

Our second application concerns the existence of universal spaces for certain classes of separable Banach spaces which are not universal for all separable Banach spaces. We start with the following definitions.

Definition 8.11. *Let $(\mathcal{P}_n)$ be a sequence of classes of separable Banach spaces, and set $\mathcal{P} := \bigcup_n \mathcal{P}_n$. We say that the sequence $(\mathcal{P}_n)$ is* stable *if the following are satisfied.*

1. *$\mathcal{P}$ is isomorphic invariant, that is, if $X \in \mathcal{P}$ and $Y \cong X$, then $Y \in \mathcal{P}$.*
2. *$\mathcal{P}$ is closed under subspaces, that is, if $X \in \mathcal{P}$ and Y is a subspace of X, then $Y \in \mathcal{P}$.*
3. *$\mathcal{P}$ contains all finite-dimensional Banach spaces.*
4. *$\mathcal{P}$ is closed under finite sums, that is, if $k \in \mathbb{N}$ and $X_1, \dots, X_k \in \mathcal{P}$, then $\sum_{i=1}^k \oplus X_i \in \mathcal{P}$.*

Definition 8.12. *Let $(\mathcal{P}_n)$ be a sequence of classes of separable Banach spaces. We say that the sequence $(\mathcal{P}_n)$ is* finitely determined *if for every separable Banach space X and every $n \in \mathbb{N}$ the following holds. If (F_k) is an increasing sequence of finite-dimensional subspaces of X with $\bigcup_k F_k$ dense in X, then we have*

$$X \in \mathcal{P}_n \Leftrightarrow F_k \in \mathcal{P}_n \text{ for every } k \in \mathbb{N}.$$

We have the following theorem.

Theorem 8.13. *Let $(\mathcal{P}_n)$ be a stable and finitely determined sequence of classes of separable Banach spaces, and set $\mathcal{P} := \bigcup_n \mathcal{P}_n$. Assume that there exists an unconditionally saturated separable Banach space X such that $X \notin \mathcal{P}$. Then there exists a separable Banach space Y with the following properties.*

1. *The space X is not contained in Y.*
2. *If $Z \in \mathcal{P}$ has a Schauder basis, then Z is contained in Y as a complemented subspace.*

We will see, later on, that a stronger version of Theorem 8.13 holds true. At this point we notice that Theorem 8.13 yields, for instance, that the class of separable Banach spaces with a Schauder basis and non-trivial type (respectively, non-trivial cotype) is not universal. More precisely, there exists a separable Banach space Y

containing all Banach spaces with a Schauder basis and non-trivial type (respectively, non-trivial cotype) and not containing a copy of ℓ_1 (respectively, c_0).

Proof of Theorem 8.13. Let (u_k) be the basis of Pelczynski's space U which is universal for all basic sequences. We may assume that (u_k) is normalized and bimonotone. Let $(u_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ be the enumeration of (u_k) as described in Example 3. The sequence $(u_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ is a normalized bimonotone Schauder tree basis of U which satisfies, additionally, the following properties.

(i) For every $L\in[\mathbb{N}]^\infty$ there exists $\sigma\in\mathcal{N}$ such that the sequence $(u_k)_{k\in L}$ coincides with $(u_{\sigma|m})$.
(ii) For every $\sigma\in\mathcal{N}$ there exists $L\in[\mathbb{N}]^\infty$ such that the sequence $(u_{\sigma|m})$ coincides with $(u_k)_{k\in L}$.

Given $\sigma\in\mathcal{N}$, let U_σ denote the space $\overline{\mathrm{span}}\{u_{\sigma|m}:m\in\mathbb{N}\}$. For every $n\in\mathbb{N}$ set $C_n\coloneqq\{\sigma\in\mathcal{N}:U_\sigma\in\mathcal{P}_n\}$. Since the sequence $(\mathcal{P}_n)$ is finitely determined and $(u_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ is a Schauder tree basis of U, we see that C_n is a closed subset of $\mathcal{N}$. Therefore, the set $C\coloneqq\bigcup_n C_n$ is F_σ. We select $F\subseteq\mathcal{N}\times\mathcal{N}$ closed such that $C=\mathrm{proj}_{\mathcal{N}}F$. As F is closed in $\mathcal{N}\times\mathcal{N}$, it is the body of a pruned tree T on $\mathbb{N}\times\mathbb{N}$. We define $(w_t)_{t\in T}$ as follows. Let $t\in T$ be arbitrary and set $n\coloneqq|t|$. There exist $(\sigma_1,\sigma_2)\in F$ such that $t=(\sigma_1|n,\sigma_2|n)$. (Note that, here, we view the nodes of T as pairs $(t_1,t_2)\in\mathbb{N}^{<\mathbb{N}}\times\mathbb{N}^{<\mathbb{N}}$ with $|t_1|=|t_2|$.) We set $w_t\coloneqq u_{\sigma_1|n}$. Observe that w_t is well-defined and independent of the choice of σ_1 and σ_2, that is, if $(\sigma_1',\sigma_2')\in F$ are such that $t=(\sigma_1|n,\sigma_2|n)=(\sigma_1'|n,\sigma_2'|n)$, then $\sigma_1|n=\sigma_1'|n$. Also set $W\coloneqq\overline{\mathrm{span}}\{w_t:t\in T\}$. The following properties are immediate consequences of the above construction.

(I) $(w_t)_{t\in T}$ is a normalized bimonotone Schauder tree basis of W.
(II) For every $\sigma\in[T]=F$ there exists $\sigma_1\in C$ such that $W_\sigma=U_{\sigma_1}$.
(III) For every $\sigma_1\in C$ there exists $\sigma\in[T]=F$ such that $U_{\sigma_1}=W_\sigma$.

The desired space Y is the HI-amalgamation of $(w_t)_{t\in T}$. We proceed to show that Y satisfies the requirements of the theorem. First notice that property (2) is an immediate consequence of (III) and property (3) in Definition 8.1. Therefore, we only need to prove that X is not isomorphic to any subspace of Y. Assume not. Then we claim that no subspace X' of X is W-singular. Indeed, if there existed an X-singular subspace X' of X, then, by property (4.a) in Definition 8.1, we would have that X' is HI, a contradiction since X is unconditionally saturated. By property (6) in Definition 8.1, there exists finite $A\subseteq[T]$ such that the operator $\tilde{P}_A\colon X\to\tilde{\mathcal{W}}_A$ is an isomorphic embedding. Let $\{\sigma_1,\dots,\sigma_k\}$ be an enumeration of A. Notice that for every $\in\{1,\dots,k\}$ there exist a final segment s_i of σ_i and a finite dimensional space F such that $\tilde{\mathcal{W}}_A=F\oplus\big(\sum_{i=1}^k\oplus\tilde{P}_{s_i}(\tilde{\mathcal{W}}_{\sigma_i})\big)$. Using the fact that the sequence $(\mathcal{P}_n)$ is stable, we conclude that $X\in\mathcal{P}$, a contradiction. Thus, the space X is not contained in Y and the proof is completed. $\square$

Remark 16. The assumption in Theorem 8.13 of the existence of an unconditional saturated separable Banach space X with $X \notin \mathcal{P}$ is not really needed. Indeed, as we shall see, for any class $\mathcal{P}$ as in Theorem 8.13 there exists an unconditionally saturated (in fact, ℓ_2-saturated) separable Banach space X with $X \notin \mathcal{P}$ provided, of course, that every space in $\mathcal{P}$ is not universal. The main point, however, in Theorem 8.13 is that *any* such space X is not contained in Y.

9. Generic classes of separable Banach spaces

Definition 9.1. *Let $\mathcal{C}$ be an isomorphic invariant class of separable Banach spaces such that every $X \in \mathcal{C}$ is non-universal.*

(1) *We say that $\mathcal{C}$ is* Bourgain generic *if every separable Banach space Y that contains all members of $\mathcal{C}$ up to isomorphism, must be universal.*
(2) *We say that $\mathcal{C}$ is* Bossard generic *if every analytic subset A of* SB *that contains all members of $\mathcal{C}$ up to isomorphism, must also contain a space $Y \in A$ which is universal.*

We proceed to discuss the relation between the different notions of genericity. We notice, first, that a class $\mathcal{C}$ is Bossard generic if and only if $\sup\{\psi_Z(Y) : Y \in \mathcal{C}\} = \omega_1$ where Z is any universal space and ψ_Z is the $\mathbf{\Pi}^1_1$-rank on NC_Z described in Section 3. This is easily seen to imply that if $\mathcal{C}$ is Bossard generic, then $\mathcal{C}$ is Bourgain generic. Concerning the opposite direction we make the following conjecture.

Conjecture. *Bourgain genericity coincides with Bossard genericity.*

We proceed to show that within the class of separable Banach spaces with the bounded approximation property, Bourgain genericity does imply Bossard genericity. To this end we start with the following proposition.

Proposition 9.2. *Let $A \subseteq$ SB be analytic such that every $X \in A$ is non-universal. Then there exists a non-universal Banach space Y with a Schauder basis which contains a complemented copy of every $X \in A$ with a Schauder basis.*

Proof. We first argue as in the proof of Theorem 8.13. Specifically, let (u_k) be the basis of Pelczynski's universal space U; as usual, we may assume that (u_k) is normalized and bimonotone. Let $(u_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ be the enumeration of (u_k) as described in Example 3. The sequence $(u_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ is a normalized bimonotone Schauder tree basis of U which satisfies properties (i) and (ii) described in Theorem 8.13. The map $\mathcal{N} \ni \sigma \mapsto U_\sigma \in \mathrm{Subs}(U)$ is easily seen to be Borel. It follows that the set

$$A_1 := \{\sigma \in \mathcal{N} : U_\sigma \in A_{\cong}\}$$

is analytic where $A_{\cong}$ denotes the isomorphic saturation of A. As in Theorem 8.13, we select a closed subset F of $\mathcal{N}\times\mathcal{N}$ such that $A_1 = \mathrm{proj}_{\mathcal{N}} F$. Let T be the (unique) downward closed pruned tree on $\mathbb{N}\times\mathbb{N}$ with $[T] = F$. Next, define $(w_t)_{t\in T}$ as in the proof of Theorem 8.13, and set $W := \overline{\mathrm{span}}\{w_t : t \in T\}$. Finally, let Y be the

HI-amalgamation of $(w_t)_{t\in T}$. We will show that Y satisfies the requirements of the proposition.

Clearly, every $X \in A$ with a Schauder basis is a complemented subspace of Y. It remains to show is that the space Y is not universal. To this end, for every $k \in \mathbb{N}$ and every $\bar{\sigma} = (\sigma_1, \dots, \sigma_k) \in [T]^k$ set $A_{\bar{\sigma}} \coloneqq \big\{\sigma_i : i \in \{1, \dots, k\}\big\}$ and observe that $|A_{\bar{\sigma}}| \leqslant k$. Also notice that the map

$$[T]^k \ni \bar{\sigma} = (\sigma_1, \dots, \sigma_k) \mapsto \tilde{\mathcal{W}}_{A_{\bar{\sigma}}} = \overline{\mathrm{span}}\{e_t : t \in A_{\bar{\sigma}}\} \in \mathrm{Subs}(Y)$$

is Borel. Therefore, the set

$$A_2 \coloneqq \{Z \in \mathrm{SB} : \exists \text{ finite } A \subseteq [T] \text{ such that } Z \cong \tilde{\mathcal{W}}_A\}$$

is analytic. Notice that if $A = \{\sigma_1, \dots, \sigma_n\} \subseteq [T]$, then there exist a finite-dimensional space F and for every $i \in \{1, \dots, n\}$ a final segment s_i of σ_i such that $\tilde{\mathcal{W}}_A = F \oplus \big(\sum_{i=1}^n \oplus \tilde{P}_{s_i}(\tilde{\mathcal{W}}_{\sigma_i})\big)$. For every $i \in \{1, \dots, n\}$ there exists $X_i \in A$ such that $X_i \cong \tilde{\mathcal{W}}_{\sigma_i}$. Hence, by our assumptions, for every $i \in \{1, \dots, n\}$ the space $\tilde{P}_{s_i}(\tilde{\mathcal{W}}_{\sigma_i})$ is not universal. By a result of Rosenthal (see [Ro2, Theorem 4.10], or [Ro3]), for every finite $A \subseteq [T]$ the space $\tilde{\mathcal{W}}_A$ is also non-universal.

Set $Z = C[0,1]$ and let (e_n) be a Schauder basis of Z. By the previous discussion, we see that $A_2 \subseteq \mathrm{NC}_Z$. Let ϕ_Z be the $\mathbf{\Pi}^1_1$-rank on NC_Z defined in Theorem 3.10. Since A_2 is analytic, by boundedness, we have $\sup\{\phi_Z(X) : X \in A_2\} = \xi < \omega_1$. By Corollary 4.15, there exists a reflexive and ℓ_2-saturated separable Banach space X_ξ such that $o\big(T(X_\xi, Z, (e_n))\big) > \xi$. We claim that X_ξ is not contained in Y. Indeed, arguing as in the proof of Theorem 8.13 and using the fact that X_ξ is ℓ_2-saturated, we see that if X_ξ was contained in Y, then there would existed finite $A \subseteq [T]$ such that X_ξ is isomorphic to a subspace of $\tilde{\mathcal{W}}_A$. This implies that

$$\xi < o\big(T(X_\xi, Z, (e_n))\big) \leqslant o\big(T(\tilde{\mathcal{W}}_A, Z, (e_n))\big) \leqslant \phi_Z(\tilde{\mathcal{W}}_A) \leqslant \xi$$

a contradiction. Therefore, X_ξ is not contained in Y and the proof is completed. □

We have the following theorem.

Theorem 9.3. *Let $A \subseteq \mathrm{SB}$ analytic such that every $X \in A$ is not universal. Then there exists a non-universal separable Banach space Y such that every $X \in A$ with the bounded approximation property is contained in Y as a complemented subspace.*

Proof. We start by recalling the definition of the space C_0 due to Johnson [J]. Let (E_n) be a sequence of finite-dimensional spaces which is dense in the Banach–Mazur distance in the family of all finite-dimensional spaces, and set

$$C_0 \coloneqq \Big(\sum_{n\in\mathbb{N}} \oplus E_n\Big)_{c_0}.$$

By a result of Lusky (see [C, Proposition 6.10], or [Lu]) for every separable Banach space with the bounded approximation property the space $X \oplus C_0$ has a Schauder basis. The map

$$\mathrm{SB} \times \mathrm{SB} \ni (X, Y) \mapsto X \oplus Y \in \mathrm{Subs}\big(C(2^{\mathbb{N}}) \oplus C(2^{\mathbb{N}})\big)$$

is Borel, and so the map $\mathrm{SB} \ni X \mapsto X \oplus C_0 \in \mathrm{SB}$ is Borel too. It follows that the set $A_1 = \{Y \in \mathrm{SB} : \exists X \in A \text{ with } Y \cong X \oplus C_0\}$ is $\mathbf{\Sigma}^1_1$. We notice the following properties of the set A_1.

(1) By Rosenthal's theorem mentioned in the previous proposition and our assumptions, every $Z \in A_1$ is not universal.
(2) If $X \in A$ has the bounded approximation property, then there exists $Z \in A_1$ with a Schauder basis such that X is isomorphic to a complemented subspace of Z.

We apply Proposition 9.2 and we obtain a non-universal separable Banach space Y such that every $Z \in A_1$ with a Schauder basis is contained in Y as a complemented subspace. Invoking (2), we see that Y is the desired space. $\square$

The notions of Bourgain and Bossard genericity can be relativized to any separable Banach space X as follows.

Definition 9.4. *Let X be a separable Banach space and let $\mathcal{C}$ be an isomorphic invariant class of separable Banach spaces such that X is not contained in any finite direct sum of members of $\mathcal{C}$.*

(1) *We say that the class $\mathcal{C}$ is* Bourgain X-generic *if for every separable Banach space Y which contains all members of $\mathcal{C}$, X is isomorphic to a subspace of a finite sum of Y.*
(2) *We say that the class is* Bossard X-generic *if for every analytic subset A of* SB *which contains all members of $\mathcal{C}$ up to isomorphism, X is isomorphic to a subspace of a finite direct sum of members of A.*

We make a few comments on the above defined notions of genericity. Assume that X is a separable Banach space with the following stability property (S).

(S) If $(Y_i)_{i=1}^n$ is a finite sequence of separable Banach spaces such that X is isomorphic to a subspace of $\sum_{i=1}^n \oplus Y_i$, then there exists $i_0 \in \{1, \dots, n\}$ such that X is isomorphic to a subspace of Y_{i_0}.

It is clear that whenever X has property (S), then the notions of Bossard and Bourgain X-genericity defined above are reduced to the corresponding analogues of Definition 9.1. Typical examples of separable Banach spaces with property (S) are the universal spaces (this is a consequence of the aforementioned result of Rosenthal) as well as the minimal spaces (such as c_0 and ℓ_p for $1 \leqslant p < \infty$). Hence, the notions of Bourgain and Bossard X-genericity are indeed generalizations of the concepts presented in Definition 9.1. Moreover, if X is a HI space, then the condition on $\mathcal{C}$ can be reduced to the following property: for every finite-codimensional subspace X' of X, the space X' is not contained in any member of $\mathcal{C}$. This follows from the following general fact (see [AT, Proposition 1.2]).

Proposition 9.5. *Let X be a HI space and let $T\colon X \to Y$ be a bounded linear operator where Y is any Banach space. If T is not strictly singular, then there exists a finite-codimensional subspace X' of X such that the operator $T\colon X' \to Y$ is an isomorphic embedding.*

We proceed to present examples showing the necessity of extra "linear" conditions in the definitions of X-genericities in the case of an arbitrary separable Banach space X.

Example 4. (1) Let A_1 and A_2 denote the isomorphic classes of ℓ_1 and ℓ_2 respectively, that is, $A_1 = \{Y \in \mathrm{SB} : Y \cong \ell_1\}$ and $A_2 = \{Y \in \mathrm{SB} : Y \cong \ell_2\}$. Both A_1 and A_2 are analytic (and, in fact, A_2 is Borel). We set $\mathcal{C} \coloneqq A_1 \cup A_2$ and $X \coloneqq \ell_1 \oplus \ell_2$. Notice that if Y is any separable Banach space containing, up to isomorphism, all members of $\mathcal{C}$, then Y must contain ℓ_1 and ℓ_2. Since these spaces are totally incomparable, we obtain that Y must also contain X. Nevertheless, the class $\mathcal{C}$ is analytic, yet no member of $\mathcal{C}$ contains X. (Note, however, that X is contained in a finite sum of members of $\mathcal{C}$.) This example was communicated to us by Rosendal and Schlumprecht.

(2) We set $Z_1 \coloneqq \mathcal{A}_{\mathrm{hi}}^{\ell_1}$ and $Z_2 \coloneqq \mathcal{A}_{\mathrm{hi}}^{\ell_2}$ (see Definition 8.8). Let $(e_t^1)_{t\in\mathbb{N}^{<\mathbb{N}}}$ and $(e_t^2)_{t\in\mathbb{N}^{<\mathbb{N}}}$ denote the Schauder bases of Z_1 and Z_2 respectively. As in the proof of Theorem 8.10, for every tree $T \in \mathrm{Tr}$ let Z_T^1 and Z_T^2 denote the subspaces of Z_1 and Z_2 spanned by the vectors $(e_t^1)_{t\in T}$ and $(e_t^2)_{t\in T}$ respectively. Next, we set $\mathcal{C}_1 \coloneqq \{Z_T^1 : T \in \mathrm{WF}\}$, $\mathcal{C}_2 \coloneqq \{Z_T^2 : T \in \mathrm{WF}\}$ and $\mathcal{C} \coloneqq \mathcal{C}_1 \cup \mathcal{C}_2$. Also set $X \coloneqq \ell_1 \oplus \ell_2$. Assume that Y is a separable Banach space which contains, up to isomorphism, all members of $\mathcal{C}$. The maps $T \mapsto Z_T^1$ and $T \mapsto Z_T^2$ are Borel, and so there exist $T_1, T_2 \in \mathrm{IF}$ such that $Z_{T_1}^1$ and $Z_{T_2}^2$ are isomorphic to subspaces of Y. Noticing that ℓ_1 (respectively, ℓ_2) is contained in Z_T^1 (respectively, Z_T^2) for any ill-founded tree T, we see that ℓ_1 and ℓ_2 are contained in Y. Therefore, as in the previous example, we conclude that X is contained in Y. Observe that X is not contained in any finite sum of members of $\mathcal{C}$, since $\mathcal{C}$ contains only HI spaces. However, setting $A \coloneqq \{Z_T^1 : T \in \mathrm{Tr}\} \cup \{Z_T^2 : T \in \mathrm{Tr}\}$, we see that A is analytic, contains all members of $\mathcal{C}$ up to isomorphism, yet no member of A contains X. (Again note that X is contained in a finite sum of members of A.)

(3) Our last example shows that if we do not impose extra conditions on the definition of Bourgain X-genericity, then it becomes incomparable with the notion of Bossard X-genericity. To this end, let W be any separable HI space. We set

$$\mathcal{C} \coloneqq \{Y : Y \text{ is isomorphic to a finite-codimensional subspace of } W\}$$

and $X \coloneqq W \oplus W$. It is clear that X is contained in a finite sum of members of $\mathcal{C}$, and so it is Bossard X-generic according to Definition 9.4. On the other hand, observe that the space W contains all members of $\mathcal{C}$, yet X is not contained in W, since W in HI and X is decomposable. (However, as in the previous examples, notice that X is contained in a finite sum of members of W.)

It is clear that for any separable Banach space X, if a class $\mathcal{C}$ of separable Banach spaces is Bossard X-generic, then it is also Bourgain X-generic. The problem concerning the converse implication for an arbitrary separable Banach space X is open, even if we restrict our attention to the class of spaces with a Schauder basis. There are, however, a number of cases where we can prove the following analogue of Proposition 9.2.

Theorem 9.6. *Let X be either an unconditionally saturated, or a HI saturated, or a minimal separable Banach space. Also let A be an analytic class of separable Banach spaces such that X is not contained in any finite sum of members of A. Then there exists a separable Banach space Y which contains an isomorphic copy of every member of A with a Schauder basis and, moreover, X is not contained in any finite sum of members of Y.*

Proof. Let A be as in the statement of the theorem. We argue as in the proof of Proposition 9.2 and we obtain a downward closed pruned tree T on $\mathbb{N}\times\mathbb{N}$, a space Z and a normalized bimonotone Schauder tree basis $(z_t)_{t\in T}$ of Z such that the following are satisfied.

(1) For every $\sigma\in[T]$ there exists $Y\in A$ such that Z_σ is isomorphic to Y.
(2) For every $Y\in A$ with a Schauder basis there exists $\sigma\in[T]$ with Y isomorphic to Z_σ.

First, we will deal with the case when X is a minimal separable Banach space. Then, by the minimality of X, there exists $1<p<\infty$ such that ℓ_p is not contained in X. The desired space Y is the p-amalgamation of $(z_t)_{t\in T}$. Clearly, we only have to show that X is not contained in any finite sum of Y. Assume not. Since X is minimal, we see that X must be contained in Y. By the properties of the p-amalgamation space, the fact that X does not contain ℓ_p and arguing as in the proof of Proposition 9.2, there exists finite $A\subseteq[T]$ such that the operator $P_A\colon X\to\tilde{\mathcal{Z}}_A$ is an isomorphic embedding. Invoking once again the minimality of X, we obtain $\sigma\in[T]$ such that X is contained in Z_σ which is a contradiction by property (1) and our assumptions.

Next, assume that X is an unconditionally saturated space. The desired space Y is the HI-amalgamation of $(z_t)_{t\in T}$. Again, we only have to show that X is not contained in any finite sum of Y. If not, then there exists $k\in\mathbb{N}$ such that X is contained in $\sum_{i=1}^k\oplus Y_i$ where $Y_i=Y$ for every $i\in\{1,\dots,k\}$. We set $Z_i:=Z$ for every $i\in\{1,\dots,k\}$, and let $\mathcal{T}_2^{Z_i}$ be the ℓ_2 Baire sum of $(z_t)_{t\in T}$ viewed as a Schauder tree basis of Z_i. We define $\bar{J}\colon\sum_{i=1}^k\oplus Y_i\to(\sum_{i=1}^k\oplus\mathcal{T}_2^{Z_i})_{\ell_2}$ by the rule $\bar{J}\big((y_1,\dots,y_k)\big)=\big(J(y_1),\dots,J(y_k)\big)$. We claim that $\bar{J}|_X$ is an isomorphic embedding. Indeed, assume on the contrary that there exists a subspace X' of X such that $\bar{J}|_{X'}$ is compact. There exist $i_0\in\{1,\dots,k\}$ and a further subspace X''

of X' such that $P_{i_0}|_{X''}$ is an isomorphic embedding where $P_{i_0}\colon \sum_{i=1}^k \oplus Y_i \to Y_{i_0}$ is the natural projection. It follows that X'' is a Z-compact subspace of Y and so, by the properties of the HI-amalgamation space, we conclude that X'' is HI which contradicts the fact that X is unconditionally saturated.

Now set $E \coloneqq \sum_{i=1}^k \oplus Z_i$. For every $i \in \{1,\dots,k\}$ let T_i be a different copy of T, and let S denote the disjoint union of the trees $(T_1,\dots,T_k)$. Clearly, S may be considered as a downward closed pruned tree on a countable set. Moreover, for every $t \in S$ there exists unique $i \in \{1,\dots,k\}$ such that $t \in T_i$. We define a Schauder tree basis $(e_t)_{t\in S}$ in E as follows. For every $t \in S$ let $i \in \{1,\dots,k\}$ be the unique i such that $t \in T_i$ and set $e_t \coloneqq z_t \in Z_i$ where we view z_t as a vector in E. Clearly, $(e_t)_{t\in S}$ is a normalized bimonotone Schauder tree basis E and satisfies properties (1) and (2) above. Precisely, the following hold.

(3) For every $\sigma \in [S]$ there exists $Y \in A$ such that E_σ is isomorphic to Y.
(4) For every $Y \in A$ with a Schauder basis there exists $\sigma \in [S]$ such that Y is isomorphic to E_σ.

Let $\mathcal{S}_2^E$ denote the ℓ_2 Baire sum of $(e_t)_{t\in S}$, and note that $(\sum_{i=1}^k \oplus \mathcal{T}_2^{Z_i})_{\ell_2} = \mathcal{S}_2^E$. By the discussion in the previous paragraph, we see that the operator $\bar{J}\colon X \to \mathcal{S}_2^E$ is an isomorphic embedding. Let W_E be the closed, bounded, convex and symmetric set defined in Definition 5.3 for the Schauder tree basis $(e_t)_{t\in S}$. We will show that the set W_E almost absorbs $B_{\bar{J}(X)}$. To this end notice, first, that

$$W_E = \mathrm{conv}\big\{ W_{Z_i} : i \in \{1,\dots,k\}\big\} \tag{9.1}$$

where W_{Z_i} denotes the set defined in Definition 5.3 for the space Z_i. (More precisely, we have $W_{Z_i} = \mathrm{conv}\big\{ \bigcup_{\sigma\in[T_i]} B_{\mathcal{E}_\sigma}\big\}$.) Since the operator $\bar{J}\colon X \to \big(\sum_{i=1}^k \mathcal{T}_2^Z\big)_2$ is an isomorphic embedding, there exists a constant $C>0$ such that if $v = \bar{J}(x) \in B_{\bar{J}(X)}$, then we have $\|x\| \leqslant C$. Write $x = x_1 + \dots + x_k$ with $x_i \in Y_i$ and $\|x_i\|_Y \leqslant C$ for every $i \in \{1,\dots,k\}$. For every $n \in \mathbb{N}$ let $\|\cdot\|_n$ be the equivalent norm on $\mathcal{T}_2^{Z_i}$ which is defined using the Minkowski gauge of the set $2^n W_{Z_i} + \frac{1}{2^n} B_{\mathcal{T}_2^{Z_i}}$. There exists a constant $C'>0$ such that for every $n \in \mathbb{N}$ and every $i \in \{1,\dots,k\}$ we have $J(x_i) \in C' 2^n W_{Z_i} + \frac{C'}{2^n} B_{\mathcal{T}_2^{Z_i}}$, and so

$$\frac{J(x_1)+\dots+J(x_k)}{k} \in (C'2^n)\,\mathrm{conv}\big\{W_{Z_i} : i \in \{1,\dots,k\}\big\} + \Big(\frac{C'}{k2^n}\Big)\sum_{i=1}^k \oplus B_{\mathcal{T}_2^{Z_i}}.$$

By (9.1), this yields that for every $n \in \mathbb{N}$ we have

$$v = \bar{J}(x) = J(x_1) + \dots + J(x_k) \in (kC'2^n)\, W_E + \Big(\frac{C'}{k2^n}\Big)\sum_{i=1}^k \oplus B_{\mathcal{T}_2^{Z_i}}$$

which is easily seen to imply that the set W_E almost absorbs $B_{\bar{J}(X)}$. By Theorem 5.16, there exists finite $A \subseteq [S]$ such that the operator $P_A\colon X \to \mathcal{E}_A$ is an isomorphic embedding. By property (3) of the Schauder tree basis $(e_t)_{t\in S}$, we obtain that X

is contained in a finite sum of members of A which contradicts our assumptions on A. This shows that X is not contained in any finite sum of Y, as desired.

The proof for the case of a HI saturated space X is similar to the proof of the previous case; the only difference is that it uses HI-amalgamations instead of p-amalgamations. The proof of the theorem is completed. $\square$

A consequence of the above theorem is the following result concerning HI Banach spaces without a Schauder basis. We recall that the existence of such spaces was established in [AKP].

Corollary 9.7. *Let X be a HI separable Banach space without a Schauder basis. Then the class $\mathcal{C}$ consisting of all subspaces of X with a Schauder basis is not Bourgain X-generic and, consequently, not Bossard X-generic.*

For the proof we need the following lemma.

Lemma 9.8. *Let X be a HI space without a Schauder basis and let $Y_1, \dots, Y_k$ be (not necessarily distinct) subspaces of X with a Schauder basis. Then X does not embed into $\sum_{i=1}^{k} \oplus Y_i$.*

Proof. Assume not. Then, by Proposition 9.5, there exist a finite-codimensional subspace X' of X and $i_0 \in \{1, \dots, k\}$ such that X' is isomorphic to a subspace of Y_{i_0}. Observe that $X \cong X' \oplus F$ for a finite dimensional space F with $\dim(F) = l$ and, moreover, that the space X/Y_{i_0} is infinite-dimensional. Therefore, there exists a finite dimensional subspace G of X such that $G \cap Y_{i_0} = \{0\}$ and $\dim(G) = l$. It follows that there exists an isomorphism $T\colon X \cong X' \oplus F \to Y_{i_0} \oplus G$. This yields to a contradiction since $Y_{i_0} \oplus G$ is a proper subspace of X (see [GM]). $\square$

We continue with the proof of Corollary 9.7.

Proof of Corollary 9.7. We first observe that for any separable Banach space X the class of all subspaces of X with a Schauder basis is an analytic subset of SB. To see this notice that the class $\mathcal{S}$ of all separable Banach spaces with a Schauder basis is analytic. Indeed, as we have indicated in Proposition 9.2, the map $\mathcal{N} \ni \sigma \mapsto U_\sigma \in \mathrm{SB}$ is Borel, and so the set $B := \{U_\sigma : \sigma \in \mathcal{N}\}$ is analytic (actually, it is Borel). Then $\mathcal{S} = B_{\cong}$ where $B_{\cong}$ denote, as usual, the isomorphic saturation of B. Clearly, this implies that $\mathcal{S}$ is analytic. Since $\mathcal{C} = \mathcal{S} \cap \mathrm{Subs}(X)$, we see that $\mathcal{C}$ is analytic. By Theorem 9.6 and Lemma 9.8, the result follows. $\square$

A natural question related to Theorems 9.3 and 9.6 is whether for an analytic subset A of SB consisting of spaces with a certain property (P), the universal space also satisfies the same property. The following definition makes this question precise.

Definition 9.9. *Let $\mathcal{C}$ be an isomorphic invariant class of separable Banach spaces. We say that $\mathcal{C}$ is* strongly bounded *if for every analytic subset A of $\mathcal{C}$ there exists $Y \in \mathcal{C}$ that contains, up to isomorphism, all members of A.*

It is clear that, under the terminology of the above definition, Theorem 9.3 states that the class of non-universal separable Banach spaces with a Schauder basis is strongly bounded. The following theorem provides further natural examples of strongly bounded classes.

Theorem 9.10. *Let $\mathcal{C}$ denote one of the following classes of Banach spaces.*

(1) *The reflexive spaces with a Schauder basis.*
(2) *The spaces with a shrinking Schauder basis.*
(3) *The ℓ_p-saturated for some $1 \leqslant p < \infty$, or c_0-saturated spaces with a Schauder basis.*
(4) *The HI saturated spaces with a Schauder basis.*
(5) *The unconditionally saturated spaces with a Schauder basis.*

Then $\mathcal{C}$ is strongly bounded.

Proof. First we will deal with the cases (1), (3), (4) and (5). Let A be an analytic subset of $\mathcal{C}$ (notice that, by definition, every $X \in A$ has a Schauder basis). We need to find a separable Banach space $Y \in \mathcal{C}$ such that every member of A is contained in Y. We first observe that there exist a downward closed pruned tree T on $\mathbb{N} \times \mathbb{N}$ and a space Z with a normalized bimonotone Schauder tree basis $(z_t)_{t\in T}$ such that the following are satisfied.

(1) For every $\sigma \in [T]$ there exists $X \in A$ such that Z_σ is isomorphic to Y.
(2) For every $X \in A$ there exists $\sigma \in [T]$ such that X is isomorphic to Z_σ.

In cases (1) and (4), the desired space Y is the HI-amalgamation of $(z_t)_{t\in T}$; indeed, for the class of reflexive spaces this follows from Proposition 8.5, while for the class of HI saturated spaces it follows from part (4.a) of Definition 8.1. In the case of unconditionally saturated spaces, the desired space Y is the p-amalgamation of $(z_t)_{t\in T}$ for any $1 < p < \infty$; the fact that Y is unconditionally saturated follows from property $(4)'$ of Definition 8.2. The p-amalgamation space can also be used if $\mathcal{C}$ is the class of ℓ_p-saturated spaces for some $1 < p < \infty$. Finally, if $\mathcal{C}$ is the class of ℓ_1-saturated (respectively, c_0-saturated) spaces with a Schauder basis, then the desired space Y is the interpolation space of $(\mathcal{T}_2^Z, W_Z)$ by considering as external norm the ℓ_1-norm (respectively, the c_0-norm). Using the same arguments as in the proof of the properties of p-amalgamations, it is easy to verify that Y is ℓ_1-saturated (respectively, c_0-saturated) and contains all members of A.

Now we consider case (2), that is, the case of spaces with a shrinking Schauder basis. Fix an analytic class A of spaces with a shrinking Schauder basis. As in the previous cases, we will obtain a downward closed pruned tree T on $\mathbb{N} \times \mathbb{N}$ and a space Z with a normalized bimonotone Schauder tree basis $(z_t)_{t\in T}$ with the following properties.

(P1) For every $\sigma \in [T]$ the sequence $(z_{\sigma|n})$ is shrinking.
(P2) For every $\sigma \in [T]$ there exists $X \in A$ such that Z_σ is isomorphic to X.
(P3) For every $X \in A$ there exists $\sigma \in [T]$ such that X is isomorphic to Z_σ.

To this end we will need some results from [B2] which we will briefly recall. Let U denote the universal space of Pelczynski for Schauder basic sequences, and let (u_k) denote the basis of U. Consider the set

$$\mathcal{S} \coloneqq \{L \in [\mathbb{N}]^\infty : (u_k)_{k\in L} \text{ is shrinking}\}.$$

In [B2, Theorem 5.4], it is shown that the set $\mathcal{S}$ is co-analytic and that the map

$$\mathcal{S} \ni L \mapsto \mathrm{Sz}\big(\overline{\mathrm{span}}\{u_k : k \in L\}\big)$$

is a co-analytic rank on $\mathcal{S}$ where $\mathrm{Sz}\big(\overline{\mathrm{span}}\{u_k : k \in L\}\big)$ denotes the Szlenk index of the space $\overline{\mathrm{span}}\{u_k : k \in L\}$. We also recall that the map $\mathrm{SD} \ni X \mapsto \mathrm{Sz}(X)$ is a $\mathbf{\Pi}^1_1$-rank on SD (see [Bo3]). Since A is an analytic subset of SD, by boundedness, we obtain that

$$\sup\{\mathrm{Sz}(X) : X \in A\} = \xi < \omega_1.$$

It follows that the set

$$\mathcal{S}_\xi \coloneqq \big\{L \in \mathcal{S} : \mathrm{Sz}\big(\overline{\mathrm{span}}\{u_k : k \in L\}\big) \leqslant \xi\big\}$$

is Borel. Let $(u_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ be the enumeration of (u_k) as described in Example 3; in particular, for every $\sigma \in \mathcal{N}$ there exists $L_\sigma = \{l_1 < l_2 < \cdots\} \in [\mathbb{N}]^\infty$ such that $(u_{\sigma|n})$ is the subsequence (u_{l_n}). The map $h\colon \mathcal{N} \to [\mathbb{N}]^\infty$ defined by $h(\sigma) = L_\sigma$ is easily seen to be continuous. It follows that the set

$$A_1 \coloneqq \{\sigma \in \mathcal{N} : h(\sigma) \in \mathcal{S}_\xi \text{ and } \exists Y \in A \text{ with } U_\sigma \cong Y\}$$

is analytic. We notice the following facts which are straightforward consequences of the universality of U, the choice of ξ and the definition of A_1.

(i) For every $\sigma \in A_1$ the sequence $(u_{\sigma|n})$ is shrinking.
(ii) For every $X \in A$ there exists $\sigma \in A_1$ such that $X \cong U_\sigma$.
(iii) For every $\sigma \in A_1$ there exists $X \in A$ such that $U_\sigma \cong X$.

Thus, as usual, we may construct a downwards closed pruned tree T on $\mathbb{N} \times \mathbb{N}$ and a Banach space Z with a normalized bimonotone Schauder tree basis $(z_t)_{t\in T}$ such that properties (P1)–(P3) are satisfied. The desired space Y is then the HI-amalgamation of $(z_t)_{t\in T}$; The fact that this space has a shrinking basis is an immediate consequence of Theorem 8.6. The proof is completed. □

We proceed to discuss another application of the above results. To this end we introduce the following definition.

Definition 9.11. *Let X be a Banach space with a Schauder basis (e_n) and let $p > 1$. We say that X has* asymptotic type p *if there exists a constant $C > 0$ such that for every $k \in \mathbb{N}$ we have that*

$$\forall x_1 \; \exists n_1 \; \forall x_2 \; \exists n_2 \; \dots \; \exists n_k \; \forall x_{k+1}$$

so that if (i) *below holds true, then so does* (ii).

(i) *For every $i \in \{2, \dots, k+1\}$ we have* $\mathrm{supp}(x_i) \subseteq \{n_{i-1}, n_{i-1}+1, \dots\}$.

(ii) *We have* $\int_0^1 \| \sum_{i=1}^{k+1} r_i(t) x_i \| \, dt \leqslant C \big(\sum_{i=1}^{k+1} \|x_i\|^p \big)^{1/p}$.

The notion of asymptotic cotype q*, where* $q < \infty$*, is defined similarly. We say that* X *has* asymptotic non-trivial type (*respectively,* asymptotic non-trivial cotype) *if it has asymptotic type for some* $p > 1$ (*respectively, some* $q < \infty$).

Remark 17. We notice that a Banach space X with a Schauder basis (e_n) and with asymptotic non-trivial type does not contain ℓ_1. To see this observe that, by a standard sliding hump argument, if ℓ_1 embeds into such a space, then there would existed a block sequence (x_n) equivalent to the ℓ_1 basis. It is easy to see, however, that in this case property (ii) in Definition 9.11 is not satisfied. Similarly we verify that if X has asymptotic non-trivial cotype, then it does not contain c_0.

As in the case of separable Banach spaces with non-trivial type, the class of Banach spaces with a Schauder basis and asymptotic non-trivial type is of low complexity.

Lemma 9.12. *Let* $(u_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$ *be the enumeration of the basis of Pelczynski's universal space* U *as described in Proposition* 9.2*. Then the sets*

$$\{\sigma \in \mathcal{N} : U_\sigma \textit{ has asymptotic non-trivial type}\}$$

and

$$\{\sigma \in \mathcal{N} : U_\sigma \textit{ has asymptotic non-trivial cotype}\}$$

are both Borel.

Proof. Fix $p > 1$ and $C > 0$. Set

$$D := \Big\{ \sum_{t \in s} a_t u_t : a_t \in \mathbb{Q} \text{ and } s \text{ is a finite segment of } \mathbb{N}^{<\mathbb{N}} \Big\}$$

and $D_l := \big\{ \sum_{t \in s} a_t u_t \in D : \text{for every } t \in s \text{ we have } |t| \geqslant l \big\}$ for every $l \in \mathbb{N}$. Notice that D is countable (hence so is every D_l), and $D = D_1$. For every $k \in \mathbb{N}$ and every $x_1, \ldots, x_{k+1} \in D$ we say that $(x_1, \ldots, x_{k+1})$ is *admissible* if property (ii) in Definition 9.11 is satisfied for this tuple of vectors and the fixed constant C. Set

$$\begin{aligned} A(x_1, \ldots, x_{k+1}) := \big\{ \sigma \in \mathcal{N} \ : \ & \exists i \in \{1, \ldots, k+1\} \text{ with } \operatorname{supp}(x_i) \nsubseteq \sigma \\ & \text{or } (x_1, \ldots, x_{k+1}) \text{ is admissible} \big\}. \end{aligned}$$

Clearly, $A(x_1, \ldots, x_{k+1})$ is closed. Now observe that the set of all $\sigma \in \mathcal{N}$ such that U_σ has asymptotic type p with constant C is equal to the set

$$\bigcap_{k \in \mathbb{N}} \Big(\bigcap_{x_1 \in D} \ \bigcup_{n_1 \in \mathbb{N}} \ \bigcap_{x_2 \in D_{n_1}} \cdots \bigcup_{n_k \in \mathbb{N}} \ \bigcap_{x_{k+1} \in D_{n_k}} A(x_1, \ldots, x_{k+1}) \Big).$$

It follows that the set of all $\sigma \in \mathcal{N}$ for which U_σ has asymptotic type p is Borel; for the class of spaces with asymptotic cotype we argue similarly. The proof is completed. $\square$

By Proposition 9.2 and its proof, Lemma 9.12 and Remark 17, we obtain the following corollary.

Corollary 9.13. *There exists a non-universal Banach space T (respectively, C) with a Schauder basis such that every Banach space with a Schauder basis and asymptotic non-trivial type (respectively, asymptotic non-trivial cotype) is contained in T (respectively, in C) as a complemented subspace.*

We close this section with the following strengthening of Theorem 3.6.

Theorem 9.14. *Let X be a separable Banach space with a Schauder basis and let A be an analytic subset of* SB *which contains (up to isomorphism) all HI spaces. Then there exists $Y \in A$ containing X as a complemented subspace.*

Proof. Let (x_n) be a Schauder basis of X. We may assume that (x_n) is normalized and bimonotone. We enumerate this basis as $(x_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ as we did in Example 1. Consider the HI-amalgamation $\mathcal{A}_{\mathrm{hi}}^X$ of $(x_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$. Following the notation in the proof of Theorem 8.10, for every well-founded tree T with infinitely many nodes let $\tilde{\mathcal{X}}_T$ be the subspace of $\mathcal{A}_{\mathrm{hi}}^X$ generated by T. Next, let $A \subseteq \mathrm{SB}$ be analytic and let $A_{\cong}$ be the isomorphic saturation of A (which is analytic too). Since the map $\Phi\colon \tilde{\mathrm{Tr}} \to \mathrm{SB}$ defined by $\Phi(T) = \tilde{\mathcal{X}}_T$ is Borel, the set $\Phi^{-1}(A_{\cong})$ is analytic and contains $\tilde{\mathrm{WF}}$. It follows that there exists an ill-founded tree T such that $\tilde{\mathcal{X}}_T \in A_{\cong}$. Noticing that X is a complemented subspace of $\tilde{\mathcal{X}}_T$, the result follows. □

10. A non-universal space with unbounded β and r_{ND} indices

10.1. **Jamesfication of a Schauder tree basis.** Let X be a separable Banach space, let Λ be a countable set, let T be a pruned subtree of $\Lambda^{<\mathbb{N}}$ and let $(x_t)_{t\in T}$ be a normalized bimonotone Schauder tree basis of X. We define the *Jamesfication* J_X of $(x_t)_{t\in T}$ to be the completion of $c_{00}(T)$ with the norm

$$\begin{aligned} \|z\|_{J_X} := \sup\Big\{ \big\| \sum_{p=1}^{k} \big(\sum_{t\in s_p} z(t) \big) x_{t_p} \big\|_X : \ & (s_p)_{p=1}^k \text{ are pairwise disjoint segments of } T, \\ & \exists \sigma \in [T] \text{ with } s_p \subseteq \sigma \text{ for all } p \in \{1,\dots,k\}, \\ & \text{and } t_p \text{ is } \sqsubset\text{-minimal node of } s_p \Big\}. \end{aligned}$$

Notice that $(e_t)_{t\in T}$ defines a normalized bimonotone Schauder tree basis of J_X. Moreover, observe that for every $\sigma \in [T]$ the space $(J_X)_\sigma$ is isometric to the Jamesfication of X_σ defined by Bellenot, Haydon and Odell in [BHO].

10.2. **The Banach space R.** We will give the definition of the space R for which both the β and the r_{ND} indices are unbounded, yet the space is not universal. We start with Pelczynski's space V which is universal for all 1-unconditional bases. Let (v_k) denote the basis of V. We enumerate (v_k) as $(v_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ as we did in Example 3. Next, we consider the Jamesfication J_V of $(v_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$. Let $(e_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ be the Schauder tree basis of J_V. The universality of V and the enumeration of the basis yield the following properties.

(I) For every $\sigma \in \mathcal{N}$ the space $(J_V)_\sigma$ is isometric to the Jamesfication of V_σ.

(II) For every space X with an unconditional basis there exists $\sigma \in \mathcal{N}$ such that the Jamesfication of X is isomorphic to $(J_V)_\sigma$.

The desired space R is the HI-amalgamation $\mathcal{A}_{\mathrm{hi}}^{J_V}$ of $(e_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$. We verify, first, that the space R is not universal. To this end, we need the following definition.

Definition 10.1. *We say that a Banach space X is* sequentially unconditional *if for every seminormalized weakly null sequence (w_n) in X there exists $L \in [\mathbb{N}]^\infty$ such that the sequence $(w_n)_{n\in L}$ is unconditional.*

We need the following result (see [BHO, Proposition 2.1]).

Proposition 10.2. *Let X be a Banach space with a Schauder basis (x_n). Then the Jamesfication J_X of X is sequentially unconditional.*

We proceed with the following lemma.

Lemma 10.3. *If $(X_i)_{i=1}^d$ are sequentially unconditional, then so is $\sum_{i=1}^d \oplus X_i$.*

Proof. Let (w_n) be a seminormalized weakly null sequence in $\sum_{i=1}^d \oplus X_i$. By our assumptions, there exists $L \in [\mathbb{N}]^\infty$ such that the following are satisfied.

(1) For every $i \in \{1,\dots,d\}$ either
 (a) $\sum_{n\in L}\|P_i(w_n)\| < 1$, or
 (b) the sequence $(P_i(w_n))_{n\in L}$ is seminormalized.

(2) If $i \in \{1,\dots,d\}$ is such that (1.b) holds true, then the sequence $(P_i(w_n))_{n\in L}$ is unconditional.

Note that there exists at least one $i \in \{1,\dots,d\}$ such that (1.b) is satisfied. It is then easy to verify that the sequence $(w_n)_{n\in L}$ is unconditional, as desired. □

We have the following proposition.

Proposition 10.4. *Neither $L_1(0,1)$ nor $C\big(\omega^{\omega^2}\big)$ are contained in R.*

Proof. Assume, on the contrary, that $L_1(0,1)$ was contained in R (the argument is symmetric for both spaces). Since $L_1(0,1)$ is unconditionally saturated, arguing as the proof of Theorem 8.13, we see that there exist a finite-dimensional space F and $(Y_i)_{i=1}^d$ such that $L_1(0,1)$ is isomorphic to a subspace of $F \oplus \big(\sum_{i=1}^d \oplus Y_i\big)$ where for every $i \in \{1,\dots,d\}$ the space Y_i is isomorphic to $(J_V)_{\sigma_i}$ for some $\sigma_i \in \mathcal{N}$. By Proposition 10.2, the space $(J_V)_{\sigma_i}$ is sequentially unconditional, and as this is a hereditary property, so is the space Y_i for every $i \in \{1,\dots,d\}$. By Lemma 10.3, it follows that $L_1(0,1)$ is sequentially unconditional which is a contradiction by a result of Johnson, Maurey and Schechtman [JMS]. (For the case of $C\big(\omega^{\omega^2}\big)$ we invoke the classical Maurey–Rosenthal example [MR].) The proof is completed. □

We will also need the fact that for every Banach space X with an unconditional basis, the Jamesfication J_X of X is contained in R as a complemented subspace. Although this is a straightforward consequence of the definition of R, it is important enough to be stated in a separate proposition.

Proposition 10.5. *Let X be a Banach space with an unconditional basis. Then the Jamesfication of X is contained in R as a complemented subspace.*

We proceed to show that the indices β and r_{ND} are unbounded on R. (We refer to [AGR, KL2] for the definitions of β and r_{ND}.) To this end, we will introduce a transfinite sequence of reflexive Banach spaces with an unconditional basis whose Jamesfications will verify that both indices are unbounded. We should point out that several authors have provided such examples (see, e.g., [HOR, F]). However, these examples are rather inconvenient for our purposes.

The aforementioned spaces will be built with the help of the Schreier families $(\mathcal{S}_\xi)_{\xi<\omega_1}$. They are compact families of finite subsets of $\mathbb{N}$ which satisfy, among others, the following properties.

(1) Each $\mathcal{S}_\xi$ is *spreading*, that is, for every $F=\{n_1<\dots<n_k\}\in\mathcal{S}_\xi$ and every $G=\{m_1<\dots<m_k\}$ with $n_i\leqslant m_i$ for all $i\in\{1,\dots,k\}$ we have $G\in\mathcal{S}_\xi$.
(2) Each $\mathcal{S}_\xi$ is *hereditary*, that is, if $F\in\mathcal{S}_\xi$ and $G\subseteq F$, then $G\in\mathcal{S}_\xi$.
(3) The Cantor–Bendixson derivative of $\mathcal{S}_\xi$ is equal to ω^ξ.

For the definition of the Schreier families and a discussion of their properties we refer to [AA, AGR].

Now for every $\xi<\omega_1$ let $X_{(\mathcal{S}_\xi,2)}$ be the completion of $c_{00}(\mathbb{N})$ with the norm

$$\|z\|_{X_{(\mathcal{S}_\xi,2)}}\coloneqq\sup\Big\{\big(\sum_{i=1}^{d}\big(\sum_{n\in F_i}|z(n)|\big)^2\big)^{\frac12} : (F_i)_{i=1}^d\in\mathcal{S}_\xi \text{ with } F_1<F_2<\dots<F_d\Big\}.$$

(Here, for every pair F,G of nonempty finite subsets of $\mathbb{N}$ we write $F<G$ if $\max(F)<\min(G)$.) We also need to introduce an auxiliary space $X_{\mathcal{S}_\xi}$ which is defined to be the completion of $c_{00}(\mathbb{N})$ with the norm

$$\|z\|_{X_{\mathcal{S}_\xi}}\coloneqq\sup\Big\{\sum_{n\in F}|z(n)| : F\in\mathcal{S}_\xi\Big\}.$$

We denote by (x_n) the standard basis of both $X_{(\mathcal{S}_\xi,2)}$ and $X_{\mathcal{S}_\xi}$ (from the context it will be clear whether we refer to $X_{(\mathcal{S}_\xi,2)}$, or $X_{\mathcal{S}_\xi}$). Notice that (x_n) is an unconditional basis of $X_{(\mathcal{S}_\xi,2)}$. It also easy to verify that the basis in $X_{(\mathcal{S}_\xi,2)}$ is boundedly complete and so, by a classical result of James (see [LT]), the space $X_{(\mathcal{S}_\xi,2)}$ does not contain c_0. On the other hand, observe that the space $X_{\mathcal{S}_\xi}$ can be realized (up to isomorphism) as a closed subspace of $C(\mathcal{S}_\xi)$. As the family $\mathcal{S}_\xi$ is countable and compact, by a result of Bessaga–Pelczynski, the space $C(\mathcal{S}_\xi)$ is c_0-saturated (see, e.g., [Ro2, Proposition 3.6]). Hence, so is the space $X_{\mathcal{S}_\xi}$. By the previous discussion, it follows that the identity operator $\mathrm{Id}\colon X_{(\mathcal{S}_\xi,2)}\to X_{\mathcal{S}_\xi}$ is strictly singular. We are now ready to state the first result concerning the space $X_{(\mathcal{S}_\xi,2)}$.

Proposition 10.6. *For every $\xi < \omega_1$ the space $X_{(\mathcal{S}_\xi,2)}$ is reflexive.*

Proof. Since the space $X_{(\mathcal{S}_\xi,2)}$ has a boundedly complete unconditional basis, by the result of James mentioned above, it is enough to show that the space $X_{(\mathcal{S}_\xi,2)}$ does not contain ℓ_1. We will show that, actually, the space $X_{(\mathcal{S}_\xi,2)}$ is ℓ_2-saturated. (Clearly, this will finish the proof.) Let Y be an arbitrary infinite-dimensional subspace of $X_{(\mathcal{S}_\xi,2)}$. Since the operator $\mathrm{Id}\colon X_{(\mathcal{S}_\xi,2)} \to X_{\mathcal{S}_\xi}$ is strictly singular, by a standard sliding hump argument, there exists a normalized block sequence (w_k) in Y such that $\|w_k\|_{X_{\mathcal{S}_\xi}} \leqslant 1/2^k$. We will show that the sequence (w_k) is equivalent to the standard unit vector basis of ℓ_2. First we show the lower estimate. For every $k \in \mathbb{N}$ set $R_k = \mathrm{range}(w_k)$. Since $\|w_k\| = 1$, we may select $(F_i^k)_{i=1}^{d_k}$ such that $F_i^k \in \mathcal{S}_\xi$, $F_1^k < \cdots < F_{d_k}^k$, $F_i^k \subseteq R_k$ and $\|w_k\| = \sum_{i=1}^{d_k}(\sum_{n\in F_i^k}|w_k(n)|)^2 = 1$. Let $l \in \mathbb{N}$ and let $a_1,\dots,a_l \in \mathbb{R}$ with $\sum_{k=1}^{l} a_k^2 = 1$. Notice that the family $\langle F_i^k : 1 \leqslant k \leqslant l, 1 \leqslant i \leqslant d_k\rangle$ consists of successive members of $\mathcal{S}_\xi$. Therefore,

$$\begin{aligned}\Big\|\sum_{k=1}^{l} a_k w_k\Big\| &\geqslant \Big(\sum_{k=1}^{l}\sum_{i=1}^{d_k}\Big(\sum_{n\in F_i^k}|a_k w_k(n)|\Big)^2\Big)^{1/2}\\ &= \Big(\sum_{k=1}^{l} a_k^2\sum_{i=1}^{d_k}\Big(\sum_{n\in F_i^k}|w_k(n)|\Big)^2\Big)^{1/2} = 1.\end{aligned}$$

Next, we argue for the upper estimate. It is convenient to work with a norming family of the dual rather than with the definition of the norm. Specifically, for every $F \in \mathcal{S}_\xi$ let $F^*(x) = \sum_{n\in F} x(n)$. Notice that $F^* \in X^*_{(\mathcal{S}_\xi,2)}$. We set

$$\mathcal{F} \coloneqq \Big\{\sum_{i=1}^{d}\beta_i F_i^* : \sum_{i=1}^{d}\beta_i^2 \leqslant 1 \text{ and } (F_i)_{i=1}^{d} \text{ are successive members of } \mathcal{S}_\xi\Big\}.$$

Since the Schreier family $\mathcal{S}_\xi$ is hereditary, by the Cauchy–Schwarz inequality, we see that $\|x\|_{X_{(\mathcal{S}_\xi,2)}} \leqslant 2\sup\{\phi(x) : \phi \in \mathcal{F}\}$. Also observe that if $\sum_{i=1}^{d}\beta_i F_i^* \in \mathcal{F}$, then for every $k \in \mathbb{N}$ we have

$$\sum_{i=1}^{d}\beta_i F_i^*(w_k) \leqslant \Big(\sum_{i=1}^{d}\beta_i^2\Big)^{1/2}. \tag{10.1}$$

(If not, then we would have that $\|w_k\| > 1$.) Let $\sum_{i=1}^{d}\beta_i F_i^* \in \mathcal{F}$ be arbitrary. For every $k \in \{1,\dots,l\}$ set $I_k = \{i \in \{1,\dots,d\} : F_i \cap R_k \neq \emptyset\}$. Notice that I_k is an interval since the set $(F_i)_{i=1}^{d}$ are successive. Let m_k and M_k be the minimum and maximum element of I_k respectively, and set $I'_k \coloneqq I_k \setminus \{m_k, M_k\}$ (of course, I'_k may be empty). Observe that $I'_{k_1} \cap I'_{k_2} = \emptyset$ for every $k_1, k_2 \in \{1,\dots,l\}$ with $k_1 \neq k_2$. We want to estimate the quantity

$$\sum_{i=1}^{d}\beta_i F_i^*\Big(\sum_{k=1}^{l} a_k w_k\Big) = \sum_{k=1}^{l} a_k \sum_{i=1}^{d}\beta_i F_i^*(w_k) = \sum_{k=1}^{l} a_k \sum_{i\in I_k}\beta_i F_i^*(w_k) \tag{10.2}$$

where, as before, $l \in \mathbb{N}$ and $a_1, \dots, a_l \in \mathbb{R}$ with $\sum_{k=1}^{l} a_k^2 = 1$. By the selection of the sequence (w_k), we have $\|w_k\|_{X_{\mathcal{S}_\xi}} \leqslant 1/2^k$. Therefore, for every $F \in \mathcal{S}_\xi$ we have $|F^*(w_k)| \leqslant 1/2^k$, and consequently, by (10.1), for every $k \in \mathbb{N}$ we see that $\sum_{i\in I_k} \beta_i F_i^*(w_k) \leqslant (\sum_{i\in I'_k} \beta_i^2)^{1/2} + 2/2^k$. By (10.2), we obtain that

$$\sum_{i=1}^{d} \beta_i F_i^*\big(\sum_{k=1}^{l} a_k w_k\big) \leqslant \sum_{k=1}^{l} a_k \big(\sum_{i\in I'_k} \beta_i^2\big)^{1/2} + 2$$

and so, by the Cauchy–Schwarz inequality, we conclude that

$$\sum_{i=1}^{d} \beta_i F_i^*\big(\sum_{k=1}^{l} a_k w_k\big) \leqslant 3.$$

This implies that $\| \sum_{k=1}^{l} a_k w_k \| \leqslant 6$ and the proof is completed. $\square$

We proceed to recall some definitions concerning spreading models.

Definition 10.7. *Let $\xi < \omega_1$ and let (x_n) be a sequence in a Banach space X. The sequence (x_n) is said to be an* ℓ_1^ξ-spreading model (*respectively,* c_0^ξ-spreading model) *if there exists $C > 0$ such that for every $F \in \mathcal{S}_\xi$ and every sequence (a_n) of scalars we have*

$$C \sum_{n\in F} |a_n| \leqslant \big\| \sum_{n\in F} a_n x_n \big\|$$

(*respectively,*

$$\big\| \sum_{n\in F} a_n x_n \big\| \leqslant C \max\{|a_n| : n \in F\}).$$

The sequence (x_n) is said to be a ξ-summing spreading model *if there exists $C > 0$ such that for every $F = \{l_1 < \dots < l_k\} \in \mathcal{S}_\xi$ and every sequence (a_n) in $\mathbb{R}$ we have*

$$\frac{1}{C} \, |||(a_n)_{n\in F}||| \leqslant \big\| \sum_{n\in F} a_n x_n \big\| \leqslant C \, |||(a_n)_{n\in F}|||$$

where $|||(a_n)_{n\in F}||| := \max\big\{ |\sum_{i\in I} a_{l_i}| : I \subseteq \{1, \dots, k\}$ *is an interval*$\big\}$.

Lemma 10.8. *Assume that X is a Banach space with an unconditional basis (x_n). Let (e_n) be the basis of the Jamesfication of X and let ξ be a countable ordinal.*

(a) *If (x_n) is an ℓ_1^ξ-spreading model, then so is every convex block sequence (g_n) of (e_n).*
(b) *If (x_n) is a c_0^ξ-spreading model, then (e_n) is a ξ-summing spreading model.*

Proof. For part (a) let (g_n) be a convex block sequence of (e_n). For every $n \in \mathbb{N}$ set $I_n := \mathrm{range}(g_n)$ and write $g_n = \sum_{k\in I_n} b_k^n e_k$ where $b_k^n \geqslant 0$ and $\sum_{k\in I_n} b_k^n = 1$. Let $F \in \mathcal{S}_\xi$ be arbitrary. For every $n \in F$ set $t_n := \min(I_n)$. Then $t_n \geqslant n$, and so $\{t_n : n \in F\} \in \mathcal{S}_\xi$ as the family $\mathcal{S}_\xi$ is spreading. Therefore,

$$\big\| \sum_{n\in F} a_n g_n \big\| \geqslant \big\| \sum_{n\in F} \big(\sum_{k\in I_n} a_n b_k^n \big) x_{t_n} \big\|_X = \big\| \sum_{n\in F} a_n x_{t_n} \big\|_X \geqslant C \sum_{n\in F} |a_n|$$

where the last inequality follows from the fact that (x_n) is an ℓ_1^ξ-spreading model. Part (b) is an immediate consequence of the definitions. The proof is completed. $\square$

Lemma 10.9. *Let $\xi < \omega_1$. Then the following hold.*

(a) *The basis (x_n) of $X_{(\mathcal{S}_\xi,2)}$ is an ℓ_1^ξ-spreading model.*

(b) *The basis (x_n^*) of $X^*_{(\mathcal{S}_\xi,2)}$ is a c_0^ξ-spreading model.*

Proof. The first part is an immediate consequence of the definition of $X_{(\mathcal{S}_\xi,2)}$ and the fact that the Schreier families are hereditary. For the second part let (x_n^*) denote the basis of $X^*_{(\mathcal{S}_\xi,2)}$. Notice that for every $F \in \mathcal{S}_\xi$ the functional $F^* = \sum_{n\in F} e_n^*$ has norm one. Since (x_n^*) is an unconditional basis of $X^*_{(\mathcal{S}_\xi,2)}$, for every sequence (a_n) in $\mathbb{R}$ and every $F \in \mathcal{S}_\xi$ we have $\|\sum_{n\in F} a_n x_n^*\| \leqslant \max\{|a_n| : n \in F\}$ and the lemma is proved. $\square$

Proposition 10.10. *Let $\xi < \omega_1$. Then the following hold.*

(1) *The basis of the Jamesfication of $X_{(\mathcal{S}_\xi,2)}$ is weakly* convergent to an element f which satisfies $\omega^\xi \leqslant \beta(f) < \omega_1$.*

(2) *The basis of the Jamesfication of $X^*_{(\mathcal{S}_\xi,2)}$ is weakly* convergent to an element g which satisfies $\omega^\xi \leqslant r_{\mathrm{ND}}(g) < \omega_1$.*

Proof. Fix $\xi < \omega_1$. In what follows, for notational simplicity, by J_ξ and J_ξ^d we shall denote the Jamesfications of $X_{(\mathcal{S}_\xi,2)}$ and $X^*_{(\mathcal{S}_\xi,2)}$ respectively. Also let (e_n) and (e_n^*) denote the bases of J_ξ and J_ξ^d. By Proposition 10.6, the spaces $X_{(\mathcal{S}_\xi,2)}$ and $X^*_{(\mathcal{S}_\xi,2)}$ are reflexive. By [BHO, Theorem 4.1], it follows that J_ξ and J_ξ^d are quasi-reflexive. Moreover, by [BHO, Theorem 2.2], the dual of J_ξ (respectively, of J_ξ^d) is generated by the biorthogonals of the basis and the "sum" functional $S = (1, 1, \dots)$. It is then clear that both (e_n) and (e_n^*) are weak* convergent, say to f and g respectively. It is also clear that f is a Baire-1 element, and so $\beta(f) < \omega_1$. On the other hand, we have $r_{\mathrm{ND}}(g) < \omega_1$ (if not, then c_0 would embed into J_ξ^d; see [HOR]). It remains to show the other inequalities.

For part (2) we observe that, by part (b) of Lemma 10.9, the basis (x_n^*) of $X^*_{(\mathcal{S}_\xi,2)}$ is a c_0^ξ spreading model. By part (b) of Lemma 10.8, the basis (e_n^*) of J_ξ^d (the Jamesfication of $X^*_{(\mathcal{S}_\xi,2)}$) is a ξ-summing spreading model. By [F, Theorem 9] (see also [AGR, Theorem II.4.8]), it follows that $r_{\mathrm{ND}}(g) \geqslant \omega^\xi$, as desired.

In order to show that $\beta(f) \geqslant \omega^\xi$ we need some results from [KL2]. First we recall the definition of the *convergence rank* γ. Let K be a compact metrizable space and let (f_n) be a sequence of continuous real-valued functions on K. For every $\varepsilon > 0$ we define a derivative operation on closed subsets of K by setting

$$F \mapsto F'_{((f_n),\varepsilon)} \coloneqq \big\{x \in F \,:\, \forall U \ni x \text{ open and } \forall n\ \exists p > q \geqslant n \\ \exists y \in U \cap F \text{ with } |f_p(y) - f_q(y)| \geqslant \varepsilon\big\}.$$

By transfinite recursion, we define the iterated derivatives $K^{(\zeta)}_{((f_n),\varepsilon)}$ $(\zeta < \omega_1)$. The convergent rank $\gamma\big((f_n)\big)$ of the sequence (f_n) is defined in the standard way, using the aforementioned derivative operation. We need the following consequence of [KL2, Theorem 2.3]: *Assume that* (f_n) *is a bounded sequence of continuous real-valued functions on* K *which is pointwise convergent to* f. *If for every convex block sequence* (g_n) *of* (f_n) *we have* $\gamma\big((g_n)\big) \geqslant \omega^\xi$, *then* $\beta(f) \geqslant \omega^\xi$.

By the previous discussion, it is enough to prove that for every convex block sequence (g_n) of the basis (e_n) of J_ξ we have $\gamma\big((g_n)\big) \geqslant \omega^\xi$. Fix such a convex block sequence (g_n). As we have shown in part (a) of Lemma 10.9, the basis (x_n) of $X_{(\mathcal{S}_\xi,2)}$ is an ℓ_1^ξ-spreading model and so, by Lemma part (a) of 10.8, the same is also true for (g_n). We have the following claim.

Claim. *Let* $I_1 < \cdots < I_d$ *be intervals of* $\mathbb{N}$ *with* $\big\{\min(I_k) : k \in \{1,\dots,d\}\big\} \in \mathcal{S}_\xi$. *Then* $\sum_{k=1}^d I_k^* \in B_{J_\xi^*}$ *where* J_ξ^* *denotes the dual of the Jamesfication of* $X_{(\mathcal{S}_\xi,2)}$.

Proof of the claim. For every $k \in \{1,\dots,d\}$ set $p_k \coloneqq \min(I_k)$. Let $z \in J_\xi$ with $\|z\| \leqslant 1$ be arbitrary. Then we have

$$\begin{aligned} \big|\sum_{k=1}^d I_k^*(z)\big| &\leqslant \sum_{k=1}^d \big|\sum_{n\in I_k} z(n)\big| \leqslant \sum_{k=1}^d \sum_{n\in I_k} |z(n)| \\ &\leqslant \big\|\sum_{k=1}^d \big(\sum_{n\in I_k} |z(n)|\big)\, x_{p_k}\big\|_{X_{(\mathcal{S}_\xi,2)}} \leqslant \|z\|_{J_\xi} \end{aligned}$$

where we used the fact that $\big\{p_k : k \in \{1,\dots,d\}\big\} \in \mathcal{S}_\xi$ and the fact that the basis (x_n) of $X_{(\mathcal{S}_\xi,2)}$ is an ℓ_1^ξ-spreading model with constant one. $\square$

For every $n \in \mathbb{N}$ set $I_n \coloneqq \mathrm{range}(g_n)$. Notice that (I_n) is a sequence of successive intervals of $\mathbb{N}$. Denote by K the unit ball of J_ξ^* equipped with the weak* topology. For every $\mathcal{F} \subseteq \mathcal{S}_\xi$ set $K_{\mathcal{F}} = \big\{\sum_{n\in F} I_n^* : F \in \mathcal{F}\big\}$. Notice that if $F \in \mathcal{S}_\xi$, then for every $n \in F$ we have $n \leqslant \min(I_n)$. Since the family $\mathcal{S}_\xi$ is spreading, we obtain that $\{\min(I_n) : n \in F\} \in \mathcal{S}_\xi$. By the previous claim, it follows that $K_{\mathcal{F}}$ is a subset of K. Denote by $\mathcal{F}^\zeta$ the ζ-th Cantor–Bendixson derivative of $\mathcal{S}_\xi$. By induction on countable ordinals, we will show that

$$K_{\mathcal{F}^\zeta} \subseteq K^{(\zeta)}_{((g_n),1)}; \tag{10.3}$$

this will finish the proof of part (1). To this end notice, first, that

$$\mathcal{F}^1 = \{F \in \mathcal{S}_\xi : F \text{ is not maximal}\}.$$

Let $G = \sum_{n\in F} I_n^* \in K_{\mathcal{F}^1}$ be arbitrary (in particular, F is not a maximal element of $\mathcal{S}_\xi$). Also let W be a weak* neighborhood of G. We may assume that there exists $n_W \in \mathbb{N}$ with the following property. If $H \in K$ satisfies $H(e_n) = G(e_n)$ for every $n \in \{1,\dots,n_W\}$, then $H \in W$. Let $n \in \mathbb{N}$, and set $n_F \coloneqq \max\{i : i \in F\}$. We select $p > q > \max\{n, n_W, n_F\}$, and we define $G' \coloneqq \sum_{n\in F} I_n^* + I_q^*$. Since $F \cup \{q\} \in \mathcal{S}_\xi$, we see that $G' \in K \cap W$. Moreover, $|g_p(G') - g_q(G')| = 1$. This implies that $G \in K^{(1)}_{((g_n),1)}$ and so $K_{\mathcal{F}^1} \subseteq K^{(1)}_{((g_n),1)}$. Using similar arguments, we can verify (10.3) for all countable ordinals (we leave the details to the interested reader). This completes the proof of part (1), and so the entire proof is completed. □

By Propositions 10.4, 10.5 and 10.10, we have the following theorem.

Theorem 10.11. *There exists a non-universal separable Banach space R for which both the β and r_{ND} indices are unbounded. In particular, the space R contains neither $L_1(0,1)$ nor $C\big(\omega^{\omega^2}\big)$.*

Remark 18. (1) By definition, the space R constructed above contains ℓ_1 and c_0. Actually, by a result of Bourgain [B2] and the c_0-index theorem [AK], *any* space for which both indices are unbounded must contain ℓ_1 and c_0.

(2) The space R is, in some sense, minimal. Namely, every subspace of R either contains a further reflexive subspace, or ℓ_1, or c_0. To see this consider a subspace Y of R not containing any further reflexive subspace. Since R is the HI-amalgamation of J_V and every J_V-singular subspace of R is HI and reflexive saturated, we conclude that no subspace of Y is J_V-singular. Hence, there exist $\sigma \in \mathcal{N}$ and a subspace Y' of Y such that the operator $P_\sigma \colon Y' \to (J_V)_\sigma$ is an isomorphic embedding. This yields that $(J_V)_\sigma$ is not quasi-reflexive and so, by [BHO, Theorem 2.2], we conclude that Y either contains ℓ_1 or c_0.

(3) Instead of using HI-amalgamations, one can obtain the same results using p-amalgamations for any $p > 2$. Indeed, one simply observes that ℓ_p does not embed into L_1 for any $p > 2$; the rest of the argument is identical.

Appendix A. The dual of $\mathcal{T}_2^X$

Let X be a Banach space, let Λ be a countable set, let T be a pruned subtree o $\Lambda^{<\mathbb{N}}$ and let $(x_t)_{t\in T}$ a normalized bimonotone Schauder tree basis of X. Also let $\mathcal{T}_2^X$ be the ℓ_2 Baire sum of $(x_t)_{t\in T}$, and set

$$W^* := \overline{\mathrm{span}}\Big\{ \bigcup_{\sigma\in[T]} B_{\mathcal{X}^*_\sigma} \Big\}. \tag{A.1}$$

Our goal in this appendix is to show that $W^* = (\mathcal{T}_2^X)^*$. To this end we need several auxiliary results. We start with the following lemma.

Lemma A.1. *Let $(x_i)_{i\in I}$ be a net in $\mathcal{T}_2^X$ with the following properties.*

- (C1) *Each x_i has finite support and $\|x_i\| = 1$.*
- (C2) *For every $w^* \in W^*$ we have $\lim_{i\in I} w^*(x_i) = 0$.*
- (C3) *There exists $x^* \in (\mathcal{T}_2^X)^*$ with $x^*(x_i) \geqslant \frac{1}{2}$ for every $i \in I$.*

Also let $F \subseteq [T]$ be finite and let $0 < \varepsilon < \frac{1}{2}$. Then there exist finite $A \subseteq [T]$, a block sequence (y_n) and a sequence (z_n) of convex combinations of $(x_i)_{i\in I}$ such that the following are satisfied.

1. *We have $\sum_{n\in\mathbb{N}} \|y_n - z_n\| < \varepsilon$.*
2. *We have $A \cap F = \emptyset$.*

(3) *For every segment s of T with $s \cap A = \emptyset$ we have $\|P_s(y_n)\| \leqslant \varepsilon$ for all $n \in \mathbb{N}$.*

Proof. We start with the following observation. Let $B \subseteq [T]$ be finite. By the definition of W^* and (C2), we see that if J is any cofinal subset of I, then $(P_B(x_j))_{j\in J}$ tends weakly to 0. Thus, for every $c > 0$ we may select a finite convex combination w of $(x_j)_{j\in J}$ such that $\|P_B(w)\| < c$.

By the above remarks and a sliding hump argument, we may select a block sequence (x_n) and a sequence (w_n) of convex combinations of $(x_i)_{i\in I}$ such that

(O1) $\lim \|P_F(w_n)\| = 0$, and
(O2) $\sum_{n\in\mathbb{N}} \|w_n - x_n\| < \varepsilon$.

Notice that (O1) and (O2) yield that $\|P_F(x_n)\| \to 0$. Moreover, since (w_n) is a sequence of convex combinations of $(x_i)_{i\in I}$, by (C1) and (C3), we obtain that $\frac{1}{2} \leqslant \|w_n\| \leqslant 1$ for every $n \in \mathbb{N}$ and so, by (O2), we have $\frac{1}{2} - \varepsilon \leqslant \|x_n\| \leqslant 1 + \varepsilon$ for every $n \in \mathbb{N}$.

Next, we argue that it cannot be the case that $\lim \|P_\sigma(x_n)\| = 0$ for every $\sigma \in [T]$. Indeed is the case, by Proposition 4.10, we would have that the sequence (x_n) is weakly null. But this would imply that the sequence (w_n) is also weakly null which contradicts (C3). It follows that there exist $L \in [\mathbb{N}]^\infty$, $r > 0$ and $\sigma \in [T]$ such that $\|P_\sigma(x_n)\| > r$ for every $n \in L$. Clearly, we may assume that $\varepsilon > r$. Also notice that, since $\lim \|P_F(x_n)\| = 0$, we may assume that for every segment s of T with $s \subseteq F$ (in the sense that there exists $\sigma \in F$ with $s \subseteq \sigma$) we have that $\|P_s(x_n)\| < \frac{r}{4}$. Thus, by passing to subsequences, we have the following properties.

(a) If $s \subseteq F$, then $\|P_s(x_n)\| < \frac{r}{4}$ for every $n \in \mathbb{N}$.
(b) There exists at least one segment s (in particular, a branch) with $s \nsubseteq F$ such that $\|P_s(x_n)\| > \varepsilon$ for every $n \in \mathbb{N}$.

Claim. *There exist finite $A \subseteq [T]$ with $A \cap F = \emptyset$ and $L \in [\mathbb{N}]^\infty$ such that for every segment s of T with $s \cap A = \emptyset$ we have $\limsup_{n\in L} \|P_s(x_n)\| < \frac{\varepsilon}{2}$.*

The proof of the above claim is identical to the proof of Lemma 4.3; the only extra condition is that $A \cap F = \emptyset$, which causes no problem in the argument.

Now applying inductively Lemma 4.8 we obtain a sequence (y_n) of block convex combinations of (x_n) such that for every segment s of T with $s \cap A = \emptyset$ we have $\|P_s(y_n)\| \leqslant \varepsilon$ for every $n \in \mathbb{N}$. (Here, A denotes the finite subset of $[T]$ obtained by the above claim.) Let (z_n) be the corresponding block convex combinations of $(w_n)_{n\in L}$. Then A, (y_n) and (z_n) are as desired. □

Lemma A.2. *Let $(x_i)_{i\in I}$ be a net in $\mathcal{T}_2^X$ which satisfies* (C1), (C2) *and* (C3) *in Lemma* A.1. *Then there exist a decreasing sequence (ε_l) with $0 < \varepsilon_l < \frac{1}{2}$ and $\varepsilon_l \to 0$, a sequence (A_l) of finite subsets of $[T]$, and for every $l \in \mathbb{N}$ sequences (y_n^l) and (z_n^l) with the following properties.*

(I) *For every $l \in \mathbb{N}$ the sequence (y_n^l) is block and the sequence (z_n^l) consists of convex combinations of $(x_i)_{i\in I}$.*

(II) *We have* $\sum_{n\in\mathbb{N}} \|z^l_n - y^l_n\| < \varepsilon_l$ *for every* $l \in \mathbb{N}$.

(III) *For every* $l_1, l_2 \in \mathbb{N}$ *with* $l_1 \neq l_2$ *we have* $A_{l_1} \cap A_{l_2} = \emptyset$.

(IV) *For every* $l, n \in \mathbb{N}$ *if* s *is a segment of* T *with* $s \cap A_l = \emptyset$, *then* $\|P_s(y^l_n)\| \leqslant \varepsilon_l$.

Proof. The selection of (ε_l), (A_l), (y^l_n) and (z^l_n) is done recursively and using Lemma A.1. Indeed, let $k \in \mathbb{N}$ and assume that $\varepsilon_l, A_l, (y^l_n)$ and (z^l_n) have been constructed for every $l < k$. We select ε_k with $\varepsilon_k < \min\{\varepsilon_{k-1}, 1/2^{k+1}\}$, and we set $F = A_1 \cup \cdots \cup A_{k-1}$. (For the first step of the selection we set $F = \emptyset$.) By Lemma A.1, we obtain A_k, (y^k_n) and (z^k_n) satisfying (1)–(3) in Lemma A.1. Clearly, ε_k, A_k, (y^k_n) and (z^k_n) are as desired. □

We proceed with the following lemma.

Lemma A.3. *Let the notation and assumptions be as in Lemma* A.2. *Then there exist sequences* (l_k) *and* (n_k) *such that the sequence* $(y^{l_k}_{n_k})$ *is block and satisfies* $\lim \|P_\sigma(y^{l_k}_{n_k})\| = 0$ *for every* $\sigma \in [T]$.

In order to select the sequences (l_k) and (n_k) described in Lemma A.3 we follow an inductive scheme described in the following sublemma. Before we state it we recall that if $t \in \Lambda^{<\mathbb{N}}$, then by $\mathcal{L}_t$ we denote the set of all *segments* s of $\Lambda^{<\mathbb{N}}$ for which there exists $t' \in s$ with $t \sqsubseteq t'$. Once again we remark that the family $\{\mathcal{L}_t : t \in \Lambda^{<\mathbb{N}}\}$ restricted to the branches of $\Lambda^{<\mathbb{N}}$, is the usual sub-basis of the topology on $\Lambda^{\mathbb{N}}$. Next, let T be the pruned subtree of $\Lambda^{<\mathbb{N}}$ which is used to define the Schauder tree basis of X. For every $t \in T$ let $\mathcal{T}_t$ denote the subset of $\mathcal{L}_t$ consisting of all segments s which belong to T.

Sublemma A.4. *Let* $L \in [\mathbb{N}]^\infty$, *and for every* $q \in L$ *let* $M_q \in [\mathbb{N}]^\infty$. *Also let* $\varepsilon > 0$. *Then the following hold.*

(I) *There exist* $l, n \in \mathbb{N}$ *with* $l \in L$ *and* $n \in M_l$.

(II) *If* $A_l = \{\sigma^l_1, \dots, \sigma^l_k\}$ *where* l *is as in* (I), *then for every* $i \in \{1, \dots, k\}$ *there exist* $t_i \sqsubset \sigma^l_i$ *and there exists* $j \in \mathbb{N}$ *such that for every* $i_1, i_2 \in \{1, \dots, k\}$ *with* $i_1 \neq i_2$ *we have that* t_{i_1} *and* t_{i_2} *are incomparable and* $|t_{i_1}| = |t_{i_2}| = j$.

(III) *There exists* $L' \in [L]^\infty$ *and for every* $q \in L'$ *there exists* $M'_q \in [M_q]^\infty$ *such that for every* $m \in M'_q$ *the following hold.*

  (a) *We have* $\max\{k : k \in h\big(\mathrm{supp}(y^l_n)\big)\} < \min\{k : k \in h\big(\mathrm{supp}(y^q_m)\big)\}$ *where* l, n *are as in* (I) *and* $h \colon T \to \mathbb{N}$ *is the fixed enumeration of* T.

  (b) *For every segment* s *of* T *with* $s \in \mathcal{T}_{t_1} \cup \mathcal{T}_{t_2} \cup \cdots \cup \mathcal{T}_{t_k}$ *we have that* $\|P_s(y^q_m)\| \leqslant \varepsilon$.

(IV) *If* l, n *are as in* (I), *then for every* $t \in \mathrm{supp}(y^l_n) \cap A_l$ *there is* $i \in \{1, \dots, k\}$ *with* $t_i \sqsubset t$ *where* $(t_i)^k_{i=1}$ *are as in* (II).

Proof. First recall that for every $q, n \in \mathbb{N}$ we have $\|y^q_n\| \leqslant 1 + \varepsilon_q < 2$ since, by Lemma A.2, we have $\varepsilon_q < 1/2$ for every $q \in \mathbb{N}$. Fix $k_0 \in \mathbb{N}$ with $\varepsilon\sqrt{k_0} \geqslant 2$. Let $\{l_1 < l_2 < \cdots\}$ be the increasing enumeration of L. Consider the set $\{l_1, \dots, l_{k_0}\}$

and the corresponding A_{l_i}'s for $i \in \{1, \dots, k_0\}$. Since the sets $(A_{l_i})_{i=1}^{k_0}$ are mutually disjoint finite subsets of $[T]$, we may select $j_0 \in \mathbb{N}$ such that if we restrict every branch σ of every A_{l_i} $(1 \leqslant i \leqslant k_0)$ after the j_0-level of T, then this collection of all these final segments of T is a collection of mutually *incomparable* final segments. Let us denote these final segments by $C_i = \{I_1^i, \dots, I_{w_i}^i\}$ $(1 \leqslant i \leqslant k_0)$. Moreover, for every $i \in \{1, \dots, k_0\}$ and every $j \in \{1, \dots, w_i\}$ let t_j^i denote the $\sqsubset$-least element of I_j^i (notice that $|t_j^i| = j_0$).

For every $i \in \{1, \dots, k_0\}$ consider the sequence $(y_n^{l_i})_{n \in M_{l_i}}$. This is a block sequence, and so we may select $n_i \in M_{l_i}$ such that for every $n \in M_{l_i}$ with $n \geqslant n_i$ the following holds. For every $t \in \mathrm{supp}(y_n^{l_i}) \cap A_{l_i}$ the length of t is greater than j_0, that is, $|t| > |t_j^i| = j_0$ for every $j \in \{1, \dots, w_i\}$. (Notice that this property corresponds to property (IV) in the statement of the sublemma.) The desired pair (l, n) described in (I) will be one of the (l_i, n_i)'s for some $i \in \{1, \dots, k_0\}$. For notational simplicity we set $U_i = \mathcal{T}_{t_1^i} \cup \dots \cup \mathcal{T}_{t_{w_i}^i}$. Observe that if $s_1 \in U_{i_1}$, $s_2 \in U_{i_2}$ and $i_1 \neq i_2$, then s_1 and s_2 are mutually incomparable provided that for every $t_1 \in s_1$ and $t_2 \in s_2$ we have $|t_1|, |t_2| \geqslant j_0$.

For every $q \in L$ with $q > l_{k_0}$ we select $n_q \in M_q$ such that for every $i \in \{1, \dots, k_0\}$ and every $n \in M_q$ with $n \geqslant n_q$ we have that

(i) if $t \in \mathrm{supp}(y_n^q)$ and $\{t\} \in U_i$, then $|t| > j_0$, and
(ii) $\max\big\{k : k \in h\big(\mathrm{supp}(y_{n_i}^{l_i})\big), 1 \leqslant i \leqslant k_0\big\} < \min\{k : k \in h\big(\mathrm{supp}(y_n^q)\big)\}$.

This is possible as the sequence $(y_n^q)_{n \in M_q}$ is block. Set $M_q^* \coloneqq \{n \in M_q : n \geqslant n_q\}$.

Claim. *For every $q \in L$ with $q > l_{k_0}$ and every $n \in M_q^*$ there exists $i \in \{1, \dots, k_0\}$ such that for every segment s with $s \in U_i$ we have $\|P_s(y_n^q)\| \leqslant \varepsilon$.*

Proof of the claim. If not, then there exist $q \in L$ with $q > l_{k_0}$ and $n \in M_q^*$ such that for every $i \in \{1, \dots, k\}$ there exists a segment s_i with $s_i \in U_i$ and $\|P_{s_i}(y_n^q)\| > \varepsilon$. By the choice of M_q^*—in particular, by (i) above—we may assume that for every $i \in \{1, \dots, k_0\}$ and every $t \in s_i$ we have $|t| > j_0$ (this is our usual restriction argument). So the s_i's can be selected to be mutually incomparable. Therefore,

$$2 > \|y_n^q\| \geqslant \Big(\sum_{i=1}^{k_0} \big\| \sum_{t \in s_i} y_n^q(t) x_t \big\|_X^2\Big)^{1/2} = \Big(\sum_{i=1}^{k_0} \|P_{s_i}(y_n^q)\|^2\Big)^{1/2} > \sqrt{\varepsilon^2 k_0} \geqslant 2$$

which yields a contradiction. The claim is proved. □

By the above claim, it follows that for every $q \in L$ with $q > l_{k_0}$ there exist $i \in \{1, \dots, k_0\}$ and $M_q^{**} \in [M_q^*]^\infty \subseteq [M_q]^\infty$ such that for every $n \in M_q^{**}$ and every segment s with $s \in U_i$ we have $\|P_s(y_n^q)\| \leqslant \varepsilon$. Hence, we may select $i_0 \in \{1, \dots, k_0\}$ and $L_1 \in [L]^\infty$ such that for every $q \in L_1$ there exists $M_q^{**} \in [M_q]^\infty$ with the property that for every $n \in M_q^{**}$ and every segment s with $s \in U_{i_0}$ we have $\|P_s(y_n^q)\| \leqslant \varepsilon$. The sublemma follows by setting $l \coloneqq l_{i_0}$, $n \coloneqq n_{i_0}$, $\{t_1^{i_0}, \dots, t_{w_{i_0}}^{i_0}\}$, $j \coloneqq j_0$, $L' \coloneqq L_1$ and $M_q' \coloneqq M_q^{**}$ for every $q \in L'$. □

We proceed to the proof of Lemma A.3.

Proof of Lemma A.3. The sequences (l_k) and (n_k) will be selected recursively with the help of Sublemma A.4. We start by applying Sublemma A.4 for $L=\mathbb{N}$, $M_q=\mathbb{N}$ for every $q\in\mathbb{N}$ and $\varepsilon=1/2$. The sublemma provides us with a pair $l,n\in\mathbb{N}$ which will be our l_1 and n_1 respectively. Also, under the notation of its proof, it provides us with a set U_1 and $L'\in[L]^\infty=[\mathbb{N}]^\infty$, which we denote by L_1, such that for every $q\in L_1$ there exists $M^1_q\coloneqq M'_q\in[M_q]^\infty=[\mathbb{N}]^\infty$ with the following property. If s is a segment with $s\in U_1$, then for every $q\in L_1$ and every $n\in M^1_q$ we have $\|P_s(y^q_n)\|\leqslant\frac{1}{2}=\varepsilon$.

Next, we apply Sublemma A.4 for $L=L_1$, $M_q=M^1_q$ for every $q\in L_1$ and $\varepsilon=\ 1/2^2$, and we proceed recursively mutatis mutandis. This completes the recursive selection.

We isolate the following crucial property established by this selection. Let s be an arbitrary segment of T (it might be a branch, of course) and let $k_0\in\mathbb{N}$. Then one of the following mutually exclusive cases must occur.

Case 1: *We have* $s\in U_{k_0}$ In this case we have $\|P_s(y^{l_k}_{n_k})\|\leqslant 1/2^{k_0}$ for every $k>k_0$. This is a consequence of part (III.b) of Sublemma A.4.

Case 2: *We have* $s\notin U_{k_0}$. Consider the set $A_{l_{k_0}}$ obtained by Lemma A.2, and set $s'\coloneqq\{t : t\in s \text{ and } t\notin\sigma \text{ for every } \sigma\in A_{l_{k_0}}\}$. Notice that s' is a subsegment of s and, clearly, $s'\cap A_{l_{k_0}}=\emptyset$. Invoking part (IV) of Sublemma A.4, we see that $\|P_s(y^{l_{k_0}}_{n_{k_0}})\|=\|P_{s'}(y^{l_{k_0}}_{n_{k_0}})\|$. Therefore, by part (IV) of Lemma A.2, it follows that $\big\|P_s(y^{l_{k_0}}_{n_{k_0}})\big\|\leqslant\varepsilon_{l_{k_0}}$.

Now let $\sigma\in[T]$ be arbitrary. Then either the set $\{k\in\mathbb{N}:\sigma\in U_k\}$ is infinite, or the set $\{k\in\mathbb{N}:\sigma\in U_k\}$ is finite. If the set $\{k\in\mathbb{N}:\sigma\in U_k\}$ is infinite, then, by Case 1, we have

$$\lim\|P_\sigma(y^{l_k}_{n_k})\|=0.$$

On the other hand, if the set $\{k\in\mathbb{N}:\sigma\in U_k\}$ is finite, then, by Case 2 and the choice of the sequence (ε_l) in Lemma A.2,

$$\lim\|P_\sigma(y^{l_k}_{n_k})\|=\lim\varepsilon_{l_k}=0.$$

The proof is completed. □

We are finally in a position to describe the dual of $\mathcal{T}^X_2$.

Theorem A.5. *We have* $(\mathcal{T}^X_2)^*=\overline{\mathrm{span}}\big\{\bigcup_{\sigma\in[T]}B_{\mathcal{X}^*_\sigma}\big\}$.

Proof. Assume not. Recall that $W^*=\overline{\mathrm{span}}\big\{\bigcup_{\sigma\in[T]}B_{\mathcal{X}^*_\sigma}\big\}$. By the Hahn–Banach theorem, there exists $x^{**}\in(\mathcal{T}^X_2)^{**}$ such that $\|x^{**}\|=1$ and $x^{**}|_{W^*}=0$. We select a net $(x_i)_{i\in I}$ in $\mathcal{T}^X_2$ such that $w^*-\lim_{i\in I}x_i=x^{**}$ and $\|x_i\|=1$ for every $i\in I$. Notice that there exists $x^*\in(\mathcal{T}^X_2)^*$ such that $x^*(x_i)\geqslant 1/2$ for every $i\in I$; in particular, if w is a convex combination of $(x_i)_{i\in I}$, then we have $\|w\|\geqslant 1/2$.

By Lemma A.3, there exists a block sequence $(y^{l_k}_{n_k})$ such that $\lim \|P_\sigma(y^{l_k}_{n_k})\| = 0$ for every $\sigma \in [T]$. By Proposition 4.10, the sequence $(y^{l_k}_{n_k})$ is weakly null. Hence so is the corresponding sequence $(z^{l_k}_{n_k})$ of convex combination of $(x_i)_{i\in}$ obtained by Lemma A.2. By Mazur's theorem, we obtain a further convex combination of $(z^{l_k}_{n_k})$ with arbitrarily small norm. Since this is also a convex combination of $(x_i)_{i\in I}$, we derive a contradiction. □

References

[AKP] G. Allexandrov, D. Kutzarova and A. Plichko, *A separable space with no Schauder decomposition*, Proc. Amer. Math. Soc. 127 (1999), 2805–2806.

[AA] D. Alspach and S. A. Argyros, *Complexity of weakly null sequences*, Dissertationes Math. 321 (1992), 1–44.

[Ar] S. A. Argyros, *A universal property of reflexive hereditarily indecomposable Banach spaces*, Proc. Amer. Math. Soc. 129 (2001), 3231–3239.

[AF] S. A. Argyros and V. Felouzis, *Interpolating hereditarily indecomposable Banach spaces*, J. Amer. Math. Soc. 13 (2001), 243–294.

[AGR] S. A. Argyros, G. Godefroy and H. P. Rosenthal, *Descriptive Set Theory and Banach Spaces*, Handbook of the Geometry of Banach Spaces, Vol. 2, Edited by W. B. Johnson and J. Lindenstrauss, Elsevier, 2003.

[AK] S. A. Argyros and V. Kanellopoulos, *Optimal sequences of continuous functions converging to a Baire-*1 *function*, Math. Annalen 324 (2002), 689–729.

[AM] S. A. Argyros and A. Manoussakis, *An indecomposable and unconditionally saturated Banach space*, Studia Math. 159 (2003), 1–32.

[ATo] S. A. Argyros and S. Todorcevic, *Ramsey Methods in Analysis*, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser, Verlag, Basel, 2005.

[AT] S. A. Argyros and A. Tolias, *Methods in the theory of hereditarily indecomposable Banach spaces*, Memoirs Amer. Math Soc., Vol. 806, 2004.

[BHO] S. F. Bellenot, R. Haydon and E. Odell, *Quasi-reflexive and tree spaces constructed in the spirit of R. C. James*, Contemp. Math. 85 (1989), 19–43.

[BL] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, American Mathematical Society Colloquium Publications, Vol. 48, American Mathematical Society, Providence, RI, 2000.

[Bo1] B. Bossard, *Codages des espaces de Banach separables. Familles analytiques ou coanalytiques d'espaces de Banach*, C. R. Acad. Sci. Paris Ser. I Math. 316 (1993), 1005–1010.

[Bo2] B. Bossard, *Théorie descriptive des ensembles en géométrie des espaces de Banach*, Thése, Univ. Paris VI, 1994.

[Bo3] B. Bossard, *A coding of separable Banach spaces. Analytic and coanalytic families of Banach spaces*, Fund. Math. 172 (2002), 117–152.

[Bo4] B. Bossard, *On a problem of H. P. Rosenthal*, Houston J. Math. 26 (2000), 1–16.

[B1] J. Bourgain, *On separable Banach spaces, universal for all separable reflexive spaces*, Proc. Amer. Math. Soc. 79 (1980), 241–246.

[B2] J. Bourgain, *On convergent sequences of continuous functions*, Bull. Soc. Math. Belg. Sér. B 32 (1980), 235–249.

[B3] J. Bourgain, *The Szlenk index and operators on $C(K)$ spaces*, Bull. Soc. Math. Belg. Sér. B 31 (1979), 87–117.

[C] P. G. Casazza, *Approximation properties*, Handbook of the Geometry of Banach Spaces, Vol. 1, Edited by W. B. Johnson and J. Lindenstrauss, Elsevier, 2003.

[DFJP] W. J. Davis, T. Fiegel, W. B. Johnson and A. Pelczynski, *Factoring weakly compact operators*, J. Funct. Anal. 17 (1974), 311–327.

[D] J. Diestel, *Sequences and Series in Banach spaces*, Graduate Texts in Mathematics, Vol. 92, Springer-Verlag, 1984.

[F] V. Farmaki, *Classifications of Baire-*1 *functions and* c_0*-spreading models*, Trans. Amer. Math. Soc. 345 (1994), 819–831.

[Go] G. Godefroy, *Renormings of Banach spaces*, Handbook of the Geometry of Banach Spaces, Vol. 1, Edited by W. B. Johnson and J. Lindenstrauss, Elsevier, 2003.

[GM] W. T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. 6 (1993), 851–874.

[Gr] A. Grothendieck, *Critères de compacité dans les espaces fonctionelles généraux*, Amer. J. Math. 74 (1952), 168–186.

[HOR] R. Haydon, E. Odell and H. P. Rosenthal, *On certain classes of Baire-*1 *functions with applications to Banach space theory*, Lecture Notes in Mathematics, Vol. 1470, Springer, 1991, 1–35.

[J] W. B. Johnson, *A complementably universal conjugate Banach space and its relation to the approximation problem*, Israel J. Math. 13 (1972), 301–310.

[JMS] W. B. Johnson, B. Maurey and G. Schechtman, *Weakly null sequences in* L_1, J. Amer. Math. Soc. 20 (2007) 25–36.

[K] A. S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, Vol. 156, Springer-Verlag, 1995.

[KL1] A. S. Kechris and A. Louveau, *Descriptive Set Theory and the Structure of Sets of Uniqueness*, London Mathematical Society Lecture Notes Series, Vol. 128, Cambridge University Press, 1989.

[KL2] A. S. Kechris and A. Louveau, *A classification of Baire class* 1 *functions*, Trans. Amer. Math. Soc. 318 (1990), 209–236.

[KW1] A. S. Kechris and W. H. Woodin, *Ranks on differentiable functions*, Mathematika 32 (1986), 252–278.

[KW2] A. S. Kechris and W. H. Woodin, *A strong boundedeness theorem for dilators*, Annals of Pure and Applied Logic 52 (1991), 93–97.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I and II*, Springer, 1996.

[Lu] W. Lusky, *A note on Banach spaces containing* c_0 *or* C_∞, J. Funct. Anal. 62 (1985), 1–7.

[MR] B. Maurey and H. P. Rosenthal, *Normalized weakly null sequences with no unconditional subsequences*, Studia Math. 61 (1977), 77–98.

[Mo] Y. N. Moschovakis, *Descriptive Set Theory*, North Holland, Amsterdam, 1980.

[N1] R. Neidinger, *Properties of Tauberian Operators*, Dissertation, University of Texas at Austin, 1984.

[N2] R. Neidinger, *Factoring operators through hereditarily* ℓ_p *spaces*, Lecture Notes in Mathematics, Vol. 1166, Springer, 1985, 116–128.

[OR] E. Odell and H. P. Rosenthal, *A double-dual characterization of separable Banach spaces not containing* ℓ_1, Israel J. Math. 20 (1975), 375–384.

[OS1] E. Odell and Th. Schlumprecht, personal communication.

[OS2] E. Odell and Th. Schlumprecht, *A separable reflexive universal for the uniformly convex Banach spaces*, Math. Annalen 335 (2006), 901–916.

[P] A. Pelczynski, *Universal bases*, Studia Math. 19 (1960), 247–268.

[Pr1] S. Prus, *Finite-dimensional decompositions of Banach spaces with* (p, q)*-estimates*, Dissertationes Math. 263 (1987), 1–45.

[Pr2] S. Prus, *Finite-dimensional decompositions with p-estimates and universal Banach spaces*, Bull. Polish Acad. Sci. Math. 31 (1983), 281–288.

[R] F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. 30 (1930), 264–286.

[Ro1] H. P. Rosenthal, *A characterization of Banach spaces not containing* ℓ_1, Proc. Natl. Acad. Sci. USA 71 (1974), 2411–2413.

[Ro2] H. P. Rosenthal, *The Banach spaces* $C(K)$, Handbook of the Geometry of Banach Spaces, Vol. 2, Edited by W. B. Johnson and J. Lindenstrauss, Elsevier, 2003.

[Ro3] H. P. Rosenthal, *On factors of* $C([0,1])$ *with non-separable dual*, Israel J. Math. 13 (1972), 361–378.

[Sch] G. Schechtman, *On Pelczynski's paper "Universal bases"*, Israel J. Math. 22 (1975), 181–184.

[Schl] Th. Schlumprecht, *An arbitrarily distortable Banach space*, Israel J. Math. 76 (1991), 81–95.

[Sz] W. Szlenk, *The non existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces*, Studia Math. 30 (1968), 53–61.

[TJ] N. Tomczak-Jaegermann, *Banach spaces of type p have arbitrarily distortable subspaces*, Geom. Funct. Anal. 6 (1996), 1074–1082.

[Z] M. Zippin, *Banach spaces with separable duals*, Trans. Amer. Math. Soc. 310 (1988), 371–379.

National Technical University of Athens, Faculty of Applied Sciences, Department of Mathematics, Zografou Campus, 157 80, Athens, Greece.

E-mail address: sargyros@math.ntua.gr, pdodos@math.ntua.gr