TREE STRUCTURES ASSOCIATED TO A FAMILY OF FUNCTIONS

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The research presented in this paper was motivated by our aim to study a problem due to Bourgain [3]. The problem in question concerns the uniform boundedness of the classical separation rank of the elements of a separable compact set of the first Baire class. In the sequel we shall refer to these sets (separable or non-separable) as Rosenthal compacta and we shall denote by $\alpha(f)$ the separation rank of a real-valued function f in $\mathcal{B}_1(X)$ with X a Polish space. Notice that in [3], Bourgain has provided a positive answer to this problem in the case of $\mathcal{K}$ satisfying $\mathcal{K}=\overline{\mathcal{K}\cap C(X)}^p$ with X a compact metric space. The key ingredient in Bourgain's approach is that whenever a sequence (f_n) of continuous functions converges pointwise to a function f, then the possible discontinuities of the limit function reflect a local ℓ^1-structure to the sequence (f_n). More precisely the complexity of this ℓ^1-structure increases as the complexity of the discontinuities of f does. This fruitful idea was extensively studied by several authors (see, e.g., [5, 7, 8]) and for an exposition of the related results we refer to [1]. It is worth mentioning that Kechris and Louveau have invented the rank $r_{ND}(f)$ which permits the link between the c_0-structure of a sequence (f_n) of uniformly bounded continuous functions and the discontinuities of its pointwise limit. Rosenthal's c_0-theorem [11] and the c_0-index theorem [2] are consequences of this interaction.

Passing to the case where either (f_n) are not continuous or X is a non-compact Polish space, this nice interaction is completely lost. Easy examples show that there exist sequences of continuous functions on $\mathbb{R}$ pointwise convergent to zero and in the same time they are equivalent to the ℓ^1 basis. Moreover, there are sequences (f_n) of Baire-1 functions, equivalent to the summing basis of c_0, pointwise convergent to a Baire-2 function. Thus if we wish to preserve the main scheme, invented by Bourgain, namely to pass from the elements of the separable Rosenthal compactum to a well-founded tree related to the dense sequence (f_n), then we have to take into account not only the finite subsets of (f_n) but also the points of the Polish space X. This is the key observation on which we have based our approach. Thus for every $\mathcal{D}$ subset of $\mathbb{R}^X$ we associate a tree $\mathcal{T}\big((f_\xi)_{\xi<\theta},a,b\big)$ where $(f_\xi)_{\xi<\theta}$ is a well-ordering of $\mathcal{D}$ and $a<b$ are reals. The elements of the tree are of the form (u,T) with u a finite increasing subsequence of $(f_\xi)_{\xi<\theta}$ and T a finite dyadic tree in X where the

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length of u and the height of T are the same and which share certain properties. The partial order of this tree is naturally defined. The basic property of this tree is described in the following proposition.

Proposition A. *For every relatively compact subset $\mathcal{D}$ of $\mathbb{R}^X$ the following are equivalent.*

1. *For every well-ordering $(f_\xi)_{\xi<\theta}$ of $\mathcal{D}$ and $a<b$ the tree $\mathcal{T}\big((f_\xi)_{\xi<\theta},a,b\big)$ is well-founded.*
2. *The accumulation points of $\mathcal{D}$ in $\mathbb{R}^X$ are Baire-1 functions.*

Motivated by this result, we introduce the class of quasi-Rosenthal compacta as the compact subsets $\mathcal{K}$ of $\mathbb{R}^X$ for which the set $Acc(\mathcal{K})$ of accumulation points of $\mathcal{K}$ is a subset of $\mathcal{B}_1(X)$. Naturally defined examples show that this class is wider than the corresponding class of Rosenthal compacta. We also present some characterizations and results for quasi-Rosenthal compacta. Next, for a sequence (f_n) in $\mathbb{R}^X$ and a function $f\in Acc\big(\{f_n\}\big)\cap\mathcal{B}_1(X)$ we compare the quantity $o\big((f_n)\big)=\sup\big\{o\big(\mathcal{T}((f_n),a,b)\big) : a<b\big\}$ with $\alpha(f)$. Specifically, we show the following theorem which corresponds to Theorem 11 in the main text.

Theorem B. *For every sequence (f_n) in $\mathbb{R}^X$ and every $f\in\mathcal{B}_1(X)$ the following are satisfied.*

1. *If $f\in Acc\big(\{f_n\}\big)$, then*
$$o\big((f_n)\big)+1\geqslant\alpha(f).$$
2. *If $f=\lim f_n$, then*
$$o\big((f_n)\big)\leqslant\omega\cdot 2\cdot\alpha(f).$$

Notice that (1) is expected and it holds for all ranks defined on a sequence and related to a rank of the limit function f. For example similar results hold for the ranks $\gamma\big((f_n)\big)$ and $\alpha(f)$ (see [7]) or the ranks $\nu\big((f_n)\big)$ and $r_{ND}(f)$ (see [2]). However, part (2) is rather unexpected since usually the ranks defined on sequences do not recognize possible noise involved in the elements of the sequence. In particular, $\gamma\big((f_n)\big)$ or $\nu\big((f_n)\big)$ could be arbitrarily larger than $\alpha(f)$ and $r_{ND}(f)$ respectively. From part (1) of the previous theorem, we obtain the following corollary.

Corollary C. *Let $\mathcal{K}=\overline{\{f_n\}}^p$ be a quasi-Rosenthal compactum. Then we have*
$$\sup\big\{\alpha(f) : f\in Acc(\mathcal{K})\big\}\leqslant o\big((f_n)\big)+1.$$

This result would yield an affirmative answer to Bourgain's question provided that $o\big((f_n)\big)<\omega_1$. This is not true in general as the examples show. However, the Kunen–Martin theorem permits us to prove it under some additional regularity properties of the sequence (f_n). Specifically, we show the following theorem (Theorem 23 in the main text) which answers Bourgain's problem.

Theorem D. *Let $\mathcal{K}$ be a Borel separable quasi-Rosenthal compactum. Then*

$$\sup\big\{\alpha(f) : f \in \mathcal{K} \cap \mathcal{B}_1(X)\big\} < \omega_1.$$

Here, Borel separable means that there exists a countable dense subset consisting of Borel functions. We notice that the above result is sharp. More precisely, we provide an example of a separable quasi-Rosenthal compactum $\mathcal{K}$ containing a countable dense subset consisting of characteristic functions of analytic sets and such that $\sup\{\alpha(f) : f \in Acc(\mathcal{K})\} = \omega_1$.

1. Preliminaries

In what follows let X be a Polish space and let d be a compatible complete metric for X. By $\mathcal{B}_1(X)$ (respectively, $\mathcal{B}(X)$) we denote the space of Baire-1 (respectively, Borel) real-valued functions on X. By $\mathbb{N}$ we denote the set of all positive integers while by ω the set of all nonnegative integers. If L is an infinite subset ω, then by $[L]^\infty$ we denote the set of all infinite subsets of L. For every well-ordered set θ by $[\theta]^{<\omega}$ we denote the set of all finite strictly increasing sequences of θ. Given $u, u' \in [\theta]^{<\omega}$, we write $u \sqsubset u'$ if u is a proper initial segment of u'. We follow the convention that all nonempty $u \in [\theta]^{<\omega}$ have terms indexed by a proper initial segment of $\mathbb{N}$ instead of ω. For notational convenience, for every $k \in \omega$ we set $D_k = 2^{\leqslant k}$ and $D = 2^{<\omega}$. For every $s \in D$ by $|s|$ we denote the length of s. For every $\mathcal{D} \subseteq \mathbb{R}^X$ by $\overline{\mathcal{D}}^p$ we denote the closure of $\mathcal{D}$ in $\mathbb{R}^X$, while by $Acc(\mathcal{D})$ we denote the set of all accumulation points of $\mathcal{D}$ in $\mathbb{R}^X$ (that is, the set of all $f \in \mathbb{R}^X$ such that f belongs to the closure of $\mathcal{D} \setminus \{f\}$ in $\mathbb{R}^X$). We recall that the topology in $\mathbb{R}^X$ is generated by the sets of the form

$$U(f, F, \varepsilon) = \{g \in \mathbb{R}^X : |g(x) - f(x)| < \varepsilon \ \forall x \in F\}$$

where f ranges over $\mathbb{R}^X$, F ranges over all finite subsets of X and $\varepsilon > 0$. Finally, for every $f \in \mathbb{R}^X$ and every $a \in \mathbb{R}$ by $[f < a]$ we denote the set $\{x \in X : f(x) < a\}$. The sets $[f \leqslant a]$, $[f > a]$ and $[f \geqslant a]$ have the obvious meaning.

1.1. **The separation rank α.** The separation rank $\alpha(f)$ of a Baire-1 function has its roots in the work of Hausdorff, Kuratowski and Lavrentiev. We recall its definition taken from [7]. Given $A, B \subseteq X$, one associates with them a derivative on closed sets by setting

$$K'_{A,B} = \overline{K \cap A} \cap \overline{K \cap B}.$$

By recursion, we define the iterated derivatives of K by the rule $K^{(0)}_{A,B} = K$, $K^{(\xi+1)}_{A,B} = \big(K^{(\xi)}_{A,B}\big)'_{A,B}$ and $K^{(\xi)}_{A,B} = \bigcap_{\zeta<\xi} K^{(\zeta)}_{A,B}$ if ξ is a limit ordinal. We set

$$\alpha(K, A, B) = \begin{cases} \text{least } \xi & : \ K^{(\xi)}_{A,B} = \emptyset \text{ if such } \xi \text{ exists} \\ \omega_1 & : \ \text{otherwise,} \end{cases}$$

and $\alpha(A,B)=\alpha(X,A,B)$. It is well-known that $\alpha(A,B)<\omega_1$ if and only if one can separate A from B by a set which is transfinite difference of closed sets (see, e.g., [6, page 177]).

Now let $f\colon X\to\mathbb{R}$ be a function. For every $a<b$ we set

$$\alpha(f,a,b)=\alpha([f<a],[f>b])$$

and, finally, we define the separation rank of f by the rule

$$\alpha(f)=\sup\{\alpha(f,a,b): a<b\}.$$

The basic fact is the following (see [7]).

Proposition 1. *A function f is Baire-1 if and only if $\alpha(f)<\omega_1$.*

The above defined rank is sightly different from the rank that Bourgain originally defined in [3]. As it is shown in [5], the two variants are equivalent. Also observe that the rank is the same if we have defined $\alpha(f,a,b)$ by $\alpha([f\leqslant a],[f\geqslant b])$. For every closed $Y\subseteq X$, every $\xi<\omega_1$ and every $a<b$ by $Y^{(\xi)}_{(f,a,b)}$ we denote the ξ-iterated derivative of Y with respect to $[f<a]$ and $[f>b]$. For the properties of α and its relations with other ordinal ranks on $\mathcal{B}_1(X)$ we refer to [7].

1.2. **Trees and well-founded relations.** By the term *tree* we mean a partial order set $(\mathcal{T},<)$ in the strict sense, such that for every $t\in\mathcal{T}$ the set $\{s\in\mathcal{T}: s<t\}$ is well-ordered. Now let $\mathcal{T}$ be a well-founded tree. As usual, we set

$$\mathcal{T}' := \{s\in\mathcal{T}: \exists t\in\mathcal{T} \text{ such that } s<t\}.$$

By recursion, we define $\mathcal{T}^{(0)}=\mathcal{T}$, $\mathcal{T}^{(\xi+1)}=\big(\mathcal{T}^{(\xi)}\big)'$ and $\mathcal{T}^{(\xi)}=\bigcap_{\zeta<\xi}\mathcal{T}^{(\zeta)}$ if ξ is a limit ordinal. The *order* $o(\mathcal{T})$ of $\mathcal{T}$ is defined to be the least ordinal ξ for which $\mathcal{T}^{(\xi)}=\emptyset$. If $(\mathcal{S},<_{\mathcal{S}})$ and $(\mathcal{T},<_{\mathcal{T}})$ are well-founded trees, then a map $\varphi\colon\mathcal{S}\to\mathcal{T}$ is called *monotone* if $s_1<_{\mathcal{S}} s_2$ implies that $\varphi(s_1)<_{\mathcal{T}}\varphi(s_2)$. Clearly in this case we have that $o(\mathcal{S})\leqslant o(\mathcal{T})$.

Let X be a set and let $\prec$ be a strict, well-founded relation on X. By recursion we define $\rho_{\prec}\colon X\to Ord$ as follows. We set $\rho_{\prec}(x)=0$ if x is minimal; otherwise, we set $\rho_{\prec}(x)=\sup\{\rho_{\prec}(y)+1: y\prec x\}$. Finally we define the *rank* of $\prec$ by setting $\rho(\prec)=\sup\{\rho_{\prec}(x)+1: x\in X\}$. We will need the following boundedness principle of analytic well-founded relations which is due to Kunen and Martin (see [6, 9]).

Theorem 2. *Let X be a Polish space and let $\prec$ be a strict and well-founded relation. If $\prec$ is analytic (as a subset of $X\times X$), then $\rho(\prec)<\omega_1$.*

Note that if $(\mathcal{T},<)$ is a well-founded tree, then the relation $\prec$ on $\mathcal{T}$ defined by $t\prec s$ if $s<t$, is strict and well-founded and $o(\mathcal{T})=\rho(\prec)$.

2. The tree $\mathcal{T}\big((f_\xi)_{\xi<\theta}, a, b\big)$.

In this section we introduce the tree $\mathcal{T}\big((f_\xi)_{\xi<\theta}, a, b\big)$ and we define the class of quasi-Rosenthal compacta. We present some results related to the above two notions as well as examples of quasi-Rosenthal compacta which are not Rosenthal compacta. We start with the following definition.

Definition 3. *Let $a<b$ reals, let θ be an infinite ordinal and let $\mathcal{D}=(f_\xi)_{\xi<\theta}$ be a long sequence of not necessarily distinct elements of $\mathbb{R}^X$. We define*

$$\mathcal{T}=\mathcal{T}\big((f_\xi)_{\xi<\theta}, a, b\big) \subseteq \bigcup_{k\in\omega} [\theta]^k \times X^{D_k}$$

to be the set of all pairs (u,T) for which the following hold. First, let $(\emptyset, t_\emptyset)\in\mathcal{T}$ for every $t_\emptyset \in X$. Moreover, if $u=(\xi_1,\dots,\xi_k)$, $T=(t_s)_{s\in D_k}$ and $k\geqslant 1$, then $(u,T)\in\mathcal{T}$ if it the following conditions are satisfied.

- (C1) *Either $t_0=t_\emptyset$ or $t_1=t_\emptyset$.*
- (C2) *For every $s\in D_k$ with $|s|<k$ the following hold.*
  - (i) *We have $d(t_{s^\frown 0}, t_{s^\frown 1}) \leqslant \frac{1}{2^{|s|+1}}$.*
  - (ii) *We have $f_{\xi_{|s|+1}}(t_{s^\frown 0})<a$ and $f_{\xi_{|s|+1}}(t_{s^\frown 1})>b$ (hence, $t_{s^\frown 0}\neq t_{s^\frown 1}$).*
  - (iii) *For every $s\neq\emptyset$ if $f_{\xi_{|s|}}(t_s)<a$, then $t_{s^\frown 0}=t_s$, while if $f_{\xi_{|s|}}(t_s)>b$, then $t_{s^\frown 1}=t_s$.*

The set $\mathcal{T}$ is a tree under the following partial ordering. If $(u_1,T_1),(u_2,T_2)\in\mathcal{T}$ with $T_1=(t^1_s)_{s\in D_{k_1}}$ and $T_2=(t^2_s)_{s\in D_{k_2}}$, then we set

$$(u_1,T_1)<(u_2,T_2) \ \text{ if } \ u_1\sqsubset u_2 \text{ and } T_1 \lhd T_2$$

where by $T_1\lhd T_2$ we mean that $t^1_s=t^2_s$ for every $s\in D_{k_1}$.

For every nonempty $Y\subseteq X$ let

$$\mathcal{T}\big(Y,(f_\xi)_{\xi<\theta}, a, b\big)\subseteq \mathcal{T}$$

be the set of all $(u,T)\in\mathcal{T}$ for which $T\in\bigcup_{k\in\omega} Y^{D_k}$. By convention, if $Y=\emptyset$, then we set $\mathcal{T}\big(Y,(f_\xi)_{\xi<\theta}, a, b\big)=\emptyset$.

Remark 1. Clearly, $\mathcal{T}\big(Y,(f_\xi)_{\xi<\theta}, a, b\big)$ equipped with the induced partial order is a subtree of $\mathcal{T}$. Also notice that if $Y'\subseteq Y$ and $a'\leqslant a<b\leqslant b'$, then $\mathcal{T}\big(Y',(f_\xi)_{\xi<\theta}, a', b'\big)$ is a subtree of $\mathcal{T}\big(Y,(f_\xi)_{\xi<\theta}, a, b\big)$.

Definition 4. *A compact subset $\mathcal{K}$ of $\mathbb{R}^X$ is said to be* quasi-Rosenthal *if the set $Acc(\mathcal{K})$ of accumulation points of $\mathcal{K}$ is a nonempty subset of $\mathcal{B}_1(X)$.*

Remark 2. Observe that for every sequence $(f_\xi)_{\xi<\theta}$ in $\mathcal{K}$ and every function $f\in Acc(\{f_\xi\}_{\xi<\theta})$ there exist $\xi_f\leqslant\theta$ and a subnet of $(f_\xi)_{\xi<\xi_f}$ converging in $\mathbb{R}^X$ to f. Indeed, we set $\xi_f:=\min\big\{\zeta\leqslant\theta : f\in\overline{\{f_\xi\}}^p_{\xi<\zeta}\big\}$. It is easy to verify that ξ_f is the desired ordinal.

Lemma 5. *Let $(f_\xi)_{\xi<\theta}$ be a pointwise bounded sequence of distinct elements of $\mathbb{R}^X$ and assume that the tree $\mathcal{T}\big((f_\xi)_{\xi<\theta}, a, b\big)$ contains an infinite chain $\big((u_k, T_k)\big)$. We set $N := \bigcup_k u_k = \{\xi_1 < \xi_2 < \cdots\}$ and $T := \bigcup_k T_k$. Then there exist $L \in [\mathbb{N}]^\infty$ and $f : T \to \mathbb{R}$ with the following properties.*

(1) *We have $\lim_{n\in L} f_{\xi_n}|_T = f$.*
(2) *We have $T = A \cup B$ with $\overline{A} = \overline{B}$, $A = [f \leqslant a]$ and $B = [f \geqslant b]$.*

In particular, every $g \in Acc\big(\{f_{\xi_n}\}_{n\in L}\big)$ is not a Baire-1 function.

Proof. We set $T := \bigcup_{k\in\mathbb{N}} T_k = (t_s)_{s\in D}$, and

$$A := \{t_{s^\frown 0} : s \in D\} \ \text{ and } \ B := \{t_{s^\frown 1} : s \in D\}.$$

Since T is countable, we may select $L \in [\mathbb{N}]^\infty$ such that (1) is satisfied. The definition of the tree $\mathcal{T}\big((f_\xi)_{\xi<\theta}, a, b\big)$ yields that $A = [f \leqslant a]$, $B = [f \geqslant b]$ and $\overline{A} = \overline{B}$. By Proposition 1, for every $g \in Acc\big(\{f_{\xi_n}\}_{n\in L}\big)$ we see that $g|_T = f$ which implies that g is not a Baire-1 function. □

We proceed to discuss characterizations of quasi-Rosenthal compacta.

Theorem 6. *Let $\mathcal{D}$ be a relatively compact subset of $\mathbb{R}^X$. Then the following are equivalent.*

(1) *$\mathcal{K} = \overline{\mathcal{D}}^p$ is a quasi-Rosenthal compactum.*
(2) *For every sequence $(f_\xi)_{\xi<\theta}$ of distinct members of $\mathcal{D}$ and every $a < b$ the tree $\mathcal{T}\big((f_\xi)_{\xi<\theta}, a, b\big)$ is well-founded.*
(3) *For every sequence (f_n) of distinct members of $\mathcal{D}$ there exists a subsequence pointwise convergent to a Baire-1 function.*
(4) *For every infinite subset $\mathcal{D}'$ of $\mathcal{D}$ there exists a Baire-1 function f belonging to $Acc(\mathcal{D}')$.*

Proof. (1)⇒(2) Assume, on the contrary, that there exists a sequence $(f_\xi)_{\xi<\theta}$ in $\mathcal{D}$ and $a < b$ such that the tree $\mathcal{T}\big((f_\xi)_{\xi<\theta}, a, b\big)$ is not well-founded. Lemma 5 yields a contradiction.

(2)⇒(1) Let $\mathcal{D} = (f_\xi)_{\xi<\theta}$ be a well-ordering of $\mathcal{D}$. Note that $Acc(\mathcal{K}) = Acc(\mathcal{D})$. Assume, towards a contradiction, that there exists $f \in Acc(\mathcal{D})$ such that f is not a Baire-1 function. Then $\alpha(f) = \omega_1$, and so there exist $a < b$ such that $\alpha(f, a, b) = \omega_1$. It follows that there exist countable subsets A, B of X with $\overline{A} = \overline{B}$ such that $A \subseteq [f < a]$ and $B \subseteq [f > b]$. We construct $T = (t_s)_{s\in D} \in X^D$ such that for every $s \in D$ the following hold.

(i) We have $t_\emptyset \in A$.
(ii) For every $s \in D$ we have $t_{s^\frown 0} \in A$ and $t_{s^\frown 1} \in B$.
(iii) For every $s \in D$ if $t_s \in A$, then $t_{s^\frown 0} = t_s$, while if $t_s \in B$, then $t_{s^\frown 1} = t_s$.
(iv) We have $d(t_{s^\frown 0}, t_{s^\frown 1}) < \frac{1}{2^{|s|+1}}$.

We proceed by induction on the length of s. For $|s| = 0$, we select $x \in A$ and we set $t_\emptyset = x$. Suppose that t_s have been defined for every $s \in D$ with $|s| \leqslant k$. For every $s \in D$ with $|s| = k$ and $t_s \in A$ we set $t_{s^\frown 0} = t_s$ and we select $y_s \in B$ such that the condition (iv) above is satisfied for $t_{s^\frown 1} = y_s$. The case $t_s \in B$ is treated similarly. The construction is completed.

Next, we select an increasing sequence $(\xi_k)_{k\in\mathbb{N}}$ such that $\xi_k < \xi_f$ (see Remark 2), and for every $s \in D$ with $|s| = k$ we have

$$f_{\xi_k}(t_s) < a \text{ if } f(t_s) < a, \text{ and } f_{\xi_k}(t_s) > b \text{ if } f(t_s) > b.$$

We set $u_k = (\xi_1, \dots, \xi_k)$ and $T_k = (t_s)_{s\in D_k}$ for every $k \in \mathbb{N}$. It is easily checked that $\big((u_k, T_k)\big)_k$ is an infinite chain of $\mathcal{T}\big((f_\xi)_{\xi<\theta}, a, b\big)$.

(3)⇒(2) It is an immediate consequence of Lemma 5.

(1)⇒(3) Let (f_n) be a sequence in $\mathcal{D}$. If (f_n) has a pointwise convergent subsequence, then the limit function belongs to the accumulation points of $\mathcal{K}$, and so it is a Baire-1 function. So assume, towards a contradiction, that there exists a sequence (f_n) in $\mathcal{D}$ with no pointwise convergent subsequence. Then, Theorem 2 in [10] yields that there exists a subsequence (f_{n_k}) of (f_n) with no accumulation point in $\mathcal{B}_1(X)$ which leads to a contradiction.

(1)⇒(4) It is obvious.

(4)⇒(2) This is also a consequence of Lemma 5. □

In the next proposition we show that quasi-Rosenthal compacta have countable tightness. This property is known for Rosenthal compacta (see [10]).

Proposition 7. *Let $\mathcal{D}$ be a subset of $\mathbb{R}^X$ such that $\mathcal{K} = \overline{\mathcal{D}}^p$ is quasi-Rosenthal. Then for every $g \in \mathcal{K}$ there exists a countable subset $\mathcal{D}'$ of $\mathcal{D}$ such that $g \in \overline{\mathcal{D}'}^p$.*

Proof. We may assume that $g \in Acc(\mathcal{D})$. We set

$$Seq(\mathcal{D}) := \{f \in Acc(\mathcal{D}) : f \text{ is the limit of a sequence of distinct members of } \mathcal{D}\}.$$

We claim that $g \in \overline{Seq(\mathcal{D})}^p$. Indeed, let $x_1, x_2, \dots, x_k \in X$ and let $\varepsilon > 0$ be arbitrary. Then there exist a sequence (f_n) of distinct members of $\mathcal{D}$ such that $|f_n(x_i) - g(x_i)| < \varepsilon$ for every $i \in \{1, \dots, k\}$ and every $n \in \mathbb{N}$. By part (3) of Theorem 6, there exists a pointwise convergent subsequence (f_{n_k}) of (f_n). If f is the pointwise limit of (f_{n_k}), then clearly $f \in Seq(\mathcal{D})$ and $|f(x_i) - g(x_i)| \leqslant \varepsilon$ for every $i \in \{1, \dots, k\}$ which proves the claim.

Now observe that $Seq(\mathcal{D})$ is a subset of $Acc(\mathcal{D}) \subseteq \mathcal{B}_1(X)$. Therefore, $Seq(\mathcal{D})$ is a relatively compact subset of $\mathcal{B}_1(X)$. By the Main Theorem in [10], the result follows. □

Remark 3. It seems to be well-known, and follows by the results in [13], that every separable Rosenthal compactum satisfies the Continuum Hypothesis. Indeed, by [13, Theorem 5], every separable Rosenthal compactum $\mathcal{K}$ either contains a discrete subspace of size continuum or it is an at most two-to-one continuous pre-image of a compact metrizable space. In any case, it is straightforward that $\mathcal{K}$ has either $\aleph_0$ or $2^{\aleph_0}$ members.

Example 1. We are about to present examples of separable quasi-Rosenthal compacta which show the variety of this class. Let $\mathcal{C} = \{0,1\}^{\mathbb{N}}$ be the Cantor set and let $X = \mathcal{C} \times \mathcal{C}$. For every $\sigma \in \mathcal{C}$ let $\mathcal{C}_\sigma = \{\sigma\} \times \mathcal{C}$. We select $\Delta_\sigma \subseteq \mathcal{C}_\sigma$ such that $\mathbf{1}_{\Delta_\sigma}$ (viewed as a real-valued function on X) is Baire-1. For every $s \in D$, where D is the dyadic tree, we set

$$\Delta_s \coloneqq \bigcup_{s \sqsubset \sigma} \Delta_\sigma$$

and $f_s \coloneqq \mathbf{1}_{\Delta_s}$. Let $\mathcal{K} = \overline{\{f_s\}}^p_{s\in D}$. Then $\mathcal{K}$ is separable, and for every sequence (f_{s_n}) in $\{f_s\}_{s\in D}$ there exists $L \in [\mathbb{N}]^\infty$ such that either $(s_n)_{n\in L}$ are pairwise incomparable, or there exists $\sigma \in \mathcal{C}$ such that $s_n \sqsubset \sigma$ for every $n \in L$. In the first case the sequence $(f_{s_n})_{n\in L}$ converges pointwise to 0, while in the second case it converges pointwise to $\mathbf{1}_{\Delta_\sigma}$. Theorem 6 yields that $\mathcal{K}$ is a separable quasi-Rosenthal compactum. Depending on the choice of $\{\Delta_\sigma : \sigma \in \mathcal{C}\}$ we obtain different spaces.

(1) We may select Δ_σ with $\alpha(\mathbf{1}_{\Delta_\sigma}) \geqslant \xi_\sigma$ and $\sup\{\xi_\sigma : \sigma \in \mathcal{C}\} = \omega_1$. This space answers in negative Bourgain's question stated for separable quasi-Rosenthal compacta.
(2) If $\aleph_0 < |\{\sigma \in \mathcal{C} : \Delta_\sigma \neq \emptyset\}| < 2^{\aleph_0}$, then the corresponding $\mathcal{K}$ satisfies that $\aleph_0 < |\mathcal{K}| < 2^{\aleph_0}$. This yields that, under the negation of CH, there exist separable quasi-Rosenthal compacta not homeomorphic to any Rosenthal compactum (see Remark 3).

A variant of this example, based on techniques of universal sets from descriptive set theory, is presented after Theorem 23.

We recall that a sequence $\big((A_k, B_k)\big)$ of subsets of a set S with $A_k \cap B_k = \emptyset$ for every $k \in \mathbb{N}$, is called *independent* provided that for every pair F, G of finite disjoint subsets of $\mathbb{N}$ we have

$$\Big(\bigcap_{k\in F} A_k\Big) \cap \Big(\bigcap_{k\in G} B_k\Big) \neq \emptyset.$$

This definition is crucial for the proof of Rosenthal's ℓ_1 theorem (see [6, 12]).

Proposition 8. *Let (f_n) be a pointwise bounded sequence of continuous real-valued functions on X, and assume that there exist $a < b$ such that the tree $\mathcal{T} = \mathcal{T}\big((f_n), a, b\big)$ is not well-founded. Let $\big((u_k, T_k)\big)$ be an infinite chain of $\mathcal{T}$, and set $N \coloneqq \bigcup_k u_k = \{n_k : k \in \mathbb{N}\}$ and $T \coloneqq \bigcup_k T_k = (t_s)_{s\in D}$. Then there exist a Cantor set $C \subseteq \overline{T}$ and a subsequence $(f_{n'_k})$ of (f_{n_k}) such that the sequence $\big(([f_{n'_k} < a] \cap C, [f_{n'_k} > b] \cap C)\big)$ is an independent sequence of disjoint pairs.*

Proof. By induction, we shall construct a subtree $T' = (t'_s)_{s\in D}$ of T, an increasing sequence $(l_k)_{k\in\omega}$ and a family of open balls $\{B_s : s \in D\}$ of X with the following properties.

(1) For every s we have $t'_s \in B_s$.
(2) If $|s| = k$, then there exists s' with $|s'| = l_k$ such that $t'_s = t_{s'}$.
(3) If $|s| = k$, then $B_{s^\frown 0} \subseteq [f_{n_{l_{k+1}}} < a]$ and $B_{s^\frown 1} \subseteq [f_{n_{l_{k+1}}} > b]$.
(4) We have $\overline{B}_{s^\frown 0} \cup \overline{B}_{s^\frown 1} \subseteq B_s$.
(5) We have $\mathrm{diam}(B_s) \leqslant \frac{1}{2^{|s|}}$.

We start by setting $t'_\emptyset = t_\emptyset$, $l_0 = 0$; also let B_0 be any open ball containing $t_\emptyset$ with $\mathrm{diam}(B_0) \leqslant \frac{1}{2}$. Suppose that the construction has been carried out for some $k \in \omega$. Let $2\varepsilon_k = \min\{\mathrm{diam}(B_s) : s \in D_k\}$ and select $m \in \omega$ such that $\frac{1}{2^{l_k+m}} < \varepsilon_k$. Let $s \in D_k$ with $|s| = k$. Then $t'_s = t_{s'}$ for some s' with $|s'| = l_k$. Hence, either $f_{n_{l_k}}(t'_s) < a$ or $f_{n_{l_k}}(t'_s) > b$. If $f_{n_{l_k}}(t'_s) < a$ (respectively, $f_{n_{l_k}}(t'_s) > b$), then we set $s'' := s'^\frown 0^m$ (respectively, $s'' = s'^\frown 1^m$) and we define $t'_{s^\frown 0} = t_{s''^\frown 0}$, $t'_{s^\frown 1} = t_{s''^\frown 1}$ and $l_{k+1} = l_k + m + 1$. Hence, $d(t'_{s^\frown 0}, t'_{s^\frown 1}) \leqslant \frac{1}{2^{l_{k+1}}} < \frac{\varepsilon_k}{2}$ and so $t'_{s^\frown 0}, t'_{s^\frown 1} \in B_s$. By the continuity of $f_{n_{l_{k+1}}}$ and the fact that $f_{n_{l_{k+1}}}(t'_{s^\frown 0}) < a$ and $f_{n_{l_{k+1}}}(t'_{s^\frown 1}) > b$, we may select $B_{s^\frown 0}$ and $B_{s^\frown 1}$ containing $t'_{s^\frown 0}$ and $t'_{s^\frown 1}$ respectively such that $B_{s^\frown 0} \subseteq [f_{n_{l_{k+1}}} < a]$, $B_{s^\frown 1} \subseteq [f_{n_{l_{k+1}}} > b]$, $\overline{B}_{s^\frown 0} \cup \overline{B}_{s^\frown 1} \subseteq B_s$ and $\mathrm{diam}(B_{s^\frown i}) < \frac{1}{2^{k+1}}$ for every $i \in \{0, 1\}$. The construction is completed.

For every $k \in \mathbb{N}$ we set $n_{l_k} := n'_k$, $A_0^k := [f_{n'_k} < a]$, $A_1^k := [f_{n'_k} > b]$ and $C := \overline{T'}$. It is easily seen that $\big((A_0^k \cap C, A_1^k \cap C)\big)$ is an independent sequence of disjoint pairs, as desired. □

Remark 4. The sequence $(f_{n'_k})$ obtained from Proposition 8 can be used to derive the following two well-known results (see [12]).

(1) The closure of $\{f_{n'_k}\}$ in $\mathbb{R}^X$ is homeomorphic to $\beta\mathbb{N}$, and every accumulation point of $\{f_{n'_k}\}$ in $\mathbb{R}^X$ is not a Borel function.
(2) The sequence $(f_{n'_k})$ contains no pointwise convergent subsequence.

The properties of the tree $\mathcal{T}\big((f_n), a, b\big)$ can also be used to derive the following well-known dichotomy (see [12]).

Theorem 9. *Let (f_n) be a pointwise bounded sequence of continuous real-valued functions on X and let $\overline{\{f_n\}}^p$ be the closure of $\{f_n\}$ in $\mathbb{R}^X$. Then one of the following, mutually exclusive, alternatives holds true.*

(i) *We have $\overline{\{f_n\}}^p \subseteq \mathcal{B}_1(X)$.*
(ii) *$\beta\mathbb{N}$ is homeomorphic to a subset of $\overline{\{f_n\}}^p$.*

Proof. Consider the following (mutually exclusive) cases.

Case 1: *for every $a < b$ the tree $\mathcal{T}\big((f_n), a, b\big)$ is well-founded.* By Theorem 6, we see that the accumulation points of $\{f_n\}$ in $\mathbb{R}^X$ belong to $\mathcal{B}_1(X)$.

Case 2: *there exists $a < b$ such that the tree $\mathcal{T}\big((f_n), a, b\big)$ is not well-founded.* In this caase, by Proposition 8 and the above remark, we obtain that $\beta\mathbb{N}$ is homeomorphic to a subset of $\overline{\{f_n\}}^p$. □

3. The tree rank $o\big((f_n)\big)$ of a pointwise bounded sequence.

This section concerns the relation between $o\big((f_n)\big)$, defined below, and the separation rank $\alpha(f)$ when f is an accumulation point of $\{f_n\}$. The exact relation is given in Theorem 11 below.

Definition 10. *Let $f_n\colon X\to\mathbb{R}$ $(n\in\mathbb{N})$ be a pointwise bounded sequence of functions. We define the* tree rank $o\big((f_n)\big)$ *of the sequence (f_n) by setting*

$$o\big((f_n)\big) := \sup\Big\{o\big(\mathcal{T}((f_n),a,b)\big) : a<b\Big\}$$

if the tree $\mathcal{T}\big((f_n),a,b\big)$ is well-founded for every $a<b$. Otherwise, we set

$$o\big((f_n)\big) = (2^{\aleph_0})^+.$$

The basic properties of the tree rank are described in the following theorem.

Theorem 11. *Let X be a Polish space. Then for every sequence (f_n) in $\mathbb{R}^X$ and every $f\in\mathcal{B}_1(X)$ the following are satisfied.*

(1) *If $f\in Acc\big(\{f_n\}\big)$, then*

$$o\big((f_n)\big)+1 \geqslant \alpha(f).$$

(2) *If $f=\lim f_n$, then*

$$o\big((f_n)\big) \leqslant \omega\cdot 2\cdot\alpha(f).$$

The proof of Theorem 11 follows from a series of lemmas.

Notation. Let $f\colon X\to\mathbb{R}$ be a function, let Y be a closed subset of X, and let $a<b$. By $\mathcal{T}(Y,f,a,b)$ we denote the tree $\mathcal{T}(Y,(f_n),a,b)$ where $f_n=f$ for every n. If $Y=X$, then we write $\mathcal{T}(f,a,b)$ instead of $\mathcal{T}(X,f,a,b)$. Note that if f is a Baire-1 function, then the tree $\mathcal{T}(f,a,b)$ is well-founded.

Lemma 12. *Let $f\colon X\to\mathbb{R}$ be Baire-1 and $a<b$. Set $\mathcal{T}=\mathcal{T}(f,a,b)$, and let ξ be a countable ordinal. Let*

$$\mathcal{S}_\xi = \mathcal{T}\big(X^{(\xi)}_{(f,a,b)}\cap([f<a]\cup[f>b]),f,a,b\big).$$

Then we have $\mathcal{S}_\xi\subseteq\mathcal{T}^{(\xi)}$.

Proof. We proceed by induction on ξ. The case $\xi=0$ is straightforward. Let $\zeta<\omega_1$ and suppose that the lemma is true for every $\xi<\zeta$. Assume that $\zeta=\xi+1$ and let $(u,T)\in\mathcal{S}_{\xi+1}$. Then $T\subseteq X^{(\xi+1)}_{(f,a,b)}\cap([f<a]\cup[f>b])$. Write $T=(t_s)_{s\in D_k}$ for some $k\in\omega$. Then for every s with $|s|=k$ we have that $t_s\in X^{(\xi+1)}_{(f,a,b)}$, and either $f(t_s)<a$ or $f(t_s)>b$. If $f(t_s)<a$, then we select $y\in X^{(\xi)}_{(f,a,b)}$ with $f(y)>b$ and $d(t_s,y)\leqslant\frac{1}{2^{k+1}}$, and we set $t_{s^\frown 0}:=t_s$ and $t_{s^\frown 1}:=y$. If $f(t_s)>b$, then with similar arguments, we select $y'\in X^{(\xi)}_{(f,a,b)}\cap[f<a]$ with $d(t_s,y)\leqslant\frac{1}{2^{k+1}}$, and we set $t_{s^\frown 0}:=y'$ and $t_{s^\frown 1}:=t_s$. We set $T':=(t_s)_{s\in D_{k+1}}$ and $u':=u^\frown n$ where $n>\max(u)$. Then $(u,T)<(u',T')$ and $(u',T')\in\mathcal{S}_\xi$. By our inductive assumption, we have

$(u',T') \in \mathcal{T}^{(\xi)}$. Therefore, $(u,T) \in \mathcal{T}^{(\xi+1)}$ which proves the case of a successor ordinal.

Finally, if ζ is a limit ordinal, then note that $\mathcal{S}_\zeta \subseteq \mathcal{S}_\xi$ for every $\xi < \zeta$. Hence,

$$\mathcal{S}_\zeta \subseteq \bigcap_{\xi<\zeta} \mathcal{S}_\xi \subseteq \bigcap_{\xi<\zeta} \mathcal{T}^{(\xi)} = \mathcal{T}^{(\zeta)}$$

and the lemma is proved. □

Lemma 13. *Let $f\colon X \to \mathbb{R}$ Baire-1 and let $a < b$. Then we have*

$$\alpha(f,a,b) \leqslant o\big(\mathcal{T}(f,a,b)\big) + 1.$$

Proof. Let $\xi < \omega_1$ such that $\alpha(f,a,b) \geqslant \xi + 2$. Then notice that

$$X^{(\xi)}_{(f,a,b)} \cap ([f < a] \cup [f > b]) \neq \emptyset$$

which, using the notation of Lemma 12, yields that $\mathcal{S}_\xi \neq \emptyset$. By Lemma 12, we obtain that $\mathcal{T}^{(\xi)} \neq \emptyset$, and so $o\big(\mathcal{T}(f,a,b)\big) \geqslant \xi+1$. Hence, $\alpha(f,a,b) \leqslant o\big(\mathcal{T}(f,a,b)\big)+1$. □

Lemma 14. *Let $\mathcal{K} = \overline{\{f_n\}}^p$ be a quasi-Rosenthal compactum, let $a < b$ and let $f \in Acc(\mathcal{K})$. Then there exists a monotone map*

$$\varphi\colon \mathcal{T}(f,a,b) \to \mathcal{T}\big((f_n),a,b\big).$$

Consequently, we have $o\big(\mathcal{T}(f,a,b)\big) \leqslant o\big(\mathcal{T}((f_n),a,b)\big)$.

Proof. Let $\mathcal{F}_2$ be the set of all finite subsets of $X\cap([f < a]\cup[f > b])$ with cardinality greater than or equal to 2. For every $F \in \mathcal{F}_2$ we set

$$N_F \coloneqq \big\{n \in \mathbb{N} : \text{ for every } x \in F \text{ if } f(x) < a \text{ then } f_n(x) < a, \\ \text{while if } f(x) > b \text{ then } f_n(x) > b\big\}.$$

Note that N_F is infinite. For every $F \in \mathcal{F}_2$ if $\{n_1 < n_2 < \cdots\}$ is the increasing enumeration of N_F, then we set $n_F = n_{|F|}$. Observe that if $F_1 \subsetneq F_2$, then we have $n_{F_1} < n_{F_2}$.

Define the map $\varphi\colon \mathcal{T}(f,a,b) \to \mathcal{T}\big((f_n),a,b\big)$ as follows. Let $(u,T) \in \mathcal{T}(f,a,b)$ be arbitrary. If $u = \emptyset$, then we set $\varphi\big((\emptyset,t_\emptyset)\big) = (\emptyset,t_\emptyset)$. If $u = (n_1,\dots,n_k)$ and $T = (t_s)_{s\in D_k}$, then we set $\varphi\big((u,T)\big) = (u',T)$ where $u' = (n'_1,\dots,n'_k)$ and

$$n'_i = n_{\{t_s:|s|=i\}} \text{ for every } i \in \{1,\dots,k\}.$$

It is easy to see that φ is a well-defined monotone map. □

By Lemmas 13 and 14, we obtain the following corollary.

Corollary 15. *Let $\mathcal{K} = \overline{\{f_n\}}^p$ be a separable quasi-Rosenthal compactum. Then*

$$\sup\{\alpha(f) : f \in Acc(\mathcal{K})\} \leqslant o\big((f_n)\big) + 1.$$

Lemma 16. *Assume that (f_n) is pointwise convergent to $f\colon X \to \mathbb{R}$. Let $Y \subseteq X$ be nonempty, let $a < b$, set $\mathcal{S} = \mathcal{T}\big(Y,(f_n),a,b\big)$ and let $m \in \mathbb{N}$. Let $(u,T) \in \mathcal{S}^{(\omega+m)}$ and $x \in T$. Then, either*

(i) *$f(x) \leqslant a$ and there exists $y \in Y$ with $d(x,y) \leqslant \frac{1}{2^{|u|+m}}$ and $f(y) \geqslant b$, or*
(ii) *$f(x) \geqslant b$ and there exists $y \in Y$ with $d(x,y) \leqslant \frac{1}{2^{|u|+m}}$ and $f(y) \leqslant a$.*

Proof. By induction on m. For $m = 1$ let $(u,T) \in \mathcal{S}^{(\omega+1)}$ be arbitrary. There exists $(u',T') \in \mathcal{S}^{(\omega)}$ such that $(u,T) < (u',T')$ and $|u'| = |u| + 1$. Let $x \in T$ and set $T' = (t'_s)_{s \in D_{|u|+1}}$. There exists $s \in D_{|u|}$ such that $x = t'_s$. We may assume that $t'_{s^\frown 0} = t'_s = x$. (The case $t'_{s^\frown 1} = t'_s = x$ is similarly treated.) We set $t'_{s^\frown 1} := y$. Then $d(x,y) \leqslant \frac{1}{2^{|u|+1}}$. Since $(u',T') \in \mathcal{S}^{(\omega)}$, we see that $(u',T') \in \mathcal{S}^{(k)}$ for every $k \in \mathbb{N}$. Hence, for every $k \in \mathbb{N}$ we may select $(u_k, T_k) \in \mathcal{S}$ with $(u',T') < (u_k,T_k)$ and $|u_k| = |u'| + k$. Set $L := \bigcup_{k \in \mathbb{N}} u_k \setminus u'$. Clearly $L \in [\mathbb{N}]^\infty$. Moreover, observe that for every $l \in L$ we have $f_l(x) < a$ and $f_l(y) > b$. Since $f = \lim f_n$, we see that $f(x) \leqslant a$ and $f(y) \geqslant b$. The proof for the case $m = 1$ is completed.

Next, assume that the lemma is true for some $m \in \mathbb{N}$. Let $(u,T) \in \mathcal{S}^{(\omega+m+1)}$ and $x \in T$. We select $(u',T') \in \mathcal{S}^{(\omega+m)}$ with $(u,T) < (u',T')$ and $|u'| = |u| + 1$. Setting $T' := (t'_s)_{s \in D_{|u'|}}$, we see that $x \in T'$ and $x = t'_s$ for some $s \in D_{|u'|}$. By our inductive assumption, there exists $y \in Y$ such that

$$d(x,y) \leqslant \frac{1}{2^{|u'|+m}} = \frac{1}{2^{|u|+(m+1)}},$$

and either $f(x) \leqslant a$ and $f(y) \geqslant b$, or $f(x) \geqslant b$ and $f(y) \leqslant a$. The proof of the lemma is completed. □

Lemma 17. *Assume that (f_n) is pointwise convergent to $f\colon X \to \mathbb{R}$. Let $Y \subseteq X$ be closed, let $a < b$, and set $\mathcal{S} = \mathcal{T}\big(Y, (f_n), a, b\big)$. Then for every $1 \leqslant \xi < \omega_1$ and every $0 < \varepsilon < \frac{b-a}{2}$ we have*

$$\mathcal{S}^{(\omega \cdot 2 \cdot \xi)} \subseteq \mathcal{T}\big(Y^{(\xi)}_{(f, a+\varepsilon, b-\varepsilon)}, (f_n), a, b\big).$$

Proof. First we deal with the case $\xi = 1$. Let $(u,T) \in \mathcal{S}^{(\omega\cdot 2)}$ and $x \in T$. Let $m \in \mathbb{N}$ be arbitrary. As $(u,T) \in \mathcal{S}^{(\omega \cdot 2)} \subseteq \mathcal{S}^{(\omega+m)}$, by Lemma 16, either

(i) $f(x) \leqslant a < a + \varepsilon$ and there exists $y \in Y$ with $d(x,y) \leqslant \frac{1}{2^{|u|+m}}$ and $f(y) \geqslant b > b - \varepsilon$, or
(ii) $f(x) \geqslant b > b - \varepsilon$ and there exists $y \in Y$ with $d(x,y) \leqslant \frac{1}{2^{|u|+m}}$ and $f(y) \leqslant a < a + \varepsilon$.

Since m was arbitrary, we obtain that

$$x \in \overline{Y \cap [f < a + \varepsilon]} \cap \overline{Y \cap [f > b - \varepsilon]} = Y^{(1)}_{(f, a+\varepsilon, b-\varepsilon)}.$$

This shows that $T \subseteq Y^{(1)}_{(f, a+\varepsilon, b-\varepsilon)}$ and, in particular, that

$$\mathcal{S}^{(\omega\cdot 2)} \subseteq \mathcal{T}\big(Y^{(1)}_{(f, a+\varepsilon, b-\varepsilon)}, (f_n), a, b\big).$$

Next, we proceed by induction on ξ. Let $\zeta < \omega_1$ and assume that the lemma is true for every $\xi < \zeta$. If $\zeta = \xi + 1$, then, by our inductive assumption, we have

$$\mathcal{S}^{(\omega\cdot 2 \cdot (\xi+1))} = \big(\mathcal{S}^{(\omega \cdot 2 \cdot \xi)}\big)^{(\omega\cdot 2)} \subseteq \Big(\mathcal{T}\big(Y^{(\xi)}_{(f, a+\varepsilon, b-\varepsilon)}, (f_n), a, b\big)\Big)^{(\omega\cdot 2)}.$$

We set $Z = Y^{(\xi)}_{(f,a+\varepsilon,b-\varepsilon)}$. By the first case of our induction,

$$\Big(\mathcal{T}\big(Y^{(\xi)}_{(f,a+\varepsilon,b-\varepsilon)},(f_n),a,b\big)\Big)^{(\omega\cdot 2)} \subseteq \mathcal{T}\big(Z^{(1)}_{(f,a+\varepsilon,b-\varepsilon)},(f_n),a,b\big).$$

By the above inclusions, we conclude that

$$\mathcal{S}^{(\omega\cdot 2\cdot(\xi+1))} \subseteq \mathcal{T}\big(Y^{(\xi+1)}_{(f,a+\varepsilon,b-\varepsilon)},(f_n),a,b\big).$$

Now suppose ζ is a limit ordinal. Then, by our inductive assumption, we have

$$\mathcal{S}^{(\omega\cdot 2\cdot\zeta)} = \bigcap_{\xi<\zeta} \mathcal{S}^{(\omega\cdot 2\cdot\xi)} \subseteq \bigcap_{\xi<\zeta} \mathcal{T}\big(Y^{(\xi)}_{(f,a+\varepsilon,b-\varepsilon)},(f_n),a,b\big).$$

Note that

$$\bigcap_{\xi<\zeta} \mathcal{T}\big(Y^{(\xi)}_{(f,a+\varepsilon,b-\varepsilon)},(f_n),a,b\big) \subseteq \mathcal{T}\big(Y^{(\zeta)}_{(f,a+\varepsilon,b-\varepsilon)},(f_n),a,b\big).$$

Indeed, if $T \subseteq Y^{(\xi)}_{(f,a+\varepsilon,b-\varepsilon)}$ for every $\xi < \zeta$, then

$$T \subseteq \bigcap_{\xi<\zeta} Y^{(\xi)}_{(f,a+\varepsilon,b-\varepsilon)} = Y^{(\zeta)}_{(f,a+\varepsilon,b-\varepsilon)}.$$

The proof is completed. □

By Lemma 17, we obtain the following corollary.

Corollary 18. *Let (f_n) be a sequence of functions which is pointwise convergent to a Baire-1 function f. Then*

$$o\big((f_n)\big) \leqslant \omega \cdot 2 \cdot \alpha(f).$$

We are ready to complete the proof of Theorem 11.

Proof of Theorem 11. Follows immediately by Corollaries 18 and 15. □

We will give some consequences of Theorem 11. To this end we introduce the following definition.

Definition 19. *For every $\xi < \omega_1$ we set*

$$\mathcal{B}_1^{\xi}(X) \coloneqq \{f \in \mathcal{B}_1(X) : \alpha(f,a,b) < \omega^{\xi} \text{ for every } a < b\}.$$

Note that in the case of compact metrizable spaces, the above defined class coincides with the class of small Baire class ξ introduced by Kechris and Louveau [7].

Corollary 20. *Let (f_n) be a sequence of real-valued functions on X which is pointwise convergent to a Baire-1 function f. Then the following are satisfied.*

(1) *If $\alpha(f)$ is a limit ordinal, then $\alpha(f) \leqslant o\big((f_n)\big) \leqslant \omega \cdot 2 \cdot \alpha(f)$.*
(2) *If $\alpha(f) < \omega^{n+1}$ (respectively, $\alpha(f) \leqslant \omega^{n+1}$) with $n \in \omega$, then we have $o\big((f_n)\big) < \omega^{n+2}$ (respectively, $o\big((f_n)\big) \leqslant \omega^{n+2}$).*
(3) *If $\alpha(f) < \omega^{\xi}$ (respectively, $\alpha(f) \leqslant \omega^{\xi}$) with $\omega \leqslant \xi < \omega_1$, then $o\big((f_n)\big) < \omega^{\xi}$ (respectively, $o\big((f_n)\big) \leqslant \omega^{\xi}$).*

(4) *If* $o\big((f_n)\big) < \omega^\xi$, *then* $\alpha(f) < \omega^\xi$ *for every countable ordinal* ξ.
(5) *For every* $\omega \leqslant \xi < \omega_1$ *we have* $f \in \mathcal{B}_1^\xi(X)$ *if and only if* $o\big(\mathcal{T}((f_n), a, b)\big) < \omega^\xi$ *for every* $a < b$ *and every sequence* (f_n) *which is pointwise convergent to* f.

4. Borel separable quasi-Rosenthal compacta

The last section is devoted to the proof of the main result of the paper, and to the presentation of an example which shows that our results are the best possible. We have also included some open questions. We start with the following definition.

Definition 21. *A quasi-Rosenthal compactum* $\mathcal{K}$ *will be called* Borel separable *if it has a countable dense subset of Borel functions.*

In the following proposition, the well-known equivalence of (ii) and (iii) (see, e.g., [1]) is connected to well-founded tree structures.

Proposition 22. *Let* (f_n) *be a pointwise bounded sequence of Borel functions. Then the following are equivalent.*

(i) *There exists a finer Polish topology* τ' *on* X *with* $B(X) = B(X, \tau')$ *such that for every* $a < b$ *the tree* $\mathcal{T}\big(X', (f_n), a, b\big)$ *is well-founded.*
(ii) *For every* $L \in [\mathbb{N}]^\infty$ *there exists* $L' \in [L]^\infty$ *such that* $(f_n)_{n\in L'}$ *is pointwise convergent.*
(iii) *The closure of* $\{f_n\}$ *in* $\mathbb{R}^X$ *is a subset of* $\mathcal{B}(X)$.

Proof. Let τ' be a finer Polish topology on X with $B(X) = B(X, \tau')$ and such that f_n is τ'-continuous for every n (see [6]). The equivalence between (i) for τ' and (ii) is precisely the equivalence of (2) and (3) in Theorem 6. Similarly, the equivalence between (i) for τ' and (iii) is the equivalence of (1) and (2) in Theorem 6. □

Remark 5. Related to the above proposition the following question is open for us. If (f_n) is a pointwise bounded sequence of functions such that $Acc\big(\{f_n\}\big) \subseteq \mathcal{B}(X)$, then does this imply that $\overline{\{f_n\}}^p$ is homeomorphic to a quasi-Rosenthal compactum? Let us observe that the stronger question of the existence of a finer Polish topology τ' on X such that $\overline{\{f_n\}}^p$ becomes a quasi-Rosenthal compactum, has a negative answer. Indeed, consider the variant of Example 1 where $\{\Delta_\sigma : \sigma \in \mathcal{C}\} = B(\mathcal{C})$. Then, as in Example 1, every $f \in Acc\big(\{f_s\}_{s\in D}\big)$ is a Borel function. As it is well-known, every finer Polish topology on X has the same Borel sets. Therefore, assuming that for a finer Polish topology τ' each $\mathbf{1}_{\Delta_\sigma}$ is a Baire-1 function, we conclude that each Δ_σ belongs to $B_\xi(\mathcal{C})$ where $\xi = \sup\{\zeta_n < \omega_1 : V_n \in B_{\zeta_n}(\mathcal{C})\} + 2$ and (V_n) is a basis for τ'. This yields a contradiction.

The following theorem is the main result of this section.

Theorem 23. *Let* X *be a Polish space and let* $\mathcal{K}$ *be a Borel separable quasi-Rosenthal compact of* $\mathbb{R}^X$. *Then we have*

$$\sup\{\alpha(f) : f \in \mathcal{K} \cap \mathcal{B}_1(X)\} < \omega_1.$$

Clearly, Theorem 23 answers in the affirmative Bourgain's question stated in [3]. We postpone its proof in order to give an example which shows that our theorem is sharp.

Example 2. Let $X = \mathcal{C} \times \mathcal{C}$ where $\mathcal{C} = \{0,1\}^{\mathbb{N}}$ is the Cantor set. Let $A \subseteq \mathcal{C} \times \mathcal{C}$ be a $\mathcal{C}$-universal set for the class of $\mathcal{F}_\sigma$ subsets of $\mathcal{C}$ (see [6]). That is, A is $\mathcal{F}_\sigma$ in $\mathcal{C} \times \mathcal{C}$, for every $\sigma \in \mathcal{C}$ the section $A_\sigma = \{x \in \mathcal{C} : (\sigma, x) \in A\}$ of A is $\mathcal{F}_\sigma$, and for every $\mathcal{F}_\sigma$ subset F of $\mathcal{C}$ there exists $\sigma \in \mathcal{C}$ such that $F = A_\sigma$. Next, set $\Pi \coloneqq \{\sigma \in \mathcal{C} : A_\sigma \text{ is } \mathcal{G}_\delta\}$. Then Π is a co-analytic subset of $\mathcal{C}$. To see this, observe that $\Pi = \{\sigma \in \mathcal{C} : (X \setminus A)_\sigma \text{ is } \mathcal{F}_\sigma\}$. Since $X \setminus A$ is Borel in $\mathcal{C} \times \mathcal{C}$, by a classical theorem of Hurewicz (see [6, page 297]), we see that Π is co-analytic. For every $s \in D$ we set, as usual, $\mathcal{C}_s \coloneqq \{\sigma \in \mathcal{C} : s \sqsubset \sigma\}$. (Clearly, $\mathcal{C}_s$ is open in $\mathcal{C}$.) Now let $A_s = A \cap \big((\Pi \cap \mathcal{C}_s) \times \mathcal{C}\big)$. Then A_s is co-analytic in X. Define $f_s = \mathbf{1}_{A_s}$ and set $\mathcal{K} \coloneqq \overline{\{f_s\}}^p_{s \in D}$. As in Example 1, it is easily verified that $\mathcal{K}$ is a separable quasi-Rosenthal compactum. Moreover, note that for every $\Delta \subseteq \mathcal{C}$ which is $\mathcal{F}_\sigma$ and $\mathcal{G}_\delta$ there exists $\sigma \in \Pi$ such that $A_\sigma = \Delta$. It follows that for every such $\Delta \subseteq \mathcal{C}$ there exists $\sigma \in \mathcal{C}$ such that, setting $\Delta_\sigma \coloneqq \{\sigma\} \times \Delta$, we have that $\mathbf{1}_{\Delta_\sigma} \in \mathcal{K}$. Since for every countable ordinal ξ there exists $\Delta \subseteq \mathcal{C}$ such that $\mathbf{1}_\Delta$ is Baire-1 (that is, Δ is $\mathcal{F}_\sigma$ and $\mathcal{G}_\delta$) and $\alpha(\mathbf{1}_\Delta) \geqslant \xi$, we obtain that

$$\sup\{\alpha(f) : f \in Acc(\mathcal{K})\} = \omega_1.$$

Note that by taking complements we obtain an example of a separable quasi-Rosenthal compactum having a dense subset consisting of characteristic functions of analytic sets and for which Theorem 23 is not valid.

Lemma 24. *Let $f_n \colon X \to \mathbb{R}$ $(n \in \mathbb{N})$ be a sequence of Borel functions and $a < b$. Assume that the tree $\mathcal{T} = \mathcal{T}\big((f_n), a, b\big)$ is well-founded. Then we have $o(\mathcal{T}) < \omega_1$.*

Proof. We enlarge the original topology of X to a Polish topology τ' in order to make the sequences of sets $\big([f_n < a]\big)$ and $\big([f_n > b]\big)$ τ'-clopen (see [6]). Then the tree $\mathcal{T}$ is a subset of the Polish space

$$Y = \bigoplus_{k \in \omega} [\mathbb{N}]^k \times (X, \tau')^{D_k}.$$

We claim that $\mathcal{T}$ is closed in Y. Indeed, let $\big((u_i, T_i)\big)$ be a sequence such that $u_i \to u$, $T_i \to T$ and $(u_i, T_i) \in \mathcal{T}$ for every i. There exists i_0 such that $u = u_i$ for every $i \geqslant i_0$. Let $k = |u|$. Then $T_i \in (X, \tau')^{D_k}$ for every $i \geqslant i_0$, and so $T \in (X, \tau')^{D_k}$. For every $i \geqslant i_0$ set $T_i \coloneqq (t^i_s)_{s \in D_k}$ and $T \coloneqq (t_s)_{s \in D_k}$, and note that $t^i_s \to t_s$ in τ' for every $s \in D_k$. It is clear that if $k = 0$, then $(u, T) \in \mathcal{T}$.

So, assume that $k \geqslant 1$ and write $u = (n_1, \dots, n_k)$. We will verify that (u, T) satisfies conditions (C1) and (C2) of the definition of $\mathcal{T}$. Since for every $i \geqslant i_0$ we have that either $t^i_0 = t^i_\emptyset$ or $t^i_1 = t^i_\emptyset$, there exists $I \in [\mathbb{N}]^\infty$ such that either $t^i_0 = t^i_\emptyset$ for every $i \in I$ or $t^i_1 = t^i_\emptyset$ for every $i \in I$. Hence, either $t_0 = t_\emptyset$ or $t_1 = t_\emptyset$. Therefore, condition (C1) is satisfied.

To verify condition (C2), let $s \in D_k$ with $|s| < k$. As τ' is finer than the original topology, we have that $t_s^i \to t_s$ in the original topology. Hence, $d(t_{s^\frown 0}, t_{s^\frown 1}) \leqslant \frac{1}{2^{|s|+1}}$ and so (C2.i) is clear. Moreover, since the sets $[f_{n_i} < a]$ and $[f_{n_i} > b]$ are τ'-closed, we obtain that $f_{n_{|s|+1}}(t_{s^\frown 0}) < a$ and $f_{n_{|s|+1}}(t_{s^\frown 1}) > b$; that is, condition (C2.ii) is satisfied. Finally, in order to show that condition (C2.iii) is satisfied, assume that $s \neq \emptyset$ and $f_{n_{|s|}}(t_s) < a$. Since $t_s^i \to t_s$ and $[f_{n_{|s|}} < a]$ is τ'-open, we see that there exists $i_s \geqslant i_0$ such that $f_{n_{|s|}}(t_s^i) < a$ for every $i \geqslant i_s$. Therefore, $t^i_{s^\frown 0} = t_s^i$ for every $i \geqslant i_s$ which implies that $t_{s^\frown 0} = t_s$. The case $f_{n_{|s|}}(t_s) > b$ is similarly treated. This completes the proof that $\mathcal{T}$ is closed in Y.

Now define the relation $\prec$ on Y by setting

$$(u', T') \prec (u, T) \ \text{ if } \ (u, T), (u', T') \in \mathcal{T} \text{ and } (u, T) < (u', T').$$

Clearly, $\prec$ is a strict well-founded relation on Y and $o(\mathcal{T}) = \rho(\prec)$. We will show that $\prec$, as a subset of $Y \times Y$, is closed. Indeed, let $\big((u_i', T_i')\big)$, $(u', T') \in Y$, $\big((u_i, T_i)\big)$ and $(u, T) \in Y$ be such that $(u_i', T_i') \to (u', T')$, $(u_i, T_i) \to (u, T)$ and $(u_i', T_i') \prec (u_i, T_i)$ for every i. Since $\mathcal{T}$ is a closed subset of Y, we see that $(u', T'), (u, T) \in \mathcal{T}$. Moreover, as $u_i \sqsubset u_i'$ and $T_i \lhd T_i'$, we obtain that $u \sqsubset u'$ and $T \lhd T'$, that is, $(u', T') \prec (u, T)$. Therefore, $\prec$ is a closed relation.

Finally, by the Kunen–Martin theorem, we conclude that $o(\mathcal{T}) = \rho(\prec) < \omega_1$ and the proof is completed. □

Corollary 25. *Let $\mathcal{K} = \overline{\{f_n\}}^p$ be a Borel separable quasi-Rosenthal compactum. Then $o\big((f_n)\big) < \omega_1$.*

Proof. Let $a < b$ and consider the tree $\mathcal{T}\big((f_n), a, b\big)$. By Theorem 6, the tree $\mathcal{T}\big((f_n), a, b\big)$ is well-founded. By Lemma 24, we obtain that $o\big(\mathcal{T}((f_n), a, b)\big) < \omega_1$. Finally, note that

$$\sup\Big\{o\big(\mathcal{T}((f_n), a, b)\big) : a < b\Big\} = \sup\Big\{o\big(\mathcal{T}((f_n), a, b)\big) : a < b \text{ rationals}\Big\}.$$

Therefore, $o\big((f_n)\big) < \omega_1$ as desired. □

Proof of Theorem 23. By Corollaries 15 and 25, it follows that

$$\sup\{\alpha(f) : f \in Acc(\mathcal{K})\} < \omega_1.$$

Since the isolated points of $\mathcal{K}$ in $\mathcal{B}_1(X)$ are at most countable, the result follows. □

Remark 6. We conclude this paper with some open problems.

(1) Even for separable Rosenthal compacta $\mathcal{K}$, it is unclear to us whether there exists an equivalence between the quantities $o\big((f_n)\big)$ and $\sup\{\alpha(f) : f \in \mathcal{K}\}$. However, the proper setting of this problem seems to be for the class of Borel separable quasi-Rosenthal compacta. We notice that in the case where the sequence (f_n) has finitely many accumulation points, then easy modifications of the proof of part (2) of Theorem 11 yield a positive answer.

(2) For an arbitrary sequence (f_n) of functions with $\mathcal{T}\big((f_n), a, b\big)$ well-founded for every $a < b$, we do not know if $\big(2^{\aleph_0}\big)^+$ is the best upper bound for $o\big((f_n)\big)$.

(3) As we have shown in Proposition 7, every quasi-Rosenthal compactum $\mathcal{K}$ has countable tightness. It remains open whether every accumulation point of a subset $\mathcal{L}$ of $\mathcal{K}$ is the limit of a sequence in $\mathcal{L}$. For quasi-Rosenthal compacta homeomorphic to a Rosenthal compactum this is a consequence of the Bourgain–Fremlin–Talagrand theorem [4].

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