ON CERTAIN REGULARITY PROPERTIES OF HAAR-NULL SETS

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Abstract. Let X be an abelian Polish group. For every analytic Haar-null set $A \subseteq X$ let $T(A)$ denote the set of test measures of A. We show that $T(A)$ is always dense and co-analytic in $P(X)$. We prove that if A is compact then $T(A)$ is G_δ dense, while if A is non-meager then $T(A)$ is meager. We also strengthen a result of Solecki and we show that for every analytic Haar-null set A, there exists a Borel Haar-null set $B \supseteq A$ such that $T(A) \setminus T(B)$ is meager. Finally, under Martin's Axiom and the negation of Continuum Hypothesis, some results concerning co-analytic sets are derived.

1. Introduction and auxiliary lemmas

A universally measurable subset A of an abelian Polish group X is called *Haar-null* if there exists a probability Borel measure μ on X, called a *test measure* of A, such that $\mu(x + A) = 0$ for every $x \in X$. This definition is due to Christensen [C] and extends the usual notion of a Haar-measure zero set. The complement of a Haar-null set is called *prevalent*. The class of Haar-null sets has been also considered by Hunt, Sauer and Yorke in [HSY]. They used that term *shy* instead of Haar-null.

If X is locally-compact, then a universally measurable set A is Haar-null if and only if there exists a Borel Haar-null set $B \supseteq A$. In the non-locally compact case the situation is different. As Dougherty observed in [D], answering a problem of Mycielski (see [My]), assuming Continuum Hypothesis (or just Martin's Axiom) there exists a universally measurable set A which is not contained in any Borel Haar-null set. However, Solecki [S1, S2] proved that if A is an analytic Haar-null set, then this can be done.

In this paper we are concerned with the properties of the set of test measures $T(A)$ of an analytic Haar-null set A. We show that this set is always dense and co-analytic in $P(X)$ (by $P(X)$ we denote the space of all probability Borel measures on X endowed with the weak topology). If A is not meager and Haar-null, then we prove that $T(A)$ is meager in $P(X)$. On the other hand we show that compact sets are tested by co-meager many measures.

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We also strengthen the above-mentioned result of Solecki. We prove that for every analytic Haar-null set A, there exists a Borel Haar-null set $B \supseteq A$ such that $T(A) \setminus T(B)$ is meager in $P(X)$. That is, almost every test measure for A is a test measure for B. Actually, we prove a general theorem which permits us to derive, besides the just mentioned result, an approximation type result for analytic sets. The proof of this theorem is combinatorial in nature.

Under stronger set-theoretic assumptions, in particular under Martin's Axiom and the negation of Continuum Hypothesis, we show that $\mathbf{\Pi}^1_2$ sets still behave well at least with respect to translations over co-meager sets. Again, under Martin's Axiom and the negation of Continuum Hypothesis, we show that for every analytic or co-analytic set $A \subseteq X$ there exists a Borel set $B \subseteq A$ such that $T(B) \setminus T(A)$ is meager in $P(X)$. Finally, an application of these ideas is given in the last section, in the theory of essentially smooth Lipschitz functions.

1.1. **Notation.** In what follows, for any Polish space X by $P(X)$ we denote (as we have already mentioned) the space of all Borel probability measures on X. The set $P(X)$ equipped with the weak topology is a Polish space (see [Ke, page 112]). If d is a compatible complete metric for X, then a compatible complete metric for $P(X)$ is the so-called *Lévy metric* ϱ, defined by the rule

$$\varrho(\mu,\nu) = \inf\{\delta \geqslant 0 : \mu(A) \leqslant \nu(A_\delta) + \delta \text{ and } \nu(A) \leqslant \mu(A_\delta) + \delta\}$$

where $A_\delta = \{x \in X : d(x,A) \leqslant \delta\}$. For every $\mu \in P(X)$ by $\mathrm{supp}(\mu)$ we denote the support of μ. All the other pieces of notation we use are standard (for more information we refer to [Ke]).

1.2. **Some auxiliary lemmas.** Let X be an abelian Polish group. For every $\mu \in P(X)$ and every $x \in X$ let $\mu_x \in P(X)$ denote the measure defined by setting $\mu_x(A) = \mu(x + A)$ for every Borel subset A of X.

Lemma 1. *The following hold.*

(i) *For every $\mu \in P(X)$ the function $x \mapsto \mu_x$ is continuous.*
(ii) *The function $(\mu, x) \mapsto \mu_x$ is (jointly) continuous.*

Proof. Part (i) follows from Lebesgue's dominated convergence theorem; part (ii) follows from part (i) and [Ke, Theorem 9.14]. □

Lemma 2. *Let $\varepsilon \geqslant 0$. Then the following hold.*

(i) *If $F \subseteq X$ is closed, then the set $\{(\mu, x) \in P(X) \times X : \mu(x + F) \geqslant \varepsilon\}$ is closed in $P(X) \times X$.*
(ii) *If $B \subseteq X$ is Borel and $\mu \in P(X)$, then the set $\{x \in X : \mu(x + B) > \varepsilon\}$ is Borel in X.*

Proof. (i) Recall that if $F \subseteq X$ is closed, then the function $\mu \mapsto \mu(F)$ is upper semicontinuous (see [Ke, page 111]). As the composition of a continuous function with an upper semicontinuous real-valued function is upper semicontinuous, the result follows by part (ii) of Lemma 1.

(ii) We recall that if $B \subseteq X$ is Borel, then the function $\mu \mapsto \mu(B)$ is Borel measurable (see [Ke, page 112]). Hence, the result follows by part (i) of Lemma 1. $\square$

2. The set of test measures

In this section X will be an uncountable abelian Polish group (locally or non-locally compact). For every universally measurable set $A \subseteq X$ we set

$$T(A) := \{\mu \in P(X) : \mu(x + A) = 0 \text{ for every } x \in X\}.$$

That is, $T(A)$ is the set of all test measures of A. Clearly, A is Haar-null if and only if $T(A) \neq \emptyset$. The following simple fact will be useful in what follows. The proof is straightforward.

Lemma 3. *Let (A_n) be a sequence of universally measurable subsets of X. Then we have $T(A) = \bigcap_n T(A_n)$ where $A = \bigcup_n A_n$.*

We have the following estimate for the complexity of the set of test measures of an analytic set.

Lemma 4. *If $A \subseteq X$ is analytic, then $T(A)$ is a co-analytic subset of $P(X)$.*

Proof. By [Ke, Theorem 29.26], the set $\{(\mu, x) \in P(X) \times X : \mu(x + A) = 0\}$ is $\mathbf{\Pi}^1_1$ and the lemma follows. $\square$

Remark 1. If A is a co-analytic subset of X, then it is also easily verified that the set $T(A)$ is $\mathbf{\Pi}^1_2$.

In the following proposition we state some properties of the set of test measures.

Proposition 5. *Let $A \subseteq X$ be a universally measurable Haar-null set. Then the following hold.*

(i) *$T(A)$ is always dense in $P(X)$.*
(ii) *If A is analytic and non-meager, then $T(A)$ is meager.*
(iii) *If A is σ-compact, then $T(A)$ is co-meager.*

Proof. (i) Fix a compatible complete metric d for X. For every $x \in X$ and every $r \geqslant 0$ we set $B(x, r) := \{y \in X : d(x, y) \leqslant r\}$. Recall that if D is a countable dense subset of X, then the set of all convex combinations of Dirac measures $(\delta_x)_{x \in D}$ is dense in $P(X)$ (see [Ke, page 110]). So, let $\nu = \sum_{i=1}^{l} a_i \delta_{x_i}$ and $r > 0$ where $x_i \in D$ and $\sum_{i=1}^{l} = 1$ with $a_i > 0$. We will find $\mu \in T(A)$ with $\varrho(\mu, \nu) < r$. As r and ν were arbitrary, this will finish the proof. Let $r' > 0$ be small enough so that $r' < r$ and $B(x_i, r') \cap B(x_j, r') = \emptyset$ for every $i, j \in \{1, \dots, l\}$ with $i \neq j$. It is easy to verify that for every $i \in \{1, \dots, l\}$ there exists a measure $\mu_i \in T(A)$ such that $x_i \in \mathrm{supp}(\mu_i)$ and $\mathrm{supp}(\mu_i) \subseteq B(x_i, r')$. We set $\mu := \sum_{i=1}^{l} a_i \mu_i$. Then $\varrho(\nu, \mu) < r$ and, clearly, $\mu \in T(A)$.

(ii) Let (x_n) be a countable dense subset of X and set $B \coloneqq \bigcup_n (x_n + A)$. Then B is co-meager in X and, moreover, $T(A) = T(B)$. We select a dense G_δ subset G of B. Note that $T(A) \subseteq P(X) \setminus \{\mu \in P(X) : \mu(G) = 1\}$. Since $\{\mu \in P(X) : \mu(G) = 1\}$ is co-meager (see, e.g., [Ke]), the result follows.

(iii) By Lemma 3, it is enough to show that if $K \subseteq X$ is compact, then $T(K)$ is co-meager. We already know from part (i) that $T(K)$ is dense in $P(X)$. We will show that it is also G_δ. Set

$$A \coloneqq \{\mu \in P(X) : \exists x \in X \text{ such that } \mu(x + K) > 0\}.$$

Also, for every $m \geqslant 1$ set

$$A_m \coloneqq \{\mu \in P(X) : \exists x \in X \text{ such that } \mu(x + K) > 1/m\}.$$

Clearly $A = \bigcup_m A_m$. Then observe that $T(K) = P(X) \setminus A$. We will show that for every m we have $\overline{A}_m \subseteq A$. This implies that $A = \bigcup_m \overline{A}_m$ and, in particular, that $T(K)$ is G_δ.

Fix m and let $(\mu_n) \subseteq A_m$ with $\mu_n \to \mu$ in $P(X)$. We will show that $\mu \in A$. Since $\mu_n \in A_m$, for every n there exists $x_n \in X$ such that $\mu_n(x_n + K) > 1/m$. We set $D \coloneqq \{x_n : n \in \mathbb{N}\}$. We distinguish two cases.

First assume that $\overline{D} = D$. In this case D is countable and closed. We set $F \coloneqq \bigcup_n (x_n + K)$. Then, as K is compact, F is a closed subset of X. Also note that for every n we have $\mu_n(F) \geqslant \mu_n(x_n + K) > 1/m$. As $\mu_n \to \mu$ and F is closed, we obtain that

$$\mu(F) = \limsup \mu_n(F) \geqslant 1/m > 0.$$

From the fact that F is the countable union of the sets $x_n + K$, we conclude that there exists n such that $\mu(x_n + K) > 0$ which implies that $\mu \in A$.

Now assume that $\overline{D} \supset D$. In this case we select $x \in \overline{D}$ and a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to x$. Set $L \coloneqq \{x\} \cup \{x_{n_k} : k \geqslant 1\}$ and $F \coloneqq \bigcup_{x \in L} (x + K)$. Then F is closed (in fact compact). Note that the corresponding subsequence (μ_{n_k}) of (μ_n) still converges to μ. Arguing as before, we conclude that $\mu \in A$ and the proof is completed. $\square$

Remark 2. (a) The fact that $T(A)$ is always dense in $P(X)$ has been prove implicitly by Christensen (see, e.g., [BL, C]).

(b) Preiss and Tišer proved the existence of non-meager Haar-null sets in separable Banach spaces with many remarkable additional properties (see [PT, BL]).

(c) Note that part (iii) of Proposition 5 immediately implies that the set of all non-atomic Borel probability measures of X is co-meager in $P(X)$.

(d) We should point out that the converse implication of part (iii) of Proposition 5 does not hold. For instance, every proper closed subspace of a separable Hilbert space is a Haar-null set (see, e.g., [BL, page 130]) for which the set of test measures is co-meager. As a matter of fact it can be shown that for every separable reflexive

Banach space X and every weakly closed Haar-null set $F \subseteq X$, the set $T(F)$ is co-meager in $P(X)$.

3. Covering analytic Haar-null sets

In [S1], Solecki proved that if A is an analytic Haar-null set and $\mu \in P(X)$ is a test measure of A, then there exists a Borel Haar-null set $B \supseteq A$ also tested by μ. The aim of this section is to show that the Borel set B may be chosen so that $T(A) \setminus T(B)$ is meager in $P(X)$. In fact we are going to prove the following theorem.

Theorem 6. *Let X be an abelian Polish group, let $r \geqslant 0$, let $A \in \mathbf{\Sigma}^1_1(X)$ and let $Z \subseteq P(X) \times X$ be Borel such that $\mu(x + A) \leqslant r$ for every $(\mu, x) \in Z$. Then there exists a Borel set $B \supseteq A$ such that $\mu(x + B) \leqslant r$ for every $(\mu, x) \in Z$.*

Theorem 6 may be proved using the arguments of Solecki (he used reflection). We will give a different proof of this fact which is combinatorial in nature.

Let $Z \subseteq P(X) \times X$ be as in Theorem 6. Enlarge the topology on $P(X) \times X$ to make Z clopen (see [Ke, page 82]). Denote this topology by τ. Then Z equipped with the τ topology becomes a Polish space. Fix a compatible complete metric d for (Z, τ). Also let δ be a compatible complete metric for the original topology of X. We introduce the following definition which is crucial for the proof.

Definition 7. *Let $A \subseteq X$, let $\varepsilon \geqslant 0$ and let $F \subseteq Z$. We say that A is (ε, F)*-covered *if there is a Borel set $B \supseteq A$ such that $\mu(x + B) \leqslant \varepsilon$ for every $(\mu, x) \in F$.*

We will need the following two simple lemmas concerning (ε, F)-covered sets. The first one is straightforward.

Lemma 8. *Let $A \subseteq X$, let $r \geqslant 0$ and let $F \subseteq Z$. Then A is (r, F)-covered if and only if A is (ε, F)-covered for every $\varepsilon > r$.*

Lemma 9. *Let $\varepsilon \geqslant 0$, let $A \subseteq X$ and let (A_m) be a sequence of subsets of X such that $A_m \uparrow A$. Also let $F \subseteq Z$ and let (F_n) be a sequence of arbitrary subsets of Z such that $F = \bigcup_n F_n$. Then the following are equivalent.*

(i) *The set A is (ε, F)-covered.*
(ii) *For every n, m the set A_m is (ε, F_n)-covered.*

Proof. Clearly only the implication (ii)$\Rightarrow$(i) needs to be proved. So assume that for every n, m the set A_m is (ε, F_n)-covered. We select $B_{n,m} \supseteq A_m$ Borel witnessing this fact. Thus $\mu(x + B_{n,m}) \leqslant \varepsilon$ for every $(\mu, x) \in F_n$. We set $C_m \coloneqq \bigcap_n \bigcap_{k \geqslant m} B_{n,k}$. Then C_m is Borel and $C_m \supseteq A_m$ for every m, since $B_{n,k} \supseteq A_k \supseteq A_m$ for every n and every $k \geqslant m$. In addition, we have $\mu(x + C_m) \leqslant \varepsilon$ for every $(\mu, x) \in F$. As the sequence (C_m) is increasing, setting $C \coloneqq \bigcup_m C_m$, we see that C is Borel, $C \supseteq A$ and $\mu(x + C) \leqslant \varepsilon$ for every $(\mu, x) \in F$ (the last inequality holds from the continuity of the measure). This implies that A is (ε, F)-covered and the proof is completed. $\square$

For every $K \subseteq X$ closed (in the original topology of X) and every $i \geqslant 1$ we set $K_i \coloneqq \{x \in X : \delta(x, K) \leqslant 1/i\}$ (recall that δ is the fixed compatible complete metric for X). Note that $K = \bigcap_i K_i$ and $K \supseteq \mathrm{Int}(K_i)$ for every i (again the interior is taken in the original topology of X).

Lemma 10. *Let (F_i) be a decreasing sequence of τ-closed subsets of Z such that $d - \mathrm{diam}(F_i) \to 0$. Also let $K \subseteq X$ be closed (in the original topology) and (r, Z)-covered. Then for every $\varepsilon > r$ there exists i such that $F_i \subseteq \{(\mu, x) : \mu(x + K_i) < \varepsilon\}$.*

Proof. Suppose not. Then there exists $\varepsilon > r$ such that for every i we have that $F_i \nsubseteq \{(\mu, x) : \mu(x + K_i) < \varepsilon\}$ or equivalently $F_i \cap \{(\mu, x) : \mu(x + K_i) \geqslant \varepsilon\} \neq \emptyset$. We set $L_i \coloneqq F_i \cap \{(\mu, x) : \mu(x + K_i) \geqslant \varepsilon\}$. By part (i) of Lemma 2, L_i is a τ-closed subset of Z (recall that the τ topology on Z is larger that the original topology). Moreover, the sequence (L_i) is decreasing, as the sequences (F_i) and (K_i) are decreasing, and clearly $d - \mathrm{diam}(L_i) \to 0$. Therefore, we have $\bigcap_i L_i = \{(\nu, y)\} \in Z$. But then $\nu(y + K) = \lim \nu(x + K_i) \geqslant \varepsilon > r$ contradicting the fact that K is (r, Z)-covered. □

We will also need the following representation of analytic sets. For a proof we refer to [Ke, page 200].

Theorem 11. *Let X be a Polish space and let A be an analytic subset of X. Then there exists a regular Suslin scheme $(P_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ with $A = \mathcal{A}_s P_s$ such that the following hold.*

- (i) *P_s is analytic.*
- (ii) *$P_\emptyset = A$, $P_s = \bigcup_n P_{s^\frown n}$ and, moreover, $P_{s^\frown n} \subseteq P_{s^\frown m}$ if $n \leqslant m$.*
- (iii) *For every $y \in \mathcal{N}$ the set $P_y = \bigcap_n P_{y|n}$ is compact.*
- (iv) *If $U \subseteq X$ is open such that $P_y \subseteq U$, then there exists n such that $P_{y|m} \subseteq U$ for every $n > m$.*

We are ready to complete the proof of Theorem 6.

Proof of Theorem 6. Assume that the conclusion of the theorem fails. So, according to our terminology, the set A is not (r, Z)-covered. By Lemma 8, there exists $\varepsilon > r$ such that A is not (ε, Z)-covered. We set $Z = \bigcup_n F_n^1$ where each F_n^1 is τ-closed and $d - \mathrm{diam}(F_n^1) \leqslant 1/2$. Let $(P_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ be the regular Suslin scheme obtained by Theorem 11 when applied to the set A. Then $A = P_\emptyset = \bigcup_m P_m$ with $P_m \uparrow A$. Since A is not (ε, Z)-covered, by Lemma 9, there exist n_1 and m_1 such that P_{m_1} is not $(\varepsilon, F_{n_1}^1)$-covered.

Continuing in this way, we define (recursively) $y \in \mathcal{N}$ and a decreasing sequence $Z \supseteq F_{n_1}^1 \supseteq \cdots \supseteq F_{n_i}^i \supseteq \cdots$ of τ-closed sets such that for every i we have that

- (i) $P_{y|i}$ is not $(\varepsilon, F_{n_i}^i)$-covered, and
- (ii) $d - \mathrm{diam}(F_{n_i}^i) \to 0$.

We set $K \coloneqq \bigcap_i P_{y|i}$ which is a compact (hence closed) subset of A. So, K is (r, Z)-covered. Applying Lemma 10 to the sequence $(F^i_{n_i})$ and K, there exists i such that $F^i_{n_i} \subseteq \{(\mu, x) : \mu(x + K_i) < \varepsilon\}$. Since $K \subseteq \mathrm{Int}(K_i)$, by the adopted representation, for this particular i there is n such that $P_{y|m} \subseteq K_i$ for every $m \geqslant n$. We set $l \coloneqq \max\{n, i\}$. Then observe that $P_{y|l} \subseteq P_{y|n} \subseteq \mathrm{Int}(K_i)$ and, moreover, $F^l_{n_l} \subseteq F^i_{n_i} \subseteq \{(\mu, x) : \mu(x + K_i) < \varepsilon\}$. This implies that $P_{y|l}$ is $(\varepsilon, F^l_{n_l})$-covered, and we derive a contradiction. $\square$

By Theorem 6, we obtain the following corollary.

Corollary 12. *Let $A \subseteq X$ be an analytic Haar-null set. Then there exists a Borel Haar-null set $B \supseteq A$ such that $T(A) \setminus T(B)$ is meager in $P(X)$.*

Proof. By Lemma 4, $T(A)$ is co-analytic in $P(X)$, and so it has the Baire property. We select $L \subseteq T(A)$ Borel such that $T(A) \setminus L$ is meager in $P(X)$. We set $Z \coloneqq L \times X$. Then Z is Borel and, clearly, $\mu(x + A) = 0$ for every $(\mu, x) \in Z$. By Theorem 6, we obtain a Borel set $B \supseteq A$ such that $\mu(x + B) = 0$ for every $(\mu, x) \in Z$. Clearly, the set B is as desired. $\square$

Let us point out that the Borel set obtained by Corollary 12 need not be a G_δ set even if the group X is locally compact. To see this, it is enough to consider an analytic Haar-null set A such that A is dense in X and $T(A)$ is co-meager (by part (iii) of Proposition 5, such sets exist). Now note that if B is a G_δ Haar-null set with $B \supseteq A$, then B must be co-meager and so, by part (ii) of Proposition 5, the set $T(B)$ is meager. Hence, B does not satisfy the conclusion of Corollary 12.

Using Theorem 6 we will also derive an approximation type result for analytic sets. We recall that by a result of Solecki (see [S1, page 208]) in every non-locally compact abelian Polish group there exists an analytic set which cannot be approximated up to Haar-null sets by Borel or even co-analytic sets.

Proposition 13. *Let $A \subseteq X$ analytic and $\mu \in P(X)$. Then there exist $B \supseteq A$ Borel and $G \subseteq X$ co-meager such that $\mu(x + A) = \mu(x + B)$ for every $x \in G$.*

Proof. Let n be a positive integer. For every $k \in \{0, \dots, n + 1\}$ we set

$$F^n_k \coloneqq \Big\{x \in X : \frac{k}{n+1} \leqslant \mu(x + A) \leqslant \frac{k+1}{n+1}\Big\}.$$

Then observe that $X = \bigcup_{k=0}^{n+!} F^n_k$. Moreover, it is easy to verify that F^n_k is the intersection of an analytic set and a co-analytic set. Therefore, each F^n_k has the Baire property. We select $G^n_k \supseteq F^n_k$ Borel such that $F^n_k \setminus G^n_k$ is meager. We set $Z^n_k = \{\mu\} \times G^n_k$. Then for every $k \in \{0, \dots, n+1\}$ the set Z^n_k is Borel in $P(X) \times X$ and $\mu(x + A) \leqslant (k + 1)/(n + 1)$ for every $x \in G^n_k$, thus for every $(\mu, x) \in Z^n_k$. By Theorem 6, we obtain a Borel set $B^n_k \supseteq A$ such that $\mu(x + B^n_k) \leqslant (k + 1)/(n + 1)$ for every $(\mu, x) \in Z^n_k$. We set $B_n = \bigcap_{k=0}^{n+1} B^n_k$ and $G_n = \bigcup_{k=0}^{n+1} G^n_k$. Then $G_n \subseteq X$ is co-meager, $B_n \supseteq A$ is Borel and $\mu(x + B_n) - \mu(x + A) \leqslant \frac{1}{n+1}$ for every $x \in G_n$. Finally, let $B = \bigcap_n B_n$ and $G = \bigcap_n G_n$. Clearly, B and G are as desired. $\square$

4. Results under Martin's axiom

The aim of this section is to show that under stronger set-theoretic assumptions (in particular, under Martin's Axiom and the negation of Continuum Hypothesis, abbreviated as MA + ¬CH) $\mathbf{\Pi}^1_2$ sets satisfy the conclusion of Proposition 13. As before, X is an abelian Polish group. For the following proposition let us recall that if $(A_\xi)_{\xi<\omega_1}$ is a transfinite sequence of universally measurable subsets of X, then under MA + ¬CH the set $\bigcap_{\xi<\omega_1} A_\xi$ is universally measurable (see, e.g., [Ku]).

Proposition 14 (MA + ¬CH)**.** *Let $A \in \mathbf{\Pi}^1_2(X)$, let $\mu \in P(X)$ and let $G \subseteq X$ be co-meager. Assume that $\mu(x+A)=0$ for every $x \in G$. Then there exists a Borel set $B \supseteq A$ and a co-meager set $G' \supseteq X$ such that $\mu(x+B)=0$ for every $x \in G'$.*

Proof. By a classical result of Sierpiński (see [Ke, page 324]), there exists a decreasing transfinite sequence $(B_\xi)_{\xi<\omega_1}$ of Borel sets such that $A=\bigcap_{\xi<\omega_1} B_\xi$. We set $C \coloneqq \bigcap_{\xi<\omega_1} C_\xi$ where $C_\xi=\{x\in X : \mu(x+B_\xi)>0\}$. By our assumption, there is $\xi_0<\omega_1$ such that $C_{\xi_0}\setminus C$ is meager. Moreover, for each $x\in G$ there is $\lambda<\omega_1$ with $\mu(x+B_\lambda)=0$, hence $G\cap C=\emptyset$. It follows that C is meager and, consequently, so is C_{ξ_0}. We set $B=B_{\xi_0}$ and $G'=X\setminus C_{\xi_0}$. □

Note the Borel set B obtained from Proposition 14 need not be Haar-null even if its translates are μ-null over a large set (in the topological sense). This phenomenon does not occur if co-meagerness is replaced by prevalence as the following proposition shows.

Proposition 15. *Let $\mu \in P(X)$ and let $A, G \subseteq X$ be universally measurable sets such that $\mu(x+A)=0$ for every $x\in G$. If G is prevalent, then A is Haar-null.*

Proof. Note that the set $-G=\{-x : x\in G\}$ is also prevalent. Let $\nu\in P(X)$ be a test measure of $-G$; that is, $\nu(x-G)=1$ for every $x\in X$. Then observe that

$$\begin{aligned}(\nu*\mu)(x+A) &= \int_X \mu(x+A-y)\,d\nu(y)\\ &= \int_{x-G}\mu(x+A-y)\,d\nu(y)+\int_{X\setminus x-G}\mu(x+A-y)\,d\nu(y)\\ &\leqslant \nu(X\setminus x-G)=1-\nu(x-G)=0\end{aligned}$$

for every $x\in X$. Therefore, A is Haar-null and $\nu*\mu$ is a test measure for A. □

We proceed to show that if $A\supseteq X$ is analytic or co-analytic, then there exists a Borel set $B\supseteq A$ such that $T(B)\setminus T(A)$ is meager in $P(X)$. Actually, for co-analytic sets one does not need the full strength of MA + ¬CH but only the statement that "all $\mathbf{\Sigma}^1_2$ sets have the Baire property" (for the relationship between these statements see [Ka]). The following proposition has been suggested to us by the referee.

Proposition 16. *Assume that all $\mathbf{\Sigma}^1_2$ sets have the Baire property. Then for any co-analytic set A there exists a Borel set $B\supseteq A$ such that $T(B)\setminus T(A)$ is meager in $P(X)$.*

Proof. Let $\mathrm{WO} \subseteq 2^{\mathbb{N}^2}$ be the set of well-orderings of $\mathbb{N}$. For every $\gamma \in \mathrm{WO}$ let $|\gamma|$ denote the ordinal isomorphic to the well-ordering coded by γ. The set WO is $\mathbf{\Pi}^1_1$-complete and the map $\gamma \mapsto |\gamma|$ is a $\mathbf{\Pi}^1_1$-rank for WO (see [Ke, pages 213 and 267]). So, there is a Borel function $f\colon X \to 2^{\mathbb{N}^2}$ such that $x \in A$ if and only if $f(x) \in \mathrm{WO}$. Let $\phi\colon A \to \omega_1$ be defined by $\phi(x) = |f(x)|$. Then ϕ is a $\mathbf{\Pi}^1_1$-rank for A. For every countable ordinal set, as usual, $A_\xi = \{x \in A : \phi(x) \leqslant \xi\}$, and note that each A_ξ is Borel and $A = \bigcup_{\xi<\omega_1} A_\xi$.

Claim 17. *We have* $T(A) = \bigcap_{\xi<\omega_1} T(A_\xi)$.

Proof of Claim 17. It is clear that $T(A) \subseteq \bigcup_{\xi<\omega_1} T(A_\xi)$. To show the other inclusion, fix $\mu \in \bigcup_{\xi<\omega_1} T(A_\xi)$. Let $x \in X$ be arbitrary. From the regularity of the measure μ there exists a Borel set $B \subseteq A$ with $\mu(x + A) = \mu(x + B)$. Since $B \subseteq A$ is Borel, by the boundedness theorem for $\mathbf{\Pi}^1_1$-ranks (see [Ke, page 288]), we have $\sup\{\phi(x) : x \in B\} = \lambda < \omega_1$, and so $B \subseteq A_\lambda \subseteq A$. From the fact that $\mu \in T(A_\lambda)$, we obtain that

$$\mu(x + A) = \mu(x + B) = \mu(x + A_\lambda) = 0.$$

Since x was arbitrary, we conclude that $\mu \in T(A)$. The claim is proved. □

Consider the relation

$$R = \big\{(x, y, \gamma) \in X \times X \times 2^{\mathbb{N}^2} : \gamma \notin \mathrm{WO} \text{ or } [f(x - y) \in \mathrm{WO} \text{ and } |f(x - y)| \leqslant |\gamma|]\big\}.$$

As the relation

$$\gamma \notin \mathrm{WO} \text{ or } (\delta \in \mathrm{WO} \text{ and } |\delta| \leqslant |\gamma|)$$

is $\mathbf{\Sigma}^1_1$ on $(\delta, \gamma) \in 2^{\mathbb{N}^2} \times 2^{\mathbb{N}^2}$ (see [Ke, page 269]), it follows that R is $\mathbf{\Sigma}^1_1$. Furthermore, if $\gamma \in \mathrm{WO}$ and $y \in X$, for the section $R_{(y,\gamma)} = \{x \in X : (x, y, \gamma) \in R\}$ we have

$$R_{(y,\gamma)} = y + A_{|\gamma|}. \tag{1}$$

For every $\mu, \nu \in P(X)$ define

$$\mu \leqslant^* \nu \quad\Leftrightarrow\quad \big(\mu, \nu \notin T(A) \text{ and } \forall \gamma \in 2^{\mathbb{N}^2}\ [\forall y \in X\ \nu\big(R_{(y,\gamma)}\big) = 0 \Rightarrow \forall y \in X\ \mu\big(R_{(y,\gamma)}\big) = 0]\big).$$

The set $P(X) \setminus T(A)$ is $\mathbf{\Sigma}^1_2$. Moreover, since R is $\mathbf{\Sigma}^1_1$, the rest of the right hand side of the above formula is $\mathbf{\Pi}^1_2$ by [Ke, Theorem 29.26]. Thus, by our assumption, $\leqslant^* \subseteq P(X) \times P(X)$ has the Baire property in $P(X) \times P(X)$. Moreover, by (1), for every $\mu, \nu \notin T(A)$, $\mu \leqslant^* \nu$ is equivalent to saying that for each $\gamma \in \mathrm{WO}$ we have

$$\nu \in T(A_{|\gamma|}) \Rightarrow \mu \in T(A_{|\gamma|}),$$

that is,

$$\mu \leqslant^* \nu \Rightarrow \xi_\mu \leqslant \xi_\nu,$$

where $\xi_\mu \coloneqq \min\{\xi < \omega_1 : \mu \notin T(A_\xi)\}$ and similarly for ξ_ν. Since $(T(A_\xi))_{\xi<\omega_1}$ is a decreasing sequence of transfinite sets with the Baire property, there is $\xi_0 < \omega_1$

such that for every $\xi \geqslant \xi_0$ the set $T(A_{\xi_0}) \setminus T(A)$ is meager. By [Ke, Exercise 8.49] applied to $\leqslant^*$ on $T(A_{\xi_0}) \setminus T(A)$, we deduce that

$$T(A_{\xi_0}) \setminus T(A) = \bigcup_{\xi_0 \leqslant \xi < \omega_1} \big(T(A_{\xi_0}) \setminus T(A_\xi)\big)$$

is meager and the proof is completed. $\square$

Under $\mathrm{MA} + \neg\mathrm{CH}$, the above result is also valid for analytic sets. First we need the following extension of Lemma 3. Again the proof is omitted.

Lemma 18 ($\mathrm{MA} + \neg\mathrm{CH}$)**.** *Let $(A_\xi)_{\xi<\omega_1}$ be an increasing transfinite sequence of Borel sets. Then $T(A) = \bigcap_{\xi<\omega_1} T(A_\xi)$ where $A = \bigcup_{\xi<\omega_1} A_\xi$. In particular, the set $T(A)$ has the Baire property.*

We also need the following lemma. Its proof is left to the reader.

Lemma 19 ($\mathrm{MA} + \neg\mathrm{CH}$)**.** *Let $(A_\xi)_{\xi<\omega_1}$ and A be as in Lemma 18. Then there exists a countable ordinal ξ such that $T(A_\xi) \setminus T(A)$ is meager in $P(X)$.*

By Lemma 19 with Sierpiński's result, we obtain the following proposition.

Proposition 20 ($\mathrm{MA} + \neg\mathrm{CH}$)**.** *Let $A \subseteq X$ be analytic. Then there exists a Borel set $B \subseteq A$ such that $T(B) \setminus T(A)$ is meager in $P(X)$.*

Remark 3. In [D], Dougherty asked whether any analytic non Haar-null set must include a Borel non Haar-null set. By Proposition 20, under $\mathrm{MA}+\neg\mathrm{CH}$ any analytic non Haar-null set A includes a Borel set B such that $T(B)$ is meager in $P(X)$. That is, the set B may be Haar-null, but only a relatively small set of measures witness this fact. We should point out that not every universally measurable non Haar-null set shares this property. For instance, Dougherty considered a set $A \subseteq X$ such that $|A \cap B| < 2^{\aleph_0}$ whenever B is a σ-compact set and $|A \cap B| = 2^{\aleph_0}$ whenever B is a Borel set not included in a σ-compact set (see [D, page 85]). Under Martin's Axiom, this set A is universally measurable and Haar-null. So its complement is prevalent. However, if B is any Borel set included in $X \setminus A$, then B must be included in a σ-compact set. Hence, by part (iii) of Proposition 5, the set $T(B)$ must be co-meager in $P(X)$.

Let us conclude this section with some remarks concerning a possible extension of Corollary 12 to co-analytic sets. To the best of our knowledge, even the problem whether for any co-analytic Haar-null set $A \subseteq X$ there exists a Borel Haar-null set $B \supseteq A$, is open. In this direction it seems quite natural to ask whether under $\mathrm{MA} + \neg\mathrm{CH}$ the conclusion of Corollary 12 is valid for co-analytic sets.

5. An application

Let X be a separable Banach space. A locally Lipschitz function $f\colon X \to \mathbb{R}$ is said to be an *essentially smooth Lipschitz function* if the set

$$S_f \coloneqq \{x \in X : \partial f(x) \text{ is not a singleton}\}$$

is contained in the σ-ideal generated by the Borel Haar-null sets (by ∂f we denote the subdifferential of f in the sense of Clarke). That is, f is said to be essentially smooth if there exists a Borel Haar-null set $B \subseteq X$ such that $B \supseteq S_f$. The class of essentially smooth Lipschitz functions was defined and studied extensively by Borwein and Moors [BM]. They proved that the members of this class possess differentiability properties similar to the ones of convex functions (for instance their subdifferential is both minimal and integrable). In addition, they showed that this class has strong closure properties (it is closed under addition, multiplication and lattice operations).

Note, however, that the set S_f is not Borel in general. So, it might belong to the σ-ideal of universally measurable Haar-null sets but not to the σ-ideal generated by the Borel Haar-null sets (as we have already mentioned, the existence of such sets was observe by Dougherty). We will show that in this particular case there is no difference.

Proposition 21. *If the set S_f is contained in the σ-ideal of universally measurable Haar-null sets, then there exists a Borel Haar-null set $B \subseteq X$ such that $B \supseteq S_f$.*

Proof. We first observe that, since X is a separable Banach space, the space $(X^*, B(X^*_{w*}))$—that is, the Borel σ-algebra of the weak* topology of X^*—is a standard Borel space (see [Ke, p. 79]). So, fix a Polish topology τ on X^* such that $B(X^*_\tau) = B(X^*_{w*})$. From the fact that ∂f has closed graph in $X \times X^*_{w*}$ we see that

$$\mathrm{Gr}(\partial f) \in B(X) \otimes B(X^*_{w*}) = B(X) \otimes B(X^*_\tau) = B(X \times X^*_\tau).$$

As $\mathrm{Gr}(\partial f)$ is a Borel subset of $X \times X^*_{w*}$, it admits a co-analytic uniformization $A \subseteq X \times X^*_\tau$ (see [Ke, p. 306]). But then observe that

$$S_f = \mathrm{proj}_X\{\mathrm{Gr}(\partial f) \cap ((X \times X^*) \setminus A)\}.$$

Clearly, the set S_f is analytic. So if S_f is Haar-null, by Corollary 12, there exists a Borel Haar-null set $B \subseteq X$ such that $S_f \subseteq B$. □

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