STABLE FAMILIES OF ANALYTIC SETS

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Abstract. We give a different proof of the well-known fact that any uncountable family of analytic subsets of a Polish space with the point-finite intersection property must contain a subfamily whose union is not analytic. Our approach is based on the Kunen–Martin theorem.

1. Introduction and notation

It is well-known that any uncountable family of analytic subsets of a Polish space with the point-finite intersection property must contain a subfamily whose union is not analytic (see [1, 2, 3, 5]). In [1], this (in fact a much stronger) result is proved but the proof heavily depends on the Axiom of Choice. In [2, 3, 5], the proofs are effective but the argument is more complicated. In this note we give a short proof using the Kunen–Martin theorem.

Notation. In what follows X and Y will be Polish spaces. By $\mathcal{N}$ we denote the Baire space. If $A \subseteq X \times Y$ and U is an arbitrary open subset of Y, then we set

$$A(U) := \mathrm{proj}_X \{A \cap (X \times U)\}.$$

All the other pieces of notation we use are standard (see, e.g., [4]).

2. Stable families of analytic sets

Departing from standard terminology, we make the following definition.

Definition 1. *A family $\mathcal{F} = (A_i)_{i \in I}$ of analytic subsets of X will be called* stable *if for every $J \subseteq I$ the set $\bigcup_{i \in J} A_i$ is an analytic subset of X.*

Clearly any subfamily of a stable family is stable. Furthermore any countable family of analytic sets is stable. There exist, however, uncountable stable families of analytic sets.

Example 1. Let $A \subset X$ be an analytic non-Borel set. By a classical result of Sierpiński (see [4, page 201]), there exists a transfinite sequence $(B_\xi)_{\xi<\omega_1}$ of Borel sets such that $A = \bigcup_{\xi<\omega_1} B_\xi$. Clearly, we may assume that the sequence $(B_\xi)_{\xi<\omega_1}$ is increasing. Since A is not Borel, there exists an uncountable subset Λ of ω_1 such that $B_\xi \subsetneq B_\zeta$ for every $\xi, \zeta \in \Lambda$ with $\xi < \zeta$. Then the family $\mathcal{F} = (B_\xi)_{\xi \in \Lambda}$

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is an uncountable stable family of mutually different analytic sets (note that the members of $\mathcal{F}$ are actually Borel sets).

Definition 2. *A family $\mathcal{F} = (A_i)_{i\in I}$ of subsets of X is said to have the* point-finite intersection property (*abbreviated as* p.f.i.p.) *provided that for every $x \in X$ the set $I_x = \{i \in I : x \in A_i\}$ is finite.*

As before, any subfamily of a family with the point-finite intersection property has the point-finite intersection property. We will show that stable families of analytic sets with the p.f.i.p. are necessarily countable. To that end we need a couple of lemmas. The one that follows is elementary.

Lemma 3. *Let X and Y be Polish spaces. If $A \in \mathbf{\Pi}^1_1(X)$ and $U \subseteq Y$ is open, then $A \times U \in \mathbf{\Pi}^1_1(X \times Y)$.*

Lemma 4. *Let X and Y be Polish spaces. Assume that $X \times Y$ has closed sections (that is, for every $x \in X$ the set $A_x = \{y \in Y : (x, y) \in A\}$ is closed) and, moreover, for every open set $U \subseteq Y$ the set $A(U)$ is analytic. Then A is also analytic.*

Proof. Let $\mathcal{B} = (V_n)$ be a countable base for Y. Observe that $(x, y) \notin A$ if and only if there exists a basic open subset V_n of Y such that $x \notin A(V_n)$. It follows that

$$(X \times Y) \setminus A = \bigcup_n \big(X \setminus A(V_n)\big) \times V_n$$

and so, by Lemma 3, the set A is analytic. □

We have the following stability result.

Lemma 5. *Let $\mathcal{F} = (A_i)_{i\in I}$ be a stable family of analytic subsets of X with the point-finite intersection property. Then for every Polish space Y and for every family $(B_i)_{i\in I}$ of analytic subsets of Y, the set*

$$A := \bigcup_{i\in I} (A_i \times B_i)$$

is an analytic subset of $X \times Y$.

Proof. Let $\mathcal{F} = (A_i)_{i\in I}$ and $(B_i)_{i\in I}$ be as above. Since every B_i is analytic, there exists $C_i \subseteq Y \times \mathcal{N}$ closed such that $B_i = \mathrm{proj}_Y C_i$. We define $\tilde{A} \subseteq X \times Y \times \mathcal{N}$ by

$$\tilde{A} = \bigcup_{i\in I} (A_i \times C_i).$$

Clearly, $A = \mathrm{proj}_{X\times Y} \tilde{A}$. Note that for every $x \in X$ the section

$$\tilde{A}_x = \{(y, z) \in Y \times \mathcal{N} : (x, y, z) \in \tilde{A}\}$$

is exactly the set $\bigcup_{i\in I_x} C_i$. As the family $\mathcal{F}$ has the point-finite intersection property, for every $x \in X$ the section $\tilde{A}_x$ of $\tilde{A}$ is closed. Observe that for every open subset U of $Y \times \mathcal{N}$ we have

$$\begin{aligned} \tilde{A}(U) &= \operatorname{proj}_X\{\tilde{A} \cap (X \times U)\} \\ &= \{x \in X : \exists i \in I_x \text{ such that } C_i \cap U \neq \emptyset\} = \bigcup\{A_i : C_i \cap U \neq \emptyset\}. \end{aligned}$$

It follows directly from the stability of the family that $\tilde{A}(U)$ is analytic and so, by Lemma 4, we see that $\tilde{A}$ is an analytic subset of $X \times Y \times \mathcal{N}$. Hence so is A. □

Let $\prec$ be a strict well-founded binary relation on X. By recursion, we define the rank function $\varrho_\prec \colon X \to \mathrm{Ord}$ as follows. We set $\varrho_\prec(x) = 0$ if x is minimal; otherwise, let $\varrho_\prec(x) = \sup\{\varrho_\prec(y) : y \prec x\} + 1$. Finally, we define the rank of $\prec$ by setting $\varrho(\prec) = \sup\{\varrho_\prec(x) + 1 : x \in X\}$. We will need the following boundedness principle of analytic well-founded relations due to Kunen and Martin (see [4, 6]).

Theorem 6. *Let $\prec$ be a strict well-founded relation and assume that $\prec$ is analytic (as a subset of $X \times X$). Then $\varrho(\prec)$ is countable.*

Lemma 7. *Let $\mathcal{F} = (A_i)_{i\in I}$ be a stable family of mutually disjoint analytic subsets of X. Then $\mathcal{F}$ is countable.*

Proof. Assume that $\mathcal{F}$ is not countable. Select an uncountable subfamily $\mathcal{F}'$ of $\mathcal{F}$ with $|\mathcal{F}'| = \aleph_1$, and let $\mathcal{F}' = (A_\xi)_{\xi<\omega_1}$ be a well-ordering of $\mathcal{F}'$. Clearly $\mathcal{F}'$ remains stable. As the sets A_ξ are pairwise disjoint, we define $\phi \colon \bigcup_{\xi<\omega_1} A_\xi \to \mathrm{Ord}$ by setting $\phi(x)$ to be the unique ξ such that $x \in A_\xi$. Define the binary relation $\prec$ by

$$x \prec y \Leftrightarrow \phi(x) < \phi(y).$$

Clearly $\prec$ is well-founded and strict. Moreover note that $\prec$, as a subset of $X \times X$, is the set

$$\bigcup_{\xi<\omega_1} (A_\xi \times B_\xi)$$

where $B_\xi = \bigcup_{\zeta>\xi} A_\zeta$. From the stability of $\mathcal{F}'$, we see that the sets B_ξ are analytic subsets of X for every $\xi < \omega_1$. Since $\mathcal{F}'$ is stable and has the p.f.i.p., by Lemma 5, we obtain that $\prec$ is a $\mathbf{\Sigma}^1_1$ relation. By Theorem 6, $\varrho(\prec)$ must be countable and we derive a contradiction. □

Finally we have the following theorem.

Theorem 8. *Let $\mathcal{F}$ be a stable family of analytic sets with the point-finite intersection property. Then $\mathcal{F}$ is countable.*

Proof. Assume not. Let $\mathcal{F}'$ be as in Lemma 7. Let Y be an arbitrary uncountable Polish space, and let $(y_\xi)_{\xi<\omega_1}$ be a transfinite sequence of distinct members of Y. For every $\xi < \omega_1$, we set $L_\xi \coloneqq A_\xi \times \{y_\xi\}$; clearly, L_ξ is an analytic subset of $X \times Y$ and, moreover, $L_\xi \cap L_\zeta = \emptyset$ if $\xi \neq \zeta$. As the family (and every subfamily of) $\mathcal{F}'$ is stable and has the p.f.i.p., by Lemma 5, we see that for every $G \subseteq \omega_1$ the set

$$\bigcup_{\xi\in G} (A_\xi \times \{y_\xi\}) = \bigcup_{\xi\in G} L_\xi$$

is an analytic subset of $X \times Y$. It follows that the family $\mathcal{L} = (L_\xi)_{\xi<\omega_1}$ is a stable family of mutually disjoint analytic subsets of $X \times Y$. By Lemma 7, the family $\mathcal{L}$ must be countable and again we derive a contradiction. $\square$

By Theorem 8, we have the following corollary (see also [7]).

Corollary 9. *Let X be a Polish space, Y a metrizable space and $A \in \mathbf{\Sigma}^1_1(X)$. Also let $f\colon X \to Y$ be a Borel measurable function. Then $f(A)$ is separable.*

Proof. Assume not. Let $C \subseteq f(A)$ be an uncountable closed discrete set with $|C| > \aleph_0$. For every $y \in C$ we set $A_y := f^{-1}(\{y\})$. Then $\mathcal{F} = (A_y)_{y\in C}$ is a stable family of mutually disjoint analytic subsets of X. By Theorem 8, the family $\mathcal{F}$ must be countable and we derive a contradiction. $\square$

Remark 1. Say that a family $\mathcal{F} = (A_i)_{i\in I}$ has the point-countable intersection property if for every $x \in X$ the set $I_x = \{i \in I : x \in A_i\}$ is countable. One can easily see that Theorem 8 is not valid for stable families with the point-countable intersection property. For instance, let $(A_\xi)_{\xi<\omega_1}$ be a strictly decreasing transfinite sequence of analytic sets with $\bigcap_{\xi<\omega_1} A_\xi = \emptyset$. As the sequence is decreasing, the family $\mathcal{F} = (A_\xi)_{\xi<\omega_1}$ is stable. Moreover, note that for every $x \in X$ there exists $\xi < \omega_1$ such that $x \notin A_\zeta$ for every $\zeta > \xi$. (For if not, there would existed $x \in X$ such that $x \in A_\xi$ for every $\xi < \omega_1$, that is, $x \in \bigcap_{\xi<\omega_1} A_\xi$.) Hence, the family $\mathcal{F}$ is an uncountable stable family of analytic sets with the point-countable intersection property.

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References

[1] J. Brzuchowski, J. Cichoń, E. Grzegorek and C. Ryll-Nardzewski, *On the existence of non-measurable unions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 27 (1979), 447–448.

[2] A. Emeryk, R. Frankiewicz and W. Kulpa, *Remarks on Kuratowski's theorem on meager sets*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 27 (1979), 493–498.

[3] R. W. Hansell, *Borel measurable mappings for non-separable metric spaces*, Trans. Amer. Math. Soc. 161 (1971), 145–169.

[4] A. S. Kechris, *Classical Descriptive Set Theory*, Grad. Texts in Math., vol. 156, Springer–Verlag, 1995.

[5] J. Kaniewski and R. Pol, *Borel measurable selectors for compact-valued mappings in the non-separable case*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 23 (1975), 1043–1050.

[6] Y. N. Moschovakis, *Descriptive Set Theory*, North Holland, Amsterdam, 1980.

[7] S. M. Srivastava, *A course on Borel sets*, Grad. Texts in Math., vol. 180, Springer–Verlag, 1998.

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