STABLE FAMILIES OF ANALYTIC SETS

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Abstract. We give a different proof of the well-known fact that any uncountable family of analytic subsets of a Polish space with the point-finite intersection property must contain a subfamily whose union is not analytic. Our approach is based on the Kunen–Martin theorem.

1. Introduction and notation

It is well-known that any uncountable family of analytic subsets of a Polish space with the point-finite intersection property must contain a subfamily whose union is not analytic (see [1, 2, 3, 5]). In [1], this (in fact a much stronger) result is proved but the proof heavily depends on the Axiom of Choice. In [2, 3, 5], the proofs are effective but the argument is more complicated. In this note we give a short proof using the Kunen–Martin theorem.

Notation. In what follows $X$ and $Y$ will be Polish spaces. By $\mathcal{N}$ we denote the Baire space. If $A \subseteq X \times Y$ and $U$ is an arbitrary open subset of $Y$, then we set

$$A(U) := \text{proj}_X \{A \cap (X \times U)\}.$$ 

All the other pieces of notation we use are standard (see, e.g., [4]).

2. Stable families of analytic sets

Departing from standard terminology, we make the following definition.

Definition 1. A family $\mathcal{F} = (A_i)_{i \in I}$ of analytic subsets of $X$ will be called stable if for every $J \subseteq I$ the set $\bigcup_{i \in J} A_i$ is an analytic subset of $X$.

Clearly any subfamily of a stable family is stable. Furthermore any countable family of analytic sets is stable. There exist, however, uncountable stable families of analytic sets.

Example 1. Let $A \subset X$ be an analytic non-Borel set. By a classical result of Sierpiński (see [4, page 201]), there exists a transfinite sequence $(B_\xi)_{\xi < \omega_1}$ of Borel sets such that $A = \bigcup_{\xi < \omega_1} B_\xi$. Clearly, we may assume that the sequence $(B_\xi)_{\xi < \omega_1}$ is increasing. Since $A$ is not Borel, there exists an uncountable subset $\Lambda$ of $\omega_1$ such that $B_\xi \subset B_\zeta$ for every $\xi, \zeta \in \Lambda$ with $\xi < \zeta$. Then the family $\mathcal{F} = (B_\xi)_{\xi \in \Lambda}$

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is an uncountable stable family of mutually different analytic sets (note that the members of $\mathcal{F}$ are actually Borel sets).

**Definition 2.** A family $\mathcal{F} = (A_i)_{i \in I}$ of subsets of $X$ is said to have the point-finite intersection property (abbreviated as p.f.i.p.) provided that for every $x \in X$ the set $I_x = \{ i \in I : x \in A_i \}$ is finite.

As before, any subfamily of a family with the point-finite intersection property has the point-finite intersection property. We will show that stable families of analytic sets with the p.f.i.p. are necessarily countable. To that end we need a couple of lemmas. The one that follows is elementary.

**Lemma 3.** Let $X$ and $Y$ be Polish spaces. If $A \in \Pi^1_1(X)$ and $U \subseteq Y$ is open, then $A \times U \in \Pi^1_1(X \times Y)$.

**Lemma 4.** Let $X$ and $Y$ be Polish spaces. Assume that $X \times Y$ has closed sections (that is, for every $x \in X$ the set $A_x = \{ y \in Y : (x,y) \in A \}$ is closed) and, moreover, for every open set $U \subseteq Y$ the set $A(U)$ is analytic. Then $A$ is also analytic.

**Proof.** Let $B = (V_n)$ be a countable base for $Y$. Observe that $(x,y) \notin A$ if and only if there exists a basic open subset $V_n$ of $Y$ such that $x \notin A(V_n)$. It follows that

$$(X \times Y) \setminus A = \bigcup_n (X \setminus A(V_n)) \times V_n$$

and so, by Lemma 3, the set $A$ is analytic. □

We have the following stability result.

**Lemma 5.** Let $\mathcal{F} = (A_i)_{i \in I}$ be a stable family of analytic subsets of $X$ with the point-finite intersection property. Then for every Polish space $Y$ and for every family $(B_i)_{i \in I}$ of analytic subsets of $Y$, the set

$$A := \bigcup_{i \in I} (A_i \times B_i)$$

is an analytic subset of $X \times Y$.

**Proof.** Let $\mathcal{F} = (A_i)_{i \in I}$ and $(B_i)_{i \in I}$ be as above. Since every $B_i$ is analytic, there exists $C_i \subseteq Y \times \mathcal{N}$ closed such that $B_i = \text{proj}_Y C_i$. We define $\hat{A} \subseteq X \times Y \times \mathcal{N}$ by

$$\hat{A} = \bigcup_{i \in I} (A_i \times C_i).$$

Clearly, $A = \text{proj}_{X \times Y} \hat{A}$. Note that for every $x \in X$ the section

$$\hat{A}_x = \{ (y,z) \in Y \times \mathcal{N} : (x,y,z) \in \hat{A} \}$$

is exactly the set $\bigcup_{i \in I_x} C_i$. As the family $\mathcal{F}$ has the point-finite intersection property, for every $x \in X$ the section $\hat{A}_x$ of $\hat{A}$ is closed. Observe that for every open
subset $U$ of $Y \times N$ we have
\[ \tilde{A}(U) = \text{proj}_X \{ \tilde{A} \cap (X \times U) \} \]
\[ = \{ x \in X : \exists i \in I_x \text{ such that } C_i \cap U \neq \emptyset \} = \bigcup \{ A_i : C_i \cap U \neq \emptyset \}. \]
It follows directly from the stability of the family that $\tilde{A}(U)$ is analytic and so, by Lemma 4, we see that $\tilde{A}$ is an analytic subset of $X \times Y \times N$. Hence so is $A$. \(\square\)

Let $\prec$ be a strict well-founded binary relation on $X$. By recursion, we define the rank function $\varrho_{\prec}: X \to \text{Ord}$ as follows. We set $\varrho_{\prec}(x) = 0$ if $x$ is minimal; otherwise, let $\varrho_{\prec}(x) = \sup \{ \varrho_{\prec}(y) : y \prec x \} + 1$. Finally, we define the rank of $\prec$ by setting $\varrho(\prec) = \sup \{ \varrho_{\prec}(x) + 1 : x \in X \}$. We will need the following boundedness principle of analytic well-founded relations due to Kunen and Martin (see [4, 6]).

**Theorem 6.** Let $\prec$ be a strict well-founded relation and assume that $\prec$ is analytic (as a subset of $X \times X$). Then $\varrho(\prec)$ is countable.

**Lemma 7.** Let $\mathcal{F} = (A_i)_{i \in I}$ be a stable family of mutually disjoint analytic subsets of $X$. Then $\mathcal{F}$ is countable.

**Proof.** Assume that $\mathcal{F}$ is not countable. Select an uncountable subfamily $\mathcal{F}'$ of $\mathcal{F}$ with $|\mathcal{F}'| = \aleph_1$, and let $\mathcal{F}' = (A_\xi)_{\xi < \omega_1}$ be a well-ordering of $\mathcal{F}'$. Clearly $\mathcal{F}'$ remains stable. As the sets $A_\xi$ are pairwise disjoint, we define $\phi: \bigcup_{\xi < \omega_1} A_\xi \to \text{Ord}$ by setting $\phi(x)$ to be the unique $\xi$ such that $x \in A_\xi$. Define the binary relation $\prec$ by
\[ x \prec y \iff \phi(x) < \phi(y). \]
Clearly $\prec$ is well-founded and strict. Moreover note that $\prec$, as a subset of $X \times X$, is the set
\[ \bigcup_{\xi < \omega_1} (A_\xi \times B_\xi) \]
where $B_\xi = \bigcup_{\xi < \xi} A_\xi$. From the stability of $\mathcal{F}'$, we see that the sets $B_\xi$ are analytic subsets of $X$ for every $\xi < \omega_1$. Since $\mathcal{F}'$ is stable and has the p.f.i.p., by Lemma 5, we obtain that $\prec$ is a $\Sigma^1_1$ relation. By Theorem 6, $\varrho(\prec)$ must be countable and we derive a contradiction. \(\square\)

Finally we have the following theorem.

**Theorem 8.** Let $\mathcal{F}$ be a stable family of analytic sets with the point-finite intersection property. Then $\mathcal{F}$ is countable.

**Proof.** Assume not. Let $\mathcal{F}'$ be as in Lemma 7. Let $Y$ be an arbitrary uncountable Polish space, and let $(y_\xi)_{\xi < \omega_1}$ be a transfinite sequence of distinct members of $Y$. For every $\xi < \omega_1$, we set $L_\xi := A_\xi \times \{ y_\xi \}$; clearly, $L_\xi$ is an analytic subset of $X \times Y$ and, moreover, $L_\xi \cap L_\zeta = \emptyset$ if $\xi \neq \zeta$. As the family (and every subfamily of) $\mathcal{F}'$ is stable and has the p.f.i.p., by Lemma 5, we see that for every $G \subseteq \omega_1$ the set
\[ \bigcup_{\xi \in G} (A_\xi \times \{ y_\xi \}) = \bigcup_{\xi \in G} L_\xi \]
is an analytic subset of $X \times Y$. It follows that the family $\mathcal{L} = (L_\xi)_{\xi < \omega_1}$ is a stable family of mutually disjoint analytic subsets of $X \times Y$. By Lemma 7, the family $\mathcal{L}$ must be countable and again we derive a contradiction.

By Theorem 8, we have the following corollary (see also [7]).

**Corollary 9.** Let $X$ be a Polish space, $Y$ a metrizable space and $A \in \Sigma^1_1(X)$. Also let $f : X \to Y$ be a Borel measurable function. Then $f(A)$ is separable.

**Proof.** Assume not. Let $C \subseteq f(A)$ be an uncountable closed discrete set with $|C| > \aleph_0$. For every $y \in C$ we set $A_y := f^{-1}(\{y\})$. Then $\mathcal{F} = (A_y)_{y \in C}$ is a stable family of mutually disjoint analytic subsets of $X$. By Theorem 8, the family $\mathcal{F}$ must be countable and we derive a contradiction. □

**Remark 1.** Say that a family $\mathcal{F} = (A_i)_{i \in I}$ has the point-countable intersection property if for every $x \in X$ the set $I_x = \{i \in I : x \in A_i\}$ is countable. One can easily see that Theorem 8 is not valid for stable families with the point-countable intersection property. For instance, let $(A_\xi)_{\xi < \omega_1}$ be a strictly decreasing transfinite sequence of analytic sets with $\bigcap_{\xi < \omega_1} A_\xi = \emptyset$. As the sequence is decreasing, the family $\mathcal{F} = (A_\xi)_{\xi < \omega_1}$ is stable. Moreover, note that for every $x \in X$ there exists $\xi < \omega_1$ such that $x \notin A_\zeta$ for every $\zeta > \xi$. (For if not, there would existed $x \in X$ such that $x \in A_\zeta$ for every $\xi < \omega_1$, that is, $x \in \bigcap_{\xi < \omega_1} A_\xi$.) Hence, the family $\mathcal{F}$ is an uncountable stable family of analytic sets with the point-countable intersection property.

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**References**


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