The Tensor Renormalization Group approach of lattice models: from exact blocking formulas to accurate numerical results

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Content of the Talk

1. Motivations
2. The Tensor Renormalization Group (TRG)
   - Exact blocking (spin and gauge, PRD 88 056005)
   - Applies to many lattice models \((O(2), O(3), \text{pure gauge models, } \ldots)\)
3. Recent numerical progress with TRG
   - Truncation methods
   - Solution of sign problems (PRD 89, 016008)
   - Critical exponents
4. \(O(2)\) model with a chemical potential (arxiv 1403.5238)
   - Phase diagram
   - Comparison with the worm algorithm
   - Microscopic control of the systematic errors
   - Optical lattice realization?
5. Towards Asymptotic scaling for the \(O(3)\) model
6. Conclusions
Motivation: study of non trivial fixed points

Irrelevant directions can be slow: problem for small volumes. Blocking?

Figure: Schematic flows for $SU(3)$ with 12 flavors (picture by Yuzhi Liu).
Block Spining in Configuration Space is difficult!

Figure: Ising 2, Step 1, Step 2, ....write the formula!
For each link, we use the $Z_2$ character expansion:

$$\exp(\beta \sigma_1 \sigma_2) = \cosh(\beta)(1 + \sqrt{\tanh(\beta)}\sigma_1 \sqrt{\tanh(\beta)}\sigma_2) = \cosh(\beta) \sum_{n_{12}=0,1} (\sqrt{\tanh(\beta)}\sigma_1 \sqrt{\tanh(\beta)}\sigma_2)^{n_{12}}.$$  

Regroup the four terms involving a given spin $\sigma_i$ and sum over its two values $\pm 1$. The results can be expressed in terms of a tensor: $T^{(i)}_{xx', yy'}$ which can be visualized as a cross attached to the site $i$ with the four legs covering half of the four links attached to $i$. The horizontal indices $x, x'$ and vertical indices $y, y'$ take the values 0 and 1 as the index $n_{12}$.

$$T^{(i)}_{xx', yy'} = f_x f_{x'} f_y f_{y'} \delta \left( \text{mod}[x + x' + y + y', 2] \right),$$

where $f_0 = 1$ and $f_1 = \sqrt{\tanh(\beta)}$. The delta symbol is 1 if $x + x' + y + y'$ is zero modulo 2 and zero otherwise.
Exact form of the partition function:

\[ Z = (\cosh(\beta))^2 \text{Tr} \prod_i T_{xx',yy'}^{(i)}. \]

Tr mean contractions (sums over 0 and 1) over the link indices. Reproduces the closed paths of the HT expansion.

Important feature of the TRG blocking:

It separates the degrees of freedom inside the block (integrated over), from those kept to communicate with the neighboring blocks.

Graphically:
(isotropic blocking)
TRG Blocking defines a new rank-4 tensor $T'_{XX'YY'}$

Exact blocking formula (isotropic):

$$T'_{X(x_1,x_2)X'(x_1',x_2')Y(y_1,y_2)Y'(y_1',y_2')} = \sum_{x_U,x_D,y_R,y_L} T_{x_1x_2y_1y_L} T_{x_2x_1'x_2'y_R} T_{x_2x_2'y_Ry_2'} T_{x_2x_Dy_Ly_1'} ,$$

where $X(x_2,x_2)$ is a notation for the product states e.g., $X(0,0) = 1$, $X(1,1) = 2$, $X(1,0) = 3$, $X(0,1) = 4$.

The partition function can again be written as

$$Z = \text{Tr} \prod_{i=1}^{\infty} T'_{XX'YY'}^{(2i)} ,$$

where $2i$ denotes the sites of the coarser lattice with twice the lattice spacing of the original lattice.
\( Z = \int \prod_i \frac{d\theta_i}{2\pi} e^{\beta \sum_{ij} \cos(\theta_i - \theta_j)} \). 

\[ e^{\beta \cos(\theta_i - \theta_j)} = \sum_{n_{ij}=-\infty}^{+\infty} e^{in_{ij}(\theta_i - \theta_j)} I_{n_{ij}}(\beta), \]

where the \( I_n \) are the modified Bessel functions. In two dimensions:

\[ T^n_{n_{ix},n_{ix}',n_{iy},n_{iy}'} = \sqrt{I_{n_{ix}}(\beta)} \sqrt{I_{n_{iy}}(\beta)} \sqrt{I_{n_{ix}'}(\beta)} \sqrt{I_{n_{iy}'}(\beta)} \delta_{n_{ix}+n_{iy},n_{ix}'+n_{iy}'} \cdot \]

The partition function and the blocking of the tensor are similar to the Ising model, but the sums run over all the integers. As the \( I_n(\beta) \) decay rapidly for large \( n \) and fixed \( \beta \) (namely like \( 1/n! \)) The generalization to higher dimensions is straightforward.
TRG formulations for other lattice models

- $O(3)$ nonlinear sigma model
- Higher dimensions
- Principal chiral models
- Abelian gauge theories ($Z_2$, $Z_N$, $U(1)$)
- $SU(2)$ gauge theories

(see Y. Liu et al. PRD 88 056005)

Yuya Shimizu and Yoshinobu Kuramashi, 1 flavor of Wilson fermion Schwinger model, arxiv 1403.0642
For numerical calculations, we restrict the indices $x, y, \ldots$ to a finite number $N_{\text{states}}$.

We use the smallest blocking: $M_{XX'yy'}^{(n)} = \sum_{y''} T_{x_1 x_1' yy''}^{(n-1)} T_{x_2 x_2' y'' y'}^{(n-1)}$ where $X = x_1 \otimes x_2$ and $X' = x_1' \otimes x_2'$ take now $N_{\text{states}}^2$ values.

We make a truncation $N_{\text{states}}^2 \rightarrow N_{\text{states}}$ using

$$T_{xx'yy'}^{(n)} = \sum_{IJ} U_{xl}^{(n)} M_{IJyy'}^{(n)} U_{x'l}^{(n)*}$$

The unitary matrix $U$ diagonalizes a matrix which is either

- $G_{XX'} = \sum_{X''yy'} M_{XX''yy'} M_{X'X''yy'}^* \ (Xie \ et \ al. \ PRB86, \ HOTRG)$

- $T_{xx'} = \sum_y M_{xx''yy} \ (YM \ PRB87, \ Transfer \ Matrix)$

and we only keep the $N_{\text{states}}$ eigenvectors corresponding to the largest eigenvalues of one of these matrices.
Overlap of the eigenvectors of $G_{XX'}$ and $T_{XX'}$

The overlap matrix $O_{ij} = \sum_X U_{iX} \tilde{U}_{Xj}^*$ seems to have remarkable properties. One example with $O(2)$ indicates that the eigenvectors are approximately the same but the eigenvalues are sometimes in a different order:

$$O_{ij} = \begin{pmatrix}
1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0.9983 & 0. & 0. & 0. & 0.0576 & 0. \\
0. & 0.9999 & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0.9997 & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\
0. & 0. & 0.0576 & 0. & 0. & 0. & 0.9983 & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.9996
\end{pmatrix}_{ij}$$

Values smaller than $10^{-7}$ in absolute value have been replaced by 0.
Comparing with Onsager-Kaufman (PRD 89, 016008)
No sign problem!

**Figure:** Zeros of Real (■) and Imaginary (□) part of the partition function of the Ising model at volume $8 \times 8$ from the HOTRG calculation with $D_s = 40$ are on the exact lines. Gray lines: MC reweighting solution. Thick Black curve: the "radius of confidence" for MC reweighting result, the error is large.
Calculated zeros confirms KT FSS \((1 + \nu = 1.5)\) for the \(O(2)\) model (PRD 89, 016008)

Figure: Zeros of XY model with linear size \(L = 4, 8, 16, 32, 64, 128\) (from up-left to down-right) calculated from HOTRG with \(D_s = 40, 50\) and zeros with \(L = 4, 8, 16, 32\) from MC. The curve is a model for trajectory of the lowest zeros. Fit: \(\text{Im}\beta_z = 1.27986 \times (1.1199 - \text{Re}\beta_z)^{1.49944}\).
For the Ising model on a square lattice, the truncation method (HOSVD) sharply singles out a surprisingly small subspace of dimension two.

In the two states limit, the transformation can be handled analytically yielding a value 0.964 for the critical exponent $\nu$ much closer to the exact value 1 than 1.338 obtained in Migdal-Kadanoff approximations. Alternative blocking procedures that preserve the isotropy can improve the accuracy to $\nu = 0.987$ (isotropic $G$) and 0.993 ($T$) respectively.

More than two states: adding a few more states does not improve the accuracy (Efrati et al. RMP 86 (2014))
The simplest example of quantum rotors ("Towards quantum simulating ...", arxiv 1403.5238)

$O(2)$ model with one space and one Euclidean time direction. The $N_x \times N_t$ sites of the lattice are labelled $(x, t)$. We assume periodic boundary conditions in space and time.

$$Z = \int \prod_{(x,t)} \frac{d\theta_{(x,t)}}{2\pi} e^{-S}$$

$$S = -\beta_t \sum_{(x,t)} \cos(\theta_{(x,t+1)} - \theta_{(x,t)} + i\mu)$$

$$-\beta_s \sum_{(x,t)} \cos(\theta_{(x+1,t)} - \theta_{(x,t)}) .$$

In the isotropic case, we have $\beta_s = \beta_t = \beta$.

In the limit $\beta_t >> \beta_s$ we reach the time continuum limit.

For $\mu \neq 0$ and real, the MC method does not work (complex action).

For large $\mu$, there is a correspondence with the Bose-Hubbard model (Sachdev, Fisher, ..)
Figure: Phase diagram for 2D $O(2)$ isotropic model in $\beta-\mu$ plane (left) and in the $\beta-e^{\mu}/2$ plane (right) which resembles the anisotropic case. The lines labeled by “3s” stand for the phase separation lines of a 3-states system.
Evolution of eigenvalue distribution with $\mu$ ($\beta = 0.3$)

$\beta=0.3$, $\mu=0$, $S=1.55$

$\beta=0.3$, $\mu=0.9$, $S=1.82$

$\beta=0.3$, $\mu=1.8$, $S=2.60$

Figure: The eigenvalues of the transfer matrix are all positive, and after normalization can be interpreted as probabilities: $\sum p_i = 1$. We can then define an invariant entropy $S = \sum p_i \ln(p_i)$ which increases with $\mu$. 
Comparing Transfer matrix based TRG with the worm algorithm for small systems

11 states for the initial tensor and then enough states in the first blocking to stabilize $\langle N \rangle$ with 5 digits (in progress, estimated error less of order 1 in the last digit, preliminary).

<table>
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<th>size</th>
<th>$\beta$</th>
<th>$\mu$</th>
<th>$\langle N \rangle$ (worm)</th>
<th>$\langle N \rangle$ (HOTRG)</th>
<th>number of states</th>
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<td>1.8</td>
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<td>18</td>
</tr>
</tbody>
</table>

Good progress 4x4, 4x8, 8x8, 8x16, 16x16 (with Li Ping Yang, Yuzhi Liu and Haiyuan Zou)
Figure: (Color Online) Two-species (green and red) bosons in optical lattice with species-dependent optical lattice (with the same green and red). The nearest neighbor interaction is coming from overlap of Wannier gaussian wave functions. We assume the difference between intra-species interactions are small $U \gg \delta$ (see arxiv 1403.5238 for details).
O(3) model, Judah Unmuth-Yockey (in progress)

- 2-d O(3) has similarities with 4-d Yang-Mills:
  - asymptotic freedom
  - no phase transition (no ordered phase)
  - topological solutions (instantons)
- Goal: check the asymptotic and finite size scaling of the mass gap \( m(\beta, L) \). For large \( L \), \( m(\beta) \propto \beta \exp(-2\pi\beta) \). FSS: Luscher 82.

Numerical results (correlations and \( \langle E \rangle \)) show apparent convergence in the number of states (with J. Unmuth-Yockey and J. Osborn).
Conclusions

- TRG: Exact blocking with controllable approximations
- Deals well with sign problems, reliable at larger $\text{Im} \beta$ than reweighting MC
- Ising case: checked with the complex Onsager-Kaufman exact solution
- Finite Size Scaling of Fisher zeros of $O(2)$ agrees with Kosterlitz-Thouless
- Towards agreement with the worm algorithm at 5 digit level
- Good understanding of the systematic errors
- $O(3)$ Asymptotic scaling in progress
- Reliable transfer matrix calculations (real time evolution?)

Thanks!