Anomalous scaling at non-thermal fixed points of Gross-Pitaevskii and KPZ turbulence

Thomas Gasenzer  Steven Mathey  Jan M. Pawlowski

ITP - Heidelberg

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Non-thermal fixed points

Non-thermal fixed point are far from equilibrium quasi stationary states of matter.

Scale invariance $\epsilon(k) \sim k^{-d+\eta}$

Depending on the initial conditions, the system may take an algebraically long time on the way to thermalisation.

Driven-Dissipative GPE

Classical field equation for the average Bose wave-function $\phi(x, t)$:

$$i\partial_t \phi(x, t) = \left[-\left(\frac{1}{2m}\right)\nabla^2 + \mu + g|\phi(x, t)|^2\right] \phi(x, t)$$
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$$i\partial_t \phi(x, t) = \left[-\left(\frac{1}{2m} - i\nu\right) \nabla^2 + \mu + g|\phi(x, t)|^2\right] \phi(x, t) + \zeta(x, t)$$

With complex parameters

$$\mu = \mu_1 + i\mu_2 \quad \quad g = g_1 - ig_2$$

Single particle pump \quad 2 particle losses

and stochastic driving

$$\langle \zeta(x, t) \rangle = 0 \quad \quad \langle \zeta(x, t) \zeta(x', t') \rangle = \gamma \delta(t - t') \delta(x - x')$$
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We focus on the kinetic energy density,

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\[ \epsilon(k) \approx \epsilon_{\text{kin}} k^{-d} (k \xi)^\eta \]
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<table>
<thead>
<tr>
<th>(d)</th>
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<tbody>
<tr>
<td>(\eta_{\text{num}})</td>
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Kardar–Parisi–Zhang equation

A model for interface growth,

$$\partial_t \theta(x, t) = \nu \nabla^2 \theta(x, t) + \frac{\lambda}{2} [\nabla \theta(x, t)]^2 + \eta(x, t)$$

with diffusion, perpendicular expansion and stochastic driving,

$$\langle \eta(x, t) \rangle = 0 \quad \langle \eta(x, t) \eta(x', t') \rangle = D \delta(t - t') \delta(x - x')$$
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Scaling in interface growth

The stationary state has scaling correlation functions,

\[ \langle \theta(t + \tau, x + r)\theta(t, x) \rangle_c = r^{2\chi} g \left( \frac{\tau}{r^z} \right) \]

with exponents given by:

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<tr>
<td>(d)</td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(\chi)</td>
<td>(z = 2 - \chi)</td>
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<tr>
<td>(\chi)</td>
<td>(1/2)</td>
<td>0.379</td>
<td>0.300</td>
<td>?</td>
<td>(3/2)</td>
<td>1.6210</td>
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Kloss et al., Phys. Rev. E 86, 051124 (2012) and references therein
From DDGPE to KPZ

Density and phase decomposition $\phi(x, t) = \sqrt{n(x, t)} e^{-i\theta(x,t)}$

\[
\begin{align*}
\partial_t \theta - \frac{1}{2m} (\nabla \theta)^2 - \nu \nabla^2 \theta &= U, \\
\partial_t n - \frac{1}{m} \nabla \cdot (n \nabla \theta) &= S,
\end{align*}
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with sources of phase and density fluctuations,

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U &= U[\theta, n] + \frac{\text{Re}(\zeta e^{i\theta})}{\sqrt{n}} \\
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Altman et al., arXiv:1311.0876v2 [cond-mat.stat-mech]
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Comparing scaling exponents

\[
\epsilon_{\text{kin}} = \frac{1}{2m} \langle |\nabla \phi|^2 \rangle \\
= \frac{n}{2m} \int_k k^2 \langle |\theta(k, t)|^2 \rangle
\]

\[
\epsilon_{\text{kin}}(k) \sim k^{-d+\eta}
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\epsilon_{\text{kin}}(k) \sim k^{z-d-\chi}
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\[\eta = z - \chi\]
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GPE simulations

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KPZ literature

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\[ d = 2 \]

\[ \epsilon(k) = k^2 n(k) \]

\[ \nu = \mu_2 = 0 \]

\[ g_2 = \zeta = 0 \]

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Compressible excitations

\[ \epsilon_c = \frac{1}{2m} \langle |\nabla \phi|^2 \rangle \]

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Conclusions

- Interface dynamics described by KPZ equation does not capture vortex dynamics.
- It does captures the rest.
- We have made an estimation of anomalous scaling exponents of the ultra-cold Bose gas at a non-thermal fixed point.