Asymptotic Safety and Limit Cycles in minisuperspace quantum gravity

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Based on arXiv:1205.4218 (and forthcoming), in collaboration with Daniel Litim

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1. Renormalization group for minisuperspace cosmology

We consider Euclidean GR restricted to spatially flat FRW metrics

\[ ds^2 = a^2(t) \left[ dt^2 + dr^2 + r^2 d\Omega^2 \right] , \quad S = \int dt \frac{3\nu}{8\pi G} \left[ -a'(t)^2 + \frac{\Lambda}{3} a(t)^4 \right] \]

and attempt to study the quantum theory through the Exact Renormalization Group.

Motivation: simplest theory that can be called “gravity”!
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As starting point we take the Conformally Reduced Einstein-Hilbert theory (CREH):

\[ g_{\mu\nu} = \chi^2(x) \hat{g}_{\mu\nu} \]

\[ \Gamma_k[\bar{f}; \chi_B] = -\frac{3}{8\pi G_k} \int d^4 x \sqrt{\hat{g}} \left[ -\left(\chi_B + \bar{f}\right)\hat{\Box}\left(\chi_B + \bar{f}\right) + \frac{1}{6} \hat{R}\left(\chi_B + \bar{f}\right)^2 - \frac{1}{3} \Lambda_k \left(\chi_B + \bar{f}\right)^4 \right] \]
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In our case we want a flat reference metric.
Dimensional reduction of fluctuations

We follow the derivation of the beta functions for $G_k, \Lambda_k$ in the CREH theory, which are obtained by expanding the RG flow equation and matching terms

$$k \partial_k (G'_k)^{-1} = \int \frac{d^4 p}{(2\pi)^4} F(p^2, k), \text{ simil. for } \Lambda_k$$

Introduce a $\delta$-function to suppress fluctuations in 4-$n$ dimensions:

$$k \partial_k (G'_k)^{-1} = \int \frac{d^4 p}{(2\pi)^4} \delta^{(4-n)} \left( \frac{p_i}{a_B k} \right) F(p^2, k)$$

Technical note: $\eta_N^{(\text{kin})}$!
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leads to flow equations:

$$\dot{g}_k = (2 + \eta)g_k, \quad \eta = -\frac{2}{3\pi} \frac{g_k \lambda_k^2}{(1 - 2\lambda_k)^4}$$

$$\dot{\lambda}_k = (\eta - 2)\lambda_k + \frac{g_k}{4\pi} \left( 1 - \frac{\eta}{n+2} \right) \frac{1}{1 - 2\lambda_k}$$

(Up to an overall scaling of $g$).
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$n = 1$ → minisuperspace approximation

$n = 4$ → full CREH theory

(Up to an overall scaling of $g$).
2. RG flow, fixed points and limit cycle

In both cases:
- [A] UV-attractive non-Gaussian fixed point at positive \((\lambda, g)\).
- [B] Gaussian fixed point.
- [C] degenerate fixed point at \((\lambda = 1/2, g = 0)\).
- [D] IR attractor at \((\lambda \rightarrow -\infty, g = 0)\).

In minisuperspace case: also a limit cycle shielding the NGFP from the semiclassical regime. There are trajectories approaching the limit cycle from inside and outside, as well as others escaping towards \(\lambda \rightarrow -\infty\).
Characterizing the limit cycle

The limit cycle at $n = 1$ has period $T \approx 1.57$.

The cycle is not traversed uniformly. The flow makes a fast turn in the vicinity of the degenerate fixed point $C$.

We can also study the flow for continuous values of $n$. Increasing $n$, the period increases and is logarithmically divergent for $n \to n_{\text{crit}} \approx 1.4715$.

$$T_n = T_0 - b \ln \left(1 - \frac{n}{n_{\text{crit}}} \right)$$

$b \approx 0.57$. 
3. Degenerate limit cycle at critical point

At $n = n_{\text{crit}}$, the limit cycle collides with the fixed points B and C and becomes degenerate:

Flow for $n = n_{\text{crit}} \approx 1.4715$.

All trajectories flowing into the IR from the NGFP approach asymptotically the degenerate limit cycle. For $n > n_{\text{crit}}$, the limit cycle has vanished and the flow qualitatively resembles the full theory.
Implications for cosmological fine-tuning

For the Asymptotic Safety research program to deliver a viable cosmology, the physical values of $G_k$ and $\Lambda_k$ must be approximately constant (and small and positive) over the wide range of scales where they are measured.

The RG flow trajectory realized in Nature must spend a large amount of RG “time” in the vicinity of the Gaussian fixed point, at $\lambda_k \gtrsim 0$ $g_k \gtrsim 0$.

For the usual EH and CREH truncations (and the minisuperspace too) this is not possible without fine-tuning the initial conditions of the RG flow.

For the critical value $n = n_{\text{crit}}$, all trajectories leaving the NGFP towards the IR achieve an extended semiclassical regime.

This suggest a new possible way in which the issue of the fine-tuning of the initial conditions for the flow might resolve itself.
5. Flow for low $n$

For $n < 1$, the size of the limit cycle keeps decreasing as $\text{Re}(\theta^*)$ decreases and the NGFP becomes less strongly IR-repulsive.

At $n \approx -0.05$, the limit cycle shrinks to a point and vanishes. $\text{Re}(\theta^*)$ becomes negative, and the NGFP becomes IR-attractive.

$n$ does not have a physical interpretation in this regime.
Flow for large $n$

At $n^+ \approx 223$, there is another bifurcation in which the critical exponents become real.

Flow for $n \gg n^+$. 
6. Beyond conformal reduction

We can do a similar dimensional truncation of fluctuations on the traces that define the beta functions of the full Einstein-Hilbert theory.

In an approximation where the anomalous dimension is linear in $g$, we get:

\[
\partial_t g = (2 + \eta_N) g,
\]

\[
\eta_N = -\frac{g}{3(n-2)} \frac{96 - 46n + 12(5n - 8)\lambda + (96 - 80n)\lambda^2}{(1 - 2\lambda)^2},
\]

\[
\partial_t \lambda = (\eta_N - 2)\lambda - 8g + 10g \left(1 - \frac{\eta_N}{n+2}\right) \frac{1}{1 - 2\lambda}.
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Limit cycle still obtained, for $n < n_{\text{crit}} \approx 2.88$. This means the limit cycle is “close in theory space” to the full theory!

But equations are singular at $n = 2$. (Can we use $\eta^{(\text{kin})}_N$?)
7. Summary and outlook

- The minisuperspace reduction of Einstein-Hilbert gravity presents a renormalization group limit cycle, absent when spatial fluctuations are preserved.

- The period of the limit cycle diverges at a critical value of the tuning parameter $n$, above which the theory resembles CREH. The critical exponents are real at large $n$. For low $n$, the limit cycle vanishes in a Hopf bifurcation.

- The theory at the critical point allows for an extended semiclassical regime with a small positive $\Lambda$ with no need for fine-tuning the initial conditions.

- While this particular model with $n = n_{\text{crit}}$ is likely unphysical, it opens the door for a new way in which fine-tuning problems might resolve themselves in the Asymptotic Safety framework. The $n$-tweaked theory is very close in “theory space” to the Einstein-Hilbert theory, as a “quick and dirty” calculation confirms, and we may hope the degenerate limit cycle may be a feature of the full theory that is lost in the standard approximation and can be found again with the “dimensional tweaking”.

Thank You!